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Spacetime Numerical Techniques for the Wave and Schrödinger Equations

by

Paulina Ester Sepúlveda Salas

A dissertation submitted in fulfillment of the
requirements for the degree of

Doctor of Philosophy
in
Mathematical Sciences

Dissertation Committee:
Jay Gopalakrishnan, Chair
Panayot Vassilevski
Jeffrey Ovall
Johnny Guzmán
Erik Sánchez

Portland State University
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Abstract

The most common tool for solving spacetime problems using finite elements is based on semidiscretization: discretizing in space by a finite element method and then advancing in time by a numerical scheme. Contrary to this standard procedure, in this dissertation we consider formulations where time is another coordinate of the domain. Therefore, spacetime problems can be studied as boundary value problems, where initial conditions are considered as part of the spacetime boundary conditions.

When seeking solutions to these problems, it is natural to ask what are the correct spaces of functions to choose, to obtain wellposedness. This motivates the study of an abstract theory for unbounded partial differential operators associated with a general boundary value problem on a bounded domain. A framework for choosing the spaces is introduced, and conditions for the solvability of weak formulations are provided. We apply this framework to study wave problems on tents and to study wellposed discontinuous Petrov-Galerkin (DPG) formulations for the Schrödinger and wave equations. Several numerical issues are also discussed.

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Introduction

A wide variety of physical problems can be described using partial differential equations involving time and space. For example, the wave equation can arise in acoustics, electromagnetism, elasticity problems, and other areas. Many numerical methods already exist to solve these time-dependent problems. The most widely used technique is based on semidiscretization: discretizing in space and advancing in time by a time stepping scheme. One of the major drawbacks of these methods when considering hyperbolic problems is that they impose a global Courant-Friedrichs-Lewy (CFL) condition (see [14]), which makes local mesh-refinement a difficult task. One alternative to deal with this problem is to consider time as another coordinate and formulate a spacetime boundary value problem. The advantages of this approach include a natural way of treating moving boundary conditions, easy spacetime adaptivity, possibility to obtain convergence even under limited spacetime regularity and exploitation of parallelism without causality constraints, among others (see for example [16], [51], [52], [64] and references therein).

The first ideas to advance a numerical solution in time by local operations in space time regions were explored in [53, 4]. Recurrence relations on multiple slabs of rectangular spacetime elements were considered in [44]. These ideas were generalized to non-regular spacetime elements for beams and plates in [5]. Closest in ancestry to one of the methods we consider in this dissertation is found in [56], where it was called explicit spacetime elements.

The spacetime discontinuous Galerkin (SDG) method was announced almost at the same time in [48, 56] and continues to see active development over the years in engineering applications [49, 55, 71].

An important numerical technique that we shall highlight in this dissertation was first introduced in two papers [33, 50]. There we can find the first tent pitching ideas that were brought into the numerical analysis community. A tent pitching algorithm, which was first introduced in [67], generates an unstructured mesh for the spacetime domain using an advancing-front scheme. Recent advances include in [39], where the mapped tent pitching (MTP) method has been introduced in higher dimensions. This technique transforms the equations within a spacetime tent to a domain where space and time are separable. In this dissertation, we are interested in the study of the properties of functions of Hilbert spaces endowed with a graph norm, and their traces in tent-shaped domains, where inflow and outflow part of the boundary meet. Moreover, we are interested in the design of an explicit finite element method of the tent-shaped type whose discrete spaces conform to the weak continuity properties of the space. We also highlight other numerical techniques that consider spacetime domains (see [68]).

In this dissertation we are also interested in studying the feasibility of using the discontinuous Petrov-Galerkin (DPG) method for the spacetime Schrödinger and wave equations. The DPG method was introduced in its current form in [17] and makes use of different trial and test spaces, where optimal test functions are considered using broken test norms that allow their local calculation and guarantees the stability of the method (see [17, 19, 22, 21]). The DPG method has been successfully applied to a wide variety of problems such as second order elliptic problems [18], convective phenomena, [15, 17, 23], elasticity [6, 10, 38, 45], etc. Among the desirable properties of the DPG method are mesh-independent stability, inheritance of discrete stability

from the wellposedness of the undiscretized problem, and a built-in error estimator that exhibits good pre-asymptotic mesh-independent stability properties. Applications of the DPG method for spacetime problems have already been computationally studied in [28] for the transient parabolic partial differential equations and [27] for the time-dependent convection-diffusion equation. We also note that a scheme that combines DPG spatial discretization with backward Euler time stepping for the heat equation has been analyzed in [34].

One of the contributions of the present dissertation is to provide a general theory for the wellposedness of ultraweak formulations for boundary value problems. Spacetime formulations that consider time as part of the domain can be studied in the context of boundary value problems, where initial and boundary conditions are incorporated into boundary conditions of the spacetime domain. In order to seek solutions in suitable spaces, allowing source terms in a non-regular space, we study the unbounded differential operators related to the problem. We state, in a general context, what are the necessary conditions for the weak formulations to be wellposed. Moreover, we show a strategy to avoid the explicit use of traces, which are difficult to define in non-standard graph spaces. We are interested in applying these numerical techniques for the Schrödinger and wave equation.

This dissertation consists of five chapters. In Chapter 1, we introduce an abstract setting for a general boundary value problem, define non-standard Hilbert spaces on graph norms related to unbounded operators that include the boundary conditions of the problems in a weak sense, and show the sufficient conditions for a broken weak formulation to be wellposed. In Chapter 2 we introduce a theory for the wellposedness of local problems in tent-shaped domains for linear hyperbolic problems, and prove trace lemmas for these operators in a weak sense. In Chapter 3 we present an explicit finite element discretization of the tent pitching type for solving the wave equation,

and study the conditions to obtain a conforming method, based on the weak continuity property of the traces of functions in the lowest order case. In Chapter 4, we apply the theory developed in Chapter 1 to explore a weak formulation and a numerical technique for the second order Schrödinger equation with Dirichlet boundary conditions. From a theoretical point of view, the study of the Schrödinger equation gives an important example of a partial differential equation that should not be reformulated as a first order system. This is the reason for going beyond traditional Friedrichs systems [30] in Chapter 1. In Chapter 5, we study a suitable DPG formulation for the wave equation, following the approach laid out in our study of the DPG method for the Schrödinger equation. One important contribution of this chapter is the density result of smooth functions with boundary conditions for the space where we seek the solutions in any d -dimensional hyperrectangle. The key results presented here can also be found in the following journal articles [40], [16] and [37], which were written during this doctoral study.

Abstract ultraweak formulation

In this chapter, we are interested in deriving and identifying the sufficient conditions for the wellposedness of a mesh-dependent weak formulation of a boundary value problem involving a general partial differential operator. To motivate this framework, we will study and apply this abstract framework to the Laplace problem throughout this chapter.

1.1. Abstract setting

Let $\Omega \subseteq \mathbb{R}^d$ be a bounded non-empty open set in $d \geq 1$ dimensions, and let the operator $A : L^2(\Omega)^m \rightarrow L^2(\Omega)^l$ be a (unbounded) linear operator such that the i th component of Au is

$$(A-a) \quad [Au]_i = \sum_{j=1}^m \sum_{|\alpha| \leq k} \partial^\alpha (a_{ij\alpha} u_j),$$

where l, m, k are integers, $a_{ij\alpha} : \Omega \rightarrow \mathbb{C}$ are complex valued functions for all $i = 1, \dots, l$, $j = 1, \dots, m$, and all multi-indices $\alpha = (\alpha_1, \dots, \alpha_d)$ whose length is $|\alpha| = \alpha_1 + \dots + \alpha_d \leq k$. As usual, $\partial^\alpha = \partial_1^{\alpha_1} \dots \partial_d^{\alpha_d}$, $\mathcal{D}(\bar{\Omega})$ denote the restrictions of functions from $\mathcal{D}(\mathbb{R}^d)$ to Ω . Here, $\mathcal{D}(\mathbb{R}^d)$ denotes the space of C^∞ scalar-valued functions with compact support, and $\mathcal{D}'(\Omega)$ denotes the space of distributions. $L^2(\Omega)^m$ represents the space of vector functions where each of their m -components is in $L^2(\Omega)$. The formal adjoint of A is given by A^* , and its j th component is given by

$$(1.1.1) \quad [A^*v]_j = \sum_{i=1}^l \sum_{|\alpha| \leq k} (-1)^{|\alpha|} \overline{a_{ji\alpha}} \partial^\alpha v_i.$$

We assume that the coefficients $a_{ij\alpha}$ are such that

$$(A-b) \quad A^*u \in \mathcal{D}'(\Omega)^m \quad \text{for all } u \in L^2(\Omega)^l.$$

EXAMPLE 1.1.1. As motivation, we will study and apply this setting to the Laplace problem throughout this chapter. We are interested in finding u such that

$$(1.1.2) \quad \begin{aligned} -\operatorname{div}(\nabla u) &= f && \text{on } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

with $f \in L^2(\Omega)$. Here, Ω is a Lipschitz domain and $\partial\Omega$ denotes its boundary. Setting $\sigma = \nabla u$, we can rewrite the second order problem (1.1.2) and obtain the following equivalent first order system. Now we seek σ and u such that

$$(1.1.3a) \quad \sigma - \nabla u = 0 \quad \text{on } \Omega,$$

$$(1.1.3b) \quad -\operatorname{div}(\sigma) = f \quad \text{on } \Omega,$$

$$(1.1.3c) \quad u = 0 \quad \text{on } \partial\Omega.$$

Introducing a new vector-unknown $w = [\sigma, u]^T$, the diagonal identity operator $I_{d \times d}$, and defining the unbounded operator $A : L^2(\Omega)^{d+1} \rightarrow L^2(\Omega)^{d+1}$ such that

$$(1.1.4) \quad Aw = \begin{bmatrix} I_{d \times d} & -\nabla \\ -\operatorname{div} & 0 \end{bmatrix} w,$$

we can rewrite the equations (1.1.3a) into the equivalent problem of finding a vector function w such that

$$Aw = \begin{bmatrix} 0 \\ f \end{bmatrix}.$$

In this example, the operator A is of the form (A-a) when setting $m = l = d + 1$ and $k = 1$. Since its formal adjoint is given by $A^*w = \begin{bmatrix} I_{d \times d} & \nabla \\ \text{div} & 0 \end{bmatrix} w$, then condition (A-b) is also satisfied.

□

1.2. The graph spaces and boundary operator

In this section, we introduce spaces of functions related to the operators defined in (A-a) and (A-b). For any non-empty open subset $S \subseteq \Omega$, we define the space

$$(1.2.1) \quad W(S) = \{u \in L^2(S)^m : Au \in L^2(S)^l\},$$

normed by $\|u\|_{W(S)} = (\|u\|_S^2 + \|Au\|_S^2)^{1/2}$, and the space

$$(1.2.2) \quad W^*(S) = \{u \in L^2(S)^l : A^*u \in L^2(S)^m\}$$

normed by $\|u\|_{W^*(S)} = (\|u\|_S^2 + \|A^*u\|_S^2)^{1/2}$. Here and throughout, $(\cdot, \cdot)_S$ and $\|\cdot\|_S$ denotes the inner product and the norm, respectively, in $L^2(S)$ or its Cartesian products. To simplify the notation, we abbreviate $W = W(\Omega)$, $W^* = W^*(\Omega)$. Clearly these are inner product spaces.

LEMMA 1.2.1. *The spaces $W(S)$ and $W^*(S)$ are Hilbert spaces.*

PROOF. Since the proofs for $W(S)$ and $W^*(S)$ are similar, we only show the first. Suppose u_n is a Cauchy sequence in $W(S)$. Then u_n is Cauchy in $L^2(S)^m$ and Au_n is Cauchy in $L^2(S)^l$. Hence there is a $u \in L^2(S)^m$ and $f \in L^2(S)^l$ such that $\|u - u_n\|_S \rightarrow 0$ and $\|f - Au_n\|_S \rightarrow 0$. We will show that u is in $W(S)$.

Let $\phi \in \mathcal{D}(S)^l$. For each $w \in L^2(S)^m$, the distributional action of Aw on $\bar{\phi}$, denoted by $\langle Aw, \phi \rangle_{\mathcal{D}(S)^l}$, equals $(w, A^*\phi)_S$ for all ϕ in $\mathcal{D}(S)^l$. Thus $Aw \in L^2(\Omega)^l$. To

complete the proof, note that

$$\langle Au, \phi \rangle_{\mathcal{D}(S)^l} = (u, A^* \phi)_S = \lim_{n \rightarrow \infty} (u_n, A^* \phi)_S = \lim_{n \rightarrow \infty} (Au_n, \phi)_S = (f, \phi)_S$$

for all ϕ in $\mathcal{D}(S)^l$. Hence $Au = f$ and u is in $W(S)$. $\square$

Next, define bounded linear operators $D_S : W(S) \rightarrow W^*(S)'$ and $D_S^* : W^*(S) \rightarrow W(S)'$ by

$$(1.2.3) \quad \langle D_S u, v \rangle_{W^*(S)} = (Au, v)_S - (u, A^* v)_S,$$

$$(1.2.4) \quad \langle D_S^* v, u \rangle_{W(S)} = (A^* v, u)_S - (v, Au)_S,$$

for all $u \in W(S)$ and $v \in W^*(S)$. When $S = \Omega$, we abbreviate D_S and D_S^* to D and D^* , respectively. Here and in the remainder, we use $\langle \cdot, \cdot \rangle_X$ to denote the action of a linear functional in X' on an element of X .

EXAMPLE 1.2.2. For the system considered in Example (1.1.1), $W = W^* = H(\operatorname{div}, \Omega) \times H^1(\Omega)$. Here, $H(\operatorname{div}, \Omega) = \{\sigma \in L^2(\Omega)^d : \operatorname{div} \sigma \in L^2(\Omega)\}$, and $H^1(\Omega) = \{u \in L^2(\Omega) : \nabla u \in L^2(\Omega)^d\}$. Using the well-known trace theorems for $H^1(\Omega)$ and $H(\operatorname{div}, \Omega)$ and Green identities, (see [7]) the identity

$$(1.2.5) \quad \langle Du, v \rangle_W = (Au, v) - (u, A^* v) = -\langle \hat{u}_{\sigma_n}, \hat{v}_\mu \rangle_{H^{1/2}(\partial\Omega)} - \langle \hat{v}_{\sigma_n}, \hat{u}_\mu \rangle_{H^{1/2}(\partial\Omega)}$$

holds for any $u = (u_\sigma, u_\mu)$ and $v = (v_\sigma, v_\mu) \in W$, where $\hat{u}_{\sigma_n}$, $\hat{v}_{\sigma_n}$ represent the well-known normal traces of u_σ, v_σ respectively, while $\hat{u}_\mu$ and $\hat{v}_\mu$ represent the traces of u_μ and v_μ respectively. Here and throughout, n represents the unit outward normal vector to the respective domain. $\square$

1.3. Selecting domains of differential operators

Next, we consider $A : \text{dom}(A) \subset L^2(\Omega)^m \rightarrow L^2(\Omega)^l$ as an unbounded linear operator whose domain (denoted by $\text{dom}(A)$) is chosen such that

$$(A-c) \quad \mathcal{D}(\Omega)^m \subseteq \text{dom}(A).$$

This implies that A is a densely defined operator, and therefore there exists a unique adjoint operator whose domain is given by

$$(1.3.1) \quad \{v \in L^2(\Omega)^l : \exists \ell \in L^2(\Omega)^m \text{ such that } (Au, v) = (u, \ell), \text{ for all } u \in \text{dom}(A)\}.$$

We take a moment to explain why we have chosen the notation A^* to denote the formal adjoint, which in the literature is used to define the the adjoint operator in the sense of unbounded operators. For reasons of clarity, we will very shortly assume that the adjoint operator is denoted by A' and show that they are equal on the domain of the adjoint operator.

Identifying the dual of (Cartesian products of) $L^2(\Omega)$ with itself, the domain of A' is given by (1.3.1) and it satisfies $(Au, v)_\Omega = (u, A'v)_\Omega$ for all $u \in \text{dom}(A)$ and for all v belonging to the set defined in (1.3.1). If we choose $u \in \mathcal{D}(\Omega)^m$, we can conclude that the distribution $A^*s = A's$, for all $s \in \text{dom}(A')$. Hence, since the adjoint operator is a closed linear operator uniquely defined after setting the domain of A on the set (1.3.1) (see [7]), we keep the notation of A^* to denote both: the differential operator in (1.1.1) and the formal adjoint of the unbounded operator A .

When $\text{dom}(A)$ is endowed with the topology of $W(\Omega)$, we call it V , i.e., although V and $\text{dom}(A)$ coincide as sets, V has the topology of $W(\Omega)$ and $\text{dom}(A)$ has the topology of $L^2(\Omega)^m$. Similarly, $\text{dom}(A^*)$ is called V^* when it is endowed with the topology of $W^*(\Omega)$. For the next result, recall that the left annihilator of any subspace

R of the dual space X' of any Banach space X is defined by ${}^\perp R = \{w \in X : \langle s', w \rangle_X = 0 \text{ for all } s' \in R\}$.

LEMMA 1.3.1. *In the setting of (A-a), (A-b) and (A-c), we have $V^* = {}^\perp D(V)$.*

PROOF. Let us first prove that $V^* \subseteq D(V)$. According to the (above-mentioned) definition of $\text{dom}(A^*)$, for any $v \in \text{dom}(A^*) = V^*$, there is an $\ell \in L^2(\Omega)^l$ such that

$$(1.3.2) \quad (Au, v)_\Omega = (u, \ell)_\Omega, \quad \forall u \in V.$$

Due to (A-c), we may choose u in $\mathcal{D}(\Omega)^m$. By (A-b), A^*v is a distribution and by (1.3.2) this distribution is in $L^2(\Omega)^m$ and equals ℓ . In particular, v is in $W^*(\Omega)$. Hence (1.2.3) is applicable, and in combination with (1.3.2) yields $\langle Du, v \rangle_{W^*(\Omega)} = (Au, v)_\Omega - (u, \ell)_\Omega = 0$ for all $u \in V$. Hence $v \in {}^\perp D(V)$ and we have proved that $V^* \subseteq {}^\perp D(V)$. The reverse containment is also easy to prove. Indeed, if $w \in {}^\perp D(V)$, then $w \in W^*$ then $\langle Du, w \rangle = 0$ for all $u \in \text{dom}(A)$. Thus, setting $\ell = A^*w$, we obtain $(Au, w)_\Omega - (u, \ell)_\Omega = 0$ for all $u \in V$. Then $w \in V^*$. $\square$

EXAMPLE 1.3.2. Let u, v be in $W = H(\text{div}, \Omega) \times H^1(\Omega)$ in our model problem (1.1.2), where $u = (u_\sigma, u_\mu)$ and $v = (v_\sigma, v_\mu)$. (1.2.5) We define $V = H(\text{div}, \Omega) \times H_0^1(\Omega)$. It is well known that (A-c) holds, and that V is a subspace of $L^2(\Omega)^{d+1}$. Therefore, the adjoint operator is uniquely defined on

$$(1.3.3) \quad \text{dom}(A^*) = \{v = (v_\sigma, v_\mu) \in L^2(\Omega)^{d+1} : \exists \ell \in L^2(\Omega)^{d+1} \text{ such that}$$

$$(Au, v) = \left(\begin{bmatrix} u_\sigma - \nabla u_\mu \\ -\text{div } u_\sigma \end{bmatrix}, v \right) = (u, \ell), \quad \forall u \in V \}.$$

Since the distribution $A^*v \in L^2(\Omega)^{d+1}$ for all $v \in \text{dom}(A^*)$, then $\text{dom}(A^*)$ is a subspace of W^* , and defined as V^* , as a space with the topology of W^* .

From (1.2.5) and (1.3.3) we have

$$V^* = H(\operatorname{div}, \Omega) \times H_0^1(\Omega).$$

Thus, $V = V^*$. □

1.4. A mesh dependent weak formulation

We are interested in the boundary value problem of finding u satisfying

$$(1.4.1) \quad Au = f, \quad u - g \in V,$$

given $f \in L^2(\Omega)^l$ and $g \in W$. Homogeneous boundary conditions are incorporated in V . In this way, spacetime problems with non-homogeneous spacetime boundary conditions can be abstracted into this form. When $l = m$, it makes sense to talk about the solvability of the adjoint problem: finding $u \in W^*$, such that $u - g^* \in V^*$, with $g^* \in W^*$, satisfying $A^*u = f$.

Consider the scenario where Ω is partitioned into a mesh Ω_h of finitely many open elements K such that $\bar{\Omega} = \cup_{K \in \Omega_h} \bar{K}$. Here the index h denotes $\max_{K \in \Omega_h} \operatorname{diam}(K)$. Recall D_K and D_K^* by replacing S by K in (1.2.3) and (1.2.4).

Additionally, set

$$(1.4.2) \quad W_h = \prod_{K \in \Omega_h} W(K), \quad (W_h^*)' = \prod_{K \in \Omega_h} W^*(K)'.$$

The spaces W_h^* and W_h' are defined similarly. The component on an element K of functions in such product spaces are indicated by placing K as a subscript, e.g., for any w in W_h , the component of w on element K is denoted by w_K . Let $D_h : W_h \rightarrow (W_h^*)'$

be the continuous linear operator defined by

$$\langle D_h w, v \rangle_{W_h^*} = \sum_{K \in \Omega_h} \langle D_K w_K, v_K \rangle_{W^*(K)}$$

for all $w \in W_h$ and $v \in W_h^*$. To simplify notation, we abbreviate $\langle D_h w, v \rangle_{W_h^*}$ to $\langle D_h w, v \rangle_h$, i.e., duality pairing in W_h^* is simply denoted by $\langle \cdot, \cdot \rangle_h$. For any $w \in W_h$, we denote by $A_h w$ the function obtained by applying A to w_K , element by element, for all $K \in \Omega_h$. The resulting function $A_h w$ is an element of $\Pi_{K \in \Omega_h} L^2(K)^l$, which is identified to be the same as $L^2(\Omega)^l$. The operator $A_h^* : W_h^* \rightarrow L^2(\Omega)^m$ is defined similarly. Thus

$$(1.4.3) \quad \langle D_h w, v \rangle_h = (A_h w, v)_\Omega - (w, A_h^* v)_\Omega$$

for all $w \in W_h$ and $v \in W_h^*$.

LEMMA 1.4.1. *For all $w \in W$ and $v \in W^*$, we have $\langle D_h w, v \rangle_h = \langle Dw, v \rangle_{W^*}$.*

PROOF. If $w \in W$ and $v \in W^*$, then $A_h w = Aw$ and $A_h^* v = A^* v$. Using this in (1.4.3), $\langle D_h w, v \rangle_h = (Aw, v)_\Omega - (w, A^* v)_\Omega = \langle Dw, v \rangle_{W^*}$ whenever w is in W and v is in W^* . $\square$

To derive the mesh-dependent weak formulation, multiply (1.4.1) by a test function $v \in W_h^*$ and apply the definition of D_K . Summing over all $K \in \Omega_h$, we obtain $(u, A_h^* v)_\Omega + \langle D_h(u - g), v \rangle_h = (f, v)_\Omega - \langle Dg, v \rangle_{W^*}$ for all v in W_h^* . Let

$$(1.4.4) \quad Q = \{r \in (W_h^*)' : \text{there is a } u \in V \text{ such that } r = D_h u\}.$$

Setting $D_h u$ to be a new unknown q in Q and defining the sesquilinear form $b((u, q), v) = (u, A_h^* v)_\Omega + \langle q, v \rangle_h$ on $(L^2(\Omega)^m \times Q) \times W_h^*$, we have thus derived the following weak formulation.

PROBLEM 1.4.2. (*Ultraweak formulation*) Given any $F \in (W_h^*)'$, find $u \in L^2(\Omega)^m$ and $q \in Q$ such that

$$(1.4.5) \quad b((u, q), v) = F(v), \text{ for all } v \in W_h^*.$$

EXAMPLE 1.4.3. For our model problem introduced in (1.1.2), an exact characterization of Q is possible. This is derived by the well known trace theorems of $H^1(\Omega)$ and $H(\operatorname{div}, \Omega)$ as mentioned in [38]. Namely,

$$(1.4.6) \quad Q \equiv H^{-1/2}(\partial\Omega_h) \times H_0^{1/2}(\partial\Omega),$$

where

$$H^{-1/2}(\partial\Omega_h) = \{\eta \in \prod_K H^{-1/2}(\partial K) : \exists q \in H(\operatorname{div}, \Omega) \text{ such that } \eta|_{\partial K} = q \cdot n|_{\partial K}, \forall K \in \Omega_h\},$$

$$H^{1/2_0}(\partial\Omega_h) = \{u \in \prod_K H^{1/2}(\partial K) : \exists w \in H_0^1(\Omega) \text{ such that } u|_{\partial K} = w|_{\partial K}, \forall K \in \Omega_h\}.$$

We define

$$(1.4.8) \quad Q^* = \{r \in (W_h)' : \text{ there is a } v \in V^* \text{ such that } r = D_h^* v^*\}.$$

In the same way, setting $D_h^* v = \tilde{q}$ and defining the sesquilinear form $b^*((v, \tilde{q}), u) = (v, A_h u)_\Omega + \langle \tilde{q}, u \rangle_{W_h}$ on $(L^2(\Omega)^l \times Q) \times W_h$, the adjoint weak formulation derived from the dual problem reads as

PROBLEM 1.4.4. (*Adjoint ultraweak formulation when $l = m$*) Given any $F^* \in (W_h)'$, find $u \in L^2(\Omega)^l$ and $\tilde{q} \in Q^*$ such that

$$(u, A_h v)_\Omega - \langle \tilde{q}, v \rangle_h = F^*(v), \quad \forall v \in W_h.$$

THEOREM 1.4.5. *In the setting of (A-a), (A-b) and (A-c), suppose*

$$(1.4.9) \quad V = {}^\perp D^*(V^*), \text{ and}$$

$$(1.4.10) \quad A : V \rightarrow L^2(\Omega)^l \text{ is a bijection.}$$

Then, Problem 1.4.2 is well posed. Moreover, if $F(v) = (f, v)_\Omega + \langle D_h g, v \rangle_h$ for some $f \in L^2(\Omega)^l$ and $g \in W$, then the solution (u, q) of Problem 1.4.2 is such that $u - g \in V$ and $q = D_h u$.

THEOREM 1.4.6. *Analogously, if (A-a), (A-b) and (A-c) hold, and*

$$(1.4.11) \quad V = {}^\perp D^*(V^*), \text{ and}$$

$$(1.4.12) \quad A^* : V^* \rightarrow L^2(\Omega)^m \text{ is a bijection,}$$

then, Problem 1.4.4 is wellposed. Moreover, if $F(v) = (f, v)_\Omega$ for some $f \in L^2(\Omega)^m$, then the unique solution u of Problem 1.4.2 is in V , solves (1.4.1), and satisfies $\tilde{q} = D_h^ u$.*

Before we prove Theorem 1.4.5, we must note how our assumptions allow a natural topology on Q and Q^* . Specifically, (1.4.9) implies that V is a closed subspace of W . It is also a closed subspace of W_h , since W is continuously embedded in W_h . The same embedding also shows that the restriction of D_h to V , denoted by $D_{h,V} : V \rightarrow (W_h^*)'$, is continuous. Note that Q is the range of $D_{h,V}$. For any r in Q , we use $D_{h,V}^{-1}(\{r\})$ to denote the pre-image of r , i.e., set of all $v \in V$ such that $r = D_h v$. The continuity of $D_{h,V}$ implies that $D_{h,V}^{-1}(\{0\})$ is a closed subspace of V . Hence

$$(1.4.13) \quad \|q\|_Q = \inf_{v \in D_{h,V}^{-1}(\{q\})} \|v\|_W$$

is a norm on Q . This quotient norm makes Q complete. Similarly,

$$(1.4.14) \quad \|q\|_{Q^*} = \inf_{v \in D_{h,V^*}^{*-1}(\{q\})} \|v\|_{W^*}$$

is a norm with which Q^* is complete. The wellposedness result of Theorem 1.4.5 is to be understood with Q endowed with this norm.

1.4.1. A proof of wellposedness. We now give a proof of Theorem 1.4.5. Recall that the right annihilator of any subspace $S \subseteq X$ is defined by $S^\perp = \{w' \in W' : \langle w', s \rangle_W = 0 \text{ for all } s \in S\}$. The next lemma is used below to prove uniqueness.

LEMMA 1.4.7. *If (1.4.9) holds, then $D_h V \subseteq (V^*)^\perp$.*

PROOF. Let $u \in V \subseteq W_h$. Then for any $v \in V^*$, the functional $D_h u \in (W_h^*)'$ satisfies $\langle D_h u, v \rangle_h = \langle Du, v \rangle_{W^*}$ by Lemma 1.4.1. But $\langle Du, v \rangle_{W^*} = -\overline{\langle D^* v, u \rangle_W} = 0$ since (1.4.9) shows that $u \in {}^\perp D^*(V^*)$. Hence $D_h u \in (V^*)^\perp$. $\square$

PROOF OF THEOREM 1.4.5. We verify the uniqueness and inf-sup conditions of the Babuška-Brezzi theory to obtain wellposedness. To verify the uniqueness condition, we must prove that if

$$(1.4.15) \quad (u, A_h^* v)_K + \langle q, v \rangle_h = 0, \quad \forall v \in W_h^*,$$

then u and q vanishes. Since $q = D_h z$ for some $z \in V$, by virtue of Lemma 1.4.7, $\langle q, v \rangle_h = 0$ for any v in V^* . Hence (3.3.2) implies

$$(1.4.16) \quad (u, A_h^* v)_\Omega = 0, \quad \forall v \in V^*.$$

In particular, since $\mathcal{D}(\Omega)^l \subseteq V^*$, this implies that $Au = 0$ and therefore $u \in W$. Hence (1.2.3) and (1.4.16) imply $\langle Du, v \rangle_{W^*} = 0$, or equivalently $\langle D^* v, u \rangle_W = 0$ for all $v \in V^*$. Thus $u \in {}^\perp D^*(V^*) = V$. The bijectivity of $A : V \rightarrow L^2(\Omega)^l$ then implies

that $u = 0$. Returning to (3.3.2) and setting $u = 0$, we see that $\langle q, v \rangle_h = 0$ for all $v \in W_h^*$, so $q = 0$ as well.

It only remains to prove the inf-sup condition

$$(1.4.17) \quad \|w\|_{W_h^*} \leq C_1 \sup_{0 \neq x \in X} \frac{|b(x, w)|}{\|x\|_X},$$

where $X = L^2(\Omega)^m \times Q$ and $b((u, q), w) = (u, A_h^* w)_\Omega + \langle q, w \rangle_h$. Given any $w \in W_h^* \subseteq L^2(\Omega)^l$, we use the bijectivity of $A : V \rightarrow L^2(\Omega)^l$ and the Banach Open Mapping theorem to obtain a v in V satisfying $Av = w$, and $\|v\|_W \leq C\|w\|_{W_h^*}$. Then, setting $z = v + A_h^* w$ and $r = D_h v$, we have $\|r\|_Q \leq \|v\|_W \leq C\|w\|_{W_h^*}$ and $\|z\|_\Omega \leq (C + 1)\|w\|_{W_h^*}$. Here and throughout this dissertation $C > 0$ denotes an arbitrary constant. Hence

$$\begin{aligned} \|w\|_{W_h^*}^2 &= (A_h v, w)_\Omega + (A_h^* w, A_h^* w)_\Omega = (v + A_h^* w, A_h^* w)_\Omega + \langle D_h v, w \rangle_h \\ &= \|(z, r)\|_X \frac{b((z, r), w)}{\|(z, r)\|_X} \leq C_1 \|w\|_{W_h^*} \sup_{0 \neq x \in X} \frac{|b(x, w)|}{\|x\|_X}, \end{aligned}$$

where C_1 depends only on C . Hence (1.4.17) follows. $\square$

REMARK 1.4.8. Various elements of the arguments used in this proof are well-known in the DPG literature – see e.g., [21, Section 6.2]. A generalization of these ideas to make a unified theory for DPG approximations of all Friedrichs systems was attempted in [8]. However, [8, equation (2.17)] is not correct (a counterexample is easily furnished by the Laplace example) and unfortunately that equation is used in [8, Lemma 2.4 and Corollary 2.5] to prove the existence of a solution for Problem 1.4.2. The above proof provides a corrigendum to [8] and shows that the results claimed there for symmetric Friedrichs systems are indeed correct for operators of the form (A-a) with $k = 1$ and with V and V^* set respectively to the null spaces of the operators $B - M$ and $B + M^*$ defined there.

REMARK 1.4.9. The above analysis is applicable beyond Friedrichs systems, as the example of Schrödinger equation in Chapter 4 will show. “Instead of working with one equation of higher than first order,” writes Friedrichs in his early work [36], “we prefer to work with a system of equations of first order.” The modern theory of Friedrichs systems (for operators of the form (A-a) with $l = m$) starts with the assumption that $\|(A + A^*)\phi\|_\Omega \leq C\|\phi\|_\Omega$ for all $\phi \in \mathcal{D}(\Omega)^l$ – see [30, equation (T2)]. This assumption does not hold always, and that is the case for the Schrödinger operator.

1.4.2. An alternate proof of wellposedness. Another proof of Theorem 1.4.5 can be obtained using the following two lemmas.

LEMMA 1.4.10. $V^* = \{y \in W_h^* : \langle q, y \rangle_h = 0 \text{ for all } q \in Q\}$.

PROOF. If $y \in V^*$, then for any $z \in V$, using Lemmas 1.4.1 and 1.3.1, we have $\langle D_h z, y \rangle_h = \langle D z, y \rangle_{W^*} = 0$, i.e., $\langle q, y \rangle_h = 0$ for all $q \in Q$.

