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AN ABSTRACT OF THE THESIS OF Vincent D. Matarrese for the Master of Science
in Electrical Engineering, presented November 3, 1992.

Title: Tapered Radio Frequency Transmission Lines

APPROVED BY THE MEMBERS OF THE THESIS COMMITTEE:

Lee Casperson, Chair

Jack Riley

Gavin Bjork

A transformation used to obtain solutions for the beam parameter equation of fiber optics is applied to the second order differential equation for nonuniform transmission lines. Methods are developed for deriving possible transmission line tapers from known solutions of the transformed equation. This study begins with a comprehensive overview of previous work done to obtain closed-form solutions for the transmission line equations. Limitations of the lumped parameter model are also discussed. As part of this thesis, a tapered transmission line is constructed, based on one of the solutions obtained from the fiber optics studies. A discussion of the design and measurement results are given in the final chapter.

TAPERED RADIO FREQUENCY TRANSMISSION LINES

by

VINCENT D. MATARRESE

A thesis submitted in partial fulfillment of the
requirements for the degree of

MASTER OF SCIENCE
in
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1992

TO THE OFFICE OF GRADUATE STUDIES:

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PREFACE

This thesis has academic appeal in a variety of areas. The basic motivation, of course, is electrical engineering. Under study here is a method for taking results from the study of tapered optical waveguides and applying them to the much lower frequency RF domain. The result is a method for obtaining mathematical results which predict the behavior of a tapered RF transmission line and are useful to the electrical engineer.

There are other academic disciplines at work here, however. The mathematician, for instance, will enjoy all the mathematical effort which is required to obtain the results of this paper, particularly those of the third chapter. The historian will appreciate the portrait of the early days of communication electronics shown through the references to Heaviside, Carson, Collin, and some of the other "fathers" of the industry, in the second chapter. The test and measurement engineer will take keen interest in the fourth and fifth chapters, which present an overview of design constraints for a tapered transmission line and describe the efforts to measure the performance of a sample line. The breadth of coverage of this project has made the work challenging and rewarding.

Special mention must be made of the many people who contributed to this paper. First and foremost, loving thanks to my wife, Robbi, for all the support and hard work, often performing the duties of two parents while I was laboring over the project. Thanks also go to Ed Wardzala, Laudie Doubrava, Bill Schell, and many others at Tektronix for their help, and to Tony Tovar, fellow student at PSU, for his assistance with this paper. Finally, heartfelt appreciation to my advisor, Dr. Lee Casperson, for his patience and support throughout my many days at PSU.

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CHAPTER I

INTRODUCTION

Since the late 19th century, when engineers first began to appreciate the phenomenon of guided electromagnetic waves, there has been interest in the structures which guide such waves. While much of the work on waveguiding structures has been motivated by purely scientific concern, a great deal of effort has been undertaken with the goal of improving communication. The fact that information can be sent from one location to another quickly, accurately, and often cheaply, by means of guided electromagnetic waves has revolutionized human life. For the past 100 years, new developments in the field of communication electronics have constantly changed the way we live, and can be expected to do so for many years to come.

This thesis focuses on a particular kind of waveguiding structure, generally referred to as a transmission line. This type of structure is made of at least two electrical conductors, as opposed to a "waveguide," which is commonly thought of as a single conductor arrangement. Transmission lines have been the object of study for nearly 150 years, due in large part to their utility in the communications industry. And, at least for the last 100 years, part of this effort has been focused on nonuniform or tapered transmission lines.

The problem of analyzing the behavior of tapered transmission lines is not simple. The mathematics used to describe a general (tapered or uniform) transmission line, a pair of differential equations known as the "telegrapher's equations" or the "transmission line equations", do not yield to solution easily for tapered lines. The

tantalizing problem of finding a general solution for the equations of a line with a completely arbitrary taper has degenerated into a search for techniques that solve for as many taper profiles as possible. This paper describes such a technique, borrowed from recent work with the beam parameter equation of fiber optics, which can be applied to the tapered transmission line.

This study begins with a review of the history of efforts to solve the transmission line equations for a tapered line. The early work of Thompson and Heaviside is discussed. The two men provided the mathematical-theoretical foundation for transmission line theory, even as it is known today. The rapidly developing communications industry of the early 20th century picked up on this early work and used these theories to improve their products. The abundance of papers from Bell Labs on transmission lines (including tapered lines) from this period of time is a testament to their leadership role in this area. In addition to studies aimed at finding solutions for different tapers, significant effort was given to the problem of optimizing performance, primarily by minimizing reflections. Most of this work took place in the 1940's and 1950's. Due to the advent of modern computing techniques (and probably to a growing frustration with the inability to find new closed form solutions), interest in solving the tapered transmission line problem decreased in the 1960's. Since then, only an occasional paper appears on the topic.

