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A Lagrangian for a system of two dyons

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Maxwell's equations for the electromagnetic field are symmetrized by introducing magnetic charges into the formalism of electrodynamics. The symmetrized equations are solved for the fields and potentials of point particles. Those potentials, some of which are found to be singular along a line, are used to formulate the Lagrangian for a system of two dyons (particles with both electric and magnetic charge). The equations of motion are derived from

the Lagrangian. It is shown that the dimensionality constants k and k^* , which were introduced to define the units of the electromagnetic fields, have to be equal in order to avoid center of mass acceleration in the two dyon system.

A LAGRANGIAN FOR A SYSTEM OF TWO DYONS

by

RAINER GEORG THIERAUF

A thesis submitted in partial fulfillment of the
requirements for the degree of

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TO THE OFFICE OF GRADUATE STUDIES:

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CONVENTIONS AND SYMBOLS

(A) SYSTEM OF COORDINATES

Unless specifically stated otherwise, cartesian coordinates are used throughout this paper. \vec{i} , \vec{j} , \vec{k} stand for the unit vectors in x, y, z direction respectively. Vectors are printed in boldface. No Greek symbols are used to represent vectors.

(B) DIFFERENTIATIONS

In some areas we are making use of the following convention regarding the derivative with respect to a vector:

$$\frac{\partial S}{\partial \vec{v}} = \left(\frac{\partial S}{\partial v_x} \vec{i} + \frac{\partial S}{\partial v_y} \vec{j} + \frac{\partial S}{\partial v_z} \vec{k} \right)$$

where S is a scalar quantity. $\vec{v} = v_x \vec{i} + v_y \vec{j} + v_z \vec{k}$.

$$\frac{\partial \vec{V}}{\partial \vec{v}} = \left(\frac{\partial v_i}{\partial v_k} \right)_{i,k}$$

where $\vec{V} = V_x \vec{i} + V_y \vec{j} + V_z \vec{k}$. $i, k = x, y, z$.

(C) INTEGRATIONS

Unless specifically stated otherwise, integrations are carried out over all space. If the quantity under the

integral sign is a vector (or 4-vector) then each component of the vector (or 4-vector) is being integrated.

(D) SUMMATIONS

Einstein's summation convention is implied if two identical Greek indices appear in one term. Greek indices run from 0 to 3. Example:

$$a^\alpha b_\alpha = a^0 b_0 + a^1 b_1 + a^2 b_2 + a^3 b_3$$

(E) SYMBOLS

c = speed of light in vacuum

p^α = 4-momentum vector

u^α = 4-velocity vector

τ = proper time

CHAPTER I

INTRODUCTION

In 1931 Dirac [1] symmetrized Maxwell's equations by introducing magnetic charges into the formalism of electrodynamics. Although no magnetic charge has ever been found the experimental search for them is still continuing today [2].

In this paper we want to arrive at a non-relativistic Lagrangian for a system of two particles, each carrying electric as well as magnetic charge, so called dyons. In doing this we want to treat the electric and magnetic charges on an equal footing.

Any interaction between a charged particle and an electric or magnetic field enters the Lagrangian as a term proportional to a scalar or vector potential. The potentials for the fields of the electric and magnetic charges will be found by solving the symmetrized Maxwell equations, which describe a universe containing both types of charges. In chapter III we will symmetrize Maxwell's equations, the equations will be solved in chapter IV.

In the process of symmetrizing the Maxwell equations, we will notice that we are dealing with two separate electromagnetic fields rather than just one. Each

electromagnetic field is defined by its respective set of Maxwell equations. The sources of the first field are the electric charges and currents, the sources of the second field are the magnetic charges and currents. Since we are dealing with two fields we introduce two different dimensionality constants k and k^* . The relation between k and k^* will be investigated in chapter V. In chapter VI we will write down a Lagrangian and a Hamiltonian for a system of two dyons.

CHAPTER II

SOME FEATURES OF MAGNETIC MONOPOLES

(A) THE PROBLEM OF FINDING A VECTOR POTENTIAL FOR A MAGNETIC MONOPOLE

In discussing the mechanics and quantum mechanics of electric charges in the presence of magnetic monopoles we wish to change the existing Lagrangian and Hamiltonian formalisms as little as possible.

Any interaction between an electric charge and a field enters a Lagrangian or a Hamiltonian as a term proportional to a potential. If the charge interacts with an electric field we are able to use a scalar potential ϕ where $\vec{E} = -\text{grad } \phi$. If the charge interacts with a magnetic field \vec{B} we have to use a vector potential \vec{A} where $\vec{B} = \text{curl } \vec{A}$. For the case of a \vec{B} -field created by an electric current a regular vector potential can be found because for such fields $\text{div } \vec{B} = 0$ holds. This statement says that the net magnetic flux coming into and out of a closed Gaussian surface anywhere in space is zero. Now suppose there is a magnetic monopole, which may be the source or sink of a radial magnetic field, inside a Gaussian surface then there is a non-zero net flux through that surface. The existence of a magnetic monopole therefore contradicts $\text{div } \vec{B} = 0$. The

statement has to be replaced by say $\text{div } \vec{B} = k$. The fact that $\text{div } \vec{B} = 0$ held, however, had enabled us to find the vector potential \vec{A} for the \vec{B} -field. In other words the following two statements are equivalent:

$$(i) \text{div } \vec{B} = 0$$

(ii) There is at least one vector potential \vec{A} such that $\vec{B} = \text{curl } \vec{A}$

Since the first statement of the equivalence no longer holds the second one does not hold either. This means there can't be an \vec{A} such that $\vec{B} = \text{curl } \vec{A}$. However, there are ways to circumvent that difficulty.

Space around a magnetic monopole does not have singularities. The vector potential necessary to describe the B-field of a magnetic monopole, however, does.

This situation is similar to that encountered in the choice of a coordinate system for the surface of a sphere. A sphere clearly possesses no intrinsic singularities, yet it is not possible to find a system of coordinates, which describes the geometry of a sphere without singularities. Therefore one divides the sphere into two or more overlapping regions, which can each be described by singularity-free coordinates.

This fact inspired Wu and Yang [3,4] to their approach of the vector potential for a magnetic monopole. Space

outside the magnetic monopole will be divided into two overlapping regions R_a and R_b . The overlap is called R_{ab} . Using spherical coordinates the regions are defined as follow:

$$\begin{aligned}
 R_a: & \quad 0 \leq \theta < \frac{1}{2}\pi + \delta, \quad 0 < r, \quad 0 \leq \phi < 2\pi \\
 R_b: & \quad \frac{1}{2}\pi - \delta < \theta \leq \pi, \quad 0 < r, \quad 0 \leq \phi < 2\pi \\
 R_{ab}: & \quad \frac{1}{2}\pi - \delta < \theta < \frac{1}{2}\pi + \delta, \quad 0 < r, \quad 0 \leq \phi < 2\pi
 \end{aligned}$$

where $0 < \delta \leq \frac{1}{2}\pi$.

An example for vector potentials in those regions R_a and R_b are:

$$\begin{aligned}
 (A_r)_a = (A_\theta)_a = 0, \quad (A_\phi)_a &= \frac{g}{r \sin\theta} (1 - \cos\theta) \\
 (A_r)_b = (A_\theta)_b = 0, \quad (A_\phi)_b &= \frac{-g}{r \sin\theta} (1 + \cos\theta)
 \end{aligned}$$

Each of those two vector potentials is singularity-free in the region in which it is defined. In regions R_a and R_b we can write $\vec{B} = \text{curl } \vec{A}$ (consider Figure 1).

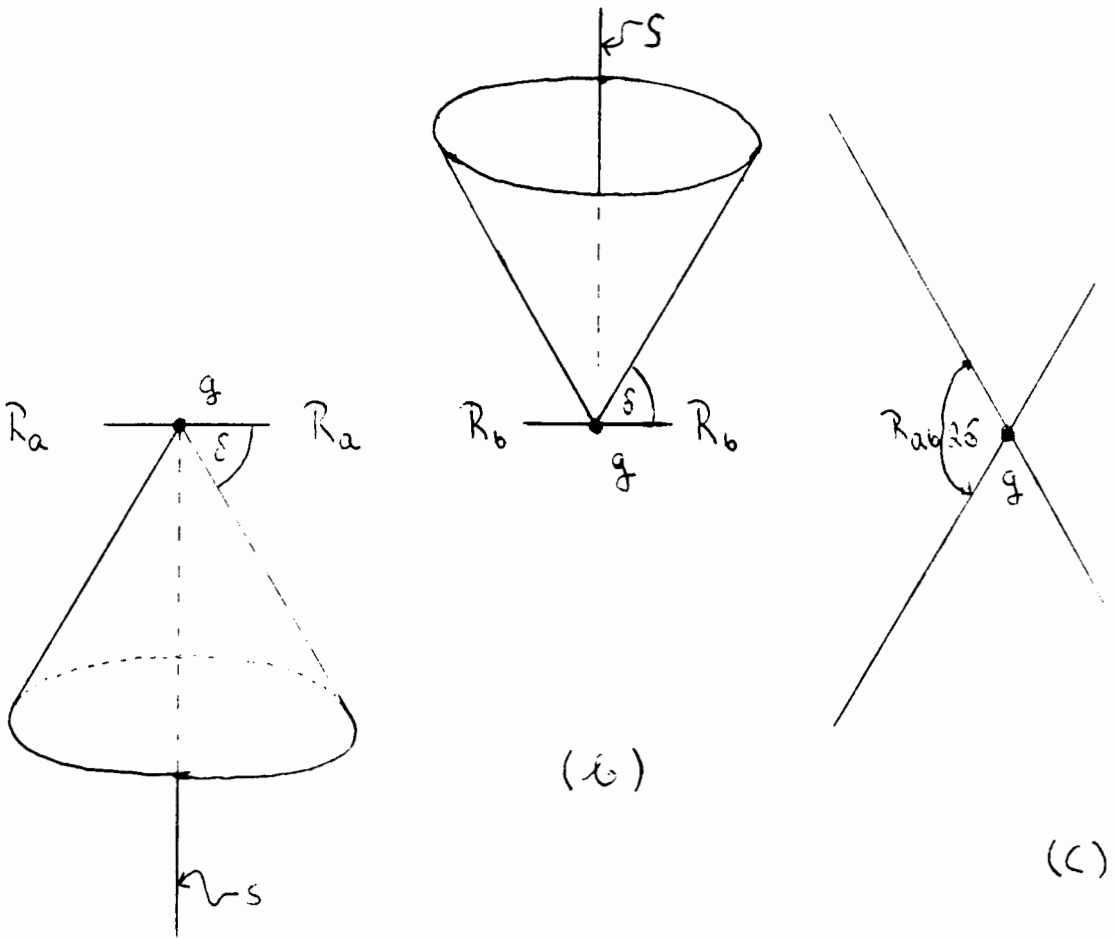


Figure 1. Area of definition of the vector potential. (a) R_a above cone. (b) R_b below cone. (c) shows area of the overlap.

As we will see in section (C) of this chapter the vector potentials in the overlap R_{ab} are related by a gauge transformation:

$$(\vec{A})_a - (\vec{A})_b = \vec{v} \text{ in } R_{ab}$$

where \vec{v} can be expressed as the gradient of a gauge function X :

$$\vec{v} = \text{grad } X$$

Another method of avoiding the singularities, which are sometimes called strings, was proposed by Edward. H. Kerner [5]. He constructed a Lagrangian into which the electric and magnetic fields enter directly rather than in form of potentials. Of course this eliminates all difficulties with the string since the string shows up only in the potential, not in the field of a magnetic monopole.

In this approach, however, the positions and velocities of the particles can not be used as canonical coordinates.

(B) SYMMETRY OF THE FIELD EQUATIONS

In the previous section we discussed the problems which the introduction of a magnetic charge into the Lagrangian formalism would cause. In this section and the following two sections we want to take a look at three theoretical arguments in favor of the existence of magnetic monopoles.