To prove the reverse containment, let $y \in W_h^*$ satisfy $\langle D_h z, y \rangle_h = 0$ for all $z \in V$. For any $\phi \in \mathcal{D}(\Omega)^m$, the distribution A^*y satisfies $\langle A^*y, \phi \rangle_{\mathcal{D}(\Omega)^m} = (y, A\phi)_\Omega = (A_h^*y, \phi)_\Omega + \overline{\langle D_h \phi, y \rangle_h}$. The last term is zero, because by (A-c), $\mathcal{D}(\Omega)^m \subseteq \text{dom}(A) = V$. Hence $A^*y = A_h^*y$ and y is in W^* . Thus by Lemma 1.4.1, $\langle D_h z, y \rangle_h = \langle D z, y \rangle_W = 0$, so $y \in {}^\perp D(V)$. Hence y is in V^* by Lemma 1.3.1. $\square$

LEMMA 1.4.11. Suppose (1.4.9) holds. Then, for all $q \in Q$,

$$\inf_{v \in D_{h,V}^{-1}(\{q\})} \|v\|_W = \sup_{0 \neq y \in W_h^*} \frac{|\langle q, y \rangle_h|}{\|y\|_{W_h^*}}.$$

PROOF. The supremum, denoted by s , is attained by the function $\tilde{u}_q$ in W_h^* satisfying

$$(1.4.18) \quad (A_h^* \tilde{u}_q, A_h^* y)_\Omega + (\tilde{u}_q, y)_\Omega = -\langle q, y \rangle_h, \quad \forall y \in W_h^*, \text{ and}$$

$$(1.4.19) \quad s = \|\tilde{u}_q\|_{W_h^*}.$$

Choosing $y \in \mathcal{D}(\Omega)^l$ in (1.4.18), we conclude that the distribution $A(A_h^* \tilde{u}_q)$ is in $L^2(\Omega)^l$. Hence (1.4.3) is applicable with $w = A_h^* \tilde{u}_q$ and we obtain

$$(1.4.20a) \quad A_h A_h^* \tilde{u}_q + \tilde{u}_q = 0$$

$$(1.4.20b) \quad D_h A_h^* \tilde{u}_q = q.$$

Now let $u_q = A_h^* \tilde{u}_q$. Then (1.4.20a) implies $A_h u_q = -\tilde{u}_q$, which implies $A_h^* A_h u_q = -A_h^* \tilde{u}_q = -u_q$. Combining with (1.4.20b), we have

$$(1.4.21a) \quad A_h^* A_h u_q + u_q = 0$$

$$(1.4.21b) \quad D_h u_q = q.$$

Next, we show that u_q is in V . By (1.4.9), it suffices to prove that $u_q \in {}^\perp D^*(V^*)$. For any $\tilde{v}$ in V^* , we have, using Lemma 1.4.1, $\langle D^* \tilde{v}, u_q \rangle_W = -\overline{\langle D u_q, \tilde{v} \rangle_{W^*}} = -\overline{\langle D_h u_q, \tilde{v} \rangle_h} = -\overline{\langle q, \tilde{v} \rangle_h}$. The last term is zero because $q = D_h z$ for some $z \in V$ and $\langle q, \tilde{v} \rangle = \langle D z, \tilde{v} \rangle_{W^*} = 0$ by Lemma 1.3.1. Hence $u_q \in {}^\perp D^*(V^*) = V$.

The infimum of the lemma is $\|q\|_Q$. By virtue of (1.4.19), to complete the proof, it suffices to show that $\|q\|_Q = \|u_q\|_W = \|\tilde{u}_q\|_{W_h^*}$. The last equality is obvious from $u_q = A_h^* \tilde{u}_q$ and $A_h u_q = -\tilde{u}_q$. Hence we need only show that $\|q\|_Q = \|u_q\|_W$. Standard variational arguments show that the infimum defining $\|q\|_Q$ is attained by a unique minimizer $v_q \in V$ satisfying $\|q\|_Q = \|v_q\|_W$, $D_h v_q = q$ and $(A_h v_q, A_h v)_\Omega + (v_q, v)_\Omega = 0$ for all $v \in D_{h,V}^{-1}(\{0\})$. Choosing a v in $\mathcal{D}(K)$ (whose extension by zero is in $D_{h,V}^{-1}(\{0\})$),

we conclude that distribution $A^*(A_h v_q)|_K$ is in $L^2(K)^m$ for any $K \in \Omega_h$. Therefore $A_h^* A_h v_q$ is in $L^2(\Omega)^m$. In view of (1.4.21), this means that $v_q = u_q$. $\square$

SECOND PROOF OF THEOREM 1.4.5. According to [9, Theorem 3.3], it suffices to prove that there are positive constants $c_0, \hat{c}$ such that

$$(1.4.22) \quad c_0 \|u\|_\Omega \leq \sup_{0 \neq y \in Y_0} \frac{|(u, A_h^* y)_\Omega|}{\|y\|_{W_h^*}} \quad \forall u \in L^2(\Omega)^m,$$

$$(1.4.23) \quad \hat{c} \|q\|_Q \leq \sup_{0 \neq y \in W_h^*} \frac{|\langle q, y \rangle_h|}{\|y\|_{W_h^*}} \quad \forall q \in Q.$$

where $Y_0 = \{y \in W_h^* : \langle q, y \rangle_h = 0 \text{ for all } q \in Q\}$.

Since (1.4.23) follows with $\hat{c} = 1$ from Lemma 1.4.11, we only need to prove (1.4.22). First note that since V is closed (by (1.4.9)), A is a closed operator. By (1.4.10), the range of A is closed. By the Closed Range Theorem for closed operators, range of A^* is closed. Also, the well-known identity $\ker(A^*) = \text{ran}(A)^\perp$, in combination with (1.4.10), implies that A^* is injective. Hence there exists a $C > 0$ such that

$$(1.4.24) \quad C \|y\|_\Omega \leq \|A^* y\|_\Omega \quad \forall y \in \text{dom}(A^*) = V^*.$$

This implies the following inf-sup condition:

$$C \|y\|_{W^*} \leq \sup_{u \in L^2(\Omega)^m} \frac{|(u, A^* y)_\Omega|}{\|u\|_\Omega} \quad \forall y \in V^*.$$

To complete the proof, we note that by standard arguments the order of arguments in the inf and sup may be reversed to get

$$\inf_{u \in L^2(\Omega)^m} \sup_{y \in V^*} \frac{|(u, A^* y)_\Omega|}{\|u\|_\Omega \|y\|_{W^*}} = \inf_{y \in V^*} \sup_{u \in L^2(\Omega)^m} \frac{|(u, A^* y)_\Omega|}{\|u\|_\Omega \|y\|_{W^*}} \geq C.$$

By Lemma 1.4.10, $Y_0 = V^*$, thus completing the proof of (1.4.22). $\square$

REMARK 1.4.12. The idea behind Lemma 1.4.11 (to consider the two related problems (1.4.20) and (1.4.21), one with essential boundary conditions and the other with natural boundary conditions) was first presented in [9, 11], tailored to the specific needs of a Maxwell problem. A generalization for first order operators was presented later in [68]. The argument to prove (1.4.22) using the Closed Range Theorem, was first presented for the case of first order Sobolev spaces in [9, Theorem 6.6].

The last part of the theorem can be obtained by expressing F in terms of f as in the setting of Theorem 1.4.5. Then choosing $\phi \in \mathcal{D}(\Omega)^l$ we have

$$(1.4.25) \quad (u, A_h^* \phi)_\Omega + \langle q, \phi \rangle_h = (f, \phi)_\Omega + \langle D_h g, \phi \rangle_h.$$

Thus, the distribution Au is in $L^2(\Omega)^l$ and, by density that $Au = f \in L^2(\Omega)^l$, and $u \in W$. Returning to Problem 1.4.2 we only need to prove that $u \in {}^\perp D^*(V^*)$.

We have, $0 = -\overline{\langle q, v^* \rangle}_h = -\overline{\langle D_h(u - g), v^* \rangle}_h = -\overline{\langle D(u - g), v^* \rangle}_{W^*} = \langle D^* v^*, u - g \rangle_W$ for all $v^* \in V^*$. Thus $u \in V = {}^\perp D^*(V^*)$ and $q = D_h u$.

REMARK 1.4.13. A similar proof follows for the adjoint problem when $l = m$.

Next, we provide some results to prove the bijectivity condition that allow us to prove the bijectivity condition. Note that in order to prove that Assumption 1.4.10 holds in Theorem 1.4.5, we could alternatively show:

THEOREM 1.4.14. *Suppose that (1.4.9) hold. If there is a constant $C > 0$ such that $A : V \rightarrow L^2(\Omega)^l$ and $A^* : V^* \rightarrow L^2(\Omega)^m$ satisfies*

$$(1.4.26a) \quad \|Au\|_\Omega \geq C\|u\|_W \quad \forall u \in V, \text{ and}$$

$$(1.4.26b) \quad \|A^*v\|_\Omega \geq C\|v\|_W \quad \forall v \in V^*,$$

then (1.4.10) holds, i.e., $A : V \rightarrow L^2$ is a bijection.

PROOF. The inequality (1.4.26a) and the closed range theorem for closed operators imply that $A : \text{dom}(A) = V \rightarrow L^2(\Omega)^l$ is injective and has closed range. Moreover, its adjoint A^* is injective (on its domain) by (1.4.26b), so the range of A must be all of $L^2(\Omega)^l$ (see e.g., [7, Corollary 2.18]). Hence A is a bijection, *i.e.*, condition (1.4.10) holds. □

REMARK 1.4.15. Note that (1.4.26a) gives *a-priori* error estimate for the strong problem, since $C\|u\|_\Omega \leq \|f\|_\Omega$.

1.5. The ideal DPG method

The ideal DPG method seeks u_h and q_h in finite-dimensional subspaces of $U_h \subset L^2(\Omega)^{d+1}$ and $Q_h \subset Q$, satisfying

$$(1.5.1) \quad b((u_h, q_h), w_h) = F(w_h), \quad \text{for all } w_h \in T(U_h \times Q_h).$$

Here $T : L^2(\Omega) \times Q \rightarrow W_h$ is defined by $(T(z, r), v)_h = b((z, r), v)$ for all $v \in W_h$ and $(z, r) \in W_h$.

The main feature of the ideal DPG method is that the wellposedness of Problem 1.4.2 implies quasioptimality of the method's error [19]. Moreover, it is well known [20] that there is a mixed method formulation that is equivalent to the above Petrov-Galerkin formulation (1.5.1). One of the variables in this mixed method is the error representation function $\varepsilon_h \in W_h^*$ defined by

$$(1.5.2) \quad (\varepsilon_h, w)_{W_h^*} = (f, w) - b((u_h, q_h), w), \quad \text{for all } w \in W_h^*.$$

One of the two equations in the mixed formulation given below is a restatement of this defining equation for $\varepsilon_h \in W_h^*$. The mixed formulation seeks $\varepsilon_h \in W_h^*$ and

$(u_h, q_h) \in U_h \times Q_h$ such that

$$\begin{aligned} (\varepsilon_h, w)_{W_h^*} + b((u_h, q_h), w) &= F(w) && \text{for all } w \in W_h, \\ b((z, r), \varepsilon_h) &= 0 && \text{for all } (z, r) \in U_h \times Q_h. \end{aligned}$$

The main difficulty of the ideal method is that in order to compute u_h , one needs a basis for W_h , which must be obtained by applying T .

1.5.1. An Example of wellposed problem under this setting. For our model problem (1.1.3a), and following [18], we have $V = V^* = H(\text{div}, \Omega) \times H_0^1(\Omega)$, and A is given in by (1.1.4).

LEMMA 1.5.1. *Problem (1.4.2) is wellposed for this example, i.e., assumptions (1.4.9) and (1.4.10) of Theorem 1.4.5 hold.*

PROOF. Since $V = V^*$. It is easy to see, from the definition of D in (1.2.5), that $V = {}^\perp D^*(V)$. Therefore, (1.4.9) is satisfied. Clearly, $A: V \rightarrow L^2(\Omega)$ satisfies the conditions 1.4.26 in Theorem 1.4.14, and therefore (1.4.10) is satisfied. Indeed, for any $u = (u_\sigma, u_\mu) \in V$ we have that $\|u_\sigma\| \leq \|u_\sigma - \nabla u_\mu\| + \|\nabla u_\mu\|$. The Poincaré inequality implies that $\|u_\mu\| \leq C\|\nabla u_\mu\|$ for all $u_\mu \in H_0^1(\Omega)$. Therefore $\|u\| \leq \|Au\|$ for any $u \in V$. The proof of $A^*: V \rightarrow L^2(\Omega)$ satisfying condition (1.4.26b) is analogous. $\square$

REMARK 1.5.2. If we are able to prove trace theorems for the non-standard Hilbert spaces, we can characterize exactly the space Q . For example, in our model problem we have (1.4.6).

1.5.1.1. *Numerical results for Example 1.1.1.* In this section we present numerical results of the ideas described in this chapter for the broken weak formulation of the Laplace problem (1.1.2), by the ideal DPG method.

For our model example, we consider $\Omega = (0, 1)^2$, and $\{\Omega_h\}_{h \in I}$ a family of conforming shape regular simplicial finite element triangulations of Ω . Define

$$U_h = \{u \in L^2(\Omega)^{d+1} : u|_K \in P_p(K)^{d+1} \text{ for all } K \in \Omega_h\},$$

$$V_h = \{u \in H_0^1(\Omega)^{d+1} \cap C^0(\bar{\Omega})^{d+1} : u|_K \in P_{p+1}(K)^{d+1} \text{ for all } K \in \Omega_h\},$$

where $P_p(K)$ is the space of polynomials of total degree $\leq p$ on K and $C^0(D)$ denote the set of continuous functions on any domain D , as in [38]. It is well known that $Q_h = D_h(V_h)$ is given by

$$Q_h = \{q \in Q : q_{\sigma_n} \in P_p(\partial K), \quad q_\mu|_{\partial K} \in \tilde{P}_{p+1}(\partial K)\}$$

where $\tilde{P}_{p+1}(\partial K) = P_p(\partial K) \cap C^0(\partial K)$ and $P_p(\partial K) = \{u : u|_F \in P_p(F) \text{ for all facets } F \in \partial K\}$.

We now provide convergence rates for the ultraweak formulation when solving

$$-\operatorname{div}(\operatorname{grad} \phi) = 32\pi^2 \sin(\pi x) \sin(\pi y),$$

on the unit square with homogeneous Dirichlet boundary conditions. The exact solution to this problem is given by

$$\phi_{exact}(x, y) = 16 \sin(\pi x) \sin(\pi y).$$

We considered a coarse triangular mesh of Ω , and successive refinements of the mesh by connecting the mid points of the edges. In Figure 1.1 we observe that the order of convergence of this experiment is optimal, see [38]. The implementation has been done using the mixed form in(1.5.2) and the finite element software NGSolve [60].

Figure 1.1. Rates of convergence for u in the L^2 -norm using a triangular mesh.

1.6. Defining the V and V^* spaces from spaces of smooth functions

Given a boundary value problem of the form (1.4.1), the first question that arises is: how to define the maximal space of functions where we can find a solution. We might think that, if the right hand side is smooth enough, then the solution will also need to be smooth. This brings the following ideas to the first attempt of finding the spaces V and V^* .

Suppose that V can be obtained by a density result from smooth functions satisfying the boundary conditions of the given problem, and that $A : V \rightarrow L^2(\Omega)^l$ is a bijection. Then we have an exact characterization of V , and many properties can be proved by density arguments.

Consider the boundary value problem $Au = f$, where $u \in L^2(\Omega)^m$ satisfies some boundary conditions in parts of the boundary of Ω . To understand this in detail, let $\{\Gamma_j\}_{1 \leq j \leq m}$ and $\{\Gamma_i^*\}_{1 \leq i \leq l}$ be closed parts of the boundary of Ω where boundary conditions are imposed for the primal and adjoint problem respectively. We define

$$(1.6.1) \quad \mathcal{V} = \{u \in \mathcal{D}(\bar{\Omega})^m : (\gamma u)_j|_{\Gamma_j} = 0, \quad 0 \leq j \leq m\}$$

and

$$(1.6.2) \quad \mathcal{V}^* = \{v \in \mathcal{D}(\bar{\Omega})^l : (\gamma^* v)_i|_{\Gamma_i^*} = 0, \quad 0 \leq i \leq l\}$$

represent the spaces of smooth functions that satisfy the boundary conditions of the initial boundary value problem, and the boundary conditions of the adjoint problem respectively.

Here we have introduced two operators related to smooth functions. These operators are related with the operators of D and D^* when applied to smooth functions and acting as distributions. When $\gamma u = u$, then we are imposing Dirichlet boundary conditions on $\mathcal{V}$. However, other types of boundary conditions can be applied, as seen in Chapter 3.

Therefore, we choose $\text{dom}(A)$ to be defined by

$$(1.6.3) \quad \text{dom}(A) = \{u \in W : \langle Du, v \rangle_{W^*} = 0, \forall v \in \mathcal{V}^*\}.$$

Since $\mathcal{D}(\Omega)^m \subset \text{dom}(A)$, the adjoint is uniquely defined on

$$(1.6.4) \quad \text{dom}(A^*) = \{s \in L^2(\Omega)^l : \exists \ell \in L^2(\Omega)^m \text{ such that } (Av, s)_\Omega = (v, \ell)_\Omega, \forall v \in \text{dom}(A)\}.$$

Finally, as in Section 1.3, we set $V = \text{dom}(A)$ and $V^* = \text{dom}(A^*)$ with the understanding that both V and V^* are endowed with the W and W^* -topology, respectively.

Observe that the choice of (1.6.1) implies that $V = {}^\perp D^*(\mathcal{V})$, since $\langle Du, v \rangle = -\overline{\langle D^*v, u \rangle}$. Since we have verified the conditions of Lemma 1.3.1, we then have

$$(1.6.5) \quad V^* = {}^\perp D(V).$$

Rewriting (2.1.5) in the same style,

$$(1.6.6) \quad V = {}^\perp D^*(\mathcal{V}^*).$$

Thus V and V^* are closed subspaces of W .

LEMMA 1.6.1. $\mathcal{V} \subset V$ and $\mathcal{V}^* \subset V^*$.

PROOF. Equation (2.1.2) implies $\langle D\phi, \phi^* \rangle_W = 0$ for all $\phi \in V$ and $\phi^* \in \mathcal{V}^*$. Hence, any $\phi \in \mathcal{V}$ is also in ${}^\perp D(\mathcal{V}^*)$, which by (4.2.4) implies that $\phi \in V$. Thus $\mathcal{V} \subset V$.

If $\phi^* \in \mathcal{V}^*$, then (4.2.2) shows that $\langle D\phi^*, u \rangle = 0$ for all $u \in V$, i.e., $\phi^* \in {}^\perp D(V)$, which by (4.2.3) implies ϕ^* is in V^* . $\square$

In (4.2.4), $\mathcal{V}^*$ may be replaced by V^* provided a density result is available, as we show next.

ASSUMPTION 1. Suppose $\mathcal{V}^*$ is dense in V^* and $\mathcal{V}$ is dense in V .

LEMMA 1.6.2. *If Assumption 1 holds, then ${}^\perp D(\mathcal{V}^*) = {}^\perp D(V^*)$ and ${}^\perp D(\mathcal{V}) = {}^\perp D(V)$.*

PROOF. Clearly $\mathcal{V}^* \subseteq V^*$ implies ${}^\perp D(\mathcal{V}^*) \supseteq {}^\perp D(V^*)$. To prove the reverse containment, suppose $w \in {}^\perp D(\mathcal{V}^*)$ and $v^* \in V^*$. By density, there is a sequence $\tilde{v}_n$ in $\mathcal{V}^*$ satisfying $\lim_{n \rightarrow \infty} \|\tilde{v}_n - v^*\|_W = 0$. Since $\langle D\tilde{v}_n, w \rangle_W = 0$, by continuity of D , we have $\langle Dv^*, w \rangle_W = 0$. Hence $w \in {}^\perp D(V^*)$. The second identity is proved similarly. $\square$

Hyperbolic systems in one space dimension

The assumptions on which the previous theory is based can be verified for several examples. We begin with the simplest example in one space dimension in Section 2.1 where all the ideas are transparent. We then generalize to the example of multidimensional advection in Section 2.2 and establish a new trace theorem for the associated graph space. The final example in Section 2.4 considers a general symmetric hyperbolic system in one space dimension and leads into the discussion on the wave equation in the subsequent chapter. Contrary to Chapter 1, here we use K to identify our domain instead of Ω , since we will be solving local problem defined on elements K of a spacetime mesh of Ω in the next chapter for the wave equation. The elements of the mesh of Ω will be generated such that they satisfy the causality condition. Note that all the ideas presented in Section 1.4 can be translated to one tent, by setting Ω_h equal to one element K . Mainly, the weak formulation of Problem 1.4.1 in one element K is given by

PROBLEM 2.0.1. (*Weak formulation on K*) Given any $F \in (W^*)'$, find $u \in L^2(K)^m$ and $q \in Q$ such that

$$b((u, q), v) = F(v) \quad \forall v \in W^*,$$

where $F(v) = (f, v) - \langle Dg, v \rangle$ and $b((u, q), v) = (u, A^*v)_\Omega + \langle q, v \rangle_{W^*}$.

2.1. The operator ∂_t

Let K be an open triangle in spacetime $(x, t) \in \mathbb{R} \times \mathbb{R}$, with vertices at $(x, t) = (0, 0)$, $(1, 0)$, and $(1, 1)$ as in Figure 2.1. We split the boundary of K into an inflow, outflow, and a characteristic part:

$$\partial_i K = \{(x, t) \in \partial K : t = 0\}, \quad \partial_o K = \{(x, t) \in \partial K : x = t\},$$

$$\partial_c K = \{(x, t) \in \partial K : x = 1\}.$$

(See Figure 2.1). We are interested in the boundary value problem of finding u such that

$$(2.1.1) \quad \begin{cases} \partial_t u = f & \text{in } K, \\ u|_{\partial_i K} = 0. \end{cases}$$

Obviously, the operator $A = \partial_t$ and the formal adjoint defined by $A^* = -\partial_t$ satisfy

Figure 2.1. Reference domain $K \subset \mathbb{R}^2$

the conditions (A-a) and (A-b) when setting $\Omega = K$, $d = 2$, and $m = l = k = 1$ in Chapter 1. The graph space is given by $W = \{u \in L^2(K) : \partial_t u \in L^2(K)\}$ and $W^* = W$. Here $\langle \cdot, \cdot \rangle_W$ represents the duality pairing in W . We define the operator D such that $D : W \rightarrow W'$ and it satisfies: $\langle Du, v \rangle_W = (Au, v) - (u, A^*v)$. Here ∂_t is taken in the distributional sense. Note that $\mathcal{D}(\bar{K}) \subset W$. Hence, the boundary

operator D on smooth functions $u, v \in \mathcal{D}(\bar{K})$ is given by

$$\begin{aligned}
(2.1.2) \quad \langle Du, v \rangle_W &= \int_0^1 \int_0^x (\partial_t u) v \, dx \, dt + \int_0^1 \int_0^x u (\partial_t v) \, dx \, dt \\
&= \int_0^1 u(x, x) v(x, x) - u(x, 0) v(x, 0) \, dx.
\end{aligned}$$

In order to define the domain of A and A^* , we introduce the following subspaces of $\mathcal{D}(\bar{K})$:

$$(2.1.3) \quad \mathcal{V} = \{u \in \mathcal{D}(\bar{K}) : u(x, 0) = 0\}$$

and

$$(2.1.4) \quad \mathcal{V}^* = \{v \in \mathcal{D}(\bar{K}) : v(x, x) = 0\}.$$

We set the $\text{dom}(A)$ to be defined as

$$\begin{aligned}
(2.1.5) \quad \text{dom}(A) &= \{u \in W : \langle Du, v \rangle_W = 0 \, \forall v \in \mathcal{V}^*\} \\
&= \{u \in W : \langle Dv, u \rangle_W = 0, \, \forall v \in \mathcal{V}^*\}.
\end{aligned}$$

Thus, it is easy to see that Assumption (A-c) is satisfied. Note that $\mathcal{D}(K) \subset \text{dom}(A)$, thus the operator A is densely defined, and therefore the domain of the adjoint is unique and given by

$$\begin{aligned}
(2.1.6) \quad \text{dom}(A^*) &= \{s \in L^2(\Omega) : \text{there is an } \ell \in L^2(\Omega) \\
&\quad \text{such that } (\partial_t v, s)_\Omega = (v, \ell)_\Omega \text{ for all } v \in \text{dom}(A)\}.
\end{aligned}$$

Finally, as in Chapter 1, we set $V = \text{dom}(A)$ and $V^* = \text{dom}(A^*)$ with the understanding that both V and V^* are endowed with the W -topology, while $\text{dom}(A)$ and $\text{dom}(A^*)$ have the topology of $L^2(\Omega)$.

Since we have verified the assumptions of Lemma 1.3.1, we then have

$$(2.1.7) \quad V^* = {}^\perp D(V).$$

Rewriting (2.1.5) in the same style,

$$(2.1.8) \quad V = {}^\perp D(\mathcal{V}^*).$$

Thus V and V^* are closed subspaces of W .

LEMMA 2.1.1. $\mathcal{V} \subset V$ and $\mathcal{V}^* \subset V^*$.

PROOF. If $v \in \mathcal{V}$, then the definition of the boundary operator applied to smooth functions in (2.1.2) and $\mathcal{V}^*$ in (2.1.4) imply that $\langle Dv, v^* \rangle_W = 0$ for all $v^* \in \mathcal{V}^*$, thus $v \in V$. In the same way, if $v^* \in \mathcal{V}^*$, then following the definition of $\text{dom}(A)$ we have $\langle Du, v^* \rangle = 0$ for all $u \in \text{dom}(A)$, where $\langle Du, v^* \rangle_W = (\partial_t u, v^*) + (u, \partial_t v^*)$. Thus, $v \in \text{dom}(A^*)$, since we had found $\ell = -\partial_t v^* \in L^2(K)$ in (2.1.6). $\square$

In (2.1.8), $\mathcal{V}^*$ may be replaced by V^* provided a density result is available, as we show next.

LEMMA 2.1.2. *If Assumption 1 holds in this example, when operator $A = \partial_t$, then ${}^\perp D(\mathcal{V}^*) = {}^\perp D(V^*)$ and ${}^\perp D(\mathcal{V}) = {}^\perp D(V)$.*

PROOF. This is an immediate conclusion from Lemma 1.6.2. $\square$

2.1.1. Density result. In order to prove Assumption 1, we will provide a trace lemma for the W space.

Because $\text{dist}(\partial_i K, \partial_o K) = 0$, although the operator D is defined on all W , we must be careful in speaking of traces of functions in W on these boundary parts. Indeed, $w(x, t) = x^{-1/2}$ is in W , since $\|w\|_K^2 = \int_0^1 \int_0^x \frac{1}{x} dt dx = 1$, and $\|\partial_t w\|_K = 0$, but its restriction to $\partial_i K$ is not in $L^2(\partial_i K)$.

To study this further, we define the maps

$$\tau_i : v(x, t) \mapsto v(x, 0) \quad \text{and} \quad \tau_o : v(x, t) \mapsto v(x, x),$$

whose application to any function gives its traces on $\partial_i K$ and $\partial_o K$, respectively. These maps are obviously well-defined for smooth functions. Below we prove that they extend to W . Let $L_\delta^2(S)$ denote the set of all measurable functions s on S with finite $\int_S \delta s^2$, with δ a strictly positive function.

LEMMA 2.1.3. *For the space W in this example, the following maps are continuous:*

$$\tau_i : W \rightarrow L_x^2(0, 1), \quad \tau_o : W \rightarrow L_x^2(0, 1), \quad \text{and} \quad \tau_i - \tau_o : W \rightarrow L_{1/x}^2(0, 1),$$

i.e., there is a constant $C_0 > 0$ such that

$$(2.1.9) \quad \int_0^1 x |\tau_i w|^2 dx + \int_0^1 x |\tau_o w|^2 dx + \int_0^1 \frac{|\tau_i w - \tau_o w|^2}{x} dx \leq C_0 \|w\|_W^2$$

for all $w \in W$.

PROOF. A general density result for first order systems in [42, Theorem 4] implies that $\mathcal{D}(\bar{K})$ is dense in W , so it suffices to prove (2.1.9) for all $w \in \mathcal{D}(\bar{K})$. Beginning with the fundamental theorem of calculus,

$$\tau_i w(x) = w(x, r) - \int_0^r \partial_t w(x, t) dt,$$

since $0 \leq x \leq 1$, squaring, integrating over r , and overestimating, we have

$$x |\tau_i w(x)|^2 = \int_0^x |\tau_i w(x)|^2 dr \leq 2 \int_0^x |w(x, r)|^2 dr + 2 \int_0^x r \int_0^r |\partial_t w(x, t)|^2 dt dr.$$

Now integrating over x and overestimating again,

$$\begin{aligned} \frac{1}{2} \int_0^1 x |\tau_1 w(x)|^2 dx &\leq \int_0^1 \int_0^x |w(x, r)|^2 dr dx + \int_0^1 \int_0^1 1 \int_0^x |\partial_t w(x, t)|^2 dt dr dx \\ &= \|w\|_W^2. \end{aligned}$$

A similar argument shows that the same inequality holds with τ_1 replaced by τ_o , using the identity

$$\tau_o w(x) = w(x, r) + \int_r^x \partial_t w(x, t) dt.$$

Since $0 \leq x \leq 1$ we have

$$\begin{aligned} x |\tau_o w(x)|^2 &= \int_0^x |\tau_o w(x)|^2 dr \leq 2 \int_0^x |w(x, r)|^2 dr + 2 \int_0^x \left(\int_r^x \partial_t w(x, t) dt \right)^2 dr \\ &\leq 2 \int_0^x |w(x, r)|^2 dr + 2 \int_0^x \left(\int_r^x 1 dr \right) \left(\int_r^x |\partial_t w(x, t)|^2 dt \right) dr \\ &\leq 2 \int_0^x |w(x, r)|^2 dr + 2 \int_0^x (x - r) \left(\int_r^x |\partial_t w(x, t)|^2 dt \right) dr. \end{aligned}$$

Thus,

$$\frac{1}{2} \int_0^1 x |\tau_o w(x)|^2 dx \leq \|w\|_W^2.$$

Finally, we prove that the inequality

$$(2.1.10) \quad \int_0^1 \frac{|\tau_1 w - \tau_o w|^2}{x} dx \leq \|w\|_W^2.$$

holds. This statement is true since

$$|\tau_1 w(x) - \tau_o w(x)|^2 = \left| \int_0^x \partial_t w(x, s) ds \right|^2 \leq x \int_0^x |\partial_t w(x, s)|^2 ds.$$

Thus, dividing through by x and integrating over x we obtain the result.

□

LEMMA 2.1.4. *The following two identities are well defined for any $u, v \in W$.*

$$(2.1.11a) \quad \langle Du, v \rangle_W = \int_0^1 \left(\frac{\tau_o u - \tau_i u}{x^{1/2}} \right) (x^{1/2} \tau_o v) dx + \int_0^1 \left(\frac{\tau_o v - \tau_i v}{x^{1/2}} \right) (x^{1/2} \tau_i u) dx,$$

and

$$(2.1.11b) \quad \langle Du, v \rangle_W = \int_0^1 \left(\frac{\tau_o v - \tau_i v}{x^{1/2}} \right) (x^{1/2} \tau_o u) dx + \int_0^1 \left(\frac{\tau_o u - \tau_i u}{x^{1/2}} \right) (x^{1/2} \tau_i v) dx.$$

PROOF. We will use the density result for first order systems as before [42, Theorem 4]. Let u, v in $\mathcal{D}(\bar{K})$. Note that expression (2.1.11a) can be obtained by adding and subtracting $(\tau_o v)(\tau_i u)$ in the second equality of (2.1.2). Thus, for any $u, v \in C^1(\bar{K})$ we have

$$\begin{aligned} \langle Du, v \rangle_W &= \int_0^1 (\tau_o v)(\tau_o u) - (\tau_o v)(\tau_i u) + (\tau_o v)(\tau_i u) - (\tau_i v)(\tau_i u) dx. \\ &= \int_0^1 \left(\frac{\tau_o u - \tau_i u}{x^{1/2}} \right) (x^{1/2} \tau_o v) dx + \int_0^1 \left(\frac{\tau_o v - \tau_i v}{x^{1/2}} \right) (x^{1/2} \tau_i u) dx. \end{aligned}$$

In the same way, by adding and subtracting $(\tau_o u)(\tau_i v)$ in the second equality of (2.1.2), we obtain (2.1.11b).

Now, using the Cauchy-Schwarz inequality, the continuity property of the traces proved in Lemma 2.1.3 and the density of $C^1(\bar{K})$ in W , we can extend these integrals continuously to W . Hence (2.1.11a) and (2.1.11b) holds for all v and w in W .