The third chapter presents a technique which has been used in solving the beam parameter equation from fiber optics and applies it to the tapered transmission line problem. The technique involves transforming the second order differential equation for transmission lines into a second order equation which does not have a first derivative term. Use of this technique is motivated by the fact that researchers working with lasers and fiber optics have, over the last eight years, found new solutions to differential equations of this form. The goal of this section is to present a method for taking

advantage of these new solutions.

In the fourth chapter, the model for describing the behavior of transmission lines (the two differential equations) is studied to determine its validity and limitations. The model uses parameters from circuit theory. The development of the transmission line equations based solely on circuit theory is presented first. Further discussion of these equations leads to a determination of their connection with electromagnetic theory and, ultimately, with Maxwell's equations. Once the model is developed, the conditions under which it is valid are discussed. This includes consideration of the inductance parameter (internal vs. external, low frequency vs. high frequency), skin effect, and cut-off frequencies for higher order mode propagation.

Applications is the subject of the final chapter. The design of a line based on solutions found in the third chapter is presented. Measurement techniques are discussed, and the results of measurements on the constructed lines are given.

CHAPTER II

HISTORY OF SOLUTIONS TO THE TRANSMISSION LINE EQUATIONS

INTRODUCTION

In this chapter, a comprehensive overview of the study of tapered transmission lines is presented. Developments are discussed in a chronological fashion and are grouped by major emphases. Some of these emphases are engineering-oriented, while others are more mathematical and theoretical. As the story of this section unfolds, special note will be made of the motivation of each area of study. The discussion which follows deals only with two-conductor, radio-frequency transmission lines and the mathematical formulae which describe their performance. Many studies have been done on tapered waveguides. Giving an adequate presentation of this area is beyond the scope of this paper. Similarly, the study of tapered multiconductor lines, though important in today's digital electronics, will not be undertaken here. Instead, this paper is restricted to developments which have been made based on the distributed parameter model of a two-conductor transmission line.

HEAVISIDE AND EARLY SOLUTIONS

The person chiefly responsible for the first thorough mathematical analysis of signal propagation along transmission lines was Oliver Heaviside (1850-1925), the British telegraph engineer and nephew of Charles Wheatstone. Heaviside developed on the model, introduced by William Thompson (Lord Kelvin) in the mid-nineteenth

century, which contains the four basic distributed circuit parameters -- series resistance (R) and inductance (L), and shunt capacitance (C) and conductance (G). His theoretical and practical studies led to a better understanding of how signal fidelity could be improved. While Thompson and others insisted that series inductance should be minimized, Heaviside's work showed that it should actually be increased. Directly from the transmission line equations,

$$-\frac{dI}{dx} = (G + j\omega C)V(x) \quad (1)$$

$$-\frac{dV}{dx} = (R + j\omega L)I(x) \quad (2)$$

he concluded that if $RC=GL$, then the frequency dependence of the characteristic impedance and the propagation factor would be eliminated. Subject to the limitations of the distributed parameter transmission line model, a line constructed with this constraint would be distortionless. This result led to the installation of low resistance loading coils in U.S. telephone lines around the turn of the century.

Heaviside was the first to state the transmission line equations in the form used today. In addition, he developed the concept and terminology for "characteristic impedance", studied numerous transmission line configurations, investigated various types of line termination and their effects on the reflection of input signals, and looked into the phenomenon now known as "skin effect." In short, his contributions to the overall development of transmission line theory should not be minimized.

Heaviside was the first person to investigate formally the tapered transmission line. In his Electromagnetic Theory [1], he derived a solution of the transmission line equations for voltage and current, given above, for the case of a linearly tapered line, that is, one in which the series resistance, R, varies inversely with distance, while the shunt capacitance, C, varies directly with distance:

$$\frac{1}{R(x)} = \frac{x}{R_0} \quad (3)$$

$$C(x) = C_0 x \quad (4)$$

By combining equations (1) and (2) and getting a second order differential equation and including the constraints given in (3) and (4), he obtained:

$$x^2 \frac{d^2 V}{dx^2} + x \frac{dV}{dx} - x^2 q_0^2 V = 0 \quad (5)$$

This is a modified Bessel function of order 0 with the solution:

$$V(x) = 1 + \frac{q_0 x^2}{2^2} + \frac{q_0 x^4}{2^2 \cdot 4^2} + \frac{q_0 x^6}{2^2 \cdot 4^2 \cdot 6^2} + \dots \quad (6)$$

This example, from Heaviside's early work on transmission lines, does not include the effects of series inductance or shunt resistance. In later articles, he incorporated these terms into a solution for the linearly tapered line. In section #336 of Electromagnetic Theory [1], he extended the above solution to include the case of a line whose parameters vary as the n^{th} power of distance.