At first glance the existence of magnetic charges symmetrizes the Maxwell field equations and since physicists always look for symmetries in equations this seems to be an argument in favor of the existence of magnetic monopoles. The symmetrized equations are:

$$\begin{aligned} \operatorname{div} \vec{E} &= k \rho_e & \operatorname{div} \vec{B} &= k^* \rho_m \\ \operatorname{curl} \vec{B} - \frac{1}{c} \frac{\partial \vec{E}}{\partial t} &= k \vec{J}_e & \operatorname{curl} \vec{E} + \frac{1}{c} \frac{\partial \vec{B}}{\partial t} &= -k^* \vec{J}_m \end{aligned}$$

where k and k^* are the dimensionality constants mentioned in section (a). In chapter (II) we will show how one can get these symmetrized equations.

These equations, however, are invariant under the following transformation:

$$\begin{aligned} \vec{e} &= \vec{E} \cos \epsilon + \vec{B} \sin \epsilon & ; & \quad \vec{b} = -\vec{E} \sin \epsilon + \vec{B} \cos \epsilon \\ \sigma_e &= \rho_e \cos \epsilon + \rho_m \sin \epsilon & ; & \quad \sigma_m = \rho_e \sin \epsilon + \rho_m \cos \epsilon \\ \vec{j}_e &= \vec{J}_e \cos \epsilon + \vec{J}_m \sin \epsilon & ; & \quad \vec{j}_m = \vec{J}_e \sin \epsilon + \vec{J}_m \cos \epsilon \end{aligned}$$

and if the ratio of magnetic to electric charge is a constant for all matter then we can always find an angle ϵ such that σ_m and \vec{j}_m are equal to zero. In other words we can find an angle ϵ such that the equations in \vec{e} , \vec{b} have

the following form:

$$\begin{aligned} \operatorname{div} \vec{e} &= k c \sigma_e & \operatorname{div} \vec{b} &= 0 \\ \operatorname{curl} \vec{b} - \frac{1}{c} \frac{\partial \vec{e}}{\partial t} &= k \vec{j}_e & \operatorname{curl} \vec{e} + \frac{1}{c} \frac{\partial \vec{b}}{\partial t} &= 0 \end{aligned}$$

These are the equations to which we are used to from regular electrodynamics without magnetic charges and they describe the same universe as the symmetric Maxwell equations above.

This means that we could have symmetric Maxwell equations if we wanted to. The asymmetry displayed in the above equations is a consequence of our choice of the angle ξ , which essentially determines the ratio of magnetic to electric charges. In regular electrodynamics we choose ξ such that this ratio is equal to zero meaning that no particle is carrying magnetic charges.

So what we mean when we are talking about a magnetic monopole is a particle whose ratio of magnetic to electric charge is different from that of other particles.

(C) CHARGE QUANTIZATION

In this section we will show how the existence of a magnetic monopole forces the quantization of electric (and magnetic charges). In order to do this we will calculate a vector potential for a magnetic monopole. The difficulties

for that, which were mentioned in section (A) of this chapter will be circumvented in the following way:

If we can find real objects which exhibit a magnetic field identical or almost identical to the one of a magnetic monopole then we can calculate their corresponding vector potentials and use those to approximate the vector potential of a magnetic monopole.

Examples for such "real" objects are a semi infinite solenoid and a semi infinite chain of magnetic dipoles.

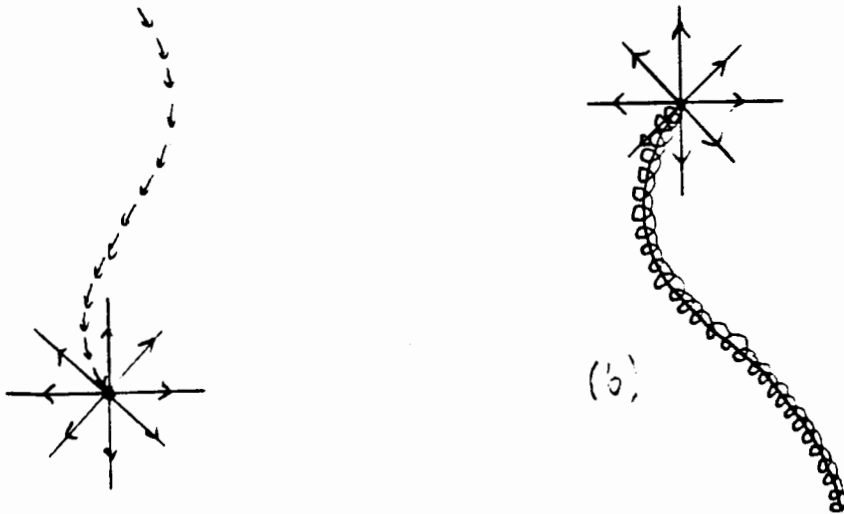


Figure 2. Representation of a magnetic monopole as a semi infinite chain of dipoles (b) or a semi infinite solenoid (a).

We will now calculate the vector potential for a string of dipoles. The vector potential for a single dipole is given by:

$$\vec{A}(\vec{r}-\vec{r}') = \frac{\vec{m} \times (\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$

To calculate the vector potential of the string we sum up over differential potentials produced by differential dipole moments:

$$\vec{A}(\vec{r}-\vec{r}') = g \int_S d\vec{l}' \times \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}$$

The integration is carried out along S.

If we calculate the vector potential for the solenoid we start with the expression:

$$d\vec{B} = \frac{I}{c} d\vec{l}' \times \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$

Integration over S and substitution of $g = (I/c)$ yields the same expression as above.

If we choose the string to be along the negative z-axis (the question of the arbitrariness of the string's position will be discussed later) then a solution for this integral in spherical coordinates is:

$$A_r = 0 ; A_\theta = 0 ; A_\phi = \frac{g(1-\cos\theta)}{r \sin\theta}$$

This vector potential yields a B-field proportional to $(r/|r|^3)$ except along the string S, where the vector potential is singular. This corresponds to a very intense field B' inside the solenoid or the chain of dipoles, which brings a magnetic flux of $4\pi g$ into any closed surface

around the end of the string. The incoming flux cancels the total outgoing flux ($\text{div } \vec{B} = 0$ holds, as it has to since we are talking about a real object).

For the purposes of classical mechanics this approximation of the vector potential for a magnetic monopole is good enough. Since the location of the string is arbitrary we can always argue that none of the interacting particles actually feels the very intense field B' , which a real magnetic monopole does not have. In quantum mechanics, however, particles are described by wave functions, which can extend over all space, thus the string would overlap with parts of the wave function and therefore be felt by the particle. Dirac thus required that the wave function of the electron vanish along the string.

Let us now look at the problem of the arbitrariness of the position of the string and the consequences resulting from it. Consider the following picture:

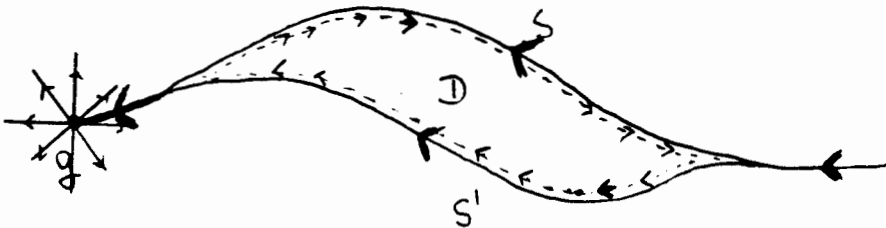


Figure 3. Position of the string.
Note the circular current in D.

The vector potential for the string S' can be expressed as the sum of the vector potentials for the string S and a magnetic dipole D , whose vector potential is given by:

$$\vec{A}_D(\vec{r}-\vec{r}') = \frac{\vec{m} \times (\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$

which can be written as:

$$\vec{A}_D(\vec{r}-\vec{r}') = g \cdot \text{grad} \Omega$$

where Ω is the solid angle for the loop subtended at the point P . Therefore \vec{A}_S' can be written as:

$$\vec{A}_S'(\vec{r}-\vec{r}') = \vec{A}_S(\vec{r}-\vec{r}') + g \cdot \text{grad} \Omega$$

From this we can see that the choice of a different string position is merely a gauge transformation. $\text{curl} \vec{A}_S' = \text{curl} \vec{A}_S = \vec{B}$. The position of the string has no influence on the magnetic field produced by the curl of the vector potential.

In quantum mechanics a gauge transformation leaves the Schrodinger equation invariant provided that the wave function is transformed in the following manner:

$$\Psi \rightarrow \Psi' = \Psi \exp\left[\frac{ie\chi}{\hbar c}\right]$$

where e is the charge of the particle, χ the gauge function (in our case $\chi=g\Omega$), \hbar is Planck's constant divided by 2π .

A change in the position of the string must be accompanied by a change in the (arbitrary) phase factor of the wave function:

$$\Psi \rightarrow \Psi' = \Psi \exp\left[\frac{ieq\Omega}{\hbar c}\right]$$

As the electron crosses the surface of $D \Omega$ suddenly changes by 4π . Therefore we have to require that the phase factor satisfies:

$$\frac{eq}{\hbar c} 4\pi = 2\pi n \quad n \text{ is an integer}$$

or the wave function would be multiple valued. Thus the requirement of gauge invariance of the Schroedinger equation leads us directly to Dirac's quantization condition.

It is this beautiful theoretical argument by Dirac which would solve one of the great mysteries of physics (charge quantization) that kept the search for magnetic monopoles alive in spite of the lack of any experimental evidence for the existence of magnetic charges.

(D) GUT MONOPOLE

According to J. M. Pasachoff magnetic monopoles are a common feature of all so called Grand Unified Theories (GUT), which are attempts to unify the four forces known in physics.

CHAPTER III

SYMMETRIZATION OF THE FIELD EQUATIONS

In this chapter we will show how one can obtain a set of Maxwell equations for electrodynamics including electric as well as magnetic charge. This will be done applying a special case of the duality transformation to the field equations of regular electrodynamics. The resulting equations describe the same universe as the original ones did, only the names of the magnetic and electric fields and charges have been exchanged. We, however, will treat the resulting field equations as describing a second electromagnetic field in the same universe with magnetic charges and currents as its sources. We will superimpose both fields and obtain a set of Maxwell equations which describes electrodynamics with electric and magnetic charges.

Since we will be dealing with two separate electromagnetic fields, each of which is defined by its set of Maxwell equations, we will introduce two proportionality constants k and k^* , which define their respective fields. In the literature available on this subject it is always automatically assumed that k must equal k^* . In chapter IV we will show that under the assumption that the ratio of

electric to magnetic charge is a constant for all matter, the field equations of regular electrodynamics without magnetic charges can be recovered through a duality transformation. No assumption for the relation between k and k^* will have to be made in order to achieve this.

We will, however, show that in order to regain the equation of motion for a charged particle in an electromagnetic field after the duality transformation one has to require that $k=k^*$. This will be done in chapter IV.

The Maxwell Equations for the electromagnetic field without magnetic charge are:

$$\operatorname{div} \vec{E}_e = kc\rho_e \quad ; \quad \operatorname{div} \vec{B}_e = 0 \quad (3.1,2)$$

$$\operatorname{curl} \vec{B}_e - \frac{1}{c} \frac{\partial \vec{E}_e}{\partial t} = k\vec{J}_e \quad ; \quad \operatorname{curl} \vec{E}_e + \frac{1}{c} \frac{\partial \vec{B}_e}{\partial t} = 0 \quad (3.3,4)$$

The subscript "e" is used to indicate that the sources of this electromagnetic field are electric charges and currents of electric charges. k is a proportionality constant which defines the units in which the electromagnetic field is measured. For example in Gaussian units k would be $4\pi/c$.

We will now perform a duality transformation (see previous chapter) on those equations with angle $\xi=\pi$. The resulting equations describe a universe with magnetic charges and without electric charges. As was shown for the

general case of a duality transformation this is merely a renaming of quantities and does not change the physical properties of our system. Since the quantities in the system after the transformation are due to magnetic charges they will carry a subscript "m":

$$\vec{E}_e \rightarrow \vec{B}_m ; \quad \rho_e \rightarrow \rho_m \quad (3.5)$$

$$k \rightarrow k^* ; \quad \vec{B}_e \rightarrow -\vec{E}_m ; \quad \vec{J}_e \rightarrow \vec{J}_m$$

The transition from k to k^* is not part of the duality rotation. It is done here because we will treat the resulting equations as describing a different electromagnetic field than the original equations.