□

LEMMA 2.1.5. *(Characterization of the spaces V and V^* using the trace lemma). For this example, and following the setting of Chapter 1, the spaces V and V^* can*

also be represented by the following sets

$$V = \{u \in W : \tau_i(u) = 0\} \quad \text{and} \quad V^* = \{v \in W : \tau_o(v) = 0\}.$$

PROOF. First we prove $V \subseteq \{u \in W : \tau_i(u) = 0\}$. If $u \in \text{dom}(A)$, then (2.1.5) and (2.1.11a) imply

$$0 = \langle Du, v \rangle_W = \int_0^1 \left(\frac{0 - \tau_i v}{x^{1/2}} \right) (x^{1/2} \tau_i u) = \int_0^1 (-\tau_i v)(\tau_i u) dx \text{ for all } v \in \mathcal{V}^*.$$

Since $\tau_i v \in \mathcal{D}(0, 1)$ for any $v \in \mathcal{V}^*$, and by the density of $\mathcal{D}(0, 1)$ in $L^2(0, 1)$, we obtain $\tau_i u = 0$ a.e. in $(0, 1)$. Thus, $u \in \{w \in W : \tau_i(w) = 0\}$. For the reverse inclusion let $u \in \{w \in W : \tau_i(w) = 0\}$. Using again (2.1.11a) we conclude that $\langle D\phi, u \rangle = 0$ for all $\phi \in \mathcal{V}^*$, thus $u \in \text{dom}(A)$.

Analogously, using (2.1.11b) we obtain $V^* = \{v \in W : \tau_o(v) = 0\}$. $\square$

LEMMA 2.1.6. *Assumption 1 holds for this example, i.e. $\mathcal{V}^*$ is dense in V^* , and $\mathcal{V}$ is a dense subset of V .*

PROOF. Given any $v \in V^*$, by the density of $C^1(\bar{K})$ in W , there is a sequence $w_n \in C^1(\bar{K})$ converging to v in W . Let $v_n(x, t) = w_n(x, t) - (\tau_o w_n)(x)$. Since $\tau_o(v_n) = 0$, we conclude that $v_n \in \mathcal{V}^*$, for all n (definition (2.1.4)). Since we also know $\partial_t(\tau_o w_n(x)) = 0$, we obtain

$$\begin{aligned} \|v_n - v\|_W &\leq \|w_n - v\|_W + \|\tau_o w_n\|_{L^2(K)} \\ &= \|w_n - v\|_W + \left(\int_0^1 \int_0^x |w_n(x, x)|^2 dt dx \right)^{1/2} \\ &= \|w_n - v\|_W + \|\tau_o w_n\|_{L_x^2(0,1)} \\ &= \|w_n - v\|_W + \|\tau_o(w_n - v)\|_{L_x^2(0,1)}. \end{aligned}$$

This, together with Lemma 2.1.3, and the convergence of w_n to v in W , imply the convergence of v_n to v in W . Thus $\mathcal{V}^*$ is dense in V^* . Similarly, $\mathcal{V}$ is dense in V . $\square$

LEMMA 2.1.7. *The conditions of Theorem 1.4.14 are satisfied for this example.*

PROOF. Following the result of Lemma 2.1.6, it suffices to prove the conditions of Theorem of (1.4.26) for the dense subsets. For any function v in $\mathcal{V}$, we have

$$v(x, t)^2 = \int_0^t \frac{\partial}{\partial s} (v(x, s)^2) ds = \int_0^t 2v(x, s) \frac{\partial}{\partial s} v(x, s) ds,$$

which implies

$$\begin{aligned} \int_0^1 \int_0^x v(x, t)^2 dt dx &\leq 2 \int_0^1 \int_0^x \left(\int_0^t v(x, s)^2 ds \right)^{1/2} \left(\int_0^t \left| \frac{\partial}{\partial s} v(x, s) \right|^2 ds \right)^{1/2} dt dx, \\ &\leq 2 \int_0^1 \left(\int_0^x \int_0^t v(x, s)^2 ds dt \right)^{1/2} \left(\int_0^x \int_0^t \left| \frac{\partial}{\partial s} v(x, s) \right|^2 ds dt \right)^{1/2} dx. \end{aligned}$$

Since $x \leq 1$, this shows that $\|v\|_\Omega \leq 2\|Av\|_\Omega$ for all $v \in \mathcal{V}$ and hence for all $v \in V$.

The proof of (1.4.26b) is analogous using the identity

$$v^*(x, t)^2 = \int_x^t \frac{\partial}{\partial s} (v^*(x, s)^2) ds = \int_x^t 2v^*(x, s) \frac{\partial}{\partial s} v^*(x, s) ds, \quad \text{for all } v^* \in \mathcal{V}^*.$$

$\square$

THEOREM 2.1.8. *Problem (1.4.2) is well posed for this example, when Ω_h consists of only one element, namely K .*

PROOF. By Lemmas 2.1.6 and 2.1.7, assumptions 1.4.9 and (1.4.26) hold. Therefore Theorem 1.4.14 implies that (1.4.10) holds. Thus, Theorem 1.4.5 gives the required result. $\square$

2.2. Unidirectional advection

The above calculations have a straightforward generalization to multidimensional tent-shaped domains. We say that K_0 is a vertex patch around a point p if it is an open polyhedron in $\mathbb{R}^{d-1}$ ($d > 1$) that can be partitioned into a finite number of $(d-1)$ -simplices with a common vertex $p \in \mathbb{R}^{d-1}$.

We first consider domains K built on (spatial) vertex patches of the form

$$(2.2.1) \quad K = \{(x, t) : x \in K_0, g_i(x) < t < g_o(x)\}$$

(and later, after Definition 2.3.1 below, specialize to tent-shaped domains). Above, $g_o(x)$ and $g_i(x)$ are Lipschitz functions on K_0 such that K is a nonempty open set in $\mathbb{R}^d$. Then the unit outward normal vector $n = (n_x, n_t)$ exists a.e. on ∂K . Continuing to consider the same operator as in Section 2.1, namely $A = \partial_t$, but on the new domain K , the following defines inflow, outflow, and characteristic parts of the boundary:

$$(2.2.2a) \quad \partial_i K = \{(x, t) \in \partial K : n_t < 0\}, \quad \partial_o K = \{(x, t) \in \partial K : n_t > 0\},$$

$$(2.2.2b) \quad \partial_c K = \{(x, t) \in \partial K : n_t = 0\}.$$

We define the space W as the graph space, and notice that $W = W^*$, and thus the operator D is defined as previously such that $D : W \rightarrow W'$ and it satisfies

$$\langle Du, v \rangle_W = (Au, v)_K - (u, A^*v)_K$$

Note that Assumptions (A-a) and (A-b) hold, when setting $l = m = 1$ in Chapter 1. The boundary operator for the unidirectional advection operator is given by

$$\langle Dv, u \rangle = \int_{K_0} \int_{g_i(x)}^{g_o(x)} \partial_t u \cdot v + u \cdot \partial_t v \, dt \, dx \quad \text{for all } u, v \in W,$$

which for smooth functions $u, v \in \mathcal{D}(\bar{K})$, is expressed by

$$(2.2.3) \quad \langle Dv, u \rangle = \int_{K_0} u(x, g_o(x)) \cdot v(x, g_o(x)) - u(x, g_i(x)) \cdot v(x, g_i(x)) \, dx.$$

We define, as in the previous section,

$$(2.2.4) \quad \mathcal{V} = \{u \in \mathcal{D}(\bar{K}) : u(x, g_i(x)) = 0\},$$

$$(2.2.5) \quad \mathcal{V}^* = \{v \in \mathcal{D}(\bar{K}) : v(x, g_o(x)) = 0\},$$

and

$$(2.2.6) \quad \text{dom}(A) = \{u \in W : \langle Dv, u \rangle_W = 0, \forall v \in \mathcal{V}^*\}.$$

Note that $\mathcal{D}(K) \subset \text{dom}(A)$, by 2.2.3 and definition 2.2.5. Thus Assumption (A-c) is satisfied. Therefore, A is a densely defined operator and the domain of the adjoint is given by

$$(2.2.7) \quad \text{dom}(A^*) = \{s \in L^2(\Omega) : \exists \ell \in L^2(\Omega) \text{ such that } (\partial_t v, s)_\Omega = (v, \ell)_\Omega \quad \forall v \in \text{dom}(A)\}.$$

We now denote $\text{dom}(A)$ by V and $\text{dom}(A^*)$ by V^* , when these spaces are endowed with the topology of W .

THEOREM 2.2.1. *Let K be as in (2.2.1) and let $A = \partial_t$, the spaces $\mathcal{V}^*$ and $\mathcal{V}$ defined by (2.2.7) and (2.2.5), and V and V^* given by (2.2.6) and (2.2.7) respectively.*

Then

(i) *The inflow and outflow trace maps,*

$$\tau_{\text{i}} : v(x, t) \mapsto v(x, g_{\text{i}}(x)) \text{ and } \tau_{\text{o}} : v(x, t) \mapsto v(x, g_{\text{o}}(x)),$$

extend to continuous linear operators

$$\tau_{\text{i}} : W \rightarrow L^2_{g_{\text{o}}-g_{\text{i}}}(K_0), \quad \tau_{\text{o}} : W \rightarrow L^2_{g_{\text{o}}-g_{\text{i}}}(K_0), \quad \text{and} \quad \tau_{\text{i}} - \tau_{\text{o}} : W \rightarrow L^2_{1/(g_{\text{o}}-g_{\text{i}})}(K_0).$$

(ii) *(Characterization of the spaces V and V^* using the trace theorem of part (i)).*

The spaces V and V^ satisfy*

$$V = \{u \in W : \tau_{\text{i}}u = 0\} \quad \text{and} \quad V^* = \{u \in W : \tau_{\text{o}}u = 0\},$$

(iii) *$\mathcal{V}$ is dense in V and $\mathcal{V}^*$ is dense in V^* .*

Therefore, conditions of Theorem 1.4.10 holds and Problem 1.4.2 is wellposed on K .

PROOF. The proof is an extension of the previous section. To prove (i), we use the fundamental fact that $\mathcal{D}(K)$ is dense in W for $K \subset \mathbb{R}^d$ (see [42, Theorem 4]). To prove the first continuity result, choose $w \in \mathcal{D}(K)$ and use the identity $\tau_{\text{i}}w(x) = w(x, r) - \int_{g_{\text{i}}(x)}^r \partial_t w(x, t) dt$, for any r such that $g_{\text{i}}(x) < r \leq g_{\text{o}}(x)$. Therefore,

$$\begin{aligned} (g_{\text{o}}(x) - g_{\text{i}}(x)) |\tau_{\text{i}}w(x)|^2 &= \int_{g_{\text{i}}(x)}^{g_{\text{o}}(x)} |\tau_{\text{i}}w(x)|^2 dr \\ &\leq 2 \left(\int_{g_{\text{i}}(x)}^{g_{\text{o}}(x)} |w(x, r)|^2 dr + \int_{g_{\text{i}}(x)}^{g_{\text{o}}(x)} (r - g_{\text{i}}(x)) \int_{g_{\text{o}}(x)}^r |\partial_t w(x, t)|^2 dt dr \right). \end{aligned}$$

Now, integrating over K_0 , we have

$$\begin{aligned}
\int_{K_0} (g_o(x) - g_i(x)) |\tau_i w(x)|^2 &= \int_{K_0} \int_{g_i(x)}^{g_o(x)} |\tau_i w(x)|^2 dr \\
&\leq 2 \int_{K_0} \int_{g_i(x)}^{g_o(x)} |w(x, r)|^2 dr \\
&\quad + 2 \int_{K_0} \int_{g_i(x)}^{g_o(x)} (r - g_i(x)) \int_{g_o(x)}^r |\partial_t w(x, t)|^2 dt dr.
\end{aligned}$$

Then, $\int_{K_0} (g_o(x) - g_i(x)) |\tau_i w(x)|^2 \leq 2 \max_{x \in K_0} \{1, g_o(x) - g_i(x)\} \|w\|_W^2$ and we extend the result to W due to the density result.

Analogously we use the identity $\tau_o w(x) = w(x, r) + \int_r^{g_o(x)} \partial_t w(x, t) dt$, to obtain

$$\begin{aligned}
(g_o - g_i)(x) |\tau_o w(x)|^2 &= \int_{g_i(x)}^{g_o(x)} |\tau_o w(x)|^2 dr \\
&\leq 2 \int_{g_i(x)}^{g_o(x)} |w(x, r)|^2 dr + 2 \int_0^x \left(\int_r^x \partial_t w(x, t) dt \right)^2 dr \\
&\leq 2 \left(\int_{g_i(x)}^{g_o(x)} |w(x, r)|^2 dr + \int_{g_i(x)}^{g_o(x)} \left(\int_r^{g_o(x)} 1 dt \right) \left(\int_r^x |\partial_t w(x, t)|^2 dt \right) dr \right) \\
&\leq 2 \left(\int_{g_i(x)}^{g_o(x)} |w(x, r)|^2 dr + \int_{g_i(x)}^{g_o(x)} (g_o(x) - r) \left(\int_r^x |\partial_t w(x, t)|^2 dt \right) dr \right).
\end{aligned}$$

Now, integrating over K_0 , overestimating the quantity, and using the density result, we obtain the second continuity result.

The third continuity results holds since

$$|\tau_i w(x) - \tau_o w(x)|^2 = \left| \int_{g_i(x)}^{g_o(x)} \partial_t w(x, s) ds \right|^2 \leq (g_i(x) - g_o(x)) \int_{g_i(x)}^{g_o(x)} |\partial_t w(x, s)|^2 ds,$$

dividing through by $(g_i(x) - g_o(x))$ and integrating over K_0 .

To prove part (ii), we use the following identities

$$(2.2.8a) \quad \begin{aligned} \langle Du, v \rangle_W &= \int_{K_0} \left(\frac{\tau_o u - \tau_i u}{(g_o(x) - g_i(x))^{1/2}} \right) ((g_o(x) - g_i(x))^{1/2} \tau_o v) dx \\ &+ \int_{K_0} \left(\frac{\tau_o v - \tau_i v}{(g_o(x) - g_i(x))^{1/2}} \right) ((g_o(x) - g_i(x))^{1/2} \tau_i u) dx \end{aligned}$$

and

$$(2.2.8b) \quad \begin{aligned} \langle Du, v \rangle_W &= \int_{K_0} \left(\frac{\tau_o u - \tau_i u}{(g_o(x) - g_i(x))^{1/2}} \right) ((g_o(x) - g_i(x))^{1/2} \tau_i v) dx \\ &+ \int_{K_0} \left(\frac{\tau_o v - \tau_i v}{(g_o(x) - g_i(x))^{1/2}} \right) ((g_o(x) - g_i(x))^{1/2} \tau_o u) dx \end{aligned}$$

for all $u, v \in W$. These identities holds using the density result and the continuity of the trace theorems proved in part (i). Thus, to prove part (ii) we first show that $V \subseteq \{u \in W : \tau_i(u) = 0\}$. Indeed, if $u \in \text{dom}(A)$, then (2.2.6) and (2.2.8a) imply

$$\begin{aligned} 0 = \langle Du, v \rangle_W &= \int_0^1 \left(\frac{0 - \tau_i v}{(g_o(x) - g_i(x))^{1/2}} \right) ((g_o(x) - g_i(x))^{1/2} \tau_i u) \\ &= \int_0^1 (-\tau_i v)(\tau_i u) dx, \quad \text{for all } v \in \mathcal{V}^*. \end{aligned}$$

Denoting $\Gamma_i = \{(x, g_i(x)) : x \in K_0\}$ and $\Gamma_0 = \{(x, g_o(x)) : x \in K_0\}$, we observe that $\tau_i v \in \mathcal{D}(\Gamma_i)$ for any $v \in \mathcal{V}^*$. Since $\mathcal{D}(\Gamma_i)$ is dense in $L^2(\Gamma_i)$, we obtain that $\tau_i u = 0$ a.e. in Γ_i . Thus, $u \in \{w \in W : \tau_i(w) = 0\}$. For the reverse inclusion let $u \in \{w \in W : \tau_i(w) = 0\}$. Using again (2.2.8a) we conclude that $\langle Du, \phi \rangle_W = 0$ for all $\phi \in \mathcal{V}^*$, thus $u \in \text{dom}(A)$. Thus, $V = \{w \in W : \tau_i(w) = 0\}$. Analogously, using (2.2.8b) we obtain $V^* = \{v \in W : \tau_o(v) = 0\}$.

For the third part of the theorem, we prove that $\mathcal{V}^*$ is dense in V^* , since the result $\mathcal{V}$ is dense in V is analogous.

Given any $v \in V^*$, by the density of $C^1(\bar{K})$ in W , there is a sequence $w_k \in C^1(\bar{K})$ converging to v in W . Let $v_k(x, t) = w_k(x, t) - (\tau_o w_k)(x)$. Since $\tau_o(v_k) = 0$, we conclude that $v_k \in \mathcal{V}^*$, for all $k \in \mathbb{N}$ (definition (2.2.5)). We also know that

$\partial_t(\tau(w_k)(x)) = 0$, thus

$$\begin{aligned}
\|v_k - v\|_W &\leq \|w_k - v\|_W + \|\tau_o w_k\|_{L^2(K)} \\
&= \|w_k - v\|_W + \left(\int_{K_0} \int_{g_i(x)}^{g_o(x)} |w_k(x, x)|^2 dt dx \right)^{1/2} \\
&= \|w_k - v\|_W + \|\tau_o w_k\|_{L^2_{g_o(x)-g_i(x)}(\Gamma_0)} \\
&= \|w_k - v\|_W + \|\tau_o(w_k - v)\|_{L^2_{g_o(x)-g_i(x)}(\Gamma_0)}.
\end{aligned}$$

This, together with trace theorem and the convergence of w_k to v in W , imply the convergence of v_k to v in W . Thus $\mathcal{V}^*$ is dense in V^* . Similarly, $\mathcal{V}$ is dense in V .

The operator A and A^* are bounded by below in their respective domain, with an argument similar to the previous section. Thus all the hypotheses of Theorem 1.4.5 are satisfied for this example, and Theorem 2.2.1 holds. $\square$

2.3. Advection operator on tent-shaped domains

DEFINITION 2.3.1. Suppose K and K_0 are as in (2.2.1). If, in addition, K can be divided into finitely many $(d+1)$ -simplices with a common edge $\{(p, t) : g_i(p) < t < g_o(p)\}$, then we call K a spacetime *tent*. We refer to the common edge as its *tent pole*. Clearly, in this case, g_o and g_i are linear on each simplex of K_0 . We split the tent's boundary into these parts:

$$(2.3.1a) \quad \partial_i K = \{(x, g_i(x)) : x \in K_0\}, \quad \partial_o K = \{(x, g_o(x)) : x \in K_0\},$$

$$(2.3.1b) \quad \partial_b K = \partial K \setminus (\partial_i K \cup \partial_o K).$$

We refer to the two parts in (2.3.1a) as the tent's *inflow* and *outflow* boundaries, respectively. (Using such terms without regard to an underlying flow operator is an abuse of terminology that we overlook for expediency.)

Figure 2.2. Spacetime tent in $\mathbb{R}^d$ with $d = 3$.

The equation modeling advection along a fixed direction $\alpha \equiv (\alpha_i) \in \mathbb{R}^{d-1}$ is of the form $Au = f$ with

$$(2.3.2) \quad Au = \frac{\partial u}{\partial t} + \sum_{i=1}^{d-1} \alpha_i \frac{\partial u}{\partial x_i}.$$

Setting $l = m = 1$, and noting that $A^* = -A$, we can easily see prerequisite (A-a) and (A-b) hold. We define the space $W = \{u \in L^2(\Omega) : Au \in L^2(\Omega)\}$ and the operator $D : W \rightarrow W'$ such that

$$\langle Du, v \rangle_W = (Au, v) + (u, Av).$$

We now assume that the tent boundaries are such that

$$(2.3.3a) \quad \partial_i K \subseteq \{(x, t) \in \partial K : n \text{ at } (x, t) \text{ satisfies } n_t + \alpha \cdot n_x < 0\}$$

$$(2.3.3b) \quad \partial_o K \subseteq \{(x, t) \in \partial K : n \text{ at } (x, t) \text{ satisfies } n_t + \alpha \cdot n_x > 0\}.$$

The vertical part of the boundary, namely $\partial_b K$, is further split into three parts $\partial_b^+ K$, $\partial_b^- K$, and $\partial_b^0 K$ where $n_t + \alpha \cdot n_x = \alpha \cdot n_x$ are positive, negative, and zero, respectively (see Figures 2.2, 2.3). Let Γ_i and Γ_o denote the closures of $\partial_i K \cup \partial_b^- K$ and $\partial_o K \cup \partial_b^+ K$, respectively, and let

$$\Gamma_{io} = \Gamma_i \cap \Gamma_o.$$

Define $\delta(z) = \text{dist}(z, \Gamma_{\text{io}})$. We will use the restrictions of this function to Γ_{i} and Γ_{o} as weight functions while describing the norm continuity of traces below. For smooth functions w on K , let

$$\tau_{\text{i}}w = w|_{\Gamma_{\text{i}}}, \quad \tau_{\text{o}}w = w|_{\Gamma_{\text{o}}}.$$

We define

$$(2.3.4) \quad \mathcal{V}^* = \{v \in \mathcal{D}(\bar{K}) : \tau_{\text{o}}v = 0\} \quad \text{and} \quad \mathcal{V} = \{v \in \mathcal{D}(\bar{K}) : \tau_{\text{i}}v = 0\}.$$

We set the $\text{dom}(A)$ to be defined as

$$(2.3.5) \quad \text{dom}(A) = \{u \in W : \langle Du, v \rangle_W = 0, \forall v \in \mathcal{V}^*\}.$$

Note that $\mathcal{D}(K) \subset \text{dom}(A)$, by 2.2.3 and definition 2.3.5. Thus Assumption (A-c) is satisfied. Then A is a densely defined operator and the domain of the adjoint is given by

$$(2.3.6) \quad \begin{aligned} \text{dom}(A^*) = & \{s \in L^2(\Omega) : \text{there is an } \ell \in L^2(\Omega) \\ & \text{such that } (\partial_t v, s)_\Omega = (v, \ell)_\Omega \text{ for all } v \in \text{dom}(A)\}. \end{aligned}$$

Note that the operator D is given by :

$$\langle Du, v \rangle = \int_{\partial K} (n_t u + (\alpha \cdot n_x)u)v, \quad \text{for all } u, v \in \mathcal{D}(\bar{K}).$$

THEOREM 2.3.2. *Let K be a tent and A be given by (2.3.2). Suppose (2.3.3) holds. Then the above-defined maps τ_{i} and τ_{o} extend to continuous linear operators*

$$\tau_{\text{i}} : W \rightarrow L^2_\delta(\Gamma_{\text{i}}) \quad \text{and} \quad \tau_{\text{o}} : W \rightarrow L^2_\delta(\Gamma_{\text{o}}).$$

Hence, setting V and V^* as in (1.6.3) and (1.6.4), we can identify these spaces to be $V = \ker(\tau_i)$ and $V^* = \ker(\tau_o)$, which are closed subspaces of W . When restricted to these subspaces, the traces have an additional continuity property, namely

$$(2.3.7) \quad \tau_i : V^* \rightarrow L^2_{1/\delta}(\Gamma_i) \quad \text{and} \quad \tau_o : V \rightarrow L^2_{1/\delta}(\Gamma_o)$$

are continuous. Finally, with this V and V^* , and $\mathcal{V}$ and $\mathcal{V}^*$ defined in (2.3.4) Assumption 1 is satisfied, and therefore the weak formulation (2.0.1) is well-posed on K .

PROOF. The idea is to use a change of variable that brings the operator to the previously analyzed operator ∂_t . The new variables are $\hat{x} = x - \alpha t$ and $\hat{t} = t$, i.e.,

$$\begin{bmatrix} x \\ t \end{bmatrix} = H \begin{bmatrix} \hat{x} \\ \hat{t} \end{bmatrix} \quad \text{where} \quad H = \begin{bmatrix} I & \alpha \\ 0 & 1 \end{bmatrix}.$$

Let $\hat{K} = H^{-1}K$. (Note that $\hat{K}$ is not a tent, in general.) Pulling back functions w on K to functions $\hat{w} = w \circ H$ on $\hat{K}$, the chain rule gives

$$(2.3.8) \quad \hat{A}\hat{w} = (Aw) \circ H, \quad \text{where} \quad \hat{A} = \frac{\partial}{\partial \hat{t}}.$$

Thus $w \in W$ if and only if $\hat{w} \in \hat{W} = \{\hat{z} \in L^2(\hat{K}) : \hat{A}\hat{z} \in L^2(\hat{K})\}$.

Next, let $\hat{n} = (\hat{n}_{\hat{x}}, \hat{n}_{\hat{t}})$ denote the unit outward normal on $\partial \hat{K}$. Then $\hat{n} = (\hat{n}_{\hat{x}}, \hat{n}_{\hat{t}}) = H^t n / \|H^t n\|_2$. Defining $\partial_i \hat{K}$, $\partial_o \hat{K}$, and $\partial_c \hat{K}$ as in (2.2.2), we claim that

$$(2.3.9a) \quad \partial_i \hat{K} \equiv \{(\hat{x}, \hat{t}) \in \partial \hat{K} : \hat{n}_{\hat{t}} < 0\} = H^{-1}(\partial_i K \cup \partial_b^- K),$$

$$(2.3.9b) \quad \partial_o \hat{K} \equiv \{(\hat{x}, \hat{t}) \in \partial \hat{K} : \hat{n}_{\hat{t}} > 0\} = H^{-1}(\partial_o K \cup \partial_b^+ K),$$

$$(2.3.9c) \quad \partial_c \hat{K} \equiv \{(\hat{x}, \hat{t}) \in \partial \hat{K} : \hat{n}_{\hat{t}} = 0\} = H^{-1}(\partial_b^0 K).$$

To prove the first identity, take any $(x, t) \in \partial_i K$, by definition it will be of the form $(x, g_i(x))$. Note that the normal n at $(x, g_i(x))$ is in the direction of $(\nabla_x g_i, -1)$ where ∇_x denotes the gradient with respect to x . Hence, because of (2.3.3), we have $\alpha \cdot \nabla_x g_i - 1 < 0$ on $\partial_i \hat{K}$. Since the mapped normal $\hat{n}$ is in the direction of

$$H^t n = \begin{bmatrix} I & 0 \\ \alpha^t & 1 \end{bmatrix} \begin{bmatrix} \nabla_x g_i \\ -1 \end{bmatrix} = \begin{bmatrix} \nabla_x g_i \\ \alpha \cdot \nabla_x g_i - 1 \end{bmatrix}$$

we conclude that $\hat{n}_t < 0$. In the same way, if we take $(x, t) \in \partial_b^- K$ the normal vector n at that point will be in the direction of $(n_x, 0)$. By mapping we get

$$(2.3.10) \quad H^t n = \begin{bmatrix} I & 0 \\ \alpha^t & 1 \end{bmatrix} \begin{bmatrix} n_x \\ 0 \end{bmatrix} = \begin{bmatrix} n_x \\ \alpha \cdot n_x \end{bmatrix},$$

hence $\hat{n}_t < 0$ because $\alpha \cdot n_x < 0$ for all $(x, t) \in \partial_b^- K$. Therefore $H^{-1}(\partial_i K \cup \partial_b^- K) \subset \partial_i \hat{K}$. The reverse inclusion has a similar proof using that any normal component n of ∂K goes in the direction of the mapped normal component $\hat{n}$ given by $n = (H^{-1})^t \hat{n} / \|(H^{-1})^t \hat{n}\|_2$. Take any $(\hat{x}, \hat{t}) \in \partial_i \hat{K}$, using the mapping we will get

$$(H^{-1})^t \hat{n} = \begin{bmatrix} I & 0 \\ -\alpha^t & 1 \end{bmatrix} \begin{bmatrix} \hat{n}_{\hat{x}} \\ \hat{n}_{\hat{t}} \end{bmatrix} = \begin{bmatrix} \hat{n}_{\hat{x}} \\ \hat{n}_{\hat{t}} - \alpha \cdot \hat{n}_{\hat{x}} \end{bmatrix}.$$

Therefore, $n_t + \alpha \cdot n_x$ goes in the direction of $\hat{n}_{\hat{t}}$, and we conclude $n_t + \alpha \cdot n_x < 0$. If $n_t = 0$, $\alpha \cdot n_x < 0$. Thus (2.3.9a) holds..

Note that using (2.3.10) we can obtain that $H^{-1}(\partial_b^+ K) \subset \partial_o \hat{K}$. Moreover, if we take $(x, t) \in \partial_o K$, the normal component n at that point will be in the direction of $(\nabla_x g_o, 1)$. By mapping we have

$$H^t n = \begin{bmatrix} I & 0 \\ \alpha^t & 1 \end{bmatrix} \begin{bmatrix} \nabla_x g_o \\ 1 \end{bmatrix} = \begin{bmatrix} \nabla_x g_o \\ 1 + \alpha \cdot \nabla_x g_o \end{bmatrix}.$$

Therefore $\hat{n}_{\hat{t}} > 0$, because $1 + \alpha \cdot \nabla_x g_o > 0$. Hence $H^{-1}(\partial_o K \cup \partial_b^+ K) \subset \partial_o \hat{K}$. We conclude as before that $\partial_o \hat{K} \subset H^{-1}(\partial_o K \cup \partial_b^+ K)$. Thus (2.3.9b) holds.

From (2.3.10), if $(x, t) \in \partial_b^0 K$ we have that the normal component at that point satisfies $\alpha \cdot n_x = 0$, which implies $\hat{n}_{\hat{t}} = 0$, and then $H^{-1}(\partial_b^0 K) \subset \partial_c \hat{K}$. If we take $(\hat{x}, \hat{t}) \in \partial_c \hat{K}$, we find that the mapped normal component goes in the direction of $(\hat{n}_{\hat{x}}, 0)$ and have $\partial_c \hat{K} \subset H^{-1}(\partial_b^0 K)$. Hence (2.3.9c) holds. Thus claim (2.3.9) is proved.

Let $\hat{K}_0$ be the projection of $\hat{K}$ on the $\hat{t} = 0$ plane. There are (continuous piecewise linear) functions $\hat{g}_o$ and $\hat{g}_i$ such that $\partial_o \hat{K}$ and $\partial_i \hat{K}$ are graphs of $\hat{g}_o$ and $\hat{g}_i$, respectively, over $\hat{K}_0$. On $\hat{K}$, since $\hat{A} = \partial_{\hat{t}}$, we apply Theorem 2.2.1 to conclude that $\hat{\tau}_i \hat{w} = \hat{w}|_{\partial_i \hat{K}}$ and $\hat{\tau}_o \hat{w} = \hat{w}|_{\partial_o \hat{K}}$ extend to continuous linear operators $\hat{\tau}_i : \hat{W} \rightarrow L_{\hat{g}_o - \hat{g}_i}^2(\hat{K}_0)$ and $\hat{\tau}_o : \hat{W} \rightarrow L_{\hat{g}_o - \hat{g}_i}^2(\hat{K}_0)$. Hence, setting $\text{dom}(A)$ and $\text{dom}(A^*)$ as in (4.2.2) and (1.6.4), and using Theorem 2.2.1 we let $\hat{V} = \ker(\hat{\tau}_i)$ and $\hat{V}^* = \ker(\hat{\tau}_o)$ which are closed subspace of $\hat{W}$. By the additional continuity of $\hat{\tau}_i - \hat{\tau}_o : \hat{W} \rightarrow L_{1/(\hat{g}_o - \hat{g}_i)}^2(\hat{K}_0)$ (also given by Theorem 2.2.1), we conclude that

$$\hat{\tau}_i : \hat{V}^* \rightarrow L_{1/(\hat{g}_o - \hat{g}_i)}^2(\hat{K}_0), \quad \text{and} \quad \hat{\tau}_o : \hat{V} \rightarrow L_{1/(\hat{g}_o - \hat{g}_i)}^2(\hat{K}_0),$$

are also continuous.

These continuity results are more conveniently mapped to K by using $\hat{\delta}(z) = \text{dist}(z, \hat{\Gamma}_{i_o})$. Note that $\hat{g}_o - \hat{g}_i$ vanishes at $\hat{\Gamma}_{i_o} = H^{-1}\Gamma_{i_o}$. To restate the continuity properties of τ_i in terms of $\hat{\delta}$, we prove that there are $c_1, c_2 > 0$ such that

$$(2.3.11) \quad c_1 \hat{\delta}(\hat{x}, \hat{g}_i(\hat{x})) \leq \hat{g}_o(\hat{x}) - \hat{g}_i(\hat{x}) \leq c_2 \hat{\delta}(\hat{x}, \hat{g}_i(\hat{x})), \quad \forall \hat{x} \in \hat{K}_0,$$

(and similarly for τ_o). When a point $N = (\hat{x}, \hat{g}_i(\hat{x}))$ on $\partial_i \hat{K}$ is sufficiently near to $\hat{\Gamma}_{i_o}$, the point P nearest to it on $\hat{\Gamma}_{i_o}$, together with $O = (\hat{x}, \hat{g}_o(\hat{x}))$ form a triangle (as

Figure 2.3. On the left is a tent K with $A = \partial_t + 0.5\partial_x$ that satisfies (2.3.3). On the right is $\hat{K}$ obtained after applying the map in the proof of Theorem 2.3.2 with mapped over operator $\hat{A} = \partial_{\hat{t}}$.

shown in Figure 2.3). Now we may restrict ourselves to the two-dimensional plane containing this triangle.

Consider the case when the segment PO lies on or below the plane of constant $\hat{t}$ passing through P , so that PO makes an angle $\theta_o \geq 0$ with that plane. Let θ be the angle made by PN and PO at P . Then, by elementary geometry,

$$(2.3.12) \quad \|P - N\|_2 = \frac{\cos \theta_o}{\sin \theta} \|O - N\|_2.$$

Note that $\theta > 0$ and $0 \leq \theta_o < \pi/2$. Therefore, observing that $\|P - N\|_2 = \hat{\delta}(N)$ and $\|O - N\|_2 = \hat{g}_o - \hat{g}_i$, (2.3.12) proves (2.3.11). For the remaining geometrical configurations, identities similar to (2.3.12) can be derived to prove (2.3.11). Having established (2.3.11), we find that after mapping back to K , the stated continuity properties of $\tau_i w = (\hat{\tau}_i \hat{w}) \circ H^{-1}$ and $\tau_o w = (\hat{\tau}_o \hat{w}) \circ H^{-1}$ are proved.