In the years which followed the publication of Heaviside's work, telegraph engineers moved in the direction of finding increasingly more general approaches to solving the transmission line equations for tapered lines. Infinite series solutions were generated by C. Ravut (1920) [2] and J. R. Carson, of AT&T (1921) [3]. Based on the transmission line equations, Ravut developed a recursive formula for the derivatives of $V(x)$ and $I(x)$. He rewrote these formulae in terms of a MacLaurin series:

$$V(x) = V_0 + xV_0' + \frac{x^2}{1 \cdot 2} V_0'' + \dots + \frac{x^n}{1 \cdot 2 \cdot 3 \dots n} V_0^{(n)} + \dots \quad (7)$$

Numerical results are realized by calculating successive derivatives of the basic

transmission line differential equations:

$$\frac{dV}{dx} = -Z(x)I(x) \quad (8)$$

$$\frac{dI}{dx} = -Y(x)V(x) \quad (9)$$

where $Z(x)$ is the series parameter, $R + j\omega L$, and $Y(x)$ is the shunt parameter, $G + j\omega C$, of the line. This solution is restricted, clearly, to cases where the variation of the line parameters is differentiable.

Carson employed the Picard method, first integrating the transmission line equations to get:

$$V(x) = V_0 - \int_0^x Z(x_1)I(x_1)dx_1 \quad (10)$$

$$I(x) = I_0 - \int_0^x Y(x_1)V(x_1)dx_1 \quad (11)$$

then presenting a recursive formula for successive terms, also obtained by integration.

He then took the summations of both series of integrals and placed the appropriate terms in the voltage and current solutions for a uniform line:

$$V = hV_0 \cosh(P) - hKI_0 \sinh(P) \quad (12)$$

$$I = \frac{1}{h}I_0 \cosh(P) - \frac{1}{hK}V_0 \sinh(P) \quad (13)$$

where h , K , and P represent sums of the derived integral series. This approach required only that the variations are integrable, thus allowing for discontinuities. While both approaches broke new ground, their utility was somewhat limited by the amount of hand calculation required to obtain a solution for a particular line profile.

These efforts were followed by others, which, though less general, were couched in terms of familiar functions, thus more useful to solve the practical problems of the day. Arnold and Bechberger [4] worked on the particular case in which L and R (the

series inductance and series resistance terms) are varied linearly, but independently:

$$L(x) = L_0 + k_L x \quad (14)$$

$$R(x) = R_0 + k_R x \quad (15)$$

and obtained a power series solution. They factored their restriction into the voltage and current equations, and combined them:

$$\frac{d^2 I}{dx^2} = (\gamma^2 + \delta^2 x) I \quad (16)$$

Using a change of variables:

$$\gamma = \vartheta \quad (17)$$

$$\text{and} \quad \frac{\delta^2}{\gamma^3} = \alpha \quad (18)$$

they obtained:

$$\frac{d^2 I}{d\vartheta^2} = (1 + \alpha \vartheta) I \quad (19)$$

from which a solution can be gotten in the form:

$$I = AC_1 + BS_1 \quad (20)$$

where C_1 and S_1 are power series in terms of α and ϑ . Similar solutions can be obtained for the voltage function. When $\alpha = 0$, the power series terms become hyperbolic cosines and sines, respectively.

M.E. Frederici (1931)[5], followed a tack taken by Heaviside and solved the case in which the series impedance, $Z(x)$, varies linearly, while the shunt admittance, $Y(x)$, is constant. His solution is given in terms of Bessel functions. A more general solution along these same lines was given by A.T. Starr (1932)[6]. Starr imposes only the restriction that the series and shunt parameters vary as arbitrary powers of distance, that is,

$$Z = z_0 x^a \quad (21)$$

$$Y = y_0 x^b \quad (22)$$

He employed a change of variables:

$$v = x^p u \quad (23)$$

$$w = x^q \quad (24)$$

to transform the equation into Bessel form:

$$\frac{d^2 u}{dw^2} + \frac{1}{w} \frac{du}{dw} + \left[\frac{-4y_0 z_0}{(2+a+b)^2} - \left\{ \frac{(1+a)^2}{(2+a+b)} \right\}^2 \frac{1}{w^2} \right] u = 0 \quad (25)$$

This can be solved, in general, and transformed back to the standard form. His solution covers the following cases: 1) the uniform line ($a = b = 0$), 2) the "Bessel" line ($b = -a$) [the case solved by Heaviside], and 3) the series tapered line ($a = 1, b = 0$). [the case solved by Frederici]. And, since a and b can be arbitrarily assigned, Starr's work is more general than previous efforts.