With those transformations the above equations take the form:

$$\text{div} \vec{E}_m = 0 ; \quad \text{div} \vec{B}_m = k^* c \rho_m \quad (3.6,7)$$

$$\text{curl} \vec{B}_m - \frac{1}{c} \frac{\partial \vec{E}_m}{\partial t} = 0 ; \quad \text{curl} \vec{E}_m + \frac{1}{c} \frac{\partial \vec{B}_m}{\partial t} = -k^* \vec{J}_m \quad (3.8,9)$$

This set of Maxwell equations describes an electromagnetic field which is created by magnetic charges and currents of magnetic charges. Now we superimpose both fields by adding the following equations: (3.1)+(3.6); (3.2)+(3.7); (3.3)+(3.8); (3.4)+(3.9) and we get:

$$\operatorname{div}(\vec{E}_e + \vec{E}_m) = k c \rho_e \quad ; \quad \operatorname{div}(\vec{B}_e + \vec{B}_m) = k^* c \rho_m \quad (3.10, 11)$$

$$\operatorname{curl}(\vec{B}_e + \vec{B}_m) - \frac{1}{c} \frac{\partial}{\partial t} (\vec{E}_e + \vec{E}_m) = k \vec{J}_e \quad (3.12)$$

$$\operatorname{curl}(\vec{E}_e + \vec{E}_m) + \frac{1}{c} \frac{\partial}{\partial t} (\vec{B}_e + \vec{B}_m) = - k^* \vec{J}_m \quad (3.13)$$

We define the total electric and magnetic fields:

$$\vec{E} = \vec{E}_e + \vec{E}_m \quad ; \quad \vec{B} = \vec{B}_e + \vec{B}_m \quad (3.14)$$

and we get the Maxwell equations for a universe with magnetic and electric charges:

$$\operatorname{div} \vec{E} = k c \rho_e \quad ; \quad \operatorname{div} \vec{B} = k^* c \rho_m \quad (3.16, 17)$$

$$\operatorname{curl} \vec{B} - \frac{1}{c} \frac{\partial \vec{E}}{\partial t} = k \vec{J}_e \quad ; \quad \operatorname{curl} \vec{E} + \frac{1}{c} \frac{\partial \vec{B}}{\partial t} = - k^* \vec{J}_m \quad (3.18, 19)$$

This derivation shows that we have essentially two separate electromagnetic fields. The sources of one of them are the electric charges and currents, the sources of the second electromagnetic field are the magnetic charges and currents. Each of those fields obeys a separate set of four Maxwell equations - two homogeneous and two inhomogeneous equations. This special structure makes it possible to add (superimpose) both electromagnetic fields and derive a set of Maxwell equations for a system with both electric and magnetic charges as sources for the electromagnetic field.

CHAPTER IV

SOLUTIONS TO THE FIELD EQUATIONS

In this chapter we will only seek solutions for the three sets of field equations, which describe the fields of point particles. We will start by solving the field equations which describe regular electrodynamics without magnetic charges, hereinafter referred to as the "first set of equations". Of course this is done in almost any textbook (Jackson [6]; Landau [7]) on electrodynamics. We will repeat the procedure here for two reasons. First to emphasize that the set of equations, which describes electrodynamics with magnetic charges and without electric charges (from now on the "second set of equations"), will be solved in the same way. Second because the solutions in the two separate cases will guide our approach for the solution of equations (3.16-19), which will be called the "combined set of equations" from now on.

(A) SOLUTIONS TO THE FIRST SET OF EQUATIONS

$$\operatorname{div} \vec{E}_e = k \rho_e \quad ; \quad \operatorname{div} \vec{B}_e = 0 \quad (3.1,2)$$

$$\operatorname{curl} \vec{B}_e - \frac{1}{c} \frac{\partial \vec{E}_e}{\partial t} = k \vec{J}_e \quad ; \quad \operatorname{curl} \vec{E}_e - \frac{1}{c} \frac{\partial \vec{B}_e}{\partial t} = 0 \quad (3.3,4)$$

Equation (3.2) guarantees that there is an \vec{A}_e such that $\vec{B}_e = \text{curl} \vec{A}_e$. We insert this into equation (3.3):

$$\text{curl} \left\{ \vec{E}_e + \frac{1}{c} \frac{\partial \vec{A}_e}{\partial t} \right\} = 0 \quad (4.1)$$

This means that there is a ϕ_e such that:

$$\vec{E}_e + \frac{1}{c} \frac{\partial \vec{A}_e}{\partial t} = - \text{grad} \phi_e$$

or:

$$\vec{E}_e = - \text{grad} \phi_e - \frac{1}{c} \frac{\partial \vec{A}_e}{\partial t} \quad (4.3)$$

We insert (4.3) into (3.1):

$$\text{div} \text{grad} \phi_e + \frac{1}{c} \frac{\partial}{\partial t} \text{div} \vec{A}_e = - k c \rho_e \quad (4.4)$$

We now insert $\vec{B}_e = \text{curl} \vec{A}_e$ and (4.3) into equation (3.4):

$$\text{curl} \{ \text{curl} \vec{A}_e \} = k \vec{J}_e + \frac{1}{c} \frac{\partial}{\partial t} \left(- \text{grad} \phi - \frac{1}{c} \frac{\partial \vec{A}_e}{\partial t} \right)$$

or:

$$\text{div} \text{grad} \vec{A}_e - \frac{1}{c^2} \frac{\partial^2 \vec{A}_e}{\partial t^2} - \text{grad} \left(\text{div} \vec{A}_e + \frac{1}{c} \frac{\partial \phi_e}{\partial t} \right) = k \vec{J}_e \quad (4.5)$$

We choose the gauge such that:

$$\text{div} \vec{\Lambda}_e + c^{-1} (\partial \phi_e / \partial t) = 0 \quad (4.6)$$

We also solve this gauge relation for $\text{div} \vec{\Lambda}_e$ and insert this into (4.4) and we get a set of equations in terms of the potentials ϕ_e and $\vec{\Lambda}_e$, which is equivalent to the Maxwell equations (3.1-3.4):

$$\text{div grad } \phi_e - \frac{1}{c^2} \frac{\partial^2 \phi_e}{\partial t^2} = -kc\rho_e \quad (4.7)$$

$$\text{div grad } \vec{\Lambda}_e - \frac{1}{c^2} \frac{\partial^2 \vec{\Lambda}_e}{\partial t^2} = -k\vec{J}_e \quad (4.8)$$

As mentioned above we are only seeking particle solutions for these equations. The partial derivatives with respect to time are therefore set to zero, the charge density and current density take the form:

$$\rho_e = e\delta(\vec{r}-\vec{r}_0) \quad ; \quad \vec{J}_e = ev\delta(\vec{r}-\vec{r}_0) \quad (4.9)$$

where r_0 is the position of the particle.

The solutions to the equations above with those specifications and their corresponding fields are:

$$\phi_e = \frac{kc}{4\pi} \int \frac{e\delta(\vec{r}-\vec{r}_0)}{|\vec{r}-\vec{r}_0|} d^3x' = \frac{kc}{4\pi} \frac{e}{|\vec{r}-\vec{r}_0|} \quad (4.10)$$

$$\vec{E}_e = -\text{grad } \phi_e = \frac{kc}{4\pi} \frac{e(\vec{r}-\vec{r}_0)}{|\vec{r}-\vec{r}_0|^3}$$

$$\vec{\lambda}_e = \frac{k}{4\pi} \int \frac{e\vec{v}\delta(\vec{r}-\vec{r}_0)}{|\vec{r}-\vec{r}_0|} d^3x' = \frac{k}{4\pi} \frac{e\vec{v}}{|\vec{r}-\vec{r}_0|} \quad (4.10)$$

$$\vec{B}_e = \text{curl}\vec{\lambda}_e = \frac{ke}{4\pi} \frac{\vec{v} \times (\vec{r}-\vec{r}_0)}{|\vec{r}-\vec{r}_0|^3}$$

(B) SOLUTIONS TO THE SECOND SET OF EQUATIONS

Now we will solve the second set of equations, which describes a system with magnetic charges only. This will be done in the same way as above. The field equations are:

$$\text{div}\vec{B}_m = k^* c \rho_m ; \quad \text{div}\vec{E}_m = 0 \quad (3.6,7)$$

$$\text{curl}\vec{E}_m + \frac{1}{c} \frac{\partial \vec{B}_m}{\partial t} = k^* c \vec{j}_m ; \quad \text{curl}\vec{B}_m - \frac{1}{c} \frac{\partial \vec{E}_m}{\partial t} = 0 \quad (3.8,9)$$

Equation (3.7) guarantees that there is an $\vec{\lambda}_m$ such that $\vec{B}_m = \text{curl}\vec{\lambda}_m$. We insert this into equation (3.8):

$$\text{curl} \left\{ \vec{B}_m - \frac{1}{c} \frac{\partial \vec{\lambda}_m}{\partial t} \right\} = 0 \quad (4.11)$$

This means that there is a ϕ_m such that:

$$\vec{B}_m - \frac{1}{c} \frac{\partial \vec{\lambda}_m}{\partial t} = - \text{grad}\phi_m$$

or:

$$\vec{B}_m = - \text{grad}\phi_m + \frac{1}{c} \frac{\partial \vec{\lambda}_m}{\partial t} \quad (4.12)$$

We insert this into equation (3.6):

$$-\text{div grad } \phi_m + \frac{1}{c} \frac{\partial}{\partial t} \text{div } \vec{\Lambda}_m = 0$$

We now insert $\vec{E}_m = \text{curl } \vec{\Lambda}_m$ and (4.12) into equation

(3.9):

$$\text{curl}\{\text{curl } \vec{\Lambda}_m\} = -k^* \vec{J}_m + \frac{1}{c} \frac{\partial}{\partial t} \left[-\text{grad } \phi_m + \frac{1}{c} \frac{\partial \vec{\Lambda}_m}{\partial t} \right]$$

or:

$$-\text{div grad } \vec{\Lambda}_m + \frac{1}{c} \frac{\partial^2 \vec{\Lambda}_m}{\partial t^2} = -k^* \vec{J}_m - \text{grad} \left(\text{div } \vec{\Lambda}_m - \frac{1}{c} \frac{\partial \phi_m}{\partial t} \right) \quad (4.14)$$

We choose the gauge such that:

$$\text{div } \vec{\Lambda}_m - c^{-1} (\partial \phi_m / \partial t) = 0 \quad (4.15)$$

We also solve the gauge relation for $\text{div } \vec{\Lambda}_m$ and insert this into (4.13). We get a set of equations in the potentials $\phi_m, \vec{\Lambda}_m$ which are equivalent to the second set of Maxwell equations (3.6-3.9).

$$\text{div grad } \phi_m - \frac{1}{c^2} \frac{\partial^2 \phi_m}{\partial t^2} = -k^* c \rho_m \quad (4.16)$$

$$\text{div grad } \vec{\Lambda}_m - \frac{1}{c^2} \frac{\partial^2 \vec{\Lambda}_m}{\partial t^2} = k^* \vec{J}_m \quad (3.17)$$

Again, we are only seeking solutions which describe point particles. The time derivatives vanish, and the

charge and current densities take the form:

$$\rho_m = q\delta(\vec{r}-\vec{r}_0) \quad ; \quad \vec{J}_m = q\vec{v}\delta(\vec{r}-\vec{r}_0) \quad (4.18)$$