It now only remains to prove the stated wellposedness of the weak formulations. By Theorem 2.2.1, for any $\hat{F} \in \hat{W}'$, there is a unique $\hat{u} \in L^2(\hat{K})$ and $\hat{q} = \hat{D}\hat{z} \in (\hat{V}^*)^\perp$

satisfying

$$(2.3.13) \quad -(\hat{u}, \hat{A}\hat{v})_{\hat{K}} + \langle \hat{D}\hat{z}, \hat{v} \rangle_{\hat{W}} = \hat{F}(\hat{v}), \quad \forall \hat{v} \in \hat{W},$$

where $\hat{D} \in \mathcal{L}(\hat{W}, \hat{W}')$ is defined as before by $\langle \hat{D}\hat{u}, \hat{v} \rangle_{\hat{W}} = (\hat{A}\hat{u}, \hat{v})_{\hat{K}} + (\hat{u}, \hat{A}\hat{v})_{\hat{K}}$. Here we have used (1.4.4) to find a $\hat{z} \in \hat{V}$ such that $\hat{q} = \hat{D}\hat{z}$. (While $\hat{q}$ is unique, $\hat{z}$ need not be unique.) It now follows from the properties of the mapping that $\hat{u}$ and $\hat{q} = \hat{D}\hat{z}$ satisfies (2.3.13) if and only if $u = \hat{u} \circ H^{-1}$ and $z = \hat{z} \circ H^{-1}$ satisfies

$$(2.3.14) \quad -(u, Av)_K + \langle Dz, v \rangle_W = \hat{F}(v \circ H), \quad \forall v \in W.$$

Here we have used the fact that (2.3.8) implies $(\hat{u}, \hat{A}\hat{v})_{\hat{K}} = (u, Av)_K$ and consequently, $\langle \hat{D}\hat{z}, \hat{v} \rangle_{\hat{W}} = \langle Dz, v \rangle_W$. This shows that the weak formulations on $\hat{K}$ and K are equivalent, so the well-posedness of the latter, namely (1.4.2), follows from the former. $\square$

REMARK 2.3.3. Under additional assumptions, including $\text{dist}(\partial_i K, \partial_o K) > 0$, a stronger trace result is proved in [30, Lemma 5.1]. However, on tents, $\text{dist}(\partial_i K, \partial_o K)$ is always zero, so we are unable to apply their result.

2.4. Linear symmetric hyperbolic system

Let $C \in \mathbb{R}^{m \times m}$ be an $m \times m$ real symmetric matrix and let K be a tent as in Section 2.2. Set $l = m$, and

$$(2.4.1) \quad Au = \frac{\partial u}{\partial t} + \mathbb{C} \frac{\partial u}{\partial x}$$

where $\partial_t u$ and $\partial_x u$ are vectors in $\mathbb{R}^m$ with their ℓ th component equal to $\partial_t u_\ell$ and $\partial_x u_\ell$, respectively. Since $\mathbb{C}$ is symmetric, $A^* = -A$ and Assumptions (A-a), (A-b) in

Chapter 1 are obviously satisfied. Note that

$$\langle Dv, u \rangle = \int_{\partial K} (n_t v + \mathbf{C} n_x \phi) \cdot u, \quad \text{for all } \phi, u \in \mathcal{D}(\bar{K})$$

Thus defining $\text{dom}(A)$ as in (4.2.2), Assumption (A-c) is satisfied. Let $\mathbf{Q}$ be an orthogonal matrix and $\Lambda = \text{diag}(\lambda_\ell)$ be a diagonal matrix such that $\mathbf{C} = \mathbf{Q}\Lambda\mathbf{Q}^T$. Let $\partial_i K$, $\partial_o K$ and $\partial_b K$ be as defined in (2.3.1). In this subsection, we assume – instead of (2.3.3) – that

$$(2.4.2a) \quad \partial_i K \subseteq \{x \in \partial K : n_t I + n_x \mathbf{C} \text{ is negative definite}\},$$

$$(2.4.2b) \quad \partial_o K \subseteq \{x \in \partial K : n_t I + n_x \mathbf{C} \text{ is positive definite}\}.$$

For each $\ell = 1, \dots, m$, we decompose $\partial_b K$ into $\partial_b^{+, \ell} K$, $\partial_b^{-, \ell} K$, and $\partial_b^{0, \ell} K$ where $\lambda_\ell n_x$ is positive, negative, and zero, respectively. Then we have the following theorem, which is proved using the diagonalization of $\mathbf{C}$ to separate each component and then appealing to the analysis in Section 2.2. We now opt for a brief statement of the theorem, leaving the tacitly used properties of the traces to the proof.

THEOREM 2.4.1. *Suppose (2.4.2) holds for the tent K and the operator A in (2.4.1). Then, the formulation (2.0.1) with*

$$(2.4.3a) \quad V = \{z \in W : [\mathbf{Q}^t z]_\ell|_{\partial_i K \cup \partial_b^{-, \ell} K} = 0, \text{ for all } \ell = 1, \dots, m\},$$

$$(2.4.3b) \quad V^* = \{z \in W : [\mathbf{Q}^t z]_\ell|_{\partial_o K \cup \partial_b^{+, \ell} K} = 0, \text{ for all } \ell = 1, \dots, m\},$$

is well-posed.

PROOF. Let $\check{A} = \mathbf{Q}^T A \mathbf{Q}$, $\check{W} = \{\check{v} \in L^2(K)^m : \check{A}\check{v} \in L^2(K)^m\}$, and $\check{D}$ be the corresponding boundary operator on $\check{W}$. Then clearly, $v \in W$ if and only if $\check{v} = \mathbf{Q}^T v$ is in $\check{W}$, since $\check{A}\check{v} = \mathbf{Q}^T A v$. Moreover, $\check{A}\check{w} = \partial_t \check{w} + \Lambda \partial_x \check{w}$, i.e., its ℓ th component

equals

$$\check{A}_\ell \check{w}_\ell \equiv \partial_t \check{w}_\ell + \lambda_\ell \partial_x \check{w}_\ell.$$

Note that $\check{A}_\ell$ is an operator of the form (2.3.2) and has its associated graph space $\check{W}_\ell$ and boundary operator $\check{D}_\ell$.

Now, the assumptions of (2.4.2) imply that (2.3.3) holds for each $\check{A}_\ell$ (with $\alpha = \lambda_\ell$) so Theorem 2.3.2 yields the continuity of the maps $\check{\tau}_i^\ell : \check{w} \mapsto \check{w}_\ell|_{\Gamma_i^\ell}$ and $\check{\tau}_o^\ell : \check{w} \mapsto \check{w}_\ell|_{\Gamma_o^\ell}$ on $\check{W}$, where $\Gamma_i^\ell = \partial_i K \cup \partial_b^{-,\ell} K$ and $\Gamma_o^\ell = \partial_o K \cup \partial_b^{+,\ell} K$. Therefore, the full trace maps $\check{\tau}_i = (\check{\tau}_i^1, \dots, \check{\tau}_i^m)$ and $\check{\tau}_o = (\check{\tau}_o^1, \dots, \check{\tau}_o^m)$ are continuous on $\check{W}$. Set $\check{V} = \{u \in \mathcal{D}(\bar{K}) : \check{\tau}_i u = 0\}$ and $\check{V}^* = \{u \in \mathcal{D}(\bar{K}) : \check{\tau}_o u = 0\}$, thus following Theorem 2.3.2 we can set $\check{V} = \ker(\check{\tau}_i)$ and $\check{V}^* = \ker(\check{\tau}_o)$. Then the following variational equation for $\check{u} \in L^2(K)^m$ and $\check{q} = \check{D}\check{z}$ with $\check{z} \in \check{V}$,

$$(2.4.4) \quad -(\check{u}, \check{A}\check{v})_{L^2(K)^m} + \langle \check{q}, \check{v} \rangle_{\check{W}} = F(\check{v}), \quad \forall \check{v} \in \check{W},$$

splits into m decoupled equations, namely

$$(2.4.5) \quad -(\check{u}_\ell, \check{A}_\ell \check{v}_\ell)_{L^2(K)} + \langle \check{D}_\ell \check{z}_\ell, \check{v}_\ell \rangle_{\check{W}_\ell} = F(\check{v}_\ell), \quad \forall \check{v}_\ell \in \check{W}_\ell, \quad \forall \ell = 1, \dots, m.$$

Here $\check{u}_\ell \in L^2(K)$ and $\check{z}_\ell \in \check{V}_\ell \equiv \ker(\check{\tau}_i^\ell)$ are the ℓ th components of $\check{u}$ and $\check{z}$, respectively. By Theorem 2.3.2, there is a unique $\check{u}_\ell \in L^2(K)$ and $\check{q}_\ell = \check{D}_\ell \check{z}_\ell \in (\check{V}_\ell^*)^\perp$ solving (2.4.5) for each ℓ . This in turn proves the well-posedness of (2.4.4).

To transfer these results for $\check{A}$ to A , we define

$$\tau_i w = \check{\tau}_i(\mathbf{Q}^T w), \quad \tau_o w = \check{\tau}_o(\mathbf{Q}^T w).$$

Then (2.4.3) is the same as $V = \ker(\tau_i)$ and $V^* = \ker(\tau_o)$. Note that $z \in V$ if and only if $\check{z} = \mathbf{Q}^T z \in \check{V}$. Also note that a $\check{u} \in L^2(K)^m$ and $\check{z} \in \check{V}$ solves (2.4.4) if and

only if $u = \mathbf{Q}\check{u}$ and $z = \mathbf{Q}\check{z}$ satisfies

$$-(u, Av)_K + \langle Dz, v \rangle_W = F(\mathbf{Q}^T v), \quad \forall v \in W.$$

Here we have used $(\check{u}, \check{A}\check{v})_K = (u, Av)_K$ and consequent identities for the corresponding boundary operators. Thus the stated well-posedness of (2.0.1) follows from the established well-posedness of (2.4.4). $\square$

REMARK 2.4.2. Consider a tent K with empty $\partial_b K$. Then, under the assumptions of Theorem 2.4.1, a function in V has all its m components equal to zero on the inflow boundary $\partial_i K$. Moreover, if $v \in \mathcal{V}$, then applying the additional continuity property (2.3.7) to the operators $\check{\tau}_o^\ell$ in the above proof, we find that the outflow trace of each component of v must approach zero as we approach Γ_{io} where the inflow and outflow boundary parts meet. Note that this is an example of (1.6.1), where γ is taken as the orthogonal matrix $\mathbf{Q}^T$.

A tent pitching scheme for the wave equation

3.1. Introduction to tent pitching

This chapter is dedicated to design an explicit spacetime finite element scheme of a tent pitching type using the spaces and weak formulations motivated by Chapter 2. The method we construct is a low order method that works on unstructured grids. On uniform grids, comparison with a standard low order finite difference method does not reveal any striking advantages for the new method. However tents of nonuniform heights can be used to handle multiple materials and wavespeeds.

To describe tent pitching schemes in general, consider the case of a hyperbolic problem posed in one space dimension in space and time. Given a spatial mesh, we pitch a tent by erecting a tent pole (vertically in time) at a vertex, as in Figure 3.1. In the plots of Figure 3.1, the horizontal and vertical dimensions are space and time, respectively. The height of the tent pole must be chosen small enough in relation to the hyperbolic propagation speed, so that the domain of dependence of all points in the tent remains within the tent's footprint. We use the given initial data to solve, by some numerical scheme, the hyperbolic problem restricted to the tent. Proceeding to the next vertex, where the second tent is pitched in Figure 3.1, we find that the initial data combined with the solution in the previous tent, provides inflow data to solve the hyperbolic problem there. Solutions in the newer tents proceeds similarly.

Figure 3.1. Tent pitching mesh (read row by row)

3.2. Model problem: wave equation

We now apply the previous ideas introduced in the previous chapter, to the example of the wave equation on spacetime tents and work out the resulting weak formulation in detail. Our model problem is to find a real-valued function ϕ on the

spacetime domain $\Omega = (0, S) \times (0, T)$, satisfying

$$(3.2.1a) \quad c^{-2} \partial_{tt} \phi - \partial_{xx} \phi = g, \quad 0 < x < S, \ 0 < t < T,$$

$$(3.2.1b) \quad \partial_t \phi = \phi = 0, \quad t = 0, \ 0 < x < S,$$

$$(3.2.1c) \quad \partial_t \phi - c \partial_x \phi = 0, \quad x = 0, \ 0 < t < T,$$

$$(3.2.1d) \quad \partial_t \phi + c \partial_x \phi = 0, \quad x = S, \ 0 < t < T,$$

where $c > 0$ is the wave speed. Here, we have imposed the outgoing impedance boundary conditions (but other boundary conditions can also be considered – see Section 3.5).

The above second order system for ϕ arises from a system of first order physical principles, which also matches the form of the problems we have been studying, namely (2.4). Set

$$u = \begin{bmatrix} c \partial_x \phi \\ \partial_t \phi \end{bmatrix}$$

and observe that $\partial_t u_1 = c \partial_{xt} \phi = c \partial_x u_2$ and $\partial_t u_2 = \partial_{tt} \phi = c \partial_x u_1 + c^2 g$. These two equations give the first order system $Au = f$ where

$$(3.2.2) \quad Au = \partial_t u - \begin{bmatrix} 0 & c \\ c & 0 \end{bmatrix} \partial_x u, \quad f = \begin{bmatrix} 0 \\ c^2 g \end{bmatrix}.$$

Note that this operator satisfies the conditions (A-a) and (A-b) after setting $d = l = m = 2$ and $k = 1$ in Chapter 1. It also fits into the framework of Section 2.4 after the diagonalization

$$\mathbf{C} \equiv - \begin{bmatrix} 0 & c \\ c & 0 \end{bmatrix} = \mathbf{Q} \mathbf{\Lambda} \mathbf{Q}^T, \quad \mathbf{Q} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

where $\lambda_1 = -c$ and $\lambda_2 = c$. Then, we define $\partial_i \Omega = (0, S) \times \{0\}$, $\partial_o \Omega = (0, S) \times \{T\}$ and $\partial_b \Omega = \partial \Omega \setminus (\partial_i \Omega \cup \partial_o \Omega)$. The vertical parts $\partial_b \Omega$ are further split into

$$\partial_b^{+,1} \Omega = \partial_b^{-,2} \Omega = \{0\} \times [0, T], \quad \partial_b^{-,1} \Omega = \partial_b^{+,2} \Omega = \{S\} \times [0, T].$$

Note that satisfy $\partial_o \Omega, \partial_i \Omega, \partial_b^{+,1} \Omega = \partial_b^{-,2} \Omega, \partial_b^{-,1} \Omega = \partial_b^{+,2} \Omega$ the previously conditions on a tent K in (2.4.2). Indeed, since

$$n = (n_x, n_t) = \begin{cases} (0, -1) & \text{if } (x, t) \in \partial_i \Omega, \\ (0, 1) & \text{if } (x, t) \in \partial_o \Omega, \\ (-1, 0) & \text{if } (x, t) \in \{0\} \times [0, T], \\ (1, 0) & \text{if } (x, t) \in \{S\} \times [0, T], \end{cases}$$

then

$$\partial_o \Omega \subseteq \{(x, t) \in \partial \Omega : n_t I + C n_x \text{ is negative definite}\},$$

$$\partial_o \Omega \subseteq \{(x, t) \in \partial \Omega : n_t I + C n_x \text{ is positive definite}\},$$

$$\partial_b^{+,1} \Omega \subseteq \{(x, t) \in \partial \Omega : \lambda_1 n_x = -c n_x > 0\},$$

$$\partial_b^{-,1} \Omega \subseteq \{(x, t) \in \partial \Omega : \lambda_1 n_x = -c n_x < 0\},$$

$$\partial_b^{+,2} \Omega \subseteq \{(x, t) \in \partial \Omega : \lambda_2 n_x = c n_x > 0\},$$

$$\partial_b^{-,2} \Omega \subseteq \{(x, t) \in \partial \Omega : \lambda_2 n_x = c n_x < 0\}.$$

Following Theorem 2.4.1, to understand the wave operator, it is sufficient to understand the diagonalized (decoupled) operator:

$$\check{A} \check{w} = \partial_t \check{w} + \Lambda \partial_x \check{w}$$

Figure 3.2. Mapping Ω to $\hat{\Omega}_1$ using the H_1 transformation.

Then, we pull back ℓ - component of the decoupled vector function using the mapping

$$H_\ell = \begin{bmatrix} 1 & (-1)^\ell c \\ 0 & 1 \end{bmatrix}.$$

Following Theorem 2.4.1, set Γ_i^ℓ and Γ_o^ℓ to the closures of $\partial_i \Omega \cup \partial_b^{-,\ell} \Omega$ and $\partial_o \Omega \cup \partial_b^{+,\ell} \Omega$, respectively, $\Gamma_{io}^\ell = \Gamma_i^\ell \cap \Gamma_o^\ell$, and $\delta_\ell(x, t) = \text{dist}((x, t), \Gamma_{io}^\ell)$ for $\ell = 1, 2$. We set the graph space $W = \{u \in L^2(\Omega)^2 : Au \in L^2(\Omega)^2\}$, and the operator D as before. Note that the conditions in (2.3.9) holds for each component and its mapping operator H_ℓ . Indeed,

$$\partial_i \hat{\Omega}_\ell \equiv \{(\hat{x}, \hat{t}) \in \partial \hat{\Omega}_\ell : \hat{n}_{\hat{t}} < 0\} = H_\ell^{-1}(\partial_i \Omega \cup \partial_b^{-,1} \Omega)$$

$$\partial_o \hat{\Omega}_\ell \equiv \{(\hat{x}, \hat{t}) \in \partial \hat{\Omega}_\ell : \hat{n}_{\hat{t}} > 0\} = H_\ell^{-1}(\partial_o \Omega \cup \partial_b^{+,1} \Omega),$$

Note that $\hat{\Omega}_1$ is a parallelogram with vertices $\{(0, 0), (S, 0), (S + cT, T), (cT, T)\}$, and $\hat{\Omega}_2$ is the parallelogram with vertices $\{(0, 0), (S, 0), (S - cT, T), (-cT, T)\}$. See Figure 3.2 the mapping of the first component of the decoupled vector function.

Thus, defining

$$(3.2.3) \quad \tau_i w = \check{\tau}_i(\mathbf{Q}^T w), \quad \tau_o w = \check{\tau}_o(\mathbf{Q}^T w).$$

we can prove, as in Section 2.4, that the global trace maps

$$(3.2.4a) \quad \tau_i \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} (\mathbb{Q}^t u)_1|_{\Gamma_i^1} \\ (\mathbb{Q}^t u)_2|_{\Gamma_i^2} \end{bmatrix} = \begin{bmatrix} (z_1 + z_2)|_{\Gamma_i^1} \\ (z_1 - z_2)|_{\Gamma_i^2} \end{bmatrix}, \quad \tau_i : W \rightarrow L_{\delta_1}^2(\Gamma_i^1) \times L_{\delta_2}^2(\Gamma_i^2)$$

$$(3.2.4b) \quad \tau_o \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} (\mathbb{Q}^t u)_1|_{\Gamma_o^1} \\ (\mathbb{Q}^t u)_2|_{\Gamma_o^2} \end{bmatrix} = \begin{bmatrix} (z_1 + z_2)|_{\Gamma_o^1} \\ (z_1 - z_2)|_{\Gamma_o^2} \end{bmatrix}, \quad \tau_o : W \rightarrow L_{\delta_1}^2(\Gamma_o^1) \times L_{\delta_2}^2(\Gamma_o^2)$$

are continuous, and we can set

$$V(\Omega) = \ker(\tau_i), \quad V^*(\Omega) = \ker(\tau_o),$$

Indeed, since $\tau_i w = \check{\tau}_i \check{w}$ and $\tau_o w = \check{\tau}_o \check{w}$ in (3.2.3), we just need to study the components of the decoupled operator $\check{\tau}_i \check{w}$, i.e. $(\check{\tau}_i \check{w})_\ell$. Mapping Ω to $\hat{\Omega}_\ell$ for $\ell = 1, 2$, by

$$(\check{\tau}_i \check{w})_\ell = (\hat{\tau}_i \hat{w}_\ell) \circ H_\ell^{-1} \quad \text{and} \quad (\check{\tau}_o \check{w})_\ell = (\hat{\tau}_o \hat{w}_\ell) \circ H_\ell^{-1},$$

we can prove that $\tau_i w$ satisfies the continuity properties by proving that operator on $\hat{\delta}_\ell$ on $\hat{\Omega}_\ell$, satisfies

$$c_1^\ell \hat{\delta}_\ell(\hat{x}, \hat{g}_i^\ell(\hat{x})) \leq \hat{g}_o^\ell(\hat{x}) - \hat{g}_i^\ell(\hat{x}) \leq c_2^\ell \hat{\delta}_\ell(\hat{x}, \hat{g}_i^\ell(\hat{x}))$$

for c_1^ℓ, c_2^ℓ constants. Then, the continuity properties of the operator $\hat{\tau}_i : \hat{W} \rightarrow L_{\hat{g}_o - \hat{g}_i}^2(\hat{K}_0)$ showed in Theorem (2.2.1) will also imply the continuity properties for the operator $(\check{\tau}_i \check{w})_\ell$, after mapping back to Ω , and therefore for (3.2.4). The proof of the continuity properties of τ_o is analogous.

Since the steps to find the constant c_1^ℓ and c_2^ℓ for $\ell = 1, 2$ are similar, we just show the case when $\ell = 1$. Since $\hat{\Gamma}_{i_o}^1 = \{(0, 0), (S + cT, T)\}$, if we choose a point $(\hat{x}, 0)$ closed to $(0, 0)$, we obtain $c_1^1 = c_2^1 = \frac{1}{c}$. If we choose $(\hat{x}, \frac{1}{c}(\hat{x} - S))$ approaching $(S + cT, T)$ we obtain $\hat{g}_o^1(\hat{x}) - \hat{g}_i^1(\hat{x}) = (cT + S - \hat{x})/c$, and $\hat{\delta}_1(\hat{x}, \hat{g}_i^1(\hat{x})) = (cT + S - \hat{x})\sqrt{c^2 + 1}/c$,

thus $c_1^1 = c_2^1 = \frac{1}{c^2+1}$. See Figure 3.2. Analogously we obtain similar results with $\hat{\Gamma}_{\text{io}}^2 = \{(-cT, T), (S, 0)\}$, and therefore the trace continuity properties are well defined for the wave equation in Ω .

These spaces can be used to give a global weak formulation on Ω , but our focus is on local solvers. Note that $\mathcal{D}(\Omega)^2 \subset V(\Omega)$, and therefore condition A-c is satisfied.

In spacetime tent pitching methods, we are required to numerically solve the wave equation on spacetime tents, ordered so that inflow data on a tent can be provided by the outflow solution on previously handled tents or through given data. Hence we now focus on the formulation and discretization on one tent K .

3.3. Weak formulation on a tent

Consider the analogue of (3.2.1) on one tent K , with zero initial data on the inflow boundaries and with boundary conditions inherited from the global boundary conditions (3.2.1c)–(3.2.1d).

Define, as before, the boundary parts of a tent K , by

$$\begin{aligned}\partial_i K &= \{(x, t) \in \partial K : n_t < 0\}, & \partial_o K &= \{(x, t) \in \partial K : n_t > 0\}, & \partial_b K &= \partial K \setminus (\partial_i K \cup \partial_o K), \\ \partial_b^{+,1} K &= \partial_b^{-,2} K = \{(x, t) \in \partial_b K : cn_x < 0\}, \\ \partial_b^{+,2} K &= \partial_b^{-,1} K = \{(x, t) \in \partial_b K : cn_x > 0\}.\end{aligned}$$

Note that the boundary part $\partial_b K$ may be empty in some tents. We consider the tent problem of solving for u satisfying

$$\begin{array}{llll} Au = f & \text{on } K, & u_1 - u_2 = 0 & \text{on } \partial_b^{+,1} K, \\ u = 0 & \text{on } \partial_i K, & u_1 + u_2 = 0 & \text{on } \partial_b^{+,2} K. \end{array}$$

To obtain a well-posed weak formulation on one tent, we proceed to use Theorem 2.4.1.

To this end, we must assume that the tent satisfies Assumption (2.4.2), which now reads

$$(3.3.1a) \quad \partial_i K \subseteq \{x \in \partial K : n_t \pm n_x c < 0\},$$

$$(3.3.1b) \quad \partial_o K \subseteq \{x \in \partial K : n_t \pm n_x c > 0\}.$$

Since $A^* = -A$ in this example, the weak formulation (2.0.1) reads

$$(3.3.2) \quad u \in L, \ q \in Q : \quad - (u, Av)_L + \langle q, v \rangle_W = F(v), \quad \forall v \in W,$$

where the spaces are set following (2.4.3), namely

$$V = \left\{ \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \in W : \begin{bmatrix} z_1 + z_2 \\ z_1 - z_2 \end{bmatrix} \Big|_{\partial_i K \cup \partial_b^{-, \ell} K} = 0, \quad \text{for } \ell = 1, 2 \right\},$$

$$V^* = \left\{ \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \in W : \begin{bmatrix} z_1 + z_2 \\ z_1 - z_2 \end{bmatrix} \Big|_{\partial_o K \cup \partial_b^{+, \ell} K} = 0, \quad \text{for } \ell = 1, 2 \right\}.$$

Theorem 2.4.1 shows that (3.3.2) is a well-posed weak formulation on K provided the tent K satisfies (3.3.1). Note that the above spaces change from tent to tent and may arguably be better denoted by $V(K), V^*(K)$, etc., but to avoid notational bulk we will suppress the K -dependence.

3.3.1. Causality condition. Let us take a closer look at (3.3.1). First note that each tent, in this application, consists of either two triangles (on either side of the tent pole), or just one triangle. The tents are thus divided into three types, as shown in Figure 3.3.

The length of the tent pole is k , the numbers p_l and p_r are such that $p_l k$ and $p_r k$ give the heights of the outflow boundaries on the left and right side of the tent pole, respectively, and the spatial mesh size are $h_r, h_l \geq 0$. Writing down the normal

Figure 3.3. Three types of tents

vector on the tent boundaries, we immediately find that condition (3.3.1) on a tent is equivalent to

$$(3.3.3) \quad \left| c \frac{kp_r}{h_r} \right| < 1 \quad \text{and} \quad \left| c \frac{kp_l}{h_l} \right| < 1.$$

Clearly, by controlling the size of the tent pole we can satisfy these inequalities.

The well-known Courant-Friedrichs-Levy (CFL) condition [14] identifies stability conditions as constraints on the time step size in terms of space mesh size in numerical discretizations. In our case, an analogue manifests itself as geometrical constraints (3.3.3) on the tent. We will refer to (3.3.1) – or (3.3.3) – as the causality condition.

3.4. The numerical scheme

In this section, continuing to work with the wave operator A defined by (3.2.2), we give an explicit numerical scheme for approximating $u(x, t)$ satisfying

$$(3.4.1a) \quad Au = f, \quad 0 < x < S, \ 0 < t < T,$$

$$(3.4.1b) \quad u_i(x, 0) = u_i^0(x), \quad 0 < x < S, \ i = 1, 2,$$

$$(3.4.1c) \quad u_1(0, t) - u_2(0, t) = 0, \quad 0 < t < T,$$

$$(3.4.1d) \quad u_1(1, t) + u_2(1, t) = 0, \quad 0 < t < T.$$

The scheme will allow varying spatial and temporal mesh sizes. Here f and u^0 are assumed to given smooth functions. We begin by describing the calculations within each tent, followed by the tent pitching technique to advance in time.

3.4.1. Conforming discretization on a tent. As seen above, a tent is comprised of one or two triangles. Let the space of continuous functions on a tent K whose restrictions to these triangles are linear be denoted by $P_1^h(K)$. We construct a conforming discretization of (3.3.2) within K using the discrete space

$$(3.4.2) \quad V_1 = V \cap (P_1^h(K))^2.$$

By definition, $V_1 \subseteq V$, and consequently, functions in V_1 must satisfy the essential boundary conditions of V . Depending on the tent geometry, different boundary conditions must be imposed on different tents.

To examine what this entails for the nodal coefficients on mesh vertices, let $\zeta \in P_1^h(K)$ be the continuous scalar function (unique Lagrange basis function) that equals one at the “apex” of the tent K , equals zero at all its other vertices. The *apex* of a tent, irrespective of whether it consists of one or two triangles, is the vertex in $\partial_o K$

that is away from $\partial_i K$. Now, suppose $\mu \equiv \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$ in $\mathbb{R}^2$ is such that

$$(3.4.3a) \quad \mu_1 - \mu_2 = 0 \quad \text{if } \partial_b^{+,1} K \text{ is nonempty,}$$

$$(3.4.3b) \quad \mu_1 + \mu_2 = 0 \quad \text{if } \partial_b^{-,1} K \text{ is nonempty.}$$

(Note that if $\partial_b K$ is empty, then μ is an arbitrary vector in $\mathbb{R}^2$.) Then, it is easy to see that

$$(3.4.4) \quad V_1 = \{\mu\zeta : \mu \text{ satisfies (3.4.3)}\}$$

provides an alternate characterization of (3.4.2). Indeed, if K of type I (see Figure 3.3). If $u \in V(K)$, then $u \in P_1^h(K)^2$. We then recall ζ , and denote ζ_b , ζ_l and ζ_r the basis functions that equals one at the bottom, left, and right respectively, and zero in the other vertices. Then, $u = \mu\zeta + \mu_b\zeta_b + \mu_l\zeta_l + \mu_r\zeta_r$, for μ, μ_b, μ_l, μ_r in $\mathbb{R}^2$ (for the other two tent types, we omit one of the terms associated to the missing vertex). Since u is also in V , then $(Qu)|_{\partial_i K} = 0$, thus $\mu_b, \mu_l, \mu_r = 0$ and therefore $u = \mu\zeta$ for any $\mu \in \mathbb{R}^2$. If K is of type II, then we also have $u \in V_1$ is given by $u = \mu\zeta$, since $(Qu)|_{\partial_i K} = 0$, but since $\partial_b^{-,1} K$ is not empty, we have $(Qu)_1 = (\mu_1 + \mu_2)\zeta = 0$ at $\partial_b^{-,1} K$ thus μ satisfies condition (3.4.3). Analogously, if K is of type III, then $u \in V^1$ implies that there exists $\mu \in \mathbb{R}^2$, such that $(Qu)_2 = (\mu_1 - \mu_2)\zeta = 0$ at $\partial_b^{-,2} K$, that equals $\partial_b^{+,1} K$, thus μ satisfies condition (3.4.3).

A computable conforming discretization of (3.3.2) additionally requires finite-dimensional subspaces of $L^2(K)^2$ and Q . For the latter, observe that when looking to $D(V_1)$ as a functional into the dual space of W_h , we have

$$D(V_1) \subset D(V) = D_{h,V} = Q.$$

Hence we choose an approximation q_1 of the solution component q in (3.3.2) to have the form

$$q_1 = Dz_1, \quad z_1 \in V_1.$$

Then q_1 is clearly in Q . Next, set $L_1 \subset L^2(K)^2$ to be the space of vector functions whose components are constant functions on K . Finally, set

$$(3.4.5) \quad W_1 = \{w : w = \kappa + \mu\zeta, \quad \kappa, \mu \in \mathbb{R}^2, \quad \kappa\zeta \in V_1\}.$$

Our discretization of (3.3.2) now takes the following form: Find u_1 in L_1 and $q_1 \in D(V_1)$ satisfying $-(u_1, Aw) + \langle q_1, w \rangle = F(w)$, for all $w \in W_1$. Clearly, $\dim(W_1)$ is four or three, depending on whether $\partial_b K$ is empty or not. This equation gives rise to an invertible discrete system, as a consequence of the unisolvency of the following slightly modified problem:

$$(3.4.6) \quad \begin{aligned} & \text{Find } u_1 \in L_1 \text{ and } z_1 \in V_1 \text{ such that} \\ & -(u_1, Aw) + \langle Dz_1, w \rangle = F(w), \quad \forall w \in W_1. \end{aligned}$$

PROPOSITION 3.4.1. *There is a unique solution for Problem (3.4.6).*

PROOF. Note the $\dim(L_1) + \dim(V_1) = \dim(W_1)$, so (3.4.6) gives a square (Petrov-Galerkin) system. Hence it suffices to set $F = 0$ and prove that $u_1 = z_1 = 0$. With $F = 0$, writing $z_1 = \alpha\zeta$ for some $\alpha \in \mathbb{R}^2$, we have $\langle D(\alpha\zeta), w \rangle = 0$ for all constant $w \in W$. Since $\alpha\zeta \in V_1$, by the definition of W_1 , we may set $w = \alpha$ in (3.4.6), to get

$$\langle D(\alpha\zeta), \alpha \rangle = \int_{\partial_o K \cup \partial_b K} D_{\mathbf{x}, \mathbf{t}} \alpha \cdot \alpha \zeta = 0$$

where

$$(3.4.7) \quad D_{\mathbf{x}, \mathbf{t}} = n_t I + \mathbb{C} n_x = \begin{bmatrix} n_t & -\mathbb{C} n_x \\ -\mathbb{C} n_x & n_t \end{bmatrix}.$$

If $\partial_b K$ is empty, then since $D_{x,t}\alpha \cdot \alpha = (\alpha_1 + \alpha_2)^2(n_t - cn_x)/2 + (\alpha_1 - \alpha_2)^2(n_t + cn_x)/2$, the causality condition (3.3.1) gives $\alpha = 0$. If $\partial_b K$ is nonempty, then whenever $\alpha\zeta \in V$ we have either $\alpha_1 - \alpha_2 = 0$ or $\alpha_1 + \alpha_2 = 0$, so we can continue to conclude that $\alpha = 0$. Of course $\alpha = 0$ implies $z_1 = 0$.