The infinite-series and Bessel-form solutions discussed thus far were used in a couple of ways to meet the communication engineering needs of their day. Two distinct applications deserve mention. A telegraph cable laid between Newfoundland and the Azores in 1928 [7] used tapered sections to increase the inductive loading in the central portion of the run. The infinite series solution for the case in which series inductance varies linearly with distance, developed by Arnold and Bechberger (described above) was used in the design of this cable. The taper was used, in this case, to increase the series inductance of the cable while holding the other parameters constant, therefore more closely approximating the distortionless line described by Heaviside. Tapering was recognized as being superior to loading with discrete inductors since reflections introduced by the impedance changes due to installing discrete inductors were greatly reduced.

A second application, this mentioned by Starr, is the use of his solutions in analyzing the effects of line droop in pole-mounted telegraph lines. The droop shape can be approximated as two linear tapers which meet at the low point between the poles.

THE EXPONENTIAL LINE

A family of tapers which lends itself to particularly easy analysis is the exponential line. This transmission line is realized when the series impedance and shunt admittance are varied proportional to the exponent of distance along the line and are inversely proportional to each other. The description given here, and the predominant case studied in the literature, is for the lossless line, in which the series R and shunt G are assumed to be much smaller than the series L and shunt C . Thus,

$$L(x) = L_1 e^{nx} \quad (26)$$

$$C(x) = C_1 e^{-nx} \quad (27)$$

The resulting differential equation is second order with constant coefficients, here shown for voltage, and easily solved.

$$\frac{d^2 V}{dx^2} - n \frac{dV}{dx} + \omega L_1 C_1 V = 0 \quad (28)$$

with the solution:

$$V(x) = A_1 e^{\frac{nx}{2}} e^{-jbx} + A_2 e^{\frac{nx}{2}} e^{jbx} \quad (29)$$

where the phase factor, b , is written:

$$b = \sqrt{\omega^2 L_1 C_1 - \frac{n^2}{4}} \quad (30)$$

Theoretical as well as practical studies have demonstrated the utility of such a line. The earliest work found in the literature is the classic article by Burrows, of Bell Labs (1938) [8]. His study includes the analysis and conclusions presented here.

Like any other tapered line, if the exponential line is terminated without reflection, it becomes an excellent impedance transformer. It also performs something like a high pass filter. Examining the voltage solution shows that below a certain

frequency, the phase factor becomes an attenuation factor. The critical frequency is obtained when:

$$\omega^2 L_1 C_1 = \frac{n^2}{4}, \quad (31)$$

giving a cutoff frequency of:

$$\omega_c = \frac{n}{2\sqrt{L_1 C_1}} \quad (32)$$

Especially for long haul lines which use this type of taper, the attenuation factor can be significant at or below the cut-off frequency.

If the voltage solution is divided by the equation for current, the impedances looking into each end of the line can be derived.

$$Z_{up} = \sqrt{\frac{L_1}{C_1}} \left[\sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2} - j \frac{\omega_c}{\omega} \right] \quad (33)$$

$$Z_{down} = \sqrt{\frac{L_1}{C_1}} \left[\sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2} + j \frac{\omega_c}{\omega} \right] \quad (34)$$

Here Z_{up} and Z_{down} indicate the impedance looking into the lower impedance terminal and upper impedance terminal respectively.

It is immediately obvious that the characteristic impedance of an exponential line is complex and that both the real and reactive components are frequency dependent. To terminate an exponential line properly, reactive components have to be employed. The literature contains a variety of schemes for doing so, summarized nicely by Wheeler (1939) [9]. To keep compensation networks at a minimum or eliminate them altogether, it is customary to operate an exponential line far above its cut-off frequency. The usefulness of such a line is demonstrated by a flurry of patents which were received in the mid- to late 1920's. Not surprisingly, most deal with the use of the exponential line as a filter or as an impedance matching device [10] - [13].

THE SEARCH FOR THE OPTIMUM TAPER

It wasn't long after tapered lines came into use that the question of improving their performance was raised. Engineers were aware of the bandpass characteristics of the exponential line and sought taper profiles with improved performance. Walker and Wax of Bell Telephone Laboratories [14] took the transmission line equations and rewrote them in terms of the reflection coefficient. They used the normal definition of reflection coefficient:

$$q_v = \frac{\frac{V}{I} - \frac{Z(x)}{Y(x)}}{\frac{V}{I} + \frac{Z(x)}{Y(x)}} \quad (35)$$

and obtained a first-order, non-linear differential equation of Ricatti form:

$$\frac{dq_v}{dx} - 2Z(x)Y(x)q_v + \frac{1}{2} \frac{d \ln \frac{Z(x)}{Y(x)}}{dx} (1 - q_v^2) = 0 \quad (36)$$

Next they made the substitution $q_v = e^{j\vartheta}$ and rewrote the equation in integral form as:

$$\vartheta = -2j \int_0^x Z(x)Y(x)dx + \int_0^x \sin(\vartheta) \frac{d}{dx} \left[\ln \frac{Z(