Then solutions to the equations (4.16-4.17) and their respective fields are:

$$\begin{aligned} \phi_m &= \frac{k^*c}{4\pi} \int \frac{q\delta(\vec{r}-\vec{r}_0)}{|\vec{r}-\vec{r}'|} d^3x' = \frac{k^*c}{4\pi} \frac{q}{|\vec{r}-\vec{r}_0|} \\ \vec{E}_m &= -\text{grad } \phi_m = \frac{k^*c}{4\pi} \frac{q(\vec{r}-\vec{r}_0)}{|\vec{r}-\vec{r}_0|^3} \\ \vec{A}_m &= \frac{k^*}{4\pi} \int \frac{qv\delta(\vec{r}-\vec{r}_0)}{|\vec{r}-\vec{r}'|} d^3x' = -\frac{k^*}{4\pi} \frac{qv}{|\vec{r}-\vec{r}_0|} \\ \vec{B}_m &= \text{curl } \vec{A}_m = -\frac{k^*q}{4\pi} \frac{\vec{v} \times (\vec{r}-\vec{r}_0)}{|\vec{r}-\vec{r}_0|^3} \end{aligned} \quad (4.19)$$

(C) SOLUTIONS TO THE COMBINED SET OF EQUATIONS

Now we will solve the "combined" field equations, which describe electrodynamics with electric and magnetic charges. The equations are:

$$\text{div } \vec{E} = kc\rho_e \quad ; \quad \text{div } \vec{B} = k^*c\rho_m \quad (3.16, 17)$$

$$\text{curl } \vec{B} - \frac{1}{c} \frac{\partial \vec{E}}{\partial t} = k\vec{J}_e \quad ; \quad \text{curl } \vec{E} + \frac{1}{c} \frac{\partial \vec{B}}{\partial t} = -k^*\vec{J}_m \quad (3.18, 19)$$

$$\text{Where } \vec{B} = \vec{B}_e + \vec{B}_m \quad ; \quad \vec{E} = \vec{E}_e + \vec{E}_m \quad (3.14)$$

Guided by our two previous ways of solving the separate field equations we will now make the following ansatz:

$$\vec{E} = - \text{grad}\phi_e - \frac{1}{c} \frac{\partial \vec{\Lambda}_e}{\partial t} + \text{curl}\vec{\Lambda}_m \quad (4.20)$$

$$\vec{B} = - \text{grad}\phi_m + \frac{1}{c} \frac{\partial \vec{\Lambda}_m}{\partial t} + \text{curl}\vec{\Lambda}_e \quad (4.21)$$

We insert (4.20) into (3.16) and get:

$$- \text{div grad}\phi_e - \frac{1}{c} \frac{\partial}{\partial t} \text{div}\vec{\Lambda}_e + \text{div}(\text{curl}\vec{\Lambda}_m) = k c \rho_e$$

It is: $\text{div}(\text{curl}\vec{\Lambda}_m) = 0$, and we use the gauge relation (4.6) to replace $\text{div}\vec{\Lambda}_e$:

$$\text{div grad}\phi_e - \frac{1}{c^2} \frac{\partial^2 \phi_e}{\partial t^2} = - k c \rho_e \quad (4.22)$$

We notice that we recovered equation (4.7) in this approach.

We now insert (4.21) into (3.17):

$$- \text{div grad}\phi_m + \frac{1}{c} \frac{\partial}{\partial t} \text{div}\vec{\Lambda}_m + \text{div}(\text{curl}\vec{\Lambda}_e) = k^* c \rho_m$$

$\text{div}(\text{curl}\vec{\Lambda}_e) = 0$, and we use the gauge relation (4.15):

$$\text{div grad}\phi_m - \frac{1}{c^2} \frac{\partial^2 \phi_m}{\partial t^2} = - k^* c \rho_m \quad (4.23)$$

and we recover equation (4.16).

Now we insert (4.20) and (4.21) into (3.19) and we get:

$$\begin{aligned} \text{curl} \left\{ -\text{grad} \phi_e - \frac{1}{c} \frac{\partial \vec{\Lambda}_e}{\partial t} + \text{curl} \vec{\Lambda}_m \right\} \\ + \frac{1}{c} \frac{\partial}{\partial t} \left\{ -\text{grad} \phi_m + \frac{1}{c} \frac{\partial \vec{\Lambda}_m}{\partial t} + \text{curl} \vec{\Lambda}_e \right\} = -k^* \vec{J}_m \end{aligned}$$

It is: $\text{curl}(\text{grad} \phi_e) = 0$; the terms proportional to $\vec{\Lambda}_e$ cancel:

$$\begin{aligned} \text{curl}(\text{curl} \vec{\Lambda}_m) - \frac{1}{c} \frac{\partial}{\partial t} \text{grad} \phi_m + \frac{1}{c^2} \frac{\partial^2 \vec{\Lambda}_m}{\partial t^2} = -k^* \vec{J}_m \\ - \text{div} \text{grad} \vec{\Lambda}_m + \text{grad} \left\{ \text{div} \vec{\Lambda}_m - \frac{1}{c} \frac{\partial \phi_m}{\partial t} \right\} + \frac{1}{c^2} \frac{\partial^2 \vec{\Lambda}_m}{\partial t^2} = -k^* \vec{J}_m \end{aligned}$$

We use the gauge relation (4.15) and get:

$$\text{div} \text{grad} \vec{\Lambda}_m - \frac{1}{c^2} \frac{\partial^2 \vec{\Lambda}_m}{\partial t^2} = -k^* \vec{J}_m \quad (4.24)$$

which is equation (4.17) again.

And finally we insert (4.20) and (4.21) into (3.18):

$$\begin{aligned} \text{curl} \left\{ -\text{grad} \phi_m + \frac{1}{c} \frac{\partial \vec{\Lambda}_m}{\partial t} + \text{curl} \vec{\Lambda}_e \right\} \\ - \frac{1}{c} \frac{\partial}{\partial t} \left\{ -\text{grad} \phi_e - \frac{1}{c} \frac{\partial \vec{\Lambda}_e}{\partial t} + \text{curl} \vec{\Lambda}_m \right\} = k^* \vec{J}_e \end{aligned}$$

as on

$\text{curl}(\text{grad} \phi_m) = 0$; the terms proportional to $\vec{\Lambda}_e$ cancel.

$$\begin{aligned} \text{curl}\{\text{curl}\vec{\Lambda}_e\} + \frac{1}{c} \frac{\partial}{\partial t} \text{grad}\phi_e + \frac{1}{c^2} \frac{\partial^2 \vec{\Lambda}_e}{\partial t^2} &= k\vec{J}_e \\ - \text{div grad}\vec{\Lambda}_e + \text{grad} \left\{ \text{div}\vec{\Lambda}_e + \frac{1}{c} \frac{\partial \phi_e}{\partial t} \right\} + \frac{1}{c^2} \frac{\partial^2 \vec{\Lambda}_e}{\partial t^2} &= -k\vec{J}_e \end{aligned}$$

Using the gauge relation (4.6) we get:

$$\text{div grad}\vec{\Lambda}_e - \frac{1}{c^2} \frac{\partial^2 \vec{\Lambda}_e}{\partial t^2} = -k\vec{J}_e \quad (4.25)$$

This section shows that the equations for the potentials for the separate set of field equations are still valid in this combined approach. Therefore their solutions (4.10,19) must also be valid in the combined approach.

(A) ADDITIONAL SOLUTIONS TO THE COMBINED SET OF EQUATIONS

The equations for the vector potential (4.24,25) with the specifications (4.9,18) have additional solutions which are proportional to:

$$\vec{D}(\vec{r}) = \frac{(\vec{r} \times \vec{n}) \vec{r} \cdot \vec{n}}{r(r^2 - (\vec{r} \cdot \vec{n})^2)} \quad (4.26)$$

where $r=|\vec{r}|$; \vec{n} is an arbitrary unit vector, in this paper usually chosen to be \vec{k} .

$\vec{A} = (k^* \text{cg}/4\pi) \vec{D}(\vec{r})$ describes the magnetic field of a magnetic monopole and is a "monopole"-solution to equation (4.24). Similarly equation (4.25) has a monopole solution

of the form $\vec{A} = (kce/4\pi)\vec{D}(\vec{r})$ which in this case describes the electric field of an electron. In the combined approach both monopole solutions satisfy their respective Maxwell equations (4.17,16) identically.

CHAPTER V

THE RELATION BETWEEN k AND k^*

(A) SYMMETRIZATION OF THE FIELD EQUATIONS USING THE LANGUAGE OF SPECIAL RELATIVITY

We will now repeat the procedure of chapter III using the language of special relativity. Using the potentials ϕ_e, ϕ_m, A_e, A_m for the fields can write down two four-vector potentials for the two electromagnetic fields:

$$A^\mu = (A^0, A^1, A^2, A^3) = (\phi_e, \vec{A}_e) \tag{5.1}$$

$$Z^\mu = (Z^0, Z^1, Z^2, Z^3) = (\phi_m, \vec{A}_m)$$

The electromagnetic field strength tensors are given by:

$$G^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha \quad ; \quad V^{\alpha\beta} = \partial^\alpha Z^\beta - \partial^\beta Z^\alpha \tag{5.2}$$

$$G^{\alpha\beta} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix}_e \quad V^{\alpha\beta} = \begin{bmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & -E_z & E_y \\ B_y & E_z & 0 & -E_x \\ B_z & -E_y & E_x & 0 \end{bmatrix}_m \tag{5.3}$$

and their duals are:

$$H^{\alpha\beta} = \begin{bmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{bmatrix}_e \quad W^{\alpha\beta} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{bmatrix}_e \quad (5.4)$$

$$\text{where: } H^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta} G_{\gamma\delta} \quad ; \quad W^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta} V_{\gamma\delta} \quad (5.5)$$

and:

$$\epsilon = \begin{cases} +1 & \text{for } \alpha=0, \beta=1, \gamma=2, \delta=3 \text{ and any even permutation} \\ -1 & \text{for any odd permutation} \\ 0 & \text{if any two indices are equal} \end{cases}$$

With those definitions the field equations for the separate fields and the equation of motion for a charged particle in an electromagnetic field can be written in the form:

$$\partial_\alpha G^{\alpha\beta} = kJ^\beta \quad ; \quad \partial_\alpha W^{\alpha\beta} = 0 \quad ; \quad \text{where } J^\beta = (c\rho_e, J_e) \quad (5.6)$$

ρ_e, J_e from (4.9)

The equation of motion in this case is:

$$\frac{dp^\alpha}{d\tau} = \int \frac{1}{c} G^{\alpha\beta} J_\beta d^3x = \frac{e}{c} G^{\alpha\beta} u_\beta \quad (5.7)$$

and

$$\partial_\alpha H^{\alpha\beta} = 0 \quad ; \quad \partial_\alpha V^{\alpha\beta} = k^* I^\beta \quad ; \quad \text{where } I^\beta = (c\rho_m, -J_m) \quad (5.8)$$

ρ_m, J_m from (4.18)

The equation of motion in this case is:

$$\frac{dp^\alpha}{d\tau} = \int \frac{1}{c} V^{\alpha\beta} I_\beta d^3x = \frac{q}{c} V^{\alpha\beta} u_\beta \quad (5.9)$$

We can again superimpose the four field equations (5.6,8) by defining:

$$F^{\alpha\beta} = G^{\alpha\beta} + W^{\alpha\beta} ; \quad U^{\alpha\beta} = V^{\alpha\beta} + H^{\alpha\beta} \quad (5.10)$$

$$\text{It is of course: } U^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta} \quad (5.11)$$

The combined set of Maxwell equations are:

$$\partial_{\alpha} F^{\alpha\beta} = k J^{\beta} ; \quad \partial_{\alpha} U^{\alpha\beta} = k^* I^{\beta} \quad (5.12)$$

The equation of motion for a dyon in an electromagnetic field is:

$$\frac{dp^{\alpha}}{d\tau} = \int \left\{ \frac{1}{c} F^{\alpha\beta} J_{\beta} + \frac{1}{c} U^{\alpha\beta} I_{\beta} \right\} d^3x = \frac{e}{c} F^{\alpha\beta} u_{\beta} + \frac{g}{c} U^{\alpha\beta} u_{\beta} \quad (5.13)$$

and the non-relativistic approximation is:

$$ma = eE + \frac{e}{c} \mathbf{v} \times \mathbf{B} + g\mathbf{B} - \frac{g}{c} \mathbf{v} \times \mathbf{E} \quad (5.14)$$

(B) DUALITY TRANSFORMATION TO RECOVER REGULAR ELECTRODYNAMICS

In this section we will show that the field equations of regular electrodynamics without magnetic charges can be recovered from the field equations for electrodynamics with magnetic charges through a duality transformation if we

assume that the ratio of magnetic charge to electric charge is a constant for all matter. No additional assumptions regarding k and k^* will have to be made.