To prove that $u_1 = 0$, we use (1.2.3) after substituting $z_1 = 0$, to get

$$\langle Dw, u_1 \rangle = 0, \quad \forall w \in W_1.$$

Since u_1 is a constant function, $u_1\zeta \in W_1$, so we may choose $w = u_1\zeta$ and conclude that $u_1 = 0$ by an argument analogous to what we used above. $\square$

REMARK 3.4.2. One can view $z_1|_{\partial_o K}$ as an interface trace variable and $q_1 = Dz_1$ as an interface flux variable. By the trace theory we developed previously, outflow trace $z_1|_{\partial_o K}$ must vanish at the points where outflow and inflow edges meet in order for z_1 to be in V . This motivates our choice (3.4.4) of V_1 to obtain a conforming method. Other non-conforming avenues to design approximations within a tent can be found in [33] and [50].

3.4.2. Advancing in time by tent pitching. We now show how the above ideas yield an explicit time marching algorithm for solving (3.4.1).

First, we mesh the spacetime domain $\Omega = (0, S) \times (0, T)$ by a collection Ω_h of tents K with these properties: The first property is that either $\partial_b K$ is empty or

$$(3.4.8a) \quad \partial_b K \subseteq \partial_b \Omega,$$

for all $K \in \Omega_h$. Second, there exists an enumeration of all tents, $K_1, K_2, \dots, K_J$, with the property that for each $j \in \{1, \dots, J\}$,

$$(3.4.8b) \quad \partial_i K_j \subseteq \bigcup_{k=1}^{j-1} \partial_o K_k \cup \partial_i \Omega.$$

Finally, for all $j \in \{1, \dots, J\}$,

$$(3.4.8c) \quad K_j \text{ satisfies the causality condition (3.3.3).}$$

It is well-known how to construct an algorithm (not only in one space dimension, but also in higher dimensions [29, 67]) that produces meshes satisfying (3.4.8), so we shall not dwell further on the meshing process.

The discrete spacetime approximation on the mesh Ω_h is developed using

$$(3.4.9a) \quad V_h = \{z \in H^1(\Omega)^2 \cap V(\Omega) : z|_K \in P_1^h(K)^2, \forall K \in \Omega_h, z(x, 0) = I_h u^0(x)\}$$

$$(3.4.9b) \quad L_h = \{\alpha : \alpha|_K \in \mathbb{R}^2 \text{ is constant on each } K \in \Omega_h\}$$

$$(3.4.9c) \quad W_h = \{w : w|_K \in W_1 \text{ on each } K \in \Omega_h\},$$

where I_h denote the linear nodal interpolant on the spatial mesh. The method finds approximations $u_h \in L_h$ and $z_h \in V_h$ satisfying

$$(3.4.10) \quad \sum_{K \in \Omega_h} \left(- \int_K u_h \cdot Aw + \int_{\partial K} D_{x,t} z_h \cdot w \right) = \sum_{K \in \Omega_h} \int_K f \cdot w, \quad \forall w \in W_h,$$

where $D_{x,t}$ is as in (3.4.7).

Because of (3.4.8), we are able to use a time-marching algorithm to solve (3.4.10): Proceed in the ordering of (3.4.8b), and for each tent K , solve for $u_h|_K$ and $z_h|_K$. Specifically, if α is the nodal (vector) value of z_h at the apex of K , then defining $z_o^K = \alpha \zeta$, the problem on one tent is to find $u_h|_K \in L_1$ and $z_o^K \in V_1$ satisfying (3.4.10), namely

$$(3.4.11) \quad - \int_K u_h \cdot Aw + \int_{\partial K} D_{x,t} z_o^K \cdot w = \int_K f \cdot w - \int_{\partial K} D_{x,t} z_1^K \cdot w, \quad \forall w \in W_1.$$

where $z_1^K = z_h - z_o^K$. Note that z_1^K on right hand side will be a known quantity if (3.4.8b) holds and if we have already solved on every K' appearing before K in

the ordering of tents in (3.4.8). Indeed, z_i^K is completely determined by its nodal values at (the three or two) vertices on $\partial_i K$, which either lie at $t = 0$ or were apex vertices of previous tents. Problem (3.4.11) is exactly of the same type we discussed in Section 3.4.1.

3.4.3. Propagation formula. Since the system (3.4.11) is small, we can explicitly calculate its solution. To see how information is propagated from inflow to outflow on a mesh of tents, we consider the case where the volume source f is zero. Write $z_h = \begin{bmatrix} z_{h,1} \\ z_{h,2} \end{bmatrix}$ in (3.4.10) and let the nodal values of the scalar Lagrange finite element functions $z_{h,1}$ and $z_{h,2}$ be $\begin{bmatrix} U^t \\ V^t \end{bmatrix}, \begin{bmatrix} U^b \\ V^b \end{bmatrix}, \begin{bmatrix} U^l \\ V^l \end{bmatrix}, \begin{bmatrix} U^r \\ V^r \end{bmatrix}$, at the top, bottom, left and right vertices, respectively, of a tent of Type I, as in Figure 3.3. For the other two tent types, we omit the nodal values at the missing vertex.

Equation (3.4.11) finds $\begin{bmatrix} U^t \\ V^t \end{bmatrix}$ as a function of the remaining nodal values. After tedious simplifications (displayed below), this relationship is found to be as follows:

$$\begin{aligned}
(3.4.12) \quad \begin{bmatrix} U^t \\ V^t \end{bmatrix} &= \begin{bmatrix} U^b \\ V^b \end{bmatrix} + w_1 \begin{bmatrix} 0 & c \\ c & 0 \end{bmatrix} \begin{bmatrix} U^r - U^l \\ V^r - V^l \end{bmatrix} + w_2 c \begin{bmatrix} U^r - U^l \\ V^r - V^l \end{bmatrix}, & \text{ for Type I,} \\
\begin{bmatrix} U^t \\ V^t \end{bmatrix} &= \begin{bmatrix} U^b \\ V^b \end{bmatrix} + w_1 \begin{bmatrix} 0 & c \\ c & 0 \end{bmatrix} \begin{bmatrix} U^r - U^b \\ V^r - V^b \end{bmatrix} + w_2 c \begin{bmatrix} U^r - U^b \\ V^r - V^b \end{bmatrix}, & \text{ for Type II,} \\
\begin{bmatrix} U^t \\ V^t \end{bmatrix} &= \begin{bmatrix} U^b \\ V^b \end{bmatrix} + w_1 \begin{bmatrix} 0 & c \\ c & 0 \end{bmatrix} \begin{bmatrix} U^b - U^l \\ V^b - V^l \end{bmatrix} + w_2 c \begin{bmatrix} U^b - U^l \\ V^b - V^l \end{bmatrix}, & \text{ for Type III,}
\end{aligned}$$

where

$$\begin{aligned}
w_1 &= \frac{(h_r + h_l)k}{(h_r + h_l)^2 - c^2 k^2 (p_r - p_l)^2}, & w_2 &= \frac{c k^2 (p_r - p_l)}{(h_r + h_l)^2 - c^2 k^2 (p_r - p_l)^2}, & \text{for Type I,} \\
w_1 &= \frac{k}{2(c k (1 - p_r) + h_r)}, & w_2 &= w_1, & \text{for Type II,} \\
w_1 &= \frac{k}{2(c k (1 - p_l) + h_l)}, & w_2 &= -w_1, & \text{for Type III.}
\end{aligned}$$

Indeed:

Tent of type I. We consider a tent of type I, with vertices at $(0, 0)$, $(h_r, k - p_r k)$, $(k, 0)$ and $(-h_l, k - p_l k)$, and define K_r the triangular domain with vertices $(0, 0)$, $(h_r, k - p_r k)$, $(k, 0)$ and K_l the triangular domain defined by $(0, 0)$, $(k, 0)$ and $(-h_l, k - p_l k)$ as in Figure 3.4. Since $L^1 \subset L^2(K)^2$ is the space of constant vector functions, we choose as basis $\{\eta_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \eta_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\}$. We define

$$\zeta(x, t) = \begin{cases} \frac{1}{k} \left(t - \frac{k(1-p_r)}{h_r} x \right) & \text{if } (x, t) \in K_r, \\ \frac{1}{k} \left(t + \frac{k(1-p_l)}{h_l} x \right) & \text{if } (x, t) \in K_l. \end{cases}$$

This function is one at the apex, zero at other vertices, and linear along the edges. See Figure 3.5. Thus, the space V_1 can be generated by the vector functions $\nu_1 = \begin{bmatrix} \zeta(x, t) \\ 0 \end{bmatrix}$ and $\nu_2 = \begin{bmatrix} 0 \\ \zeta(x, t) \end{bmatrix}$, and the space W_1 defined in (3.4.5) can be generated by the basis $\{\chi_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \chi_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \chi_3 = \begin{bmatrix} \zeta(x, t) \\ 0 \end{bmatrix}, \chi_4 = \begin{bmatrix} 0 \\ \zeta(x, t) \end{bmatrix}\}$.

Note that $A(w_1) = A(w_2) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and

$$\begin{aligned}
A \begin{bmatrix} \zeta \\ 0 \end{bmatrix} &= \left\{ \begin{bmatrix} \frac{1}{k} \\ c \frac{1-p_r}{h_r} \end{bmatrix} \text{ if } (x, t) \in K_r, \text{ and } \begin{bmatrix} \frac{1}{k} \\ -c \frac{1-p_l}{h_l} \end{bmatrix} \text{ if } (x, t) \in K_l \right\}, \\
A \begin{bmatrix} 0 \\ \zeta \end{bmatrix} &= \left\{ \begin{bmatrix} c \frac{1-p_r}{h_r} \\ \frac{1}{k} \end{bmatrix} \text{ if } (x, t) \in K_r, \text{ and } \begin{bmatrix} -c \frac{1-p_l}{h_l} \\ \frac{1}{k} \end{bmatrix} \text{ if } (x, t) \in K_l \right\}.
\end{aligned}$$

Now we compute the left hand side. We define

$$[\mathbb{B}_0]_{ij} = - \int_K \eta_j A(\chi_i) \text{ and } [\mathbb{B}_1]_{ij} = \int_{\partial K} \mathbf{D}_{\mathbf{x}, \mathbf{t}} \nu_j \cdot \chi_i.$$

Since the entries of the first two rows of $\mathbf{B}_0$ are zero, we only display

$$\begin{aligned}
[\mathbf{B}_0]_{31} &= - \int_K \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot A \begin{bmatrix} \zeta \\ 0 \end{bmatrix} = - \int_K \frac{1}{k} = - \frac{h_r + h_l}{2}, \\
[\mathbf{B}_0]_{32} &= - \int_K \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot A \begin{bmatrix} \zeta \\ 0 \end{bmatrix} = \int_{K_l} c \frac{1 - p_l}{h_l} - \int_{K_r} c \frac{1 - p_r}{h_r} = \frac{ck(p_r - p_l)}{2}, \\
[\mathbf{B}_0]_{41} &= - \int_K \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot A \begin{bmatrix} 0 \\ \zeta \end{bmatrix} = \int_{K_l} c \frac{1 - p_l}{h_l} - \int_{K_r} c \frac{1 - p_r}{h_r} = [\mathbf{B}_0]_{32}, \\
[\mathbf{B}_0]_{42} &= - \int_K \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot A \begin{bmatrix} 0 \\ \zeta \end{bmatrix} = - \int_K \frac{1}{k} = [\mathbf{B}_0]_{31}.
\end{aligned}$$

Next, the calculation of the outward normal vector n at different parts of the boundary gives us the following result

$$(3.4.13) \quad n = (n_x, n_t) = \begin{cases} \frac{1}{\sqrt{h_r^2 + p_r^2 k^2}} (p_r k, h_r) & \text{if } (x, t) \in \partial_o K_r, \\ \frac{1}{\sqrt{h_l^2 + p_l^2 k^2}} (-p_l k, h_l) & \text{if } (x, t) \in \partial_o K_l, \\ \frac{1}{\sqrt{h_r^2 + (k - p_r k)^2}} (k - p_r k, -h_r) & \text{if } (x, t) \in \partial_i K_r, \\ \frac{1}{\sqrt{h_l^2 + (k - p_l k)^2}} (p_l k - k, -h_l) & \text{if } (x, t) \in \partial_i K_l. \end{cases}$$

We introduce the functions $r_o(x) = (x, g_o(x))$ and $r_i(x) = (x, g_i(x))$, where

$$g_o(x) = \begin{cases} \frac{p_r k}{h_r} x + k & \text{if } 0 < x < h_r, \\ -\frac{p_l k}{h_l} x + k & \text{if } 0 < x < h_l, \end{cases} \quad \text{and} \quad g_i(x) = \begin{cases} \frac{k - p_r k}{h_r} x & \text{if } 0 < x < h_r, \\ -\frac{k - p_l k}{h_l} x & \text{if } 0 < x < h_l. \end{cases}$$

are functions introduced to parametrize the segments $\partial_o K_r$ and $\partial_o K_l$. From the def-

inition of ζ , it is easy to see that $\zeta(x, g_o(x)) = \begin{cases} 1 - \frac{x}{h_r} & \text{if } 0 < x < h_r, \\ 1 + \frac{x}{h_l} & \text{if } h_l < x < 0. \end{cases}$

Since ζ vanishes at $\partial_i K$, then

$$[\mathbf{B}_1]_{ij} = \int_{\partial_o K_r} \mathbf{D}_{\mathbf{x}, \mathbf{t}} \nu_j \cdot \chi_i + \int_{\partial_o K_l} \mathbf{D}_{\mathbf{x}, \mathbf{t}} \nu_j \cdot \chi_i.$$

Since $D_{\mathbf{x},\mathbf{t}}$ is a constant matrix when restricted to specific parts of the boundary,

$$\int_{\partial_o K_r} D_{\mathbf{x},\mathbf{t}} \nu_j \cdot \chi_i = \int_0^{h_r} \frac{1}{h_r} \begin{bmatrix} h_r & -cp_r k \\ -cp_r k & h_r \end{bmatrix} \nu_j(r_o(x)) \cdot \chi_i(r_o(x)) dx$$

and

$$\int_{\partial_o K_l} D_{\mathbf{x},\mathbf{t}} \nu_j \cdot \chi_i = \int_{-h_l}^0 \frac{1}{h_l} \begin{bmatrix} h_l & cp_l k \\ cp_l k & h_l \end{bmatrix} \nu_j(r_o(x)) \cdot \chi_i(r_o(x)) dx.$$

Note that the values of n_x and n_t are the values of the components of the normal vector of $\partial_o K_r$ and $\partial_o K_l$ respectively, see (3.4.13). Then,

$$\begin{aligned} [B_1]_{11} &= [B_1]_{22} = \int_0^{h_r} \left(1 - \frac{x}{h_r}\right) dx + \int_{-h_l}^0 \left(1 + \frac{x}{h_l}\right) dx = \frac{h_l + h_r}{2}, \\ [B_1]_{12} &= [B_1]_{21} = \int_0^{h_r} \frac{-cp_r k}{h_r} \left(1 - \frac{x}{h_r}\right) dx + \int_{-h_l}^0 \frac{cp_l k}{h_l} \left(1 + \frac{x}{h_l}\right) dx = \frac{ck}{2}(p_l - p_r), \\ [B_1]_{31} &= [B_1]_{42} = \int_0^{h_r} \left(1 - \frac{x}{h_r}\right)^2 dx + \int_{-h_l}^0 \left(1 + \frac{x}{h_l}\right)^2 dx = \frac{h_l + h_r}{3}, \\ [B_1]_{32} &= [B_1]_{41} = \int_0^{h_r} \frac{-cp_r k}{h_r} \left(1 - \frac{x}{h_r}\right)^2 dx + \int_{-h_l}^0 \frac{cp_l k}{h_l} \left(1 + \frac{x}{h_l}\right)^2 dx = \frac{ck}{3}(p_l - p_r). \end{aligned}$$

Thus,

$$B_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -\frac{h_r+h_l}{2} & \frac{ck(p_r-p_l)}{2} \\ \frac{ck(p_r-p_l)}{2} & -\frac{h_r+h_l}{2} \end{bmatrix} \quad \text{and} \quad B_1 = \begin{bmatrix} \frac{h_r+h_l}{2} & -\frac{ck(p_r-p_l)}{2} \\ -\frac{ck(p_r-p_l)}{2} & \frac{h_r+h_l}{2} \\ \frac{h_r+h_l}{3} & -\frac{ck(p_r-p_l)}{3} \\ -\frac{ck(p_r-p_l)}{3} & \frac{h_r+h_l}{3} \end{bmatrix}.$$

We define $B = [B_0, B_1]$.

Moreover, we can write $\mathbf{B} = \begin{bmatrix} 0 & \frac{1}{2}M \\ \frac{-1}{2}M & \frac{1}{3}M \end{bmatrix}$, by setting

$$\mathbf{M} = \begin{bmatrix} h_r + h_l & -ck(p_r - p_l) \\ -ck(p_r - p_l) & h_r + h_l \end{bmatrix}.$$

Thus,

$$\mathbf{B}^{-1} = \begin{bmatrix} \frac{4}{3}M^{-1} & -2M^{-1} \\ 2M^{-1} & 0 \end{bmatrix}, \quad \text{where} \quad \mathbf{M}^{-1} = \frac{1}{\det(\mathbf{M})} \begin{bmatrix} h_r + h_l & ck(p_r - p_l) \\ ck(p_r - p_l) & h_r + h_l \end{bmatrix}$$

and $\det(\mathbf{M}) = (h_r + h_l)^2 - c^2k^2(p_r - p_l)^2$.

To calculate the right hand side we choose

$$\zeta_r(x, t) = \begin{cases} \frac{x}{h_r} & \text{if } x \in K_r, \\ 0 & \text{if } x \in K_l. \end{cases}, \quad \zeta_l(x, t) = \begin{cases} -\frac{x}{h_l} & \text{if } x \in K_l, \\ 0 & \text{if } x \in K_r \end{cases} \quad \text{and}$$

$$\zeta_b(x, t) = \begin{cases} -\frac{p_r}{h_r}x + 1 - \frac{t}{k} & \text{if } (x, t) \in K_r, \\ \frac{p_l}{h_l}x + 1 - \frac{t}{k} & \text{if } (x, t) \in K_l. \end{cases}$$

From the structure of matrix $\mathbf{B}^{-1}$, it is easy to see that

$$\begin{bmatrix} U^t \\ V^t \end{bmatrix} = 2M^{-1} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix},$$

where, following (3.4.11),

$$F_1 = - \int_{\partial K} \mathbf{D}_{\mathbf{x}, \mathbf{t}} \left(\begin{bmatrix} U_b \\ V_b \end{bmatrix} \zeta_b + \begin{bmatrix} U_l \\ V_l \end{bmatrix} \zeta_l + \begin{bmatrix} U_r \\ V_r \end{bmatrix} \zeta_r \right) \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$F_2 = - \int_{\partial K} \mathbf{D}_{\mathbf{x}, \mathbf{t}} \left(\begin{bmatrix} U_b \\ V_b \end{bmatrix} \zeta_b + \begin{bmatrix} U_l \\ V_l \end{bmatrix} \zeta_l + \begin{bmatrix} U_r \\ V_r \end{bmatrix} \zeta_r \right) \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Figure 3.4. Type I: $h_r > 0, h_l > 0$

We use the following results

$$-\int_{\partial K} D_{\mathbf{x}, \mathbf{t}} \begin{bmatrix} U_i \\ V_i \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = -\int_{\partial K} n_t U_i - c \int_{\partial_o K} n_x V_i,$$

,

$$-\int_{\partial K} D_{\mathbf{x}, \mathbf{t}} \begin{bmatrix} U_i \\ V_i \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -\int_{\partial_i K} n_t U_i - c \int_{\partial_i K} n_x V_i$$

for $i = b, l, r$ to define and calculate

$$S_i = \begin{bmatrix} \int_{\partial K} n_t \zeta_i & -c \int_{\partial K} n_x \zeta_i \\ -c \int_{\partial K} n_x \zeta_i & \int_{\partial K} n_t \zeta_i \end{bmatrix}, \quad \text{for } i = b, l, r.$$

Thus, we can rewrite (3.4.14) as follows

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = S_b \begin{bmatrix} U_b \\ V_b \end{bmatrix} + S_l \begin{bmatrix} U_l \\ V_l \end{bmatrix} + S_r \begin{bmatrix} U_r \\ V_r \end{bmatrix}.$$

Since $\zeta_b(x, g_i(x)) = \begin{cases} 1 - \frac{x}{h_r} & \text{if } 0 < x < h_r \\ 1 + \frac{x}{h_l} & \text{if } h_l < x < 0 \end{cases}$, we calculate:

$$\begin{aligned} \int_{\partial K} n_t \zeta_b &= \int_{-h_l}^0 \left(1 + \frac{x}{h_l}\right) dx + \int_0^{h_r} \left(1 - \frac{x}{h_r}\right) dx = \frac{h_l + h_r}{2}, \\ \int_{\partial K} n_x \zeta_b &= \frac{k - kp_r}{h_r} \int_0^{h_r} \left(1 + \frac{x}{h_r}\right) dx + \frac{p_l k - k}{h_l} \int_{-h_l}^0 \left(1 + \frac{x}{h_l}\right) dx = \frac{k(p_l - pr)}{2}, \\ \int_{\partial K_r} n_t \zeta_r &= \int_{\partial K_l} n_t \zeta_l = 0, \\ \int_{\partial K_r} n_x \zeta_r &= - \int_{\partial K_l} n_x \zeta_l = \frac{k}{2} \end{aligned}$$

to find the matrices S_i . Hence,

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \frac{1}{2} M \begin{bmatrix} U_b \\ V_b \end{bmatrix} + \begin{bmatrix} 0 & k \\ k & 0 \end{bmatrix} \begin{bmatrix} U_l - U_r \\ V_l - V_r \end{bmatrix}.$$

Multiplying by M^{-1} to the left, we obtain the desired result. Indeed,

$$\begin{bmatrix} U_t \\ V_t \end{bmatrix} = \begin{bmatrix} U_b \\ V_b \end{bmatrix} + M^{-1} \begin{bmatrix} 0 & k \\ k & 0 \end{bmatrix} \begin{bmatrix} U_l - U_r \\ V_l - V_r \end{bmatrix}$$

equals (3.4.12).

Tent of Type II. We denote $K_r = K$ to present the results for tents of Type II in (3.4.12). We consider the basis of V_1 to be $\nu_1 = \begin{bmatrix} \zeta(x,t) \\ \zeta(x,t) \end{bmatrix}$. Thus, a basis for W_1 is given by $\{\chi_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \chi_2 = \begin{bmatrix} \zeta(x,t) \\ 0 \end{bmatrix}, \chi_3 = \begin{bmatrix} 0 \\ \zeta(x,t) \end{bmatrix}\}$, where $\zeta = \frac{1}{k} \left(t - \frac{k(1-pr)}{h_r} x \right)$. As before, the basis functions for $L^1 \subset L^2(K_r)$ are chosen to be $\{\eta_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \eta_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\}$.

Defining

$$[B_0]_{ij} = - \int_{K_r} \eta_j A(\chi_i) \text{ and } [B_1]_{ij} = \int_{\partial K_r} D_{x,t} \nu_j \cdot \chi_i,$$

Figure 3.5. $\zeta(x, t)$ basis function for tent of Type I

as before, we obtain

$$\begin{aligned} [\mathbf{B}_0]_{21} &= - \int_{K_r} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot A \begin{bmatrix} \zeta \\ 0 \end{bmatrix} = - \int_{K_r} \frac{1}{k} = - \frac{h_r}{2}, \\ [\mathbf{B}_0]_{22} &= - \int_{K_r} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot A \begin{bmatrix} \zeta \\ 0 \end{bmatrix} = - \int_{K_r} c \frac{1 - p_r}{h_r} = \frac{ck(p_r - 1)}{2}, \\ [\mathbf{B}_0]_{31} &= [\mathbf{B}_0]_{22} \quad \text{and} \quad [\mathbf{B}_0]_{32} = [\mathbf{B}_0]_{31} \end{aligned}$$

and

$$\begin{aligned} [\mathbf{B}_1]_{11} &= \int_{\partial_o K_r \cup \partial_b^{+,1} K_r} \mathbf{D}_{\mathbf{x}, \mathbf{t}} \begin{bmatrix} \zeta \\ \zeta \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = ck(1 - p_r) + h_r, \\ [\mathbf{B}_1]_{21} &= \int_{\partial_o K_r \cup \partial_b^{+,1} K_r} \mathbf{D}_{\mathbf{x}, \mathbf{t}} \begin{bmatrix} \zeta \\ \zeta \end{bmatrix} \cdot \begin{bmatrix} \zeta \\ 0 \end{bmatrix} = - \frac{ckp_r - ck - h_r}{3}, \\ [\mathbf{B}_1]_{31} &= \int_{\partial_o K_r \cup \partial_b^{+,1} K_r} \mathbf{D}_{\mathbf{x}, \mathbf{t}} \begin{bmatrix} \zeta \\ \zeta \end{bmatrix} \cdot \begin{bmatrix} 0 \\ \zeta \end{bmatrix} = - \frac{ckp_r - ck - h_r}{3}. \end{aligned}$$

Thus

$$(3.4.15) \quad \mathbf{B} = \begin{bmatrix} 0 & 0 & ck(1 - p_r) + h_r \\ -\frac{h_r}{2} & \frac{ck(p_r - 1)}{2} & -\frac{ck(p_r - 1) - h_r}{3} \\ \frac{ck(p_r - 1)}{2} & -\frac{h_r}{2} & -\frac{ck(p_r - 1) - h_r}{3} \end{bmatrix}$$

and its inverse is given by

$$(3.4.16) \quad B^{-1} = \begin{bmatrix} \frac{2}{3(ck(p_r-1)-h_r)} & \frac{2h_r}{(h_r-ck(1-p_r))(h_r+ck(1-p_r))} & \frac{-2ck(p_r-1)}{(h_r-ck(1-p_r))(h_r+ck(1-p_r))} \\ \frac{-2}{3(ck(p_r-1)-h_r)} & \frac{-2ck(p_r-1)}{(h_r-ck(1-p_r))(h_r+ck(1-p_r))} & \frac{2h_r}{(h_r-ck(1-p_r))(h_r+ck(1-p_r))} \\ \frac{-1}{ck(p_r-1)-h_r} & 0 & 0 \end{bmatrix}.$$

Since we are only interested in the last component of $B^{-1} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}$, which gives the value of $U_t = V_t$, we only need to calculate

$$\begin{aligned} F_1 &= - \int_{\partial K} D_{x,t} \left(\begin{bmatrix} U_b \\ V_b \end{bmatrix} \zeta_b + \begin{bmatrix} U_r \\ V_r \end{bmatrix} \zeta_r \right) \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= -(U_b + V_b) \int_{\partial K} (n_t - cn_x) \zeta_b - (U_r + V_r) \int_{\partial K} (n_t - cn_x) \zeta_r. \end{aligned}$$

Thus, computing

$$\int_{\partial_1 K} n_t \zeta_b, \int_{\partial_1 K \cup \partial_b^{+,1}} n_x \zeta_b, dt, \int_{\partial K_r} n_t \zeta_r, \text{ and } \int_{\partial K_r} n_x \zeta_r,$$

we obtain that

$$U_t = V_t = \frac{1}{ck(1-p_k) + h_r} F_1.$$

Rewriting this formula we obtain (3.4.12). Similar calculations are done to obtain the result for tents of Type III.

3.4.4. Error analysis on uniform grids. We now work out the stencil given by the method on a uniform grid where all tents are shaped the same (see Figure 3.6). The stencil translates (3.4.12) into an equation that gives the nodal values of the outflow apex vertex, given the nodal values at the inflow vertices. Let $h > 0$ be the uniform spatial mesh size, $k > 0$ be the time step size measured, as before, by the height of the tent pole. At a point $(jh/2, kn/2)$ in the lattice $(h/2)\mathbb{Z} \times (k/2)\mathbb{Z}$, let (U_j^n, V_j^n) denote the nodal value of the approximation to z_h there. As shown in

Figure 3.6. The stencil

Figure 3.6, the scheme uses only a subset of lattice points in $h\mathbb{Z} \times k\mathbb{Z}$. Each grid point involved in the scheme has an associated U value (indicated in the figure by “•”) and a V value (indicated by “o”).

Equation (3.4.12) now simplifies to

$$(3.4.17a) \quad U_j^{n+1} = U_j^{n-1} + ac(V_{j+1}^n - V_{j-1}^n)$$

$$(3.4.17b) \quad V_j^{n+1} = V_j^{n-1} + ac(U_{j+1}^n - U_{j-1}^n)$$

where $a = k/h$. This is simply the non-staggered leapfrog scheme (studied extensively for scalar equations) applied to the first order system. By a simple Taylor expansion about the stencil center, we see that the scheme is consistent and that the local truncation error is of second order (see [57, 65] for definitions of these and related terminology).

To examine stability, introduce a new vector variable X_j^n and rewrite the scheme (3.4.17) as follows:

$$X_j^{n+1} = \begin{bmatrix} [X_j^n]_3 + ac[X_{j+1}^n - X_{j-1}^n]_2 \\ [X_j^n]_4 + ac[X_{j+1}^n - X_{j-1}^n]_1 \\ [X_j^n]_1 \\ [X_j^n]_2 \end{bmatrix}, \quad \text{where} \quad \begin{bmatrix} U_j^n \\ V_j^n \\ U_j^{n-1} \\ V_j^{n-1} \end{bmatrix}.$$

Thus,

$$[X_j^{n+1}] = A_{-1}[X_j^n] + A_0[X_j^n] + A_1[X_{j+1}^n]$$

with

$$A_0 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & ac & 0 & 0 \\ ac & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ and } A_{-1} = A_1.$$

To this one-step scheme, we now apply von Neumann analysis [57, 65].

Letting $X^n = \{X_j^n\}$ be the vector-valued grid function, then we can write

$$X^{n+1} = A * X^n$$

Applying Fourier transform, we have $\widehat{X^{n+1}}(s) := \hat{A}(s)\widehat{X^n}(s)$. All of these transforms are defined componentwise. Thus, the amplification matrix $\hat{A}(s)$ or $G(s)$, connecting X_j^{n+1} to X_j^n is given by

$$G = \begin{bmatrix} 0 & 2is & 1 & 0 \\ 2is & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

where $\hat{i}$ denotes the imaginary unit and $s = ac \sin(\theta)$. We write $G = R\Lambda R^{-1}$, by setting

$$R = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ g_1^{-1} & g_2^{-1} & g_3^{-1} & g_4^{-1} \\ g_1^{-1} & g_2^{-1} & -g_3^{-1} & -g_4^{-1} \end{bmatrix}, \quad \Lambda = \begin{bmatrix} g_1 & 0 & 0 & 0 \\ 0 & g_2 & 0 & 0 \\ 0 & 0 & g_3 & 0 \\ 0 & 0 & 0 & g_4 \end{bmatrix},$$

the eigenvalues of G are $g_1 = \hat{i}s - \sqrt{1-s^2}$, $g_2 = \hat{i}s + \sqrt{1-s^2}$, $g_3 = -\hat{i}s - \sqrt{1-s^2}$, $g_4 = -\hat{i}s + \sqrt{1-s^2}$ and $\theta \in [-\pi, \pi]$ gives the frequency in von Neumann analysis. If

$$(3.4.18) \quad |ac| < 1$$

then all eigenvalues satisfy $|g_i| = 1$. Furthermore, since

$$\begin{aligned} \det R &= 4(g_1^{-1} - g_2^{-1})(g_4^{-1} - g_3^{-1}) \\ &= -8\sqrt{1-s^2} \end{aligned}$$

remains away from zero whenever (3.4.18) holds, the powers $G^n = R\Lambda^n R^{-1}$ are uniformly bounded for all n and all θ . Hence (3.4.18) implies that the scheme is stable.

We thus conclude, by the Lax-Richtmyer theorem, that the scheme is convergent and is of second order. Note that the causality condition we previously found on general meshes, namely (3.3.3), when restricted to uniform meshes, gives exactly the same CFL condition (3.4.18) obtained above from von Neumann analysis.

3.5. Numerical results

3.5.1. Convergence study. First, we report numerical results from our tent pitching (TP) scheme and compare it with the well-known “central-time central-space” (CTCS) finite difference scheme (see [14, 57], sometimes also known as the

Figure 3.7. Comparison with CTCS scheme

Yee scheme [41, 70]). The only difference between the two is that while the TP scheme sets the U and V nodes on the same location (exactly as indicated in Figure 3.6), the CTCS scheme sets them on staggered locations on the same grid. Both schemes are applied to the model problem (3.4.1) on uniform grids with $S = 1$. We use a grid like that in Figure 3.6 for both methods.

To impose the outgoing impedance boundary conditions within the CTCS scheme, we use the standard finite difference technique of introducing ghost points to the left and right of the finite grid and eliminating the unknown values at those points using the boundary condition. In contrast, in the TP scheme, the impedance boundary conditions are essentially imposed within the finite element spaces, as we have already seen previously. We set $c = 1$ and impose the initial condition so that the exact solution is

$$u_1(x, t) = u_2(x, t) = e^{-1000((x+t)-1/2)^2},$$

i.e., the solution is a smooth pulse moving to the left at unit speed, eventually clearing out of the simulation domain. At every other time step (in the uniform spacetime grid) we compute the $L^2(0, 1)$ -norm of the difference between the computed and exact

solution. The evolution of these errors in time on a grid of spatial mesh size $h = 0.0025$ and $k = 0.9h$ is shown in Figure 3.7a.