The field equations for electrodynamics including both types of charges in our notation are:

$$\partial_{\alpha} F^{\alpha\beta} = k J^{\beta} \quad ; \quad \partial_{\alpha} U^{\alpha\beta} = k^* I^{\beta} \quad (5.12)$$

To perform the duality transformation we introduce the tensor:

$$f^{\alpha\beta} = F^{\alpha\beta} \cos t + U^{\alpha\beta} \sin t \quad ; \quad t \text{ real angle} \quad (5.15)$$

and its dual:

$$\begin{aligned} u^{\alpha\beta} &= \frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} f_{\gamma\delta} = \frac{1}{2} \epsilon^{\alpha\beta}{}_{\gamma\delta} f^{\gamma\delta} \\ &= \frac{1}{2} \epsilon^{\alpha\beta}{}_{\gamma\delta} [F^{\gamma\delta} \cos t + U^{\gamma\delta} \sin t] \\ &= U^{\alpha\beta} \cos t - F^{\alpha\beta} \sin t \end{aligned}$$

where we used: $U^{\alpha\beta} = \frac{1}{2} \epsilon^{\alpha\beta}{}_{\gamma\delta} F^{\gamma\delta}$; and $\frac{1}{2} \epsilon^{\alpha\beta}{}_{\gamma\delta} U^{\gamma\delta} = - F^{\alpha\beta}$

We also introduce the 4-vectors:

$$j^\alpha = kJ^\alpha \cos t + k^* I^\alpha \sin t \quad ; \quad i^\alpha = -kJ^\alpha \sin t + k^* I^\alpha \cos t \quad (5.16)$$

The field equations can be expressed in terms of $f^{\alpha\beta}$ and $u^{\alpha\beta}$:

$$\begin{aligned} \partial_\alpha f^{\alpha\beta} &= \partial_\alpha F^{\alpha\beta} \cos t + \partial_\alpha U^{\alpha\beta} \sin t \\ &= kJ^\beta \cos t + k^* I^\beta \sin t \\ &= j^\alpha \end{aligned} \quad (5.17)$$

and:

$$\begin{aligned} \partial_\alpha u^{\alpha\beta} &= \partial_\alpha U^{\alpha\beta} \cos t - \partial_\alpha F^{\alpha\beta} \sin t \\ &= k^* I^\beta \cos t - kJ^\beta \sin t \\ &= i^\alpha \end{aligned} \quad (5.18)$$

We note that the field equations are invariant under a duality transformation.

If the ratio of magnetic to electric charge is the same for all matter then we can write:

$$I^\beta = aJ^\beta \quad (5.19)$$

where a is a real number. Then:

$$\begin{aligned}\partial_{\alpha} f^{\alpha\beta} &= (k \cos \xi + a k^* \sin \xi) J^{\beta} \\ \partial_{\alpha} u^{\alpha\beta} &= (k^* a \cos \xi - k \sin \xi) J^{\beta}\end{aligned}\tag{5.20}$$

In order to recover the form of the field equations for regular electrodynamics without magnetic monopoles we have to require that the second equation in (5.20) is a homogeneous one. We have to find ξ such that:

$$k^* a \cos \xi - k \sin \xi = 0$$

$$a = (k/k^*) \tan \xi \tag{5.21}$$

$\tan \xi$ can be any real number. Therefore this relation does not give any constraints for the ratio of k and k^* .

Let us now insert (5.21) into equations (5.20):

$$\begin{aligned}\partial_{\alpha} f^{\alpha\beta} &= [k \cos \xi + k^* (k/k^*) \tan \xi \sin \xi] J^{\beta} \\ &= (k/\cos \xi) (\cos^2 \xi + \sin^2 \xi) J^{\beta} \\ &= (k/\cos \xi) J^{\beta} \\ &= k j^{\beta}\end{aligned}$$

and:

$$\partial_{\alpha} u^{\alpha\beta} = 0$$

We find that the combined set of equations is invariant under a duality transformation. For any ratio of magnetic to electric charge we can find an angle ϵ such that the combined set of equations take the form of the first set of equations (of course there is also an angle ϵ which turns the combined set into the second set of equations). We do not have to make any requirements for the relation between k and k^* in order to do this.

(C) RECOVERING THE EQUATION OF MOTION

In the previous section we showed that the field equations of regular electrodynamics without magnetic charges can be obtained from the field equations for electrodynamics with both types of charges provided that the ratio of electric to magnetic charge is the same for all matter. It was also shown that no assumption for the relation between k and k^* had to be made in order to achieve this.

In this section we will show that it is necessary to require that $k=k^*$ in order to recover the equations of motion in the correct form.

The equation of motion for dyons in an electromagnetic

field is:

$$\frac{dp^\alpha}{d\tau} = \int \left\{ \frac{1}{c} F^{\alpha\beta} J_\beta + \frac{1}{c} U^{\alpha\beta} I_\beta \right\} d^3x \quad (5.13)$$

In order to regain the equation of motion for electrodynamics without magnetic charges we perform again a duality transformation. We use the expressions for the tensors $f^{\alpha\beta}$ and $u^{\alpha\beta}$ from the previous section:

$$\begin{aligned} f^{\alpha\beta} &= F^{\alpha\beta} \cos t + U^{\alpha\beta} \sin t \\ u^{\alpha\beta} &= -F^{\alpha\beta} \sin t + U^{\alpha\beta} \cos t \end{aligned} \quad (5.22)$$

We solve this system of two linear equations for $F^{\alpha\beta}$ and $U^{\alpha\beta}$:

$$\begin{aligned} F^{\alpha\beta} &= f^{\alpha\beta} \cos t - u^{\alpha\beta} \sin t \\ U^{\alpha\beta} &= f^{\alpha\beta} \sin t + u^{\alpha\beta} \cos t \end{aligned} \quad (5.23)$$

We insert these expressions into the equation of motion (5.13):

$$\frac{dp^\alpha}{d\tau} = \frac{1}{c} \int \left\{ (f^{\alpha\beta} \cos t - u^{\alpha\beta} \sin t) J_\beta + (f^{\alpha\beta} \sin t + u^{\alpha\beta} \cos t) I_\beta \right\} d^3x$$

We use the relationship $I_\beta = aJ_\beta$ (5.13):

$$\frac{dp^\alpha}{d\tau} = \frac{1}{c} \int \left\{ f^{\alpha\beta} (\cos t + a \sin t) + u^{\alpha\beta} (a \cos t - \sin t) \right\} J_\beta d^3x$$

We replace $a=(k/k^*)(\sin t/\cos t)$

$$\begin{aligned} \frac{dp^\alpha}{d\tau} &= \frac{1}{c} \int \left\{ f^{\alpha\beta} \left(\cos t + \frac{k}{k^*} \frac{\sin^2 t}{\cos t} \right) + u^{\alpha\beta} \left(\frac{k}{k^*} \frac{\sin t}{\cos t} \cos t - \sin t \right) \right\} J_\beta d^3x \\ &= \frac{1}{c} \int \left\{ \frac{f^{\alpha\beta}}{\cos t} (\cos^2 t + \frac{k}{k^*} \sin^2 t) + u^{\alpha\beta} (\frac{k}{k^*} \sin t - \sin t) \right\} J_\beta d^3x \quad (5.24) \end{aligned}$$

In order to regain the shape of the equation of motion for electrodynamics without magnetic charges we have to require that the term proportional to $u^{\alpha\beta}$ vanish. Or:

$$(k/k^*)\sin t - \sin t = 0$$

$$k^* = k \quad (5.25)$$

If we insert this relation into (5.24) we get

$$\begin{aligned} \frac{dp^\alpha}{d\tau} &= \frac{1}{c} \int \frac{f^{\alpha\beta}}{\cos t} J_\beta d^3x \\ &= \int \frac{1}{c} f^{\alpha\beta} J_\beta d^3x \quad (5.26) \end{aligned}$$

We discover that we have to require that $k=k^*$ at this point in order to regain regular electrodynamics from electrodynamics including both types of charges.

CHAPTER VI

LAGRANGIAN AND HAMILTONIAN FOR A SYSTEM OF TWO DYONS

CHANGE IN NOTATION: From now on we will omit the subscript "e" and replace the subscript "m" by superscript "*".

We will now write down a Lagrangian for a system of two particles each carrying electric as well as magnetic charge (dyons) and show its validity by deriving the equations of motion for the two particles. Those equations will be shown to be the same as the Lorentz equations (4.14) for this system.

In deriving the Lagrangian we will treat the electric and magnetic charges on equal footing.

(A) INTERACTIONS AND POTENTIALS

For the interaction of the electric charge of particle one (e_1) with the electric charge of particle 2 (e_2) the usual scalar potential ϕ will be used. Therefore we will use a scalar potential ϕ^* for the interaction between the magnetic charges of particle 1 and 2 ($g_1;g_2$).

The terms due to the electric charges ($e_1;e_2$) in the

magnetic field of the magnetic charges ($g_2; g_1$) enter the Lagrangian as terms proportional to a vector potential \vec{A}^* , which is singular along a string. The difficulties with the string will be all but ignored for the purpose of classical mechanics, since we can always choose the positions of the involved strings such that they do not interfere with the particles.

Therefore, keeping the idea of symmetry in mind, we will introduce the interactions of the magnetic charges ($g_1; g_2$) with the electric fields of the electric charges ($e_2; e_1$) into the Lagrangian also as terms proportional to a vector potential \vec{A} .

List of the interactions involved:

Particle 1 and its potentials	Particle 2 and its potentials	Interaction terms
$ \begin{array}{c} \phi_1 \\ / \quad \backslash \\ e_1 \quad \vec{A}_1 \end{array} $	$ \begin{array}{c} \phi_2 \\ / \quad \backslash \\ e_1 \quad \vec{A}_2 \end{array} $	$e_1 \phi_2$
		$e_2 \phi_1$
		$e_1 \vec{A}_2^*$
		$e_2 \vec{A}_1^*$
$ \begin{array}{c} \phi_1^* \\ / \quad \backslash \\ g_1 \quad \vec{A}_1^* \end{array} $	$ \begin{array}{c} \phi_2^* \\ / \quad \backslash \\ g_2 \quad \vec{A}_2^* \end{array} $	$g_1 \phi_2^*$
		$g_2 \phi_1^*$
		$- g_1 \vec{A}_2^*$
		$- g_2 \vec{A}_1^*$

Starred quantities are due to magnetic charge. We note that we will have eight interaction terms in the Lagrangian. Two of those terms are due to the fact that we symmetrized the scalar potential terms.

The explicit form of the potentials is:

$$\begin{aligned} \phi_1(|\vec{r}_1 - \vec{r}_2|) &= h \frac{e_1}{2|\vec{r}_1 - \vec{r}_2|} ; & \phi_2(|\vec{r}_1 - \vec{r}_2|) &= h \frac{e_2}{2|\vec{r}_1 - \vec{r}_2|} \\ \phi_1^*(|\vec{r}_1 - \vec{r}_2|) &= h^* \frac{g_1}{2|\vec{r}_1 - \vec{r}_2|} ; & \phi_2^*(|\vec{r}_1 - \vec{r}_2|) &= h^* \frac{g_2}{2|\vec{r}_1 - \vec{r}_2|} \end{aligned} \quad (6.1)$$

$$\begin{aligned} \vec{\lambda}_1(\vec{r}_2 - \vec{r}_1) &= e_1 h \vec{D}(\vec{r}_2 - \vec{r}_1) ; & \vec{\lambda}_2(\vec{r}_1 - \vec{r}_2) &= e_2 h \vec{D}(\vec{r}_1 - \vec{r}_2) \\ \vec{\lambda}_1^*(\vec{r}_2 - \vec{r}_1) &= g_1 h^* \vec{D}(\vec{r}_2 - \vec{r}_1) ; & \vec{\lambda}_2^*(\vec{r}_1 - \vec{r}_2) &= g_2 h^* \vec{D}(\vec{r}_1 - \vec{r}_2) \end{aligned}$$

where: $h = kc(4\pi)^{-1}$ and $h^* = k^* c(4\pi)^{-1}$; \vec{D} is a "generic" vector potential and is of the form:

$$\vec{D}(\vec{r}) = \frac{(\vec{r} \times \vec{n})\vec{r} \cdot \vec{n}}{r(r^2 - (\vec{r} \cdot \vec{n})^2)} \quad (6.2)$$

$D(r)$ is singular along the line $\vec{l} = t\vec{n}$ where $-\infty < t < \infty$. Also: $\text{curl } \vec{D}(\vec{r}) = (\vec{r}/|\vec{r}|^3)$. (6.3) (see appendix for proof).