We observe from Figure 3.7a that the errors of both methods are comparable and remain low throughout the simulated time. Note also that after the pulse clears the simulation domain reflectionlessly (and the solution within $[0, 1]$ vanishes), the errors for both methods decrease markedly. In Figure 3.7, we display a log-log plot of the $L^2(0, 1)$ -norm of the errors at $t = 0.5$ for $h = 1/2^3, \dots, 1/2^{13}$, and $k = 0.9h$. The rate of decrease of this error is clearly seen to be of the order $O(h^2)$. This is in accordance with our von Neumann analysis of § 3.4.4 (although we did not take into account boundary conditions in that analysis).

Thus we conclude from Figure 3.7 that there is negligible difference between the performance of the two methods on uniform grids.

3.5.2. Material interfaces and other boundary conditions. Next, we consider a generalization of (3.4.1) given by

$$(3.5.1a) \quad \partial_t \begin{bmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \begin{bmatrix} 0 & c \\ c & 0 \end{bmatrix} \partial_x \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = f, \quad 0 < x < 1, 0 < t < T,$$

$$(3.5.1b) \quad u_1(x, 0) = u_1^0(x), \quad 0 < x < 1,$$

$$(3.5.1c) \quad u_2(x, 0) = u_2^0(x), \quad 0 < x < 1,$$

$$(3.5.1d) \quad z_0 u_1 - u_2 = 0, \quad x = 0, 0 < t < T,$$

$$(3.5.1e) \quad z_1 u_1 + u_2 = 0, \quad x = 1, 0 < t < T.$$

where $\kappa_1(x)$ and $\kappa_2(x)$ are time-independent material parameters and c , z_0 and z_1 are constants. Such systems arise from electromagnetics or acoustics [41, 47] on layered media and the differential equation is often written in the following equivalent, but

non-symmetric form

$$\partial_t \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \begin{bmatrix} 0 & \beta_1 \\ \beta_2 & 0 \end{bmatrix} \partial_x \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \tilde{f}$$

where $\beta_i(x) = c/\kappa_i(x)$ and $\tilde{f} = \text{diag}(\kappa_1^{-1}, \kappa_2^{-1})f$ obtained by scaling the equations of (3.5.1a) by κ_1^{-1} and κ_2^{-1} . When $\kappa_1(x) \equiv \kappa_2(x) \equiv 1$ and $z_0 = z_1 = 1$, we obtain the model formulation we discussed previously in detail. Dirichlet boundary conditions can be imposed by putting $z_0 = z_1 = 0$, while exact outgoing impedance conditions can be imposed using $z_0 = \sqrt{\kappa_1/\kappa_2}$ and $z_1 = \sqrt{\kappa_1/\kappa_2}$. Intermediate values of z_i give damped impedance boundary conditions.

Whenever κ_i is a constant on each spatial mesh interval, a tent pitching scheme is suggested by a simple generalization of the previous algorithm for homogeneous media. We define the discrete spaces exactly as in (3.4.9), but noting that $V(\Omega)$ now has different essential boundary conditions – stemming from (3.5.1d)–(3.5.1e) – which are inherited by the spaces on tents with its tent pole on the boundary. The generalization of the scheme is derived by merely setting the A and $D_{x,t}$ in (3.4.11) by

$$A = \begin{bmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{bmatrix} \partial_t - \begin{bmatrix} 0 & c \\ c & 0 \end{bmatrix} \partial_x, \quad D_{x,t} = \begin{bmatrix} n_t \kappa_1 & -cn_x \\ -cn_x & n_t \kappa_2 \end{bmatrix}.$$

Note that this A , appearing on the left hand side of (3.5.1a), satisfies (A-a), (A-b) and (A-c). By solving this general version of (3.4.11) one can obtain propagation formulas similar to (3.4.12), but we omit these details and report only the numerical results.

First we consider the case

$$\kappa_1 = \begin{cases} 2, & 0 < x < 1/2, \\ 1, & 1/2 < x < 1, \end{cases} \quad \kappa_2 = \begin{cases} 2, & 0 < x < 1/2, \\ 1, & 1/2 < x < 1, \end{cases}$$

Figure 3.8. Wave propagation through an impedance-matched interface

and $c = 1$. The wave speed (equaling $c/\sqrt{\kappa_1\kappa_2}$), jumps from 0.5 in the left half to 1 in the right half. However, the impedance (equaling κ_1/κ_2 – see [47]) is one in both regions. Thus $x = 0.5$ is an impedance-matched interface about which we do not expect to see any reflection.

We use the tent pitching method to simulate a wave propagating to the right starting near $x = 0.2$. To this end, define a smooth pulse $g(x) = e^{-5000(x-0.2)^2}$ and set the data in (3.5.1) by

$$(3.5.2) \quad f = 0, \quad u_1^0(x) = (c/\kappa_1)g(x), \quad u_2^0(x) = -(c/\sqrt{\kappa_1\kappa_2})g(x),$$

and $z_0 = \sqrt{\kappa_1/\kappa_2}$ and $z_1 = \sqrt{\kappa_1/\kappa_2}$. We use a spatial mesh of mesh size $h = 10^{-3}$ in the left half and $h = 2 \times 10^{-3}$ in the right half. A simple tent meshing algorithm then produces a mesh of spacetime tents based on this non-uniform spatial mesh that

satisfies the CFL condition (3.3.3). The meshing algorithm proceeds as illustrated as in Figure 3.1 by simply picking a point with the lowest time coordinate to pitch a tent. When multiple locations have the minimal time coordinate, the algorithm picks a tent pitching location among them randomly, thus giving an unstructured mesh. To minimize the overhead in constructing the mesh of tents, instead of meshing the entire spacetime domain at once, we first mesh a thin time slab $\{(x, t) : 0 < t < 0.002, 0 < x < 1\}$ and then repeatedly stack this mesh in time to cover the entire region of time simulation. The mesh of the initial slab is shown in Figure 3.8c.

One of the two components of the computed solution is shown in the remaining two plots of Figure 3.8. Clearly, the simulated wave packet travels left across the $x = 0.5$ interface without any reflected wave and expands as it enters the region of higher wave speed. In further (unreported) numerical experiments, we have noticed changes in the discrete wave speed depending on the spacetime mesh. For example, the wave speed differs if one uses uniform space time meshes with positively sloped diagonals only or negatively sloped diagonals only. Such wave speed differences appear to approach to zero slowly as h is made smaller. High order methods may be needed to reduce these dispersive errors.

Our next and final example involves an interface where we expect both reflection and transmission. We set $c = 1$ and

$$\kappa_1 = \begin{cases} 4, & 0 < x < 1/2, \\ 1/2, & 1/2 < x < 1, \end{cases} \quad \kappa_2 = \begin{cases} 1, & 0 < x < 1/2, \\ 1/2, & 1/2 < x < 1, \end{cases}$$

Both the wave speed and the impedance jumps from the left region to the right region (from 0.5 and 4 to 2 and 1, respectively). We set f and initial data as in the last simulation by (3.5.2), but in order to impose Dirichlet boundary condition, we set $z_0 = z_1 = 0$. This time, instead of using a non-uniform mesh, we use a spatially

Figure 3.9. Case of reflected and transmitted waves

uniform mesh of $h = 10^{-3}$ and let the tent pitching algorithm adjust k to satisfy the CFL condition (3.3.3) in each tent. We found that the mesh obtained, displayed in Figure 3.9a, while not ideal due to the thin triangles, is adequate for the simulation. (Better tent pitched meshes can be obtained using non-uniform spatial mesh spacing, as we saw in the previous example and Figure 3.8c.) The solution components u_1 and u_2 obtained from the simulation are displayed in Figure 3.9b. The computed waves are transmitted as well as reflected both from the interface and the Dirichlet boundaries. The expected features of the solution are therefore recovered by the method.

A spacetime DPG method for the Schrodinger equation

4.1. Introduction to the problem

This chapter is devoted to exploring a weak formulation and an accompanying numerical technique for the Schrödinger equation with Dirichlet boundary conditions. Let $\Omega_0 \subset \mathbb{R}^{d-1}$ ($d > 1$) be an open bounded domain with Lipschitz boundary. The space variable x lies in Ω_0 while the time variable t lies in the open interval $(0, T)$ with $T < \infty$. The classical form of Schrödinger initial boundary value problem reads as follows:

$$(4.1.1a) \quad i\partial_t u - \Delta_x u = f, \quad x \in \Omega_0, \ 0 < t < T,$$

$$(4.1.1b) \quad u(x, t) = 0, \quad x \in \partial\Omega_0, \ 0 < t < T,$$

$$(4.1.1c) \quad u(x, 0) = 0, \quad x \in \Omega_0,$$

where ∂_t denotes the time derivative and Δ_x denotes the Laplacian with respect to the spatial variable x and i is the imaginary unit. Here f is any given function in $L^2(\Omega)$ and $\Omega = \Omega_0 \times (0, T)$ throughout.

Many applications of interest come as a first order system even if they are often displayed as second order partial differential equations. For example, the second order heat equation is really a combination of two first order equations, namely the Fourier law of heat conduction and the conservation of energy. Similarly, the linear elasticity equation while often displayed as a second order equation for displacement, is really a combination of the two first order equations, the constitutive (Hooke's) law

and the equation of static equilibrium. Thus its no surprise that it makes physical sense to bring back the heat equation or the elasticity equation to first order form before discretizing. However, it makes no physical sense to do this for the Schrödinger equation as its not derived from first order physical laws.

It makes no mathematical sense either. One might be tempted to introduce a “flux” τ , formulate the first order system $\hat{i}\partial_t u - \operatorname{div}_x \tau = f$ and $\nabla_x u - \tau = g$, and claim its equivalence to (4.1.1) when $g = 0$. This claim is false because while the Schrödinger problem (4.1.1) is wellposed for $f \in L^2(\Omega)$, *the first order system cannot be wellposed in $L^2(\Omega)$* . Indeed, denoting the norm of $L^2(\Omega)$ by $\|\cdot\|_\Omega$, if the first order system were wellposed, then there are constants $C_1, C_2 > 0$ such that $\|u\|_\Omega + \|\tau\|_\Omega \leq C_1\|f\|_\Omega + C_2\|g\|_\Omega$. But then the second equation of the system implies that $\|\nabla_x u\|_\Omega = \|g + \tau\|_\Omega \leq C_1\|f\|_\Omega + 2C_2\|g\|_\Omega$ for any solution u , which is false: In the next two paragraphs we will exhibit a Schrödinger solution for which $\|\nabla_x u\|_\Omega = \infty$ even when $g = 0$ and $f \in L^2(\Omega)$.

First observe that given any $f(x, t)$ in $L^2(\Omega)$, it is possible to solve (4.1.1) by the “method of Galerkin approximations” [32] (distinct from the Galerkin finite element method). Let $e_k(x)$ in $H_0^1(\Omega_0)$ and $\omega_k^2 > 0$ be an eigenpair of Δ_x satisfying

$$(4.1.2) \quad -\Delta_x e_k = \omega_k^2 e_k \quad \text{a.e. in } \Omega_0,$$

normalized so that $\|e_k\|_{\Omega_0} = 1$ for all natural numbers $k \geq 1$. Since Fubini’s theorem for product measures implies that $f(\cdot, t)$ is in $L^2(\Omega_0)$, the following definitions make sense:

$$(4.1.3a) \quad f_k(t) = \int_{\Omega_0} f(x, t) \bar{e}_k(x) dx, \quad u_k(t) = -\hat{i} \int_0^t e^{\hat{i}\omega_k^2(t-s)} f_k(s) ds,$$

$$(4.1.3b) \quad F_M(x, t) = \sum_{k=1}^M f_k(t) e_k(x), \quad U_M(x, t) = \sum_{k=1}^M u_k(t) e_k(x).$$

Figure 4.1. Real part of $U_M(x, t)$ with different values of M .

It is not difficult to show (see the proof of Theorem 4.2.3 below) that $u = \lim_{M \rightarrow \infty} U_M$ exists in $L^2(\Omega)$ and solves (4.1.1).

Now consider the one-dimensional case $\Omega_0 = (0, 1)$ where $\omega_k = k\pi$ and choose

$$f(x, t) = \sum_{k=1}^{\infty} \frac{1}{k} e^{i\omega_k^2 t} e_k(x) \quad \text{in } L^2(\Omega).$$

Then by the orthonormality of e_k , we have that $f_k(s) = e^{i\omega_k^2 s}/k$, $u_k(t) = -ite^{i\omega_k^2 t}/k$,

$$\begin{aligned} \|U_M\|_{\Omega}^2 &= \sum_{k=1}^M \int_0^T |u_k(t)|^2 dt = \sum_{k=1}^M \int_0^T \left| \frac{-it}{k} e^{i\omega_k^2 t} \right|^2 dt = \frac{T^3}{3} \sum_{k=1}^M \frac{1}{k^2} \\ \|\nabla_x U_M\|_{\Omega}^2 &= \sum_{k=1}^M \omega_k^2 \int_0^T |u_k(t)|^2 dt = T^3 \sum_{k=1}^M \frac{\omega_k^2}{3k^2} = \frac{\pi^2}{3} T^3 M. \end{aligned}$$

The solution u is the limit of U_M . The above calculations clearly show that as $M \rightarrow \infty$, while $\|u\|_{\Omega} = \lim_{M \rightarrow \infty} \|U_M\|_{\Omega} = (T^3 \pi^2 / 18)^{\frac{1}{2}}$, the limit of $\|\nabla_x U_M\|_{\Omega}$ diverges. *Thus it is possible to obtain a Schrödinger solution u whose H^1 -norm is infinite even when $f \in L^2(\Omega)$.* Note that finer arguments are needed to understand the regularity of Schrödinger solutions in unbounded domains, which although a topic of wide mathematical interest [66], is not our concern here.

The second order form necessitates formulations in the non-standard graph spaces of the second order Schrödinger operator. The general spaces and arguments required

for this analysis are collected in a standalone Chapter 1 anticipating uses outside of the Schrödinger example. Borrowing the approach of [30], we are able to prove wellposedness without developing a trace theory for the graph spaces. Numerical methods using the strong formulation must use conforming finite element subspaces of the graph spaces. On the other hand, numerical methods using the weak formulation need only use existing standard finite element spaces. In either case, an interpolation theory in the Schrödinger graph norm is needed to estimate convergence rates. We address this issue in one space dimension.

In the next section, we investigate wellposedness (in the sense of Hadamard) for a strong and weak variational formulation for the Schrödinger problem. This will require an abstract definition of a boundary operator and duality pairings in a graph space. Such abstract definitions that apply beyond the Schrödinger setting are in Chapter 1.

We now provide a functional setting within which a strong and a weak formulation of the spacetime Schrödinger problem can be proved to be wellposed (i.e., *inf-sup* stable). The analysis is an application of the general theory detailed in Chapter 1.

4.2. Wellposed formulation

The classical form of the problem is already presented in (4.1.1). Recalling that $\Omega = \Omega_0 \times (0, T)$, define these parts of $\partial\Omega$:

$$\Gamma = \partial\Omega_0 \times [0, T] \cup \Omega_0 \times \{0\}, \quad \Gamma^* = \partial\Omega_0 \times [0, T] \cup \Omega_0 \times \{T\}$$

(see Fig. 4.2). Then the initial and boundary conditions together can be written as $u|_\Gamma = 0$. We want to write (4.1.1) as an operator equation (see (1.4.1)) to apply the

general results of Chapter 1. To this end, consider the setting of Chapter 1 with

$$A = A^* = \hat{i}\partial_t - \Delta_x, \quad k = l = m = 1.$$

and moreover, Assumptions (A-a) and (A-b) are satisfied for these operators. The space $W = W^*$ is then defined by (1.2.1)–(1.2.2); namely $W = W^* = \{u \in L^2(\Omega) : \hat{i}\partial_t u - \Delta_x u \in L^2(\Omega)\}$. The operator $D = D^* : W \rightarrow W'$ is defined by (1.2.3)–(1.2.4); namely $\langle Dw, \tilde{w} \rangle_W = (Aw, \tilde{w})_\Omega - (w, A\tilde{w})_\Omega$ for all $w, \tilde{w} \in W$. As usual, let $\mathcal{D}(\bar{\Omega})$ denote the restrictions of functions from $\mathcal{D}(\mathbb{R}^{n+1})$ to Ω . The operator D (often called the “boundary operator” in the theory of Friedrichs systems [30]), satisfies

$$(4.2.1) \quad \langle D\phi, \psi \rangle_W = \int_{\partial\Omega} \hat{n}_t \phi \bar{\psi} + \int_{\partial\Omega} \phi (n_x \cdot \nabla_x \bar{\psi}) - \int_{\partial\Omega} (n_x \cdot \nabla_x \phi) \bar{\psi}$$

for all $\phi, \psi \in \mathcal{D}(\bar{\Omega})$. Note that although the integrals on the right hand side need not exist for all functions in W , D is defined on all W through (1.2.3).

Although we set the differential operators A and A^* to be equal above, note that we consider each as an unbounded operator with its own domain. We set the domain of A to

$$(4.2.2) \quad \text{dom}(A) = \{u \in W : \langle Dv, u \rangle_W = 0, \forall v \in \mathcal{V}^*\},$$

where $\mathcal{V}^* = \{\varphi \in \mathcal{D}(\bar{\Omega}) : \varphi|_{\Gamma^*} = 0\}$. Since $\mathcal{D}(\Omega) \subset \text{dom}(A)$, then Assumption (A-c) holds. An therefore there exists a unique The domain of the adjoint operator is given as in (1.3.1), and satisfy . Therefore is a by $\text{dom}(A^*) = \{s \in L^2(\Omega) : \text{there is an } \ell \in L^2(\Omega) \text{ such that } (Av, s)_\Omega = (v, \ell)_\Omega \text{ for all } v \in \text{dom}(A)\}$. Here $(\cdot, \cdot)_\Omega$ denotes the (complex) inner product in $L^2(\Omega)$. Finally, as in Chapter 1, set $V = \text{dom}(A)$ and $V^* = \text{dom}(A^*)$ with the understanding that both V and V^* are endowed with the W -topology, while $\text{dom}(A)$ and $\text{dom}(A^*)$ have the topology of $L^2(\Omega)$.

Figure 4.2. Schematic of the spacetime domain

Since conditions (A-a), (A-b) and (A-c) are easily verified using (4.2.1). Hence Lemma 1.3.1 shows that $\text{dom}(A^*)$ equals

$$(4.2.3) \quad V^* = {}^\perp D(V).$$

Rewriting (4.2.2) in the same style,

$$(4.2.4) \quad V = {}^\perp D(\mathcal{V}^*).$$

Thus V and V^* are closed subspaces of W . Let $\mathcal{V} = \{\varphi \in \mathcal{D}(\bar{\Omega}) : \varphi|_I = 0\}$.

LEMMA 4.2.1. $\mathcal{V} \subset V$ and $\mathcal{V}^* \subset V^*$.

PROOF. Equation (4.2.1) implies $\langle D\phi, \phi^* \rangle_W = 0$ for all $\phi \in V$ and $\phi^* \in \mathcal{V}^*$. Hence, any $\phi \in \mathcal{V}$ is also in ${}^\perp D(\mathcal{V}^*)$, which by (4.2.4) implies that $\phi \in V$. Thus $\mathcal{V} \subset V$.

If $\phi^* \in \mathcal{V}^*$, then (4.2.2) shows that $\langle D\phi^*, u \rangle_W = 0$ for all $u \in V$, i.e., $\phi^* \in {}^\perp D(V)$, which by (4.2.3) implies ϕ^* is in V^* . $\square$

In (4.2.4), $\mathcal{V}^*$ may be replaced by V^* provided a density result is available, as it is shown in Lemma 1.

4.2.1. Strong formulation. The strong formulation of the Schrödinger problem (4.1.1) is based on these sesquilinear and conjugate linear forms:

$$a(u, v) = (Au, v)_\Omega, \quad l(v) = (f, v)_\Omega.$$

PROBLEM 4.2.2 (Strong formulation). *Given any $f \in L^2(\Omega)$, find $u \in V$ satisfying*

$$a(u, v) = l(v), \quad \forall v \in L^2(\Omega).$$

THEOREM 4.2.3. *Suppose Assumption 1 holds. Then the linear Schrödinger operator $A : V \rightarrow L^2(\Omega)$ is a continuous bijection. Hence Problem 4.2.2 is well posed.*

PROOF. To prove the surjectivity of A , suppose $f \in L^2(\Omega)$. Recall the definitions of e_k , u_k and f_k from (4.1.2) and (4.1.3). Clearly, $AU_M = F_M$. Since U_M and any $\varphi \in \mathcal{V}^*$ are smooth enough for integration by parts using $\varphi|_{\Gamma^*} = 0$ and $U_M|_\Gamma = 0$, we have

$$(i\partial_t U_M, \varphi)_\Omega = (U_M, i\partial_t \varphi)_\Omega$$

$$(\Delta U_M, \varphi)_\Omega = (U_M, \Delta \varphi)_\Omega.$$

Hence $\langle D\varphi, U_M \rangle_W = (A\varphi, U_M)_\Omega - (\varphi, AU_M)_\Omega = 0$ for all $\varphi \in \mathcal{V}^*$. By (4.2.4), this implies that U_M is in V .

Next, we show that U_M is a Cauchy sequence in V . For any $N > M$,

$$\|U_M - U_N\|_\Omega^2 = \sum_{k=M+1}^N \int_0^T |u_k(t)|^2 dt \leq \frac{1}{2} T^2 \sum_{k=M+1}^\infty \int_0^T |f_k(t)|^2 dt,$$

$$\|A(U_M - U_N)\|_\Omega^2 = \|F_M - F_N\|_\Omega^2 \leq \sum_{k=M+1}^\infty \int_0^T |f_k(t)|^2 dt,$$

both of which converge to 0 as $M \rightarrow \infty$, because $f \in L^2(\Omega)$. Thus U_M is Cauchy. It must therefore have an accumulation point u in V . Moreover, since Au and f are

$L^2(\Omega)$ -limits of the same sequence $F_M = AU_M$, we have $Au = f$. Thus $A : V \rightarrow L^2(\Omega)$ is surjective.

We use a similar argument (with u_k defined by integrals from T to t) to show that $A = A^* : V^* \rightarrow L^2(\Omega)$ is also surjective. We omit the details, but note that the only difference is that instead of (4.2.4), we must now use

$$V^* = {}^\perp D(\mathcal{V}),$$

which follows from (4.2.3), Assumption 1 and Lemma 2.1.2. Finally, since $\ker(A) = {}^\perp \text{ran}(A^*)$, the surjectivity of $A^* : V^* \rightarrow L^2(\Omega)$ shows that $A : V \rightarrow L^2(\Omega)$ is injective, thus completing the proof of the stated bijectivity. $\square$

REMARK 4.2.4. An example of a standard wellposedness result for the Schrödinger equation obtained using semigroup theory is [46, Theorem 4.8.1], which proves that there is one and only one solution to (4.1.1) whenever $f \equiv 0$ and $u(x, 0)$ is in $H^2(\Omega) \cap H_0^1(\Omega)$. In Theorem 4.2.3, we have shown (by a different method) that the existence of a unique solution holds for any $f \in L^2(\Omega)$. Note that in the above proof, we used Assumption 1 only to obtain injectivity. If one opts to use the results of [46] (with $u(x, 0) \equiv 0$ and $f \equiv 0$) to conclude injectivity, then there is no need to place Assumption 1 in Theorem 4.2.3.

4.2.2. A weak formulation. Now we consider a mesh-dependent weak formulation that is the basis of the DPG method. This formulation, sometimes called the “ultraweak” formulation, is given in a general setting in Problem 1.4.2 of Chapter 1. We apply this to our example of the Schrödinger equation.

The spacetime domain Ω is partitioned into a mesh Ω_h of finitely many open elements K such that $\bar{\Omega} = \cup_{K \in \Omega_h} \bar{K}$ where $h = \max_{K \in \Omega_h} \text{diam}(K)$. Particularizing the general definitions in Chapter 1 (see (1.4.2) through (1.4.4)) to the Schrödinger

example, we let $A_h = A_h^*$ be the Schrödinger operator applied element by element and let $W_h = W_h^* = \{w \in L^2(\Omega) : A(w|_K) \in L^2(K) \text{ for all } K \in \Omega_h\}$. The operator $D_h : W_h \rightarrow W_h'$ is defined by $\langle D_h w, v \rangle_{W_h} = (A_h w, v)_\Omega - (w, A_h v)_\Omega$ for all $w, v \in W_h$ and let $D_{h,V} : V \rightarrow W_h'$ denote $D_h|_V$. The range of $D_{h,V}$, denoted by Q , is made into a complete space by the norm $\|q\|_Q = \inf_{v \in D_{h,V}^{-1}(\{q\})} \|v\|_W$. Abbreviating the duality pairing $\langle \cdot, \cdot \rangle_{W_h}$ by $\langle \cdot, \cdot \rangle_h$, define the sesquilinear form $b((u, q), v) = (u, A_h v)_\Omega + \langle q, v \rangle_h$ on $(L^2(\Omega) \times Q) \times W_h$.

PROBLEM 4.2.5 (Ultraweak formulation). *Given $F \in W_h'$, find $u \in L^2(\Omega)$ and $q \in Q$ such that*

$$b((u, q), v) = F(v), \quad \forall v \in W_h.$$

THEOREM 4.2.6. *Suppose Assumption 1 holds. Then Problem 4.2.5 is well posed, i.e., there is a $C > 0$ such that given any $F \in W_h'$, there is a unique solution $(u, q) \in L^2(\Omega) \times Q$ to Problem 4.2.5 and it satisfies*

$$\|u\|_\Omega^2 + \|q\|_Q^2 \leq C \|F\|_{W_h'}^2.$$

Moreover, if $F(v) = (f, v)_\Omega$ for some $f \in L^2(\Omega)$, then u is in V and $q = D_h u$.

PROOF. The result follows from Theorem 1.4.5. Since Lemma 2.1.2 together with (4.2.4) implies (1.4.9) and Theorem 4.2.3 implies (1.4.10), the assumptions of Theorem 1.4.5 are verified. $\square$

4.2.3. Verification of the density assumption. In the next three sections, Ω_0 is set to be the interval $(0, L)$ and $\Omega = (0, L) \times (0, T)$ where $L, T > 0$. The purpose of this section is to verify the density assumption (Assumption 1) in this case of one space dimension.

THEOREM 4.2.7. *Let $\Omega = (0, L) \times (0, T)$. Then $\mathcal{V}^* = \{\varphi \in \mathcal{D}(\bar{\Omega}) : \varphi|_{\Gamma^*} = 0\}$ is dense in V^* and $\mathcal{V} = \{\varphi \in \mathcal{D}(\bar{\Omega}) : \varphi|_{\Gamma} = 0\}$ is dense in V .*

PROOF. Since the proofs of both the stated density results are similar, we will only show the proof of density of $\mathcal{V}^*$ in V^* .

Step 1. Extend: Let $\Omega_l = (-L, 0] \times (0, T)$ and $\Omega_r = [L, 2L) \times (0, T)$. Define an operator G that extends functions on Ω to $\hat{\Omega} \equiv \Omega_l \cup \Omega \cup \Omega_r$ by

$$Gw(x, t) = \begin{cases} -w(-x, t), & (x, t) \in \Omega_l, \\ -w(2L - x, t), & (x, t) \in \Omega_r, \end{cases}$$

(and $Gw(x, t) = w(x, t)$ for all $(x, t) \in \Omega$). Let G' be the reverse operator that maps functions on $\hat{\Omega}$ to Ω by $G'\bar{w}(x, t) = \bar{w}(x, t) - \bar{w}(-x, t) - \bar{w}(2L - x, t)$, for all $(x, t) \in \Omega$ (see Figure 4.3). Such definitions are to be interpreted a.e., so that for example, Gw is well defined for any w in $L^2(\Omega)$. It is easy to see by a change of variable that

$$(4.2.5) \quad (Gf, g)_{\hat{\Omega}} = (f, G'g)_{\Omega}, \quad \forall f \in L^2(\Omega), \ g \in L^2(\hat{\Omega}).$$

Next, we claim that

$$(4.2.6) \quad AGv = GAv, \quad \forall v \in V^*.$$

Clearly, Gv is in $L^2(\hat{\Omega})$. Let $\varphi \in \mathcal{D}(\hat{\Omega})$. Let $\langle AGv, \varphi \rangle_{\mathcal{D}(\hat{\Omega})}$ denote the action of the distribution AGv on φ . Then $\langle AGv, \varphi \rangle_{\mathcal{D}(\hat{\Omega})} = (Gv, A\varphi)_{\hat{\Omega}} = (v, G'A\varphi)_{\Omega}$ because of (4.2.5). By the chain rule applied to the smooth function φ , we find that

$$(4.2.7) \quad G'A\varphi = AG'\varphi.$$

Figure 4.3. Extension and translation in the proof of Theorem 4.2.7.

Hence,

$$(4.2.8) \quad \langle AGv, \varphi \rangle_{\mathcal{D}(\hat{\Omega})} = (v, AG'\varphi)_\Omega = (Av, G'\varphi)_\Omega - \langle Dv, G'\varphi \rangle_W.$$

Now observe that $G'\varphi|_\Gamma = 0$. Hence by Lemma 4.2.1, $G'\varphi$ is in V . Since $v \in V^*$ and $G'\varphi \in V$, the last term of (4.2.8) must vanish by (4.2.3). Thus $\langle AGv, \varphi \rangle_{\mathcal{D}(\hat{\Omega})} = (Av, G'\varphi)_\Omega = (GA v, \varphi)_{\hat{\Omega}}$, completing the proof of the claim (4.2.6). In view of (4.2.6), we conclude that Gv is in $W(\hat{\Omega})$ whenever $v \in V^*$.

Step 2. Translate: Let $\tilde{G}v$ denote the extension of Gv by zero to $\mathbb{R}^2$, i.e., $\tilde{G}v$ equals Gv on $\hat{\Omega}$ and equals zero elsewhere. Let τ_δ be the translation operator in the $-t$ direction by δ , i.e., $(\tau_\delta w)(x, t) = w(x, t + \delta)$. It's well known ([7]) that

$$(4.2.9) \quad \lim_{\delta \rightarrow 0} \|\tau_\delta g - g\|_{\mathbb{R}^2} = 0, \quad \forall g \in L^2(\mathbb{R}^2).$$

Let $\hat{\Omega}_\delta = (-L, 2L) \times (-\delta, T + \delta)$ and let H_δ denote the restriction of functions on $\mathbb{R}^2$ to $\hat{\Omega}_\delta$. By a change of variable,

$$(4.2.10) \quad (\tau_\delta \tilde{G}f, g)_{\hat{\Omega}_\delta} = (Gf, \tau_{-\delta}g)_{\hat{\Omega}}, \quad \forall f \in L^2(\Omega), g \in L^2(\hat{\Omega}_\delta).$$

We now claim that

$$(4.2.11) \quad AH_\delta \tau_\delta \tilde{G}v = H_\delta \tau_\delta \tilde{G}Av, \quad \forall v \in V^*.$$

Indeed, for any $\varphi \in \mathcal{D}(\hat{\Omega}_\delta)$, the action of the distribution $AH_\delta\tau_\delta\tilde{G}v$ on $\bar{\varphi}$ equals

$$\begin{aligned}\langle AH_\delta\tau_\delta\tilde{G}v, \varphi \rangle_{\mathcal{D}(\hat{\Omega}_\delta)} &= (\tau_\delta\tilde{G}v, A\varphi)_{\hat{\Omega}_\delta} = (Gv, A\tau_{-\delta}\varphi)_{\hat{\Omega}} = (v, G'A\tau_{-\delta}\varphi)_\Omega \\ &= (v, AG'\tau_{-\delta}\varphi)_\Omega \\ &= (Av, G'\tau_{-\delta}\varphi)_\Omega - \langle Dv, G'\tau_{-\delta}\varphi \rangle_W,\end{aligned}$$

where we have used (4.2.10), (4.2.5), and (4.2.7) consecutively. Since $(G'\tau_{-\delta}\varphi)|_\Gamma = 0$, Lemma 4.2.1 shows that $G'\tau_{-\delta}\varphi$ is in V , and consequently the last term above vanishes for all $v \in V^*$. Continuing and using (4.2.5) and (4.2.10) once more, $\langle AH_\delta\tau_\delta\tilde{G}v, \varphi \rangle_{\mathcal{D}(\hat{\Omega}_\delta)} = (GA v, \tau_{-\delta}\varphi)_{\hat{\Omega}} = (\tau_\delta\tilde{G}A v, \varphi)_{\hat{\Omega}_\delta}$. This proves (4.2.11).