(B) THE LAGRANGIAN FOR A SYSTEM OF DYONS

Now we can proceed and write down the Lagrangian including all interaction terms, where the gravitational interactions between the masses of the two particles will

be neglected.

$$\begin{aligned}
 L = & \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \frac{e_1}{c}A_2^*(r_1-r_2) \cdot v_1 - \frac{g_1}{c}A_2(r_1-r_2) \cdot v_1 + \\
 & \frac{e_2}{c}A_1^*(r_2-r_1) \cdot v_2 - \frac{g_2}{c}A_1(r_2-r_1) \cdot v_2 \\
 & - e_1\phi_2 - g_1\phi_2^* - e_2\phi_1 - g_2\phi_1^*
 \end{aligned}
 \tag{6.4}$$

Having written down the Lagrangian containing all eight interaction terms, with the electric fields of electrons in some places represented also by a singular vector potential we will now introduce some abbreviations which will save a lot of writing, but which "hide" the use of the singular vector potential for the electric field of the electrons.

$$T = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$$

$$\phi = e_1\phi_2 + g_1\phi_2^* + e_2\phi_1 + g_2\phi_1^* \tag{6.5}$$

$$r = r_1 - r_2$$

We will also use the "generic" vector potential D from now on. The Lagrangian can then be written as:

$$L = T + \frac{1}{c} [e_1 g_2 h^* + g_1 e_2 h] \vec{D}(\vec{r}) \cdot \vec{v}_1 + \frac{1}{c} [e_2 g_1 h^* + g_2 e_1 h] \vec{D}(\vec{r}) \cdot \vec{v}_2 - \Phi \quad (6.6)$$

We define two constants c_1 and c_2 :

$$c_1 = (e_1 g_2 h^* - g_1 e_2 h) c^{-1} ; \quad c_2 = (e_2 g_1 h^* - g_2 e_1 h) c^{-1} \quad (6.7)$$

With those definitions L takes the form:

$$L = T + c_1 \vec{D}(\vec{r}) \cdot \vec{v}_1 + c_2 \vec{D}(-\vec{r}) \cdot \vec{v}_2 - \Phi \quad (6.8)$$

From this Lagrangian we calculate the equations of motion in the usual manner:

$$\frac{d}{dt} \frac{\partial L}{\partial \vec{v}_i} - \frac{\partial L}{\partial \vec{r}_i} = 0 \quad ; \quad i = 1, 2 \quad (5.9)$$

The canonical momenta are:

$$\frac{\partial L}{\partial \vec{v}_1} = \vec{p}_1 = m_1 \vec{v}_1 + c_1 \vec{D}(\vec{r}) \quad (6.10)$$

$$\frac{\partial L}{\partial \vec{v}_2} = \vec{p}_2 = m_2 \vec{v}_2 + c_2 \vec{D}(-\vec{r})$$

$$\frac{d\vec{p}_1}{dt} = m_1 \vec{a}_1 + c_1 \frac{d\vec{D}(\vec{r})}{dt} \quad (6.11)$$

$$\frac{d\vec{p}_2}{dt} = m_2 \vec{a}_2 + c_2 \frac{d\vec{D}(-\vec{r})}{dt}$$

The derivatives with respect to the canonical

coordinates are:

$$\frac{\partial L}{\partial \vec{r}_1} = c_1 \frac{\partial \vec{D}(\vec{r})}{\partial \vec{r}_1} \cdot \vec{v}_1 + c_1 \frac{\partial \vec{D}(-\vec{r})}{\partial \vec{r}_1} \cdot \vec{v}_2 - \frac{\partial \phi}{\partial \vec{r}_1} \quad (6.12)$$

$$\frac{\partial L}{\partial \vec{r}_2} = c_1 \frac{\partial \vec{D}(\vec{r})}{\partial \vec{r}_2} \cdot \vec{v}_1 + c_1 \frac{\partial \vec{D}(-\vec{r})}{\partial \vec{r}_2} \cdot \vec{v}_2 - \frac{\partial \phi}{\partial \vec{r}_2}$$

where:

$$\frac{\partial \phi}{\partial \vec{r}_1} = - [e_1 e_2 h + g_1 g_2 h^*] \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^3} \quad (6.13)$$

$$\frac{\partial \phi}{\partial \vec{r}_2} = [e_1 e_2 h + g_1 g_2 h^*] \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^3}$$

Let us now collect all terms and write down the equations of motion according to (6.9):

$$m_1 \vec{a}_1 = - c_1 \frac{d\vec{D}(\vec{r})}{dt} + c_1 \frac{\partial \vec{D}(\vec{r})}{\partial \vec{r}} \cdot \vec{v}_1 + c_2 \frac{\partial \vec{D}(\vec{r})}{\partial \vec{r}} \cdot \vec{v}_2 - c_3 \frac{\vec{r}}{|\vec{r}|^3} \quad (6.14)$$

$$m_2 \vec{a}_2 = - c_2 \frac{d\vec{D}(\vec{r})}{dt} + c_1 \frac{\partial \vec{D}(\vec{r})}{\partial \vec{r}} \cdot \vec{v}_1 + c_2 \frac{\partial \vec{D}(\vec{r})}{\partial \vec{r}} \cdot \vec{v}_2 + c_3 \frac{\vec{r}}{|\vec{r}|^3}$$

where we used the fact that $\vec{D}(\vec{r}_1 - \vec{r}_2) = \vec{D}(\vec{r}_2 - \vec{r}_1)$ (6.15)

and introduced an additional constant:

$$c_3 = e_1 e_2 h + g_1 g_2 h^* \quad (6.16)$$

Collection of terms yields:

$$\begin{aligned}
 m_1 \vec{a}_1 &= -c_1 \frac{d\vec{D}(\vec{r})}{dt} + \frac{\partial \vec{D}(\vec{r})}{\partial \vec{r}} \cdot (c_1 \vec{v}_1 + c_2 \vec{v}_2) + c_3 \frac{\vec{r}}{|\vec{r}|^3} \\
 m_2 \vec{a}_2 &= -c_2 \frac{d\vec{D}(\vec{r})}{dt} - \frac{\partial \vec{D}(\vec{r})}{\partial \vec{r}} \cdot (c_1 \vec{v}_1 + c_2 \vec{v}_2) - c_3 \frac{\vec{r}}{|\vec{r}|^3}
 \end{aligned}
 \tag{6.17}$$

At this point it is useful to calculate the center of mass acceleration. This will give us an important piece of information for the constants c_1 and c_2 .

$$m_1 \vec{a}_1 + m_2 \vec{a}_2 = - (c_1 + c_2) \frac{d\vec{D}(\vec{r})}{dt} \tag{6.18}$$

Since there are no external forces acting upon the system the center of mass acceleration should be zero.

Therefore

$$c_1 = -c_2 \tag{6.19}$$

or:

$$e_1 g_2 h^* - g_1 e_2 h = - (e_2 g_1 h^* - g_2 e_1 h)$$

$$h^* (e_1 g_2 + e_2 g_1) = h (e_1 g_2 + e_2 g_1)$$

$$h^* = h$$

$$\text{Thus:} \quad k^* = k \tag{5.25}$$

Note at this point that the relation between k^* and k

follows from the requirement that there be no acceleration of the center of mass in a two particle system.

Let us now go back to the derivation of the equations of motion using the fact that $k^* = k$ or $c_1 = -c_2$:

$$m_1 \vec{a}_1 = -c_1 \frac{d\vec{D}(\vec{r})}{dt} + c_1 \frac{\partial \vec{D}(\vec{r})}{\partial \vec{r}} (\vec{v}_1 - \vec{v}_2) + c_3 \frac{\vec{r}}{|\vec{r}|^3} \quad (6.20)$$

$$m_2 \vec{a}_2 = c_1 \frac{d\vec{D}(\vec{r})}{dt} - c_1 \frac{\partial \vec{D}(\vec{r})}{\partial \vec{r}} (\vec{v}_1 - \vec{v}_2) - c_3 \frac{\vec{r}}{|\vec{r}|^3}$$

We now utilize the relationship:

$$-\frac{d\vec{D}(\vec{r})}{dt} + \frac{\partial \vec{D}(\vec{r})}{\partial \vec{r}} \cdot \vec{v} = \vec{v} \times \text{curl } \vec{D}(\vec{r}) \quad (6.21)$$

and we can write:

$$m_1 \vec{a}_1 = c_1 (\vec{v}_1 - \vec{v}_2) \times \text{curl } \vec{D}(\vec{r}) + c_3 \frac{\vec{r}}{|\vec{r}|^3} \quad (6.22)$$

$$m_2 \vec{a}_2 = -c_1 (\vec{v}_1 - \vec{v}_2) \times \text{curl } \vec{D}(\vec{r}) - c_3 \frac{\vec{r}}{|\vec{r}|^3}$$

where $\text{curl } \vec{D}(\vec{r}) = \vec{r}/(|\vec{r}|^3)$ and $\vec{r} = \vec{r}_1 - \vec{r}_2$

$$m_1 \vec{a}_1 = c_1 (\vec{v}_1 - \vec{v}_2) \times \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^3} + c_3 \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^3} \quad (6.23)$$

$$m_2 \vec{a}_2 = c_2 (\vec{v}_1 - \vec{v}_2) \times \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^3} - c_3 \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^3}$$

We note that these equations have Galilean invariance,

since the velocity terms depend on the relative velocity only.

The terms proportional to c_3 are the Coulomb interactions between the electric and magnetic charges of particle 1 and particle 2. The terms in the equations of motion which are proportional to the relative velocities will be interpreted in the following section.

(C) INTERPRETATION OF THE VELOCITY DEPENDENT TERMS IN THE EQUATIONS OF MOTION FOR A TWO DYON SYSTEM

In order to interpret those terms we will rewrite the equation of motion for particle 1:

$$m_1 \vec{a}_1 = c_1 \vec{v}_1 \times \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^3} - c_1 \vec{v}_2 \times \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^3} + c_3 \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^3} \quad (6.23)$$

For simplicity let us assume that we are looking at a system made up of a pure monopole (g_2) and a pure electric charge (e_1).

This means that there is no Coulomb interaction between the particles; the term proportional to c_3 vanishes.

We will also reinsert the expressions for c_1 and c_2 .

$$m_1 \vec{a}_1 = \frac{h^*}{c} e_1 \vec{v}_1 \times \frac{g_1 (\vec{r}_1 - \vec{r}_2)}{|\vec{r}_1 - \vec{r}_2|^3} - \frac{h^*}{c} e_1 g_2 \vec{v}_2 \times \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^3} \quad (6.24)$$

The first term is easily recognized as the Lorentz force on a electric charge in the a magnetic field:

$$F = \oint (1/c) \vec{J}_e \times \vec{B} d^3x \quad \text{where in this case } \vec{B} = g(\vec{r}/|\vec{r}|^3) \quad (6.25)$$

The second term must be a Biot-Savart-force term corresponding to an electric charge in an electric field created by a current of a magnetic monopoles. To demonstrate we will calculate the electric field of a moving magnetic monopole. It is:

$$\vec{E}^* = \text{curl } \vec{j}_A^*$$

where \vec{j}_A^* is the vector potential for a current \vec{j} of magnetic charges. It is not to be confused with \vec{A}^* , the singular vector potential for a magnetic monopole at rest. The calculation for this problem was done in Chapter 4 Section (B), we are just quoting the results:

$$\vec{E}^* = - \frac{gk^*}{4\pi} (\vec{v} \times \frac{\vec{r}}{|\vec{r}|^3}) \quad (4.19)$$

Therefore the force on particle 1 can be written as:

$$\vec{F}_1 = m_1 \vec{a}_1 = e_1 \vec{E}_2^* + (e_1/c) \vec{v}_1 \times \vec{B}_2^* \quad (6.26)$$

where :

$$\vec{E}_2^* = - \frac{k^*}{4\pi} \frac{g_2 \vec{v}_2 (\vec{r}_1 - \vec{r}_2)}{|\vec{r}_1 - \vec{r}_2|^3} \quad \text{and} \quad \vec{B}_2^* = \frac{g_2 k^*}{4\pi} \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^3}$$

We note at this point that the Lagrangian formulated

in this chapter (with strings for electric charges) yields the same equations of motion as the Lorentz approach which was used in this section. The above Lagrangian for magnetic and electric charges is therefore justified.