Step 3. Mollify: Consider the mollifier $\rho_\varepsilon \in \mathcal{D}(\mathbb{R}^2)$, for each $\varepsilon > 0$, defined by $\rho_\varepsilon(x, t) = \varepsilon^{-2}\rho_1(\varepsilon^{-1}x, \varepsilon^{-1}t)$, where

$$\rho_1(x, t) = \begin{cases} k e^{-\frac{1}{1-x^2-t^2}} & \text{if } x^2 + t^2 < 1, \\ 0 & \text{if } x^2 + t^2 \geq 1, \end{cases}$$

and k is a constant chosen so that $\int_{\mathbb{R}^2} \rho_1 = 1$. It is well known ([7]) that when any function w in $L^2(\mathbb{R}^2)$ is convolved with ρ_ε , the result $\rho_\varepsilon * w$ is infinitely smooth and satisfies

$$(4.2.12) \quad \lim_{\varepsilon \rightarrow 0} \|w - \rho_\varepsilon * w\|_{\mathbb{R}^2} = 0, \quad \forall w \in L^2(\mathbb{R}^2).$$

Consider any small enough $\delta > 0$, say $\delta < \min(L/2, T/2)$, and define two functions $v_\varepsilon = \rho_\varepsilon * \tau_\delta\tilde{G}v$ and $a_\varepsilon = \rho_\varepsilon * \tau_\delta\tilde{G}A v$. Note that the two smooth functions Av_ε and a_ε need not coincide everywhere. However, because of (4.2.11), they coincide on Ω whenever $\varepsilon < \delta/2$:

$$Av_\varepsilon = a_\varepsilon \quad \text{on } \Omega.$$

Let us therefore set δ to, say, $\delta = 3\varepsilon$ and let $\varepsilon < \min(L/2, T/2)/3$ go to zero. Note that

$$\begin{aligned}
\|Av_\varepsilon - Av\|_\Omega &= \|a_\varepsilon - Av\|_\Omega = \|\rho_\varepsilon * \tau_\delta \tilde{G}Av - Av\|_\Omega \\
&\leq \|\rho_\varepsilon * \tau_\delta \tilde{G}Av - \tau_\delta \tilde{G}Av\|_{\mathbb{R}^2} + \|\tau_\delta \tilde{G}Av - \tilde{G}Av\|_{\mathbb{R}^2}, \\
\|v_\varepsilon - v\|_\Omega &\leq \|\rho_\varepsilon * \tau_\delta \tilde{G}v - \tau_\delta \tilde{G}v\|_\Omega + \|\tau_\delta \tilde{G}v - v\|_\Omega \\
&\leq \|\rho_\varepsilon * \tau_\delta \tilde{G}v - \tau_\delta \tilde{G}v\|_{\mathbb{R}^2} + \|\tau_\delta \tilde{G}v - \tilde{G}v\|_{\mathbb{R}^2}.
\end{aligned}$$

Using (4.2.12) and (4.2.9) it now immediately follows that

$$\lim_{\varepsilon \rightarrow 0} \|v_\varepsilon - v\|_W = 0.$$

To conclude, examine the value of v_ε at points $z = (0, t)$ for any $0 < t < T$, namely

$$v_\varepsilon(0, t) = \int_{\mathbb{R}} \int_{\mathbb{R}} \rho_\varepsilon(-x', t - t') (\tau_\delta \tilde{G}v)(x', t') dx' dt'.$$

The integrand of the inner integral is the product of an even function (ρ_ε) of x' and an odd function $\tau_\delta \tilde{G}v$ of x' . Hence $v_\varepsilon(0, t) = 0$. The same holds for points $z = (L, t)$. Moreover, since $\tau_\delta \tilde{G}v(y)$ is identically zero in a neighborhood of $z = (T, x)$ for all $0 < x < L$, we conclude that $v_\varepsilon|_{\Gamma^*} = 0$. $\square$

4.3. Error Estimates for the ideal DPG method

Continuing to consider the set Ω as defined in Section 4.2.3, we now proceed to analyze the convergence of the ideal DPG method for Problem 4.2.5. The ideal DPG method finds u_h and q_h in finite dimensional subspaces $U_h \subset L^2(\Omega)$ and $Q_h \subset Q$ respectively, satisfying

$$(4.3.1) \quad b((u_h, q_h), v) = F(v), \quad \text{for all } v \in T(U_h \times Q_h).$$

Here $T : L^2(\Omega) \times Q \rightarrow W_h$ is defined by $(T(z, r), v) = b((z, r), v)$ for all $v \in W_h$ and any $(z, r) \in L^2(\Omega) \times Q$ as in Section 1.5. The wellposedness of Problem 4.2.5 follows from Theorem 4.2.6, now that we have verified Assumption 1 in Theorem 4.2.7. Hence to obtain convergence rates for specific subspaces, we need only develop interpolation error estimates. Since the interpolation properties of the L^2 -conforming U_h are standard, we need only discuss those of Q_h . To study this, we will create a spacetime finite element space $V_h \subset V$, then identify Q_h as $D_h(V_h)$, and finally establish interpolation estimates for Q_h using those for V_h . Note that V_h will be used only in the proof (and not in the computations).

To transparently present the ideas, we shall limit ourselves to the very simple case of a uniform mesh Ω_h of spacetime square elements of side length h . Let $\mathcal{E}_h$ denote the set of edges of Ω_h . On any $E \in \mathcal{E}_h$, let $P_p(E)$ denote the space of polynomials on the edge of degree at most p . On any $K \in \Omega_h$, let $Q_p(K)$ denote the space of polynomials of degree at most p in x and at most p in t . To begin the finite element construction, we consider the reference element $\hat{K} = (0, 1) \times (0, 1)$ and the element space $Q_p(\hat{K})$, endowed with the following degrees of freedom: For any $w \in H^3(K)$, and for each $i \in \{0, 1, \dots, p-2\}$ and $j \in \{0, 1, 2, \dots, p\}$, write $x_i = i/(p-2)$ and $t_j = j/p$ and set

$$\sigma_{ij}(w) = w(x_i, t_j), \quad \sigma_j^0(w) = \partial_x w(0, t_j), \quad \sigma_j^1(w) = \partial_x w(1, t_j).$$

Together, these form a set Σ with $(p-1)(p+1) + 2(p+1)$ linear functionals. The triple $(\hat{K}, Q_p(\hat{K}), \Sigma)$ is a unisolvent finite element, in the sense of [13], as we show next.

LEMMA 4.3.1. *Suppose $p \geq 3$. Then any polynomial $w \in Q_p(\hat{K})$ is uniquely defined by the values of its degrees of freedom σ in Σ .*

Figure 4.4. Degrees of freedom in the $p = 3$ (left) and $p = 5$ (right) cases.

PROOF. Suppose $w \in Q_p(\hat{K})$ and $\sigma(w) = 0$ for all $\sigma \in \Sigma$. Then $w_j(x) = w(x, t_j)$ is a polynomial of degree p in one variable (x). The Hermite and Lagrange degrees of freedom on $t = t_j$ imply $w_j = 0$. Now, fixing x , observe that the polynomial $w(x, t)$ is of degree at most p in the variable t and has $p + 1$ zeros. Hence $w \equiv 0$ and the proof is complete since $\dim Q_p(\hat{K})$ equals the number of degrees of freedom. $\square$

Next, consider the global finite element space $W_h^p(\Omega) = \{w \in L^2(\Omega) : \partial_t w \text{ and } \partial_{xx} w \text{ are in } L^2(\Omega) \text{ and } w|_K \in Q_p(K) \text{ for all } K \in \Omega_h\}$. Each element $K \in \Omega_h$ is obtained by mapping the reference element $\hat{K}$ by $T_K : \hat{K} \rightarrow K$, $T_K(\hat{x}, \hat{t}) = (h\hat{x} + x_K, h\hat{t} + t_K)$, where (x_K, t_K) is the lower left corner vertex of K , and the element space $Q_p(K)$ is the pull back of the reference element space $Q_p(\hat{K})$ under this map. The space $W_h^p(\Omega)$ can be controlled by a global set of degrees of freedom obtained by mapping the reference element degrees of freedom and, as usual, coalescing those that coincide at the mesh element interfaces.

On the reference element $\hat{K}$, the degrees of freedom define an interpolation operator

$$\hat{\Pi}w = \sum_{\sigma \in \Sigma} \sigma(w) \varphi_\sigma$$

where, as usual, $\{\varphi_\eta \in Q_p(\hat{K}) : \eta \in \Sigma\}$ is the set of shape functions obtained as the dual basis of Σ . By the Sobolev inequality in two dimensions, $\hat{\Pi} : H^3(\hat{K}) \rightarrow Q_p(\hat{K})$ is continuous. Similarly, the global degrees of freedom define an interpolation operator $\Pi : H^3(\Omega) \rightarrow W_h^p(\Omega)$ satisfying

$$(4.3.2) \quad (\Pi w) \circ T_K = \hat{\Pi}(w \circ T_K).$$

LEMMA 4.3.2. *If $w \in H^{p+1}(\Omega)$, then for all $p \geq 3$,*

$$\begin{aligned} \|w - \Pi w\|_\Omega &\leq Ch^{p+1}|w|_{H^{p+1}(\Omega)} \\ \|\partial_t(w - \Pi w)\|_\Omega &\leq Ch^p|w|_{H^{p+1}(\Omega)} \\ \|\partial_{xx}(w - \Pi w)\|_\Omega &\leq Ch^{p-1}|w|_{H^{p+1}(\Omega)}. \end{aligned}$$

PROOF. Changing variables $(x, t) = T_K(\hat{x}, \hat{t})$ as $(\hat{x}, \hat{t})$ runs over $\hat{K}$, integrating, and using (4.3.2),

$$(4.3.3a) \quad \|w - \Pi w\|_K = h\|\hat{w} - \hat{\Pi}\hat{w}\|_{\hat{K}}$$

$$(4.3.3b) \quad \|\partial_t(w - \Pi w)\|_K = \|\partial_{\hat{t}}(\hat{w} - \hat{\Pi}\hat{w})\|_{\hat{K}}$$

$$(4.3.3c) \quad \|\partial_{xx}(w - \Pi w)\|_K = h^{-1}\|\partial_{\hat{x}\hat{x}}(\hat{w} - \hat{\Pi}\hat{w})\|_{\hat{K}}.$$

On the reference element, since $H^{p+1}(\hat{K}) \hookrightarrow H^3(\hat{K})$, the interpolation operator $\hat{\Pi} : H^{p+1}(\hat{K}) \rightarrow Q_p(\hat{K})$ is continuous. Moreover $\hat{\Pi}\hat{w} = \hat{w}$ for all $\hat{w} \in Q_p(\hat{K})$. Hence, the Bramble-Hilbert Lemma yields a $\hat{C} > 0$ such that $\|\hat{w} - \hat{\Pi}\hat{w}\|_{H^3(\hat{K})} \leq \hat{C}|\hat{w}|_{H^{p+1}(\hat{K})}$ for all $\hat{w} \in H^{p+1}(\hat{K})$. Since $|\hat{w}|_{H^{p+1}(\hat{K})} \leq Ch^p|w|_{H^{p+1}(K)}$, combining with (4.3.3) and summing over all the elements in Ω_h , we obtain the result. $\square$

Now we are ready to present the main result of this section. Set $V_h = W_h^p(\Omega) \cap V$ and

$$(4.3.4) \quad Q_h = D_h(V_h), \quad U_h = \{u \in L^2(\Omega) : u|_K \in Q_{p-1}(K) \text{ for all } K \in \Omega_h\}.$$

THEOREM 4.3.3. *Let $p \geq 3$. Suppose $u \in V \cap H^{p+1}(\Omega)$ and $q = D_h u$ solve Problem 4.2.5 and suppose $U_h \times Q_h$ is set by (4.3.4). Then, there exists a constant C independent of h such that the discrete solution $u_h \in U_h$ and $q_h \in Q_h$ solving (4.3.1) satisfies*

$$(4.3.5) \quad \|u - u_h\|_\Omega + \|q - q_h\|_Q \leq Ch^r |u|_{H^{r+2}(\Omega)}$$

for $2 \leq r \leq p - 1$.

PROOF. By [19, Theorem 2.2] the ideal DPG method is quasioptimal:

$$\begin{aligned} \|(u, q) - (u_h, q_h)\|_{U \times Q}^2 &\leq C \inf_{(z_h, r_h) \in U_h \times Q_h} \|(u, q) - (z_h, r_h)\|_{U \times Q}^2 \\ &= C \inf_{(z_h, r_h) \in U_h \times Q_h} (\|u - z_h\|_\Omega^2 + \|q - r_h\|_Q^2). \end{aligned}$$

Because of the standard approximation estimate $\inf_{z_h \in U_h} \|u - z_h\|_\Omega \leq Ch^r |u|_{H^r(\Omega)}$ for $0 \leq r \leq p - 1$, it suffices to focus on $\|q - r_h\|_Q$. Since $q = D_h u$, by the definition of Q -norm (1.4.14), and the fact that any r_h in Q_h equals $D_h v_h$ for some $v_h \in V_h$, we have

$$\inf_{r_h \in Q_h} \|q - r_h\|_Q \leq \inf_{v_h \in V_h} \|u - v_h\|_W \leq \|u - \Pi u\|_W.$$

Applying Lemma 4.3.2, the result follows. $\square$

We conclude this chapter by examining a property of Q_h that is useful for computations. Let $\mathcal{E}_h^|$ and $\mathcal{E}_h^-$ denote the set of vertical and horizontal (closed) mesh edges,

respectively, and $\mathcal{E}_h^+ = \mathcal{E}_h^- \cup \mathcal{E}_h^-$. Let $E_h^!$ and E_h^+ denote the closed set formed by the union of all edges in $\mathcal{E}_h^!$ and $\mathcal{E}_h^+$, respectively. Let $Q_h^! = \{r \in L^2(E_h^!) : r|_F \in P_p(F) \text{ for all } F \in \mathcal{E}_h^!\}$ and $Q_h^+ = \{r \in L^2(E_h^+) : r \text{ is continuous on } E_h^+ \text{ and } r|_F \in P_p(F) \text{ for all } F \in \mathcal{E}_h^+ \text{ and } r|_\Gamma = 0\}$. For any $v_h \in V_h$, since v_h is a polynomial on each element, we may integrate by parts element by element to get

$$\begin{aligned} \langle D_h v_h, \psi \rangle_h &= (A_h v_h, \psi)_h - (v_h, A_h \psi)_h \\ &= \sum_{K \in \Omega_h} \int_{\partial K} \hat{n}_t v_h \bar{\psi} + \int_{\partial K} v_h n_x (\partial_x \bar{\psi}) - \int_{\partial K} n_x (\partial_x v_h) \bar{\psi}, \end{aligned}$$

for all $\psi \in \mathcal{D}(\bar{\Omega})$. Thus $q = D_h v_h$ satisfies

$$\langle q, \psi \rangle_h = \sum_{K \in \Omega_h} \int_{\partial K} q^+ (\hat{n}_t \bar{\psi}) + \int_{\partial K} q^+ n_x (\partial_x \bar{\psi}) - \int_{\partial K} q^! (n_x \bar{\psi}),$$

where $q^+ = v_h|_{E_h^+}$ and $q^! = \partial_x v_h|_{E_h^!}$. In computations, one may therefore identify Q_h with the interfacial polynomial space $Q_h^+ \times Q_h^!$ whose components are of degree at most p .

A spacetime DPG method for the first order wave equation

In this chapter we apply the abstract ultraweak formulation of Chapter 1 to the transient wave problem. The goal is to study the feasibility of the DPG method for the first order spacetime acoustic wave system.

5.1. Model problem: The transient wave problem

Let Ω_0 be an open subset of $\mathbb{R}^{d-1}$, with $d > 1$, representing the spatial domain, with boundary $\partial\Omega_0$, and let $\Omega = \Omega_0 \times (0, T)$ be the spacetime domain, where $T > 0$ represents the final time. We consider the first order system for the wave equation given by

$$(5.1.1a) \quad \partial_t q - c \nabla_x \mu = g,$$

$$(5.1.1b) \quad \partial_t \mu - c \operatorname{div}_x q = f,$$

where $f \in L^2(\Omega)$ and $g \in L^2(\Omega)^{d-1}$ and $c > 0$ denotes the constant wave speed of the material. Here the differential operators div_x and ∇_x represent the divergence and gradient (distributional) operators that differentiate only along the space variable x components. We add homogeneous initial and boundary conditions:

$$(5.1.1c) \quad \mu|_{t=0} = 0, \quad q|_{t=0} = 0, \quad \mu|_{\partial\Omega \times (0, T)} = 0.$$

Here, q represents the velocity and μ the pressure. We now cast this problem in the framework of the abstract setting in Chapter 1.

5.1.1. The formal wave operator. Formally, the wave operator generated by the above system may be considered as a first order distributional derivative operator. Namely, set $A : L^2(\Omega)^d \rightarrow \mathcal{D}'(\Omega)^d$ by

$$(5.1.2) \quad Au = \begin{bmatrix} \partial_t u_q - c \nabla_x u_\mu \\ \partial_t u_\mu - c \operatorname{div}_x u_q \end{bmatrix}$$

where u in $L^2(\Omega)^d$ is block partitioned into

$$(5.1.3) \quad u = \begin{bmatrix} u_q \\ u_\mu \end{bmatrix}, \quad u_q \in L^2(\Omega)^{d-1}, \quad u_\mu \in L^2(\Omega).$$

Note that this operator satisfies Assumptions A-a and A-b in Chapter 1 when setting $m = l = d$ and $k = 1$. Following the framework of Chapter 1 we define

$$W(\Omega) = \{u \in L^2(\Omega)^d : Au \in L^2(\Omega)^d\},$$

As in the previous chapters, we abbreviate $W(\Omega) = W$. The formal adjoint of A is the distributional differentiation operator $-A$, whose graph space is equal to W and satisfies

$$(Aw, \tilde{w}) = -(w, A\tilde{w}) \quad \text{for all } w, \tilde{w} \in \mathcal{D}(\Omega)^d.$$

We define the operator $D : W \rightarrow W'$ by

$$(5.1.4) \quad \langle Du, v \rangle_W = (Au, v)_\Omega + (u, Av)_\Omega \quad \text{for all } u, v \in W.$$

where W' is the dual space of W , and $\langle \cdot, \cdot \rangle_W$ represents the duality pairing of a functional in W' with a element of W . For smooth functions $u, v \in \mathcal{D}(\bar{\Omega})^d$, integration

by parts shows that

$$(5.1.5) \quad \langle Du, v \rangle_W = \int_{\partial\Omega} u_q \cdot (n_t v_q - c n_x v_\mu) + u_\mu (n_t v_\mu - c n_x \cdot v_q).$$

Here and throughout, $n = (n_x, n_t)$ represents the unit outward normal component to Ω in $\mathbb{R}^d$ and functions in $L^2(\Omega)^d$, like the u and v above, are block partitioned as in (5.1.3).

5.1.1.2. The unbounded wave operator. In order to consider the boundary and initial conditions, we now proceed as suggested in Chapter 1, to define an unbounded operator with a domain that takes these conditions into account. Below, by abusing the notation, we shall denote this unbounded operator also by A .

First, let us partition the spacetime boundary $\partial\Omega$ into

$$\Gamma_0 = \Omega_0 \times \{0\}, \quad \Gamma_T = \Omega_0 \times \{T\}, \quad \Gamma_b = \partial\Omega_0 \times [0, T].$$

We define the following sets of smooth functions:

$$(5.1.6) \quad \mathcal{V} = \{u \in \mathcal{D}(\bar{\Omega})^d : u|_{\Gamma_0} = 0, u_\mu|_{\Gamma_b} = 0\},$$

$$(5.1.7) \quad \mathcal{V}^* = \{v \in \mathcal{D}(\bar{\Omega})^d : v|_{\Gamma_T} = 0, v_\mu|_{\Gamma_b} = 0\}.$$

Next, let $A : \text{dom}(A) \subset L^2(\Omega)^d \rightarrow L^2(\Omega)^d$ be the unbounded operator in $L^2(\Omega)^d$ defined by the right hand side of (5.1.2) with

$$(5.1.8) \quad \text{dom}(A) = \{u \in W : \langle Du, v \rangle_W = 0 \text{ for all } v \in \mathcal{V}^*\}.$$

From (5.1.5), we see that the set of smooth functions $\mathcal{D}(\Omega)^d$ is contained in $\text{dom}(A)$, thus Assumption A-c holds. Hence, A is a densely defined operator in $L^2(\Omega)^d$. Therefore, it has a uniquely defined adjoint A^* , which is again an unbounded operator. The adjoint A^* equals the distributional derivative operator $-A$ when applied to $\text{dom}(A^*)$.

This domain is prescribed as in standard functional analysis [7] by

$$\begin{aligned} \text{dom}(A^*) = \{ \tilde{w} \in L^2(\Omega)^d : \exists \ell \in L^2(\Omega)^d \text{ such that } (Au, \tilde{w}) = (u, \ell) \\ \text{for all } u \in \text{dom}(A) \}. \end{aligned}$$

By definition, $\text{dom}(A)$ is a subset of $W(\Omega)$. When this subset is given the topology of $W(\Omega)$, we obtain a closed subset of $W(\Omega)$, which we call V , i.e., V and $\text{dom}(A)$ coincide as sets or vector spaces, but not as topological spaces. Note that $\text{dom}(A^*)$ is also a subset of W , since for any $\tilde{w} \in \text{dom}(A^*)$, the distribution $-A\tilde{w}$ is in $L^2(\Omega)$. Observe that since V is closed, A is a closed operator.

Since A-a, A-b and A-c hold, Lemma 1.3.1 implies

$$(5.1.9) \quad V^* = {}^\perp D(V).$$

Thus V^* is also a closed subset of W .

The next observation is that from the definitions of V and the operator D (namely (5.1.8) and (5.1.4)) it immediately follows that $\mathcal{V} \subset V$. Note also that if $v^* \in \mathcal{V}^*$, then $(Au, v^*) = -(u, Av^*) + \langle Du, v \rangle_W = -(u, Av^*)$ for all $u \in V$, since $\langle Du, v \rangle_W = 0$ by the definition of V . Therefore v^* is in V^* . To summarize these observations, we have introduced $\mathcal{V}, \mathcal{V}^*, V$ and V^* , satisfying

$$(5.1.10) \quad \mathcal{V} \subset V \quad \text{and} \quad \mathcal{V}^* \subset V^*.$$

These are the abstract ingredients in the framework of Chapter 1 applied to the wave problem.

5.2. The broken weak formulation

Following the settings of Chapter 1, we partition the spacetime domain Ω into a mesh Ω_h of finitely many open elements K such that $\bar{\Omega} = \cup_{K \in \Omega_h} \bar{K}$ where $h = \max_{K \in \Omega_h} \text{diam}(K)$. The DPG method is based on a variational formulation in a “broken” analogue of W , which we call W_h , defined as in 1.4.2.

We let A_h be the wave operator applied element by element, i.e.,

$$(A_h w)|_K = A(w|_K), \quad w \in W(K), \quad K \in \Omega_h.$$

Let $W_h = \{w \in L^2(\Omega)^d : A_h w \in L^2(\Omega)^d\}$. The operator $D_h : W_h \rightarrow W'_h$ is defined as in (1.4.3) by

$$\langle D_h w, v \rangle_{W_h} = (A_h w, v)_\Omega + (w, A_h v)_\Omega$$

for all $w, v \in W_h$, where $\langle \cdot, \cdot \rangle_{W_h}$ denotes the duality pairing in W_h in accordance with our previous notation. Below we abbreviate $\langle \cdot, \cdot \rangle_{W_h}$ to $\langle \cdot, \cdot \rangle_h$. Let $D_{h,V} : V \rightarrow W'_h$ denote the restriction of D_h to V , i.e., $D_{h,V} = D_h|_V$. The range of $D_{h,V}$, denoted by Q , is made into a complete space by the quotient norm

$$(5.2.1) \quad \|\rho\|_Q = \inf_{v \in D_{h,V}^{-1}(\{\rho\})} \|v\|_W, \quad \rho \in Q \equiv \text{ran}(D_{h,V}).$$

We define the bilinear form on $(L^2(\Omega)^d \times Q) \times W_h$ by

$$b((v, \rho), w) = -(v, A_h w)_\Omega + \langle \rho, w \rangle_h.$$

Therefoe “broken” variational formulation for the wave problem now reads as in Problem 1.5.1. Given any F in the dual space W'_h , find $u \in L^2(\Omega)^d$ and $\lambda \in Q$ such that

$$(5.2.2) \quad b((u, \lambda), w) = F(w) \quad \text{for all } w \in W_h.$$

Critical to the success of any numerical approximation of this formulation, in particular, the DPG approximation, is its wellposedness. By Theorem 1.4.5, this formulation is well-posed, provided we verify

$$(5.2.3) \quad V = {}^\perp D(V^*),$$

$$(5.2.4) \quad A : V \rightarrow L^2(\Omega)^d \text{ is a bijection.}$$

Therefore our next focus is on proving (5.2.3) and (5.2.4). Recall from (5.1.10) that $\mathcal{V}$ and $\mathcal{V}^*$ are subspaces of smooth functions within V and V^* . We now show that (5.2.3) and (5.2.4) follow if these are dense subspaces.

THEOREM 5.2.1. *Suppose*

$$(5.2.5) \quad \mathcal{V} \text{ is dense in } V \text{ and } \mathcal{V}^* \text{ is dense in } V^*.$$

Then (5.2.3) and (5.2.4) holds. Consequently, the broken weak formulation (5.2.2) is well posed.

PROOF. First observe that another way to write (5.1.8) is

$$V = {}^\perp D(\mathcal{V}^*).$$

In view of the continuity of D , (5.1.10), and the assumption that $\mathcal{V}^*$ is dense in V^* , the condition (5.2.3) now immediately follows.

Next, we will prove that

$$(5.2.6a) \quad \|u\| \leq 2T\|Au\|, \quad \text{for all } u \in \mathcal{V},$$

$$(5.2.6b) \quad \|v\| \leq 2T\|A^*v\|, \quad \text{for all } v \in \mathcal{V}^*.$$

These inequalities follow by well-known energy arguments, as shown in [25, Lemma 3].

We briefly include the proof for completeness. Let $v \in \mathcal{V}^*$. Then

$$\begin{aligned} \|v\|^2 &= \int_0^T \left(\int_{\Omega_0} |v(x, t)|^2 dx \right) dt = 2 \int_0^T \int_t^T \int_{\Omega_0} \partial_s v(x, s) \cdot v(x, s) dx ds dt \\ &= 2 \int_0^T \int_t^T \int_{\Omega_0} v(x, s) \cdot A^* v(x, s) dx ds dt - 2c \int_0^T \int_t^T \int_{\partial\Omega_0} (v_q \cdot n_x) v_\mu dx ds dt \\ &\leq 2T \|v\| \|A^* v\|. \end{aligned}$$

The inequality for $\mathcal{V}$ is similarly proved by using its boundary conditions instead of those of $\mathcal{V}^*$.

Using the density assumptions, we conclude that (5.2.6) implies

$$(5.2.7a) \quad \|u\| \leq 2T \|Au\| \quad \text{for all } u \in V \text{ and}$$

$$(5.2.7b) \quad \|v\| \leq 2T \|A^* v\| \quad \text{for all } v \in V^*.$$

The conditions of Theorem 1.4.14 then holds, and this A is a bijection. $\square$

Note that the wellposedness result of Theorem 5.2.1, in particular, implies that

$$(5.2.8) \quad \beta = \inf_{0 \neq (v, \rho) \in L^2(\Omega)^d \times Q} \sup_{0 \neq w \in W_h} \frac{b((v, \rho), w)}{\|(v, \rho)\|_{L^2(\Omega)^{d+1} \times Q}} > 0.$$

5.3. Verification of the density condition

In this section, we verify (5.2.5) for a simple domain, namely a hyperrectangle (or an orthotope). Accordingly, throughout this section, we fix $\Omega = \Omega_0 \times (0, T)$ and

$$\Omega_0 = \prod_{i=1}^d (0, a_i),$$

for some $a_i > 0$. While density of smooth functions in general graph spaces can be proved by standard Sobolev space techniques [2], to obtain the density of smooth

functions with boundary conditions (like those in $\mathcal{V}$) we need more arguments. The following proof has some similarities with the proof of Section 4.2.3, an analogous density result for the one-dimensional Schrödinger operator. The main differences from Section 4.2.3 in the proof below include the consideration of multiple spatial dimensions and the construction of extension operators for vector functions in the wave graph space by combining even and odd reflections appropriately.

THEOREM 5.3.1. *On the above Ω , $\mathcal{V}^*$ is dense in V^* and $\mathcal{V}$ is dense in V .*

PROOF. We shall only prove that $\mathcal{V}$ is dense in V , since the proof of the density of $\mathcal{V}^*$ in V^* is similar. We divide the proof into three main steps.

Step 1. Extension: In this step, we will extend a function in V using spatial reflections to a domain which has larger spatial extent than Ω (see Figure 5.1).

Let e_i denote the standard unit basis vectors in $\mathbb{R}^d$. The following operations

$$R_{i,-}x = x - 2x_ie_i, \quad R_{i,+}x = x + 2(a_i - x_i)e_i$$

perform reflections of the spatial coordinate vector x about $x_i = 0$ and $x_i = a_i$, for $i = 1, \dots, d$. We set $Q_0 \equiv \overline{\Omega}$ and then define extended domains Q_i in a recursive way, starting from $i = 1$ through $i = d$ as follows.

$$Q_{i,-} = R_{i,-}^{-1}Q_{i-1}, \quad Q_{i,+} = R_{i,+}^{-1}Q_{i-1}, \quad Q_i = Q_{i,-} \cup Q_{i-1} \cup Q_{i,+}.$$

The final extended domain is $Q \equiv Q_d$.

Next, we introduce even and odd extensions (in the x_i -direction) of scalar functions. Namely, let $G_{i,e}, G_{i,o} : L^2(Q_{i-1}) \rightarrow L^2(Q_i)$ be defined by

$$(5.3.1) \quad G_{i,e}f(x, t) = \begin{cases} f(R_{i,-}x, t) & \text{if } (x, t) \in Q_{i,-}, \\ f(R_{i,+}x, t) & \text{if } (x, t) \in Q_{i,+}, \\ f(x, t) & \text{if } (x, t) \in Q_{i-1}, \end{cases}$$

and

$$(5.3.2) \quad G_{i,o}f(x, t) = \begin{cases} -f(R_{i,-}x, t) & \text{if } (x, t) \in Q_{i,-}, \\ -f(R_{i,+}x, t) & \text{if } (x, t) \in Q_{i,+}, \\ f(x, t) & \text{if } (x, t) \in Q_{i-1}. \end{cases}$$

In the case of a vector function $v \in L^2(Q_{i-1})^d$, we define $G_i v(x, t)$ to be the extended vector function obtained by extending (in the i th direction) all the components of v using the odd scalar extension, except the i th component, which is extended using the even scalar extension. In other words, for any $i = 1, \dots, d-1$, we define $G_i : L^2(Q_{i-1})^d \rightarrow L^2(Q_i)^d$ by

$$(5.3.3) \quad G_i v = (G_{i,e}v_i)e_i + \sum_{j \neq i} (G_{i,o}v_j)e_j$$

where the sum runs over all $j = 1, \dots, d$ except i . Let $E_k = G_k \circ G_{k-1} \circ \dots \circ G_1$. The cumulative extension over all spatial directions is thus obtained using $E = E_d$. It extends functions in $\overline{\mathcal{D}}$ to Q .

By change of variable formula for integration, we obtain

$$\begin{aligned} (G_{i,o}f, g)_{Q_i} &= (f, G'_{i,o}g)_{Q_{i-1}}, & \text{for all } f \in L^2(Q_{i-1}), \quad g \in L^2(Q_i), \\ (G_{i,e}f, g)_{Q_i} &= (f, G'_{i,e}g)_{Q_{i-1}}, & \text{for all } f \in L^2(Q_{i-1}), \quad g \in L^2(Q_i), \end{aligned}$$

where the “folding” operators $G'_{i,e/o} : L^2(Q_i) \rightarrow L^2(Q_{i-1})$, that go the reverse direction of the extension operators, are defined by

$$(5.3.4) \quad G'_{i,o}g(x, t) = g(x, t) - g(R_{i,-}^{-1}x, t) - g(R_{i,+}^{-1}x, t),$$

$$(5.3.5) \quad G'_{i,e}g(x, t) = g(x, t) + g(R_{i,-}^{-1}x, t) + g(R_{i,+}^{-1}x, t).$$

These scalar folding operators combine to form an analogue for vector functions as in (5.3.3), namely

$$G'_i w = (G'_{i,e} w_i) e_i + \sum_{j \neq i} (G'_{i,o} w_j) e_j.$$

It satisfies $(G_i v, w)_{Q_i} = (v, G'_i w)_{Q_{i-1}}$ for all $v \in L^2(Q_{i-1})^d$, $w \in L^2(Q_i)^d$, and for each i from 1 to $d-1$. Let $E'_k = G'_k \circ G'_{k+1} \cdots \circ G'_d$. Then $E' = E'_1$ folds functions in Q to $\overline{\Omega}$ and is the adjoint of the extension E in the following sense.

$$(5.3.6) \quad (Eu, w)_Q = (u, E'w)_\Omega, \quad \text{for all } u \in L^2(\Omega)^d, \quad w \in L^2(Q)^d.$$

We want to prove that Ev is in $W(Q)$ for any $v \in V$. Note that if $w \in L^2(\Omega)^d$, then Ew in $L^2(Q)^d$, because each G_i maps L^2 functions into L^2 per (5.3.3). Therefore, in order to prove Ev is in $W(Q)$, it only remains to prove that $A(Ev)$ is in $L^2(Q)^d$. Let $\varphi \in \mathcal{D}(Q)^d$ (where we abuse the notation and write $\mathcal{D}(Q)$ for $\mathcal{D}(Q^0)$ whenever Q^0 is the interior of Q). Using (5.3.6), the action of the distribution AEv on φ equals

$$(5.3.7) \quad \langle AEv, \varphi \rangle_{\mathcal{D}(Q)^d} = -(Ev, A\varphi)_Q = -(v, E'A\varphi)_\Omega.$$

To analyze the last term, first observe that by the chain rule applied to a smooth scalar function ϕ on Q_1 ,

$$\begin{aligned}\partial_t(G'_{i,o}\phi) &= G'_{i,o}\partial_t\phi, & \partial_i(G'_{i,o}\phi) &= G'_{i,o}(\partial_i\phi), & \partial_j(G'_{i,o}\phi) &= G'_{i,o}(\partial_j\phi), \\ \partial_t(G'_{i,e}\phi) &= G'_{i,e}\partial_t\phi, & \partial_i(G'_{i,e}\phi) &= G'_{i,o}(\partial_i\phi), & \partial_j(G'_{i,e}\phi) &= G'_{i,e}(\partial_j\phi),\end{aligned}$$

for all $j \neq i$. Combining these appropriately for smooth vector function ψ on Q_i , we find that

$$\partial_t(G'_i\psi) = G'_i\partial_t\psi, \quad c\nabla_x G'_{i,o}\psi_\mu = G'_i(c\nabla_x\psi_\mu), \quad c\operatorname{div}_x G'_i\psi_q = G'_{i,o}(c\operatorname{div}\psi_q).$$

Thus, for any $\varphi \in \mathcal{D}(Q)^d$ we have $E'_i A\varphi = AE'_i\varphi$ for all $i = 1, \dots, d-1$, and in particular

$$(5.3.8) \quad E'A\varphi = AE'\varphi.$$

Returning to (5.3.7) and using (5.3.8) and (5.1.4),

$$(5.3.9) \quad \langle AEv, \varphi \rangle_{\mathcal{D}(Q)^d} = (Av, E'\varphi)_\Omega - \langle Dv, E'\varphi \rangle_{W(\Omega)}.$$

We shall now show that the last term above vanishes. Since v is in V , the last term will vanish by the definition of V , provided $E'\varphi$ is in $\mathcal{V}^*$. To prove that $E'\varphi$ is in $\mathcal{V}^*$, we only need to verify that $E'\varphi$ satisfies the boundary conditions in (5.1.7). Since φ is compactly supported in Q , we obviously have $(E'\varphi)|_{\Gamma_T} = 0$ as E' only involves spatial folding.