(D) THE HAMILTONIAN FOR A SYSTEM OF TWO DYONS

In classical mechanics the Hamiltonian for a system of particles is related to the Lagrangian for the same system through a Legendre transformation of the following kind:

$$H = \sum_1 \dot{v}_1 p_1 - L \quad ; \quad \text{where } p_1 = \frac{\partial L}{\partial \dot{r}_1}$$

In our case i goes from 1 to 2, the number of particles and we have:

$$\begin{aligned} H &= \dot{v}_1 \cdot \vec{p}_1 + \dot{v}_2 \cdot \vec{p}_2 - L & (6.20) \\ &= \dot{v}_1 \cdot \{m_1 \dot{v}_1 + c_1 \vec{D}(\vec{r})\} + \dot{v}_2 \cdot \{m_2 \dot{v}_2 + c_2 \vec{D}(\vec{r})\} - .5m_1 \dot{v}_1^2 - .5m_2 \dot{v}_2^2 \\ &\quad - c_1 \vec{D}(\vec{r}) \cdot \dot{v}_1 - c_2 \vec{D}(\vec{r}) \cdot \dot{v}_2 + \phi \end{aligned}$$

In this expression we have to replace the velocities by their corresponding terms which contain the canonical momenta. From (5.10) this is:

$$\dot{v}_1 = \frac{1}{m_1} \{ \vec{p}_1 - c_1 \vec{D}(\vec{r}) \} \quad ; \quad \dot{v}_2 = \frac{1}{m_2} \{ \vec{p}_2 - c_2 \vec{D}(\vec{r}) \}$$

inserting this into the above expression yields:

$$H = \frac{1}{m_1} \{ \vec{p}_1^2 - c_1 \vec{D}(\vec{r}) \cdot \vec{p}_1 \} + \frac{1}{m_2} \{ \vec{p}_2^2 - c_2 \vec{D}(\vec{r}) \cdot \vec{p}_2 \} - \frac{1}{2m_1} \{ \vec{p}_1^2 - 2c_1 \vec{p}_1 \cdot \vec{D}(\vec{r}) \}$$

$$\begin{aligned}
& + c_1^2 \vec{D}(\vec{r})^2 \Big] - \frac{1}{2m_2} \{ \vec{p}_2^2 - 2\vec{p}_2 c_2 \vec{D}(-\vec{r}) + c_2^2 \vec{D}(-\vec{r})^2 \} \\
& - \frac{1}{m_1} \{ \vec{p}_1 \vec{D}(\vec{r}) - c_1^2 \vec{D}(\vec{r})^2 \} - \frac{1}{m_2} \{ \vec{p}_2 \vec{D}(-\vec{r}) - c_2^2 \vec{D}(-\vec{r})^2 \} + \Phi \\
& = \frac{1}{m_1} \{ \vec{p}_1 - c_1 \vec{D}(\vec{r}) \} \cdot \vec{p}_1 + \frac{1}{m_2} \{ \vec{p}_2 - c_2 \vec{D}(-\vec{r}) \} \cdot \vec{p}_2 - \frac{1}{2m_1} \{ \vec{p}_1 - c_1 \vec{D}(\vec{r}) \}^2 - \\
& \frac{1}{2m_2} \{ \vec{p}_2 - c_2 \vec{D}(-\vec{r}) \}^2 - \frac{c_1}{m_1} \vec{D}(\vec{r}) \cdot \{ \vec{p}_1 - c_1 \vec{D}(\vec{r}) \} - \frac{c_2}{m_2} \vec{D}(-\vec{r}) \cdot \{ \vec{p}_2 - c_2 \vec{D}(-\vec{r}) \} + \Phi \\
& = \frac{1}{2m_1} \vec{p}_1^2 - \frac{c_1}{m_1} \vec{p}_1 \cdot \vec{D}(\vec{r}) + c_1^2 \vec{D}(\vec{r})^2 + \frac{1}{2m_2} \vec{p}_2^2 - \frac{c_2}{m_2} \vec{p}_2 \cdot \vec{D}(-\vec{r}) + \\
& c_2^2 \vec{D}(-\vec{r})^2 + \Phi \\
& = \frac{1}{2m_1} \{ \vec{p}_1 - c_1 \vec{D}(\vec{r}) \}^2 + \frac{1}{2m_2} \{ \vec{p}_2 - c_2 \vec{D}(-\vec{r}) \}^2 + \Phi \tag{6.29}
\end{aligned}$$

We note at this point that the Hamiltonian has the same form like a Hamiltonian for the interaction between two electrically charged particles.

This Hamiltonian is also the same one at which D. Zwanzinger [8,9] arrived "by trial and error" in his papers on monopole dynamics.

We will now replace all abbreviations by their original meaning and write down the Hamiltonian explicitly:

$$\begin{aligned}
H = & \frac{1}{2m_1} \{ \vec{p}_1 - \frac{1}{c} (e_1 g_2 h^* + g_1 e_2 h) \vec{D}(\vec{r}_2 - \vec{r}_1) \}^2 + \\
& \frac{1}{2m_2} \{ \vec{p}_2 - \frac{1}{c} (e_2 g_1 h^* + g_2 e_1 h) \vec{D}(\vec{r}_1 - \vec{r}_2) \}^2 + (e_1 e_2 h + g_1 g_2 h^*) \frac{1}{|\vec{r}_1 - \vec{r}_2|}
\end{aligned} \tag{6.30}$$

or using the potentials:

$$\begin{aligned}
 H = & \frac{1}{2m_1} \left[\vec{p}_1 - \frac{1}{c} e_1 \vec{A}_2^* (\vec{r}_1 - \vec{r}_2) - \frac{1}{c} g_1 \vec{A}_2 (\vec{r}_1 - \vec{r}_2) \right]^2 + \\
 & \frac{1}{2m_2} \left[\vec{p}_2 - \frac{1}{c} e_2 \vec{A}_1^* (\vec{r}_2 - \vec{r}_1) - \frac{1}{c} g_2 \vec{A}_2 (\vec{r}_2 - \vec{r}_1) \right]^2 + \phi
 \end{aligned} \quad (6.31)$$

In the next section we will calculate the equations of motion from the Hamiltonian.

(E) THE EQUATIONS OF MOTION FROM THE HAMILTONIAN

We will now calculate the equations of motion using the Hamiltonian from above. For brevity we will use all abbreviations, which were introduced previously:

$$H = \frac{1}{2m_1} \left[\vec{p}_1 - c_1 \vec{D}(\vec{r}) \right]^2 + \frac{1}{2m_2} \left[\vec{p}_2 - c_2 \vec{D}(-\vec{r}) \right]^2 + \phi \quad (6.29)$$

The equations of motion are calculated in the usual manner:

$$\frac{d\vec{r}_i}{dt} = \frac{\partial H}{\partial \vec{p}_i} ; \quad \frac{d\vec{p}_i}{dt} = - \frac{\partial H}{\partial \vec{r}_i} \quad (6.32)$$

The equations for the coordinates:

$$\frac{d\vec{r}_1}{dt} = \frac{\vec{p}_1 - c_1 \vec{D}(\vec{r})}{m_1} = \vec{v}_1 ; \quad \frac{d\vec{r}_2}{dt} = \frac{\vec{p}_2 - c_2 \vec{D}(-\vec{r})}{m_2} = \vec{v}_2$$

The equations for the canonical momenta:

$$\frac{d\vec{p}_1}{dt} = \frac{c_1}{m_1} \frac{\partial \vec{D}(\vec{r})}{\partial \vec{r}_1} [\vec{p}_1 - c_1 \vec{D}(\vec{r})] + \frac{c_2}{m_2} \frac{\partial \vec{D}(-\vec{r})}{\partial \vec{r}_1} [\vec{p}_2 - c_2 \vec{D}(-\vec{r})] + \frac{\partial \phi}{\partial \vec{r}_1} \quad (65.33)$$

$$\frac{d\vec{p}_2}{dt} = -\frac{c_1}{m_1} \frac{\partial \vec{D}(\vec{r})}{\partial \vec{r}_2} (\vec{p}_1 - c_1 \vec{D}(\vec{r})) - \frac{c_2}{m_2} \frac{\partial \vec{D}(-\vec{r})}{\partial \vec{r}_2} (\vec{p}_2 - c_2 \vec{D}(-\vec{r})) + \frac{\partial \phi}{\partial \vec{r}_2}$$

(6.33) are the Hamilton equations of motion for the two dyon system. If we insert (6.11) into (6.33), replace the canonical momenta with the expression containing the kinetic momenta and use the fact that $c_2 = -c_1$ then we get:

$$\begin{aligned} \frac{d}{dt} \vec{p}_1 &= m_1 \vec{a}_1 + c_1 \frac{d\vec{D}(\vec{r})}{dt} = c_1 \frac{\partial \vec{D}(\vec{r})}{\partial \vec{r}} (\vec{v}_1 - \vec{v}_2) + \frac{\partial \phi}{\partial \vec{r}} \\ \frac{d}{dt} \vec{p}_2 &= m_2 \vec{a}_2 - c_1 \frac{d\vec{D}(\vec{r})}{dt} = -c_1 \frac{\partial \vec{D}(\vec{r})}{\partial \vec{r}} (\vec{v}_1 - \vec{v}_2) - \frac{\partial \phi}{\partial \vec{r}} \end{aligned}$$

We move the terms with the time derivative of the generic vector potential over to the right hand side of the equations and use the relation (6.21):

$$\begin{aligned} m_1 \vec{a}_1 &= c_1 (\vec{v}_1 - \vec{v}_2) \times \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^3} - c_3 \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^3} \\ m_2 \vec{a}_2 &= -c_1 (\vec{v}_1 - \vec{v}_2) \times \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^3} + c_3 \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^3} \end{aligned} \quad (6.23)$$

These equations are of course the same as the equations which we derived from the Lagrangian.

(β) SUMMARY

In this chapter we arrived at a Lagrangian for a system of dyons. In doing this our main guidelines were to change the existing Lagrangian formalism as little as

possible and to treat the electric and magnetic charges as equally as possible. We therefore introduced the new concept of a vector potential to describe the electric field of the electric charges. This vector potential was of the same form as the vector potential for the magnetic field of a monopole and therefore singular along a string. The vector potential for the electric charge, however did not replace the scalar potential for the same field. The vector potential was only used for the interaction terms with the magnetic charge. The scalar potential was used for the interaction between the electric charges of the particles. Conversely a scalar potential for the interaction between the magnetic charges and a singular vector potential for the interaction between magnetic and electric charge was introduced. Thus the electric (magnetic) field of the electric (magnetic) charge is represented twice in the Lagrangian which was subsequently put together. From this Lagrangian we derived equations of motion which were found to be identical to the ones derived from the more basic Lorentz-force approach for the same problem. Thus the Lagrangian was justified.

In the process of deriving the Lagrangian we found a physical argument for the equality between k and k^* ; we had to assume this in order to avoid center of mass acceleration in the two dyon system.

From the Lagrangian we derived the Hamiltonian via

Legendre transformation. We found the form of the Hamiltonian similar to the form of a Hamiltonian for a system of two particles with only electric charges. It was also the same one which D. Zwanzinger [8,9] assumed for a system of two dyons.