Next, we claim that $[E'\varphi]_\mu|_{\Gamma_b} = 0$ also. To see this, let Γ^j denote the two facets of ∂Q_j where x_j is constant and γ^j denote the two facets of ∂Q_{j-1} where x_j is constant. The value of $G'_{d-1,o}\varphi_\mu(x, t)$ for any (x, t) in γ^{d-2} is the sum of the three terms in (5.3.4), two of which cancel each other, and one of which vanishes because

$\varphi_\mu|_{\Gamma^{d-1}} = 0$. Thus $\varphi_\mu|_{\Gamma^{d-1}} = 0 \implies (G'_{d,o}\varphi_\mu)|_{\partial Q_{d-2}} = 0$ (where we have also used the fact that φ_μ vanishes on the remainder $\partial Q_{d-2} \setminus \gamma^{d-2}$). The same argument can now be repeated to get that $(G'_{d-1,o}\varphi_\mu)|_{\Gamma^{d-2}} = 0 \implies (G_{d-2,o}(G_{d-1,o}\varphi_\mu))|_{\partial Q_{d-3}} = 0$. Continuing similarly, we obtain that $[E'\varphi]_\mu = G'_{1,o} \circ G'_{2,o} \circ \cdots \circ G'_{d-1,o}\varphi_\mu$ vanishes on $\partial Q_0 = \Gamma_b$. Thus, the last term in (5.3.9) is zero and by (5.3.6) we conclude that

$$(5.3.10) \quad \langle AEv, \varphi \rangle_{\mathcal{D}(Q)^d} = (EA v, \varphi)_Q$$

for all φ in $\mathcal{D}(Q)^d$.

By virtue of (5.3.10), we have proved that for any $v \in V$, AEv is in $L^2(Q)^d$, AEv coincides with $EA v$, and Ev is in $W(Q)$.

Step 2. Translation: In this step, we will translate up the previously obtained extension in time coordinate. This will give us room to mollify in the next step. Such a translation step is standard in many density proofs (see e.g., [2]).

Let $v \in V$ and let $\tilde{E}v$ denote the extension of Ev by zero to $\mathbb{R}^d$, i.e., $\tilde{E}v$ equals Ev in Q and it is zero elsewhere. Denote by τ_δ the translation operator in the t -direction by $\delta > 0$; i.e. $(\tau_\delta w)(x, t) = w(x, t - \delta)$ for scalar or vector functions w . It is well

Figure 5.1. *Left:* Extended domains Q_1 and Q_2 when $\Omega \subseteq \mathbb{R}^3$. *Right:* Translation by δ in the t direction.

known [7] that

$$(5.3.11) \quad \lim_{\delta \rightarrow 0} \|\tau_\delta g - g\|_{L^2(\mathbb{R}^d)} = 0, \quad \forall g \in L^2(\mathbb{R}^d).$$

Let $Q_\delta = \prod_{i=1}^{i=d-1} (-a_i, 2a_i) \times (-\delta, T + \delta)$ and let H_δ be the restriction from $\mathbb{R}^d$ to Q_δ .

We will now show that

$$(5.3.12) \quad AH_\delta \tau_\delta \tilde{E}v = H_\delta \tau_\delta \tilde{E}Av.$$

By a change of variable,

$$(5.3.13) \quad (\tau_\delta \tilde{E}w, \tilde{w})_{Q_\delta} = (Ew, \tau_{-\delta} \tilde{w})_Q$$

for all $w \in L^2(\Omega)^d$ and $\tilde{w} \in L^2(Q_\delta)^d$. Note that $\tau_\delta \tilde{E}v|_\Omega \in L^2(\Omega)$. The distribution $AH_\delta \tau_\delta \tilde{E}v$ applied to a smooth function $\varphi \in \mathcal{D}(Q_\delta)^d$ equals

$$\langle AH_\delta \tau_\delta \tilde{E}v, \varphi \rangle_{\mathcal{D}(Q_\delta)^d} = -(\tau_\delta \tilde{E}v, A\varphi)_{Q_\delta} = -(Ev, A\tau_{-\delta} \varphi)_Q$$

due to (5.3.13) and the fact that $\tau_{-\delta} A\varphi = A\tau_{-\delta} \varphi$. Using also (5.3.6) and (5.3.8),

$$\begin{aligned} \langle AH_\delta \tau_\delta \tilde{E}v, \varphi \rangle_{\mathcal{D}(Q_\delta)^d} &= -(v, E' A\tau_{-\delta} \varphi)_\Omega = -(v, AE' \tau_{-\delta} \varphi)_\Omega \\ &= (Av, E' \tau_{-\delta} \varphi)_\Omega - \langle Dv, E' \tau_{-\delta} \varphi \rangle_W. \end{aligned}$$

where we have used (5.3.13), (5.3.6) and (5.3.8) consecutively. Note that $E' \tau_{-\delta} \varphi$ satisfies all the boundary conditions required for it to be in $\mathcal{V}^*$. Hence the last term in the above display is zero. We therefore conclude that $\langle AH_\delta \tau_\delta \tilde{E}v, \varphi \rangle_{\mathcal{D}(Q_\delta)^d} = (\tau_\delta EAv, \varphi)_{Q_\delta}$, which proves (5.3.12). In particular, $H_\delta \tau_\delta \tilde{E}v \in W(Q_\delta)$ whenever $v \in V$.

Step 3. Mollification: In this step we finish the proof by considering a $v \in V$ and mollifying the time-translated extension $\tau_\delta \tilde{E}v$ constructed above.

To recall the standard symmetric mollifier, let $\rho_\varepsilon \in \mathcal{D}(\mathbb{R}^d)$, for each $\varepsilon > 0$ be defined by

$$\rho_\varepsilon(x, t) = \varepsilon^{-d} \rho_1(\varepsilon^{-1}x, \varepsilon^{-1}t),$$

where

$$\rho_1(x, t) = \begin{cases} k \exp\left(-\frac{1}{1 - |x|^2 - t^2}\right) & \text{if } |x|^2 + t^2 < 1, \\ 0 & \text{if } |x|^2 + t^2 \geq 1, \end{cases}$$

and k is a constant chosen so that $\int_{\mathbb{R}^d} \rho_1 = 1$. Here $|\cdot|$ is the euclidean norm in $\mathbb{R}^d$. Let $\rho_\varepsilon * v$ denote the function obtained by component-wise convolution, i.e, $[\rho_\varepsilon * v]_j = [v]_j * \rho_\varepsilon$ for all j -components. Then $\rho_\varepsilon * v$ is a infinitely smooth vector function that satisfies

$$(5.3.14) \quad \lim_{\varepsilon \rightarrow 0} \|v - \rho_\varepsilon * v\|_{\mathbb{R}^d} = 0, \quad \forall v \in L^2(\mathbb{R}^d)^d.$$

Consider any

$$0 < \delta < \min_{1 \leq i \leq d-1} (a_i/2, T/2),$$

and define two functions $v_\varepsilon = \rho_\varepsilon * \tau_\delta \tilde{E}v$ and $a_\varepsilon = \rho_\varepsilon * \tau_\delta \tilde{E}Av$. Note that the $Av_\varepsilon = a_\varepsilon$ on Ω whenever $\varepsilon < \delta/2$, thanks to (5.3.12).

We now proceed to show that

$$(5.3.15) \quad \lim_{\varepsilon \rightarrow 0} \|v_\varepsilon|_\Omega - v\|_W = 0.$$

Set $\delta = 3\varepsilon$ and let $\varepsilon < \min_{1 \leq i \leq d-1} (a_i/2, T/2)/3$ go to zero. Note that

$$\begin{aligned}
\|Av_\varepsilon - Av\| &= \|a_\varepsilon - Av\| = \|\rho_\varepsilon * \tau_\delta \tilde{E}Av - Av\|_\Omega \\
&\leq \|\rho_\varepsilon * \tau_\delta \tilde{E}Av - \tilde{E}Av\|_{\mathbb{R}^d} \\
&\leq \|\rho_\varepsilon * \tau_\delta \tilde{E}Av - \tau_\delta \tilde{E}Av\|_{\mathbb{R}^d} + \|\tau_\delta \tilde{E}Av - \tilde{E}Av\|_{\mathbb{R}^d}, \\
\|v_\varepsilon - v\| &\leq \|\rho_\varepsilon * \tau_\delta \tilde{E}v - \tau_\delta \tilde{E}v\|_\Omega + \|\tau_\delta \tilde{E}v - v\|_\Omega \\
&\leq \|\rho_\varepsilon * \tau_\delta \tilde{E}v - \tau_\delta \tilde{E}v\|_{\mathbb{R}^d} + \|\tau_\delta \tilde{E}v - \tilde{E}v\|_{\mathbb{R}^d}.
\end{aligned}$$

Using (5.3.11) and (5.3.14) it now immediately follows that (5.3.15) holds.

To conclude, it suffices to prove that $v_\varepsilon|_\Omega$ is in $\mathcal{V}$. Clearly, $\tau_\delta \tilde{E}v$ is identically zero in a neighborhood of Γ_0 . Hence we conclude that $v_\varepsilon = \rho_\varepsilon * \tau_\delta \tilde{E}v$ vanishes on Γ_0 for small enough ε . Next, let us examine the value of $[v_\varepsilon]_\mu$ at points (x, t) on Γ_b , namely

$$[v_\varepsilon]_\mu(x, t) = \int_{\mathbb{R}} \int_{\mathbb{R}^{d-1}} \rho_\varepsilon(x - x', t - t') [\tau_\delta \tilde{E}v]_\mu(x', t') dx' dt'.$$

Note that $\rho_\varepsilon(x - x', t - t')$ is a symmetric function of x' about x . The other term in the integrand, namely $[\tau_\delta \tilde{E}v]_\mu(x', t')$, is odd about every facet of Γ_b . Hence the integral of their product vanishes whenever $(x, t) \in \Gamma_b$. Thus, $[v_\varepsilon]_\mu|_{\Gamma_b} = 0$ and $v_\varepsilon \in \mathcal{V}$. $\square$

5.4. The method and its error estimates

In this section, we present the approximation of the previously described broken weak formulation by the (ideal) DPG method and provide *a priori* and *a posteriori* error estimates.

5.4.1. The DPG method. We recall the ideal DPG setting of Section 1.5 in terms of u_h, λ_h and w_h . Namely, we seek u_h and λ_h in finite-dimensional subspaces

$U_h \subset L^2(\Omega)^d$ and $Q_h \subset Q$, respectively, satisfying

$$(5.4.1) \quad b((u_h, \lambda_h), w_h) = F(w_h), \quad \text{for all } w_h \in T(U_h \times Q_h).$$

where $T : L^2(\Omega)^d \times Q \rightarrow W_h$ is such that $(T(v, \rho), w)_{W_h} = b((v, \rho), w)$ for all $w \in W_h$ and any $(v, \rho) \in L^2(\Omega)^d \times Q$. Hereon we denote U to be $L^2(\Omega)^d$ and abbreviate the W_h inner product $(\cdot, \cdot)_{W_h}$ to simply $(\cdot, \cdot)_h$.

We rewrite the mixed system introduced in (1.5.1) in terms of $\varepsilon_h \in W_h, u_h, \lambda_h$ given by

$$(5.4.2) \quad (\varepsilon_h, w)_h = (f, w) - b((u_h, \lambda_h), w), \quad \text{for all } w \in W_h.$$

One of the two equations in the mixed formulation given below is a restatement of this defining equation for ε_h . The mixed formulation seeks $\varepsilon_h \in W_h$ and $(u_h, \lambda_h) \in (U_h \times Q_h)$ such that

$$(5.4.3) \quad \begin{aligned} (\varepsilon_h, w)_h + b((u_h, \lambda_h), w) &= F(w) & \text{for all } w \in W_h, \\ b((v, \rho), \varepsilon_h) &= 0 & \text{for all } (v, \rho) \in U_h \times Q_h. \end{aligned}$$

We think of

$$\eta = \|\varepsilon_h\|_{W_h} \equiv \left(\sum_{K \in \Omega_h} \|\varepsilon_h\|_{W(K)}^2 \right)^{1/2}$$

as an *a posteriori* error estimator because ε_h can be computed from (5.4.2) after u_h and λ_h has been computed. Alternately, one can view ε_h as one of the unknowns together with u_h and λ_h as in (5.4.3). Note that (5.4.2) implies

$$\eta = \sup_{w \in W_h} \frac{b((u - u_h, \lambda - \lambda_h), w)}{\|w\|_{W_h}},$$

so it immediately follows that the estimator is globally reliable and efficient, namely

$$\beta \|(u - u_h, \lambda - \lambda_h)\|_{U \times Q} \leq \eta \leq \|b\| \|(u - u_h, \lambda - \lambda_h)\|_{U \times Q}$$

where β is as in (5.2.8). In practice, we use element-wise norms of ε_h as a *posteriori* element error indicator.

To give an *a priori* error estimate with convergence rates, we need to specify all the approximation subspaces. We choose the space $Q_h \subset Q$ by first selecting a finite element space $V_h \subset V$ and then applying D_h to all functions in it, namely

$$Q_h = D_h V_h.$$

This way we guarantee that Q_h is a subspace of Q . The definition of V_h and the finite element subspaces of U are based on the type of elements in Ω_h . We consider two cases:

Case A: Ω_h is a geometrically conforming mesh of $(d + 1)$ -simplices:

$$(5.4.4a) \quad V_h = \{u \in V \cap C(\bar{\Omega})^d : u|_K \in P_{p+1}(K)^d \text{ for all } K \in \Omega_h\}$$

$$(5.4.4b) \quad U_h = \{u \in L^2(\Omega)^d : u|_K \in P_p(K)^d \text{ for all } K \in \Omega_h\},$$

where $P_p(K)$ is the space of polynomials of total degree $\leq p$ on K .

Case B: Ω_h is a geometrically conforming mesh of hyperrectangles.

$$(5.4.5a) \quad V_h = \{u \in V \cap C(\bar{\Omega})^d : u|_K \in Q_{p+1}(K)^d \text{ for all } K \in \Omega_h\}$$

$$(5.4.5b) \quad U_h = \{u \in L^2(\Omega)^d : u|_K \in Q_p(K)^d\},$$

where $Q_p(K)$ is the space of polynomials on K that are of degree at most p in each variable.

Since the wave operator A is a first order differential operator, $H^1(\Omega)^d \subset W(\Omega)$. Hence, the Lagrange finite element space V_h is contained in W . The space V_h has a nodal interpolation operator $I_h : H^{s+1}(\Omega)^d \rightarrow V_h$ which is bounded for $s+1 > d/2$, which we shall use in the proof below. We will use c to denote a generic mesh-independent constant whose value at different occurrences may differ. Note that in the estimate of the theorem below, h is the discretization parameter in both space and time.

THEOREM 5.4.1. *Suppose $u \in V \cap H^{s+1}(\Omega)^d$ and $\lambda = D_h u$ solve (5.2.2). Suppose also that U_h and V_h are set as in (5.4.4) or (5.4.5) depending on the mesh type, and $Q_h = D_h V_h$. Then, there exists a constant $C > 0$ independent of h such that the discrete solution $u_h \in U_h$ and $\lambda_h \in Q_h$ solving (5.4.1) satisfies*

$$(5.4.6) \quad \|u - u_h\| + \|\lambda - \lambda_h\|_Q \leq Ch^s |u|_{H^{s+1}(\Omega)^d}$$

for $d/2 < s \leq p+1$.

PROOF. The ideal DPG method is quasioptimal, i.e., by [19, Theorem 2.2],

$$\begin{aligned} \|(u, \lambda) - (u_h, \lambda_h)\|_{U \times Q}^2 &\leq C \inf_{(v_h, \rho_h) \in U_h \times Q_h} \|(u, \lambda) - (v_h, \rho_h)\|_{U \times Q}^2 \\ &\leq C \inf_{(v_h, \rho_h) \in U_h \times Q_h} \|u - v_h\|^2 + \|\lambda - \rho_h\|_Q^2. \end{aligned}$$

The well-known best approximation estimates for U_h imply

$$(5.4.7) \quad \inf_{v_h \in U_h} \|u - v_h\| \leq Ch^s |u|_{H^s(\Omega)^{d+1}}, \text{ for all } 0 < s \leq p+1.$$

To estimate the remaining term, choose $\rho_h = D_h I_h u$. Then, since $\lambda = D_h u$, by the definition of the Q -norm in (5.2.1) and the Bramble-Hilbert lemma,

$$(5.4.8) \quad \inf_{\rho_h \in Q_h} \|\lambda - \rho_h\|_Q \leq \|u - I_h u\|_W \leq C \|u - I_h u\|_{H^1(\Omega)^d} \leq Ch^s |u|_{H^{s+1}(\Omega)^{d+1}}$$

for any $u \in H^{s+1}(\Omega)$, for $(d-1)/2 < s \leq p+1$. Thus, from (5.4.7) and (5.4.8), we have that (5.4.6) holds. $\square$

5.5. Implementation and numerical results

We finish this chapter with numerical results of the DPG discretization in the form (5.4.3) with the following change. Since W_h is infinite-dimensional, in order to get a practical method, we must replace W_h by a sufficiently rich finite-dimensional space Y_h^r . A full theoretical analysis of this practical realization of the ideal DPG method is currently open, but we will provide numerical studies showing its efficacy in this section. For some integer r , set Y_h^r as follows.

- In Case A (see (5.4.4)) we set $Y_h^r = \{w \in W_h(\Omega) : w|_K \in P_r(K)^d\}$,
- In Case B (see (5.4.5)) we set $Y_h^r = \{w \in W_h(\Omega) : w|_K \in Q_r(K)^d\}$.

Then, we compute $e_h \in Y_h^r$, $u_h \in U_h$ and $\lambda_h \in Q_h$ satisfying

$$(5.5.1) \quad \begin{aligned} (e_h, w)_h + b((u_h, \lambda_h), w) &= F(w) & \text{for all } w \in Y_h^r, \\ b((v, \rho), e_h) &= 0 & \text{for all } (v, \rho) \in U_h \times Q_h. \end{aligned}$$

In our numerical experience, the choice $r = p + d + 1$ gave optimal convergence rates (as reported in detail below). This choice is motivated by the study in [38]. The choice $r = p + d - 1$ did not give optimal convergence rates for $p > 2$ and $d = 2$. A brief report of the performance of an adaptive algorithm is also included in the $d = 2$ case. Here again, we observed marked deterioration of adaptivity if $r = p + d - 1$

is used instead of $r = p + d$ for higher degrees. Beyond these comments, we shall not describe these negative results further, but will henceforth focus solely on the $r = p + d + 1$ case. All the numerical results have been implemented using the NGSolve [60] finite element software and the codes used for the experiments below are available in [26].

5.5.1. A null space. In order to implement (5.5.1), one strategy is to set $\lambda_h = D_h z_h$ for some $z_h \in V_h$ and solve

$$(5.5.2) \quad \begin{aligned} (e_h, w)_h + b((u_h, D_h z_h), w) &= F(w) & \text{for all } w \in Y_h^r, \\ b((v, D_h r), e_h) &= 0 & \text{for all } v \in U_h, r \in V_h. \end{aligned}$$

We can decompose V_h into interior “bubbles” in $V_h^0 = \{z \in V_h : z|_{\partial K} = 0 \text{ for all } K \in \Omega_h\}$, and a remainder $V_h^1 \equiv V_h/V_h^0$. Since $b((v, D_h V_h^0), w) = 0$, we may replace V_h by V_h^1 in (5.5.2) (and compute a $z_h \in V_h^1$). Let $\{y_k\}, \{u_i\}$, and $\{z_j\}$ denote a local finite element basis for Y_h^r , U_h and V_h^1 , respectively. Using this basis, the system (5.5.2) with V_h replaced by V_h^1 , yields a matrix equation of the following form

$$(5.5.3) \quad \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{e} \\ \mathbf{x} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ 0 \end{bmatrix},$$

where $\mathbf{e}$ and $\mathbf{x}$ are the vectors of coefficients in the basis expansion of $e_h \in Y_h^r$ and $(u_h, z_h) \in U_h \times V_h$, respectively, $\mathbf{A}_{kl} = (y_l, y_k)_h$, $[\mathbf{B}_0]_{ki} = b((u_i, 0), y_k)$, $[\mathbf{B}_1]_{kj} = b((0, D_h z_j), y_k)$, and $\mathbf{B} = [\mathbf{B}_0, \mathbf{B}_1]$. In all our numerical experiments, for the above-mentioned choice of $r = p + d + 1$, we observed that the matrices $\mathbf{A}$ and $\mathbf{B}_0$ have trivial null spaces.

However, we caution that $\mathbf{B}_1$ may have a null space. This runs contrary to our experience with DPG methods on non-spacetime problems, so we expand on it. Note

that (cf. (5.1.5))

$$[\mathbf{B}_1]_{kj} = b((0, D_h z_j), y_k) = \sum_{K \in \Omega_h} \int_{\partial K} \mathbf{D}_{\mathbf{x}, \mathbf{t}} z_j \cdot y_k$$

where

$$\mathbf{D}_{\mathbf{x}, \mathbf{t}} = \begin{bmatrix} n_t & -cn_x \\ -cn_x^T & n_t \end{bmatrix}.$$

It is immediate that on mesh facets with certain combinations of n_x and n_t , the matrix $\mathbf{D}_{\mathbf{x}, \mathbf{t}}$ is singular. Then $\mathbf{B}_1$ will have a nontrivial kernel.

As an example, in Figure 5.2, we show one of the z_j that is in the null space of $\mathbf{B}_1$ on a triangular mesh for $p = 1$ and $c = 1$. In fact, on the mesh shown, there are 8 basis functions of V_h^1 that are in the null space of $\mathbf{B}_1$, two for each diagonal edge. Recall that the wave speed is 1 for our model wave problem, so these edges align with the light cone for $d = 2$. In the case of $d = 3$ space dimensions, we continued to find a nontrivial null space for $\mathbf{B}_1$ on analogous meshes.

Figure 5.2. Example of a spacetime shape function z_j in the kernel

This null space problem occurs because the interface variable λ_h is set indirectly by applying the singular operator D_h on V_h . If one could directly construct a basis for $Q_h = D_h V_h$, then one can directly implement (5.5.1) (instead of (5.5.2)). However,

we do not know how to construct such a basis easily on general simplicial meshes. Hence we continue on to describe how to solve (5.5.2) despite its kernel.

5.5.2. Techniques to solve despite the null space. Despite the above-mentioned problem, one may solve the DPG system using one of the following approaches.

5.5.2.1. *Technique 1: Remaining orthogonal to null space in conjugate gradients.*

The matrix system (5.5.3) can be solved by reducing it to its Schur complement

$$(5.5.4) \quad \mathbf{B}^T \mathbf{A}^{-1} \mathbf{B} \mathbf{x} = \mathbf{B}^T \mathbf{A}^{-1} \mathbf{f}$$

first. Let $\mathbf{C} = \mathbf{B}^T \mathbf{A}^{-1} \mathbf{B}$ and $\mathbf{g} = \mathbf{B}^T \mathbf{A}^{-1} \mathbf{f}$. The matrix $\mathbf{C}$ is symmetric and positive semi-definite. Its easy to see that

$$\ker \mathbf{C} = \ker \mathbf{B}.$$

Thus solutions of (5.5.4) are defined only up to this kernel. Note however that since $\ker \mathbf{B} = \ker \mathbf{B}_1$ and $\mathbf{B}_0$ has only the trivial kernel, the U_h -component of the DPG solution is uniquely defined independently of $\ker \mathbf{B}_1$.

One may obtain one solution of (5.5.4) using the conjugate gradient method, which computes its n th iterate $\mathbf{x}_n$ in the Krylov space

$$K_n(\mathbf{C}, \mathbf{r}_0) = \text{span}\{\mathbf{C}^k \mathbf{r}_0 : k = 0, \dots, n-1\}$$

where $\mathbf{r}_0 = \mathbf{g} - \mathbf{A} \mathbf{x}_0$ is the initial residual. This iteration will converge if $K_n(\mathbf{C}, \mathbf{r}_0)$ remains (ℓ^2) orthogonal to $\ker(\mathbf{C})$ for all n . A simple prescription to guarantee this orthogonality is to choose the initial iterate $\mathbf{x}_0 = 0$. Indeed, if $\mathbf{x}_0 = 0$, then $\mathbf{r}_0 = \mathbf{g} = \mathbf{B}^T \mathbf{A}^{-1} \mathbf{f}$ is in the range of $\mathbf{B}^T$ which equals the orthogonal complement of $\ker \mathbf{B} = \ker \mathbf{C}$. Then for all $n \geq 1$, its obvious that $\mathbf{C}^n \mathbf{r}_0$ is also orthogonal to $\ker \mathbf{C}$. Thus $K_n(\mathbf{C}, \mathbf{r}_0)$ is orthogonal to $\ker \mathbf{C}$.

To summarize this technique, we use the conjugate gradient algorithm to compute one solution orthogonal to $\ker(\mathbf{C})$ and extract the unique U_h -component from that solution for reporting the errors.

5.5.2.2. *Technique 2: Regularization of the linear system.* Another technique to solve the singular system (5.5.4) approximately is regularization. First, we rewrite (5.5.4) in block form as

$$\begin{bmatrix} \mathbf{B}_0^T \mathbf{A}^{-1} \mathbf{B}_0 & \mathbf{B}_0^T \mathbf{A}^{-1} \mathbf{B}_1 \\ \mathbf{B}_1^T \mathbf{A}^{-1} \mathbf{B}_0 & \mathbf{B}_1^T \mathbf{A}^{-1} \mathbf{B}_1 \end{bmatrix} \mathbf{x} = \mathbf{g}.$$

Since only $\mathbf{B}_1$ may have a nontrivial kernel in V_h^1 , we can convert this to an invertible system by adding a small positive-definite term in V_h^1 . Namely, let $\mathbf{M}$ be the mass matrix $\mathbf{M}_{jl} = (z_l, z_j)$. Instead of solving (5.5.4), we solve for

$$(5.5.5) \quad \begin{bmatrix} \mathbf{B}_0^T \mathbf{A}^{-1} \mathbf{B}_0 & \mathbf{B}_0^T \mathbf{A}^{-1} \mathbf{B}_1 \\ \mathbf{B}_1^T \mathbf{A}^{-1} \mathbf{B}_0 & \mathbf{B}_1^T \mathbf{A}^{-1} \mathbf{B}_1 + \alpha \mathbf{M} \end{bmatrix} \mathbf{x} = \mathbf{g}.$$

where α is a positive regularization parameter, usually set much smaller than the order of the expected discretization errors. In all our reported experiments it was set to 10^{-9} . The regularized system (5.5.5) is invertible and can be solved using any direct or iterative methods.

5.5.3. Convergence rates in two-dimensional spacetime. Let $\Omega = (0, 1)^2$. We consider a problem with homogeneous boundary and initial conditions where the exact solution to the second order wave equation is given by $\phi(x, t) = \sin(\pi x) \sin^2(\pi t)$. Then, the exact solution for the first order system is

$$u = \begin{bmatrix} c\pi \cos(\pi x) \sin^2(\pi t) \\ \pi \sin(\pi x) \sin(2\pi t) \end{bmatrix}$$

Figure 5.3. Iterates from the adaptive algorithm. Numerical pressure μ is shown for $p = 3$. Time axis is vertical.

and the corresponding source terms are

$$g = 0, \quad f = \pi^2 \sin(\pi x)(2 \cos(2\pi t) + c^2 \sin^2(\pi t)).$$

In each experiment, a (non-uniform) coarse triangular mesh of Ω was constructed, with element diameters not exceeding a reported mesh size h , with $c = 1$. Successive refinements of the mesh were obtained by connecting the mid points of the edges.

We observe in Table 5.5 that the order of convergence for u_h in the L^2 norm is $O(h^{p+1})$ in accordance with Theorem 5.4.1. Similarly in Table 5.5, we observe the same convergence rates for rectangular meshes. All results in both tables were obtained using Technique 1.

Figure 5.4. Convergence rates for $\|u - u_h\|$ on triangular meshes using Technique 1 on the left, and on rectangular meshes using Technique 1 on the right

5.5.4. Adaptivity. Let $\Omega = (0, 1)^2$. We consider the same model problem (5.1.1) but now with zero sources $f = g = 0$ and the non-zero initial condition

$$\mu|_{t=0} = -\phi_0, \quad q|_{t=0} = \phi_0$$

in place of (5.1.1c), where $\phi_0 = \exp(-1000((x - 0.5)^2))$. The boundary condition $\mu = 0$ continues to remain the same. This simulates a beam reflecting off the Dirichlet boundary.

In Figure 5.3, we display a few iterates from the standard adaptive refinement algorithm using $p = 3$ and the DPG error estimator. We started with the extremely coarse mesh shown in Figure 5.3(a), used the element-wise norms of e_h to compute the DPG element error indicator, marked elements with more than 50% of the total indicated error, refined the marked elements (and more for conformity) by bisection, and repeated this adaptivity loop. The few iterates from the adaptivity loop shown in Figure 5.3 show the potential of the spacetime DPG method to easily capture localized features in spacetime.

5.5.5. Convergence rates in three-dimensional spacetime. On $\Omega = (0, 1)^3$, we consider the problem where the exact solution to the second order wave equation is given by $\phi(x, t) = \sin(\pi x) \sin(\pi y) t^2$. This corresponds to

$$u = \begin{bmatrix} c\pi \cos(\pi x) \sin(\pi y) t^2 \\ c\pi \cos(\pi y) \sin(\pi x) t^2 \\ 2 \sin(\pi x) \sin(\pi y) t \end{bmatrix},$$

$f = \sin(\pi x) \sin(\pi y)(2 + 2c^2\pi^2 t^2)$ and $g = 0$.

In Figure 5.5, we show the convergence rates of u_h for successively refined tetrahedral meshes, obtained using Technique 2 for $p = 0, 1, 2, 3$ to the right it shows analogous results obtained for successively refined hexahedral meshes using Technique 1. In all these cases, we observe $O(h^{p+1})$ convergence rates for u_h .

Figure 5.5. Convergence rates for $\|u - u_h\|$ on tetrahedral meshes using Technique 2 on the left, and on hexahedral meshes using Technique 1 on the right

Conclusions

We have developed a framework providing sufficient conditions for the wellposedness of certain weak formulations of boundary value problems arising from linear spacetime partial differential equations in bounded domains. This includes not only first order formulations, as in Friedrichs systems, but also higher order partial differential equations. These ideas are essential for the proof of the wellposedness of the ultraweak formulations for evolution operators on spacetime cylinders (Chapters 4 and 5), as well as on spacetime tents (Chapter 2 and Chapter 3). We then studied the application of such formulations in spacetime DPG methods as well as in tent pitching schemes.

When studying the space of functions related to advective problems on domains where the inflow and outflow parts meet, we observed that traces of these functions satisfy a weak continuity property. We designed a new explicit finite element scheme of the tent pitching type for the wave equation that conforms to this weak continuity property for the lowest order case.

One of the contributions in this work is the proof of density of smooth functions that satisfy certain boundary conditions in a subspace of the graph space. These subspaces satisfy the boundary conditions only in a weak sense. It turns out that for hyperbolic problems in tent-shaped domains we have an exact characterization of these spaces through a trace lemma (see 2.4.1), for these spaces. Using this fact, it is easy to prove that a density result holds. In the case of a hyper-rectangular domain, we construct an extension operator that commutes with the spacetime operator such as

the Schrödinger and wave operator, which allowed us to prove that the extensions are in graph space of the extension, and such that the smooth functions after convolution satisfy the corresponding boundary conditions.

The formulations we considered are applicable to multidimensional problems. In Chapter 5, numerical results confirmed the predicted optimal convergence rates of the DPG formulation for the case of the wave equation. A few adaptive iterates for the spacetime wave equation show the potential of the spacetime method to easily capture localized features in spacetime meshes.

Some future works that could build off those presented in this dissertation could include the study of other finite element methods, such as the the Constrained First Order System Least Squares (CFOSLS) method. Other future directions include the use of parallel techniques and fast solvers for spacetime formulations.

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