CHAPTER VII

THE RESULTS OF THIS PAPER

1.) In chapter (IV) we showed that the first set of equations can be obtained from the symmetrized set of equations using a duality transformation. We discovered that we do not have to impose any constraints on the relation between k and k^* at this point.

However, if the equation of motion is to be recovered in the correct form we have to require that $k = k^*$.

The duality transformation is only possible if one assumes a constant ratio of magnetic to electric charge for all matter. If particles exist, for which that ratio differs from that constant ("true" magnetic monopoles) then a duality rotation to recover the asymmetric shape of the field equations and the Lorentz force equation is no longer possible regardless if k is equal to k^* or not. Thus we cannot draw any information about the relation between k and k^* from duality transformation arguments. However, in chapter (VI) we discovered that we have to require that $k=k^*$ in order to avoid center of mass acceleration in a two dyon system.

2.) In chapter (VI) we showed that a non-relativistic

Lagrangian (and Hamiltonian) for a system of two dyons can be found much in the same way as one constructs a Lagrangian for particles which carry electric charges only. We proved the validity of this Lagrangian by deriving the equations of motion from it. Those equations were shown to be the same as Lorentz equations for the same system.

In the Lagrangian we are using scalar and vector potentials to describe the \vec{E} and \vec{B} fields of both the electric as well as the magnetic charges. Scalar potentials were used to describe the interaction between charges of the same type; vector potentials were used to describe the interactions between charges of different type. We want to point out that a vector potential was used to describe the electric field of an electron, this vector potential is of the same singular form as the vector potential for the magnetic field of a magnetic monopole. We also notice that in this Lagrangian the electric field of the electron and the magnetic field of the monopole are represented in two different ways once by a scalar potential, once by a vector potential, depending on what type of charge the particle is interacting with.

Both types of charges are treated equally in this Lagrangian.

3.) In deriving the equations of motion from the Lagrangian we discovered that we have to use vector

potentials which are even functions (invariant under space reflection). This is significant for the charge quantization condition, which according to P. G. H. Sandars [10] and J. Schwinger [11] has to be modified in the following way if one uses an even vector potential:

$$\frac{e\mathbf{q}}{\hbar c} = n \quad ; \quad n \text{ integer}$$

The Lagrangian does not produce the correct equations of motion if one uses a vector potential of the form:

$$\vec{D}(\vec{r}) = \frac{\vec{r} \times \vec{n}}{r[r - (\vec{r} \cdot \vec{n})]}$$

although the curl of this vector potential is equal to $\vec{r}/(|\vec{r}|^3)$ and therefore yields the correct \vec{B} -field.

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APPENDIX

SOME PROPERTIES OF THE VECTOR POTENTIAL $\vec{D}(\vec{r})$

In this appendix we will prove some of the statements that were made about the generic vector potential $\vec{D}(\vec{r})$. All of the following calculations are done in cartesian coordinates ($\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$), the string is chosen to be along the z-axis ($\vec{n} = \vec{k}$). The vector potential is:

$$\vec{D}(\vec{r}) = \frac{(\vec{r} \times \vec{n}) \vec{r} \cdot \vec{n}}{r(r^2 - (\vec{r} \cdot \vec{n})^2)} \quad \text{A.1}$$

First we calculate the cross and the dot products separately:

$$\vec{r} \times \vec{n} = \det \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z \\ 0 & 0 & 1 \end{vmatrix} = (y\vec{i} - x\vec{j}) ; \quad \vec{r} \cdot \vec{n} = z \quad \text{A.2}$$

With those expressions the vector potential can be written in this form:

$$\vec{D}(\vec{r}) = \frac{yz}{r(x^2 + y^2)} \vec{i} - \frac{xz}{r(x^2 + y^2)} \vec{k} \quad \text{A.3}$$

$$(a) \text{ curl } \vec{D} = \vec{r} / (|\vec{r}|^3)$$

$$\text{curl } \vec{D}(\vec{r}) = \det \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ D_x & D_y & 0 \end{vmatrix}$$

$$= \left[-\frac{\partial D}{\partial z} y \vec{i} + \frac{\partial D}{\partial z} x \vec{j} + \left(\frac{\partial D}{\partial x} y - \frac{\partial D}{\partial y} x \right) \vec{k} \right]$$

We calculate the derivatives separately:

$$\frac{\partial D}{\partial z} x = \frac{xr(x^2+y^2) - z^2x(x^2+y^2)r^{-1}}{r^2(x^2+y^2)^2}$$

$$= \frac{x(r^2-z^2)}{r^3(x^2+y^2)}$$

$$= \frac{x(x^2+y^2)}{r^3(x^2+y^2)}$$

$$= x/(|r|^3)$$

A.4

The derivative of $-D_y$ with respect to z is formally the same as above if we replace x with y :

$$-\frac{\partial D}{\partial z} y = y/(|r|^3)$$

A.5

$$\frac{\partial D}{\partial x} y = -\frac{zr(x^2+y^2) - [x(x^2+y^2)r^{-1} + 2xr]xz}{r^2(x^2+y^2)^2}$$

$$= \frac{z\{-r^2(x^2+y^2) - [x(x^2+y^2) + 2xr^2]y\}}{r^3(x^2+y^2)^2}$$

$$\frac{\partial D}{\partial y} x = \frac{zr(x^2+y^2) - [y(x^2+y^2)r^{-1} + 2yr]yz}{r^2(x^2+y^2)^2}$$

$$= \frac{z\{r^2(x^2+y^2) - [y(x^2+y^2) + 2yr]y\}}{r^3(x^2+y^2)^2}$$

$$\frac{\partial D_y}{\partial x} - \frac{\partial D_x}{\partial y} = \frac{-2zr^2(x^2+y^2) + [(x^2+y^2) + 2r^2]x^2z + [(x^2+y^2) + 2r^2]y^2z}{r^3(x^2+y^2)^2}$$

$$= \frac{-2zr^2 + z[(x^2+y^2) + 2r^2]}{r^3(x^2+y^2)}$$

$$= \frac{z(x^2+y^2)}{r^3(x^2+y^2)}$$

$$= z/(|\vec{r}|^3)$$

A.7

Therefore $\text{curl } \vec{D}(\vec{r}) = \vec{r}/(|\vec{r}|^3)$ everywhere in space except on the string.

(b) $\vec{D}(\vec{r})$ is an even function (invariant under space inversion)

Let \vec{v} be an arbitrary vector in space which can not be written as $\vec{v} = c\vec{n}$ for any real number c . Then:

$$\vec{D}(-\vec{v}) = \frac{(-\vec{v} \times \vec{n})(-\vec{v}) \cdot \vec{n}}{v[v^2 - (-\vec{v} \cdot \vec{n})^2]} = \frac{(\vec{v} \times \vec{n})\vec{v} \cdot \vec{n}}{v[v^2 - (\vec{v} \cdot \vec{n})^2]} = \vec{D}(\vec{v})$$

Use of this statement was made in the form $\vec{D}(\vec{r}_1 - \vec{r}_2) = \vec{D}(\vec{r}_2 - \vec{r}_1)$ in equation (6.15). This relation was also used in calculating the derivatives of \vec{D} with respect to \vec{r}_1 and \vec{r}_2

in (6.14):

$$\frac{\partial \vec{D}(\vec{r}_1 - \vec{r}_2)}{\partial \vec{r}_1} = \frac{\partial \vec{D}(\vec{r})}{\partial \vec{r}} \frac{\partial \vec{r}}{\partial \vec{r}_1} = \frac{\partial \vec{D}(\vec{r})}{\partial \vec{r}}$$

$$\frac{\partial \vec{D}(\vec{r}_2 - \vec{r}_1)}{\partial \vec{r}_1} = \frac{\partial \vec{D}(\vec{r}_1 - \vec{r}_2)}{\partial \vec{r}_1} = \frac{\partial \vec{D}(\vec{r})}{\partial \vec{r}}$$

$$\frac{\partial \vec{D}(\vec{r}_1 - \vec{r}_2)}{\partial \vec{r}_2} = \frac{\partial \vec{D}(\vec{r})}{\partial \vec{r}} \frac{\partial \vec{r}}{\partial \vec{r}_2} = - \frac{\partial \vec{D}(\vec{r})}{\partial \vec{r}}$$

$$\frac{\partial \vec{D}(\vec{r}_2 - \vec{r}_1)}{\partial \vec{r}_2} = \frac{\partial \vec{D}(\vec{r}_1 - \vec{r}_2)}{\partial \vec{r}_2} = - \frac{\partial \vec{D}(\vec{r})}{\partial \vec{r}}$$

$$(c) \vec{v} \times \text{curl} \vec{D} = \frac{\partial}{\partial \vec{r}} (\vec{v} \cdot \vec{D}) - \frac{d\vec{D}}{dt}$$

The proof for this statement is given in Goldstein [12] for the case of a regular vector potential (one created by an electric current). The proof for the singular vector potentials \vec{D} is essentially the same.

We calculate the x-component of $\vec{v} \times \text{curl} \vec{D}$:

$$\begin{aligned} (\vec{v} \times \text{curl} \vec{D})_x &= v_y \left(\frac{\partial D_y}{\partial x} - \frac{\partial D_x}{\partial y} \right) - v_z \left(\frac{\partial D_x}{\partial z} - \frac{\partial D_z}{\partial x} \right) \\ &= v_y \frac{\partial D_y}{\partial x} + v_z \frac{\partial D_z}{\partial x} - v_y \frac{\partial D_x}{\partial y} - v_z \frac{\partial D_x}{\partial z} \end{aligned}$$

We now add and subtract the term $v_x (\partial D_x / \partial x)$:

$$= v_x \frac{\partial D_x}{\partial x} + v_y \frac{\partial D_y}{\partial x} + v_z \frac{\partial D_z}{\partial x} - v_x \frac{\partial D_x}{\partial x} - v_y \frac{\partial D_x}{\partial y} - v_z \frac{\partial D_x}{\partial z}$$

We recognize the last three terms as the total time

derivative of \vec{D} (\vec{D} has no explicit time dependence in our case). The first three terms can be written as the x-derivative of the dot product between \vec{v} and \vec{D} , since \vec{D} does not depend on \vec{v} .

$$= \frac{\partial}{\partial x}(\vec{v} \cdot \vec{D}) - \frac{dD_x}{dt}$$

Similarly:

$$(\vec{v} \times \text{curl} \vec{D})_y = \frac{\partial}{\partial y}(\vec{v} \cdot \vec{D}) - \frac{dD_y}{dt}$$

and:

$$(\vec{v} \times \text{curl} \vec{D})_z = \frac{\partial}{\partial z}(\vec{v} \cdot \vec{D}) - \frac{dD_z}{dt}$$

or in vector notation:

$$\vec{v} \times \text{curl} \vec{D} = \frac{\partial}{\partial \vec{r}}(\vec{v} \cdot \vec{D}) - \frac{d\vec{D}}{dt}$$

The proof in Goldstein also allows for the possibility that D possesses an explicit time dependence.

$$(d) \text{div grad } \vec{D}(\vec{r}) = 0$$

Calculating $\text{div grad} D$ explicitly would involve the calculation of at least twelve partial derivatives. This is a very tedious calculation. We will use a trick to avoid it:

$$\text{curl}\{\text{curl}\vec{D}\} = -\text{div grad}\vec{D} + \text{grad}(\text{div}\vec{D})$$

or:

$$\text{div grad } \vec{D} = -\text{curl}\{\text{curl}\vec{D}\} + \text{grad}(\text{div}\vec{D})$$

From the gauge relations (4.6) and (4.15) we see that the divergence of the vector potentials A_m and A_e must vanish since we are assuming that the time derivatives of ϕ_m and ϕ_e are zero. Therefore the divergence of D must be zero, too.

$$\begin{aligned} \text{div grad } \vec{D} &= -\text{curl}\{\text{curl}\vec{D}\} \\ &= -\text{curl}\{\vec{r}/(|\vec{r}|^3)\} \\ &= \text{curl}\{\text{grad}|\vec{r}|^{-1}\} \\ &= 0 \end{aligned}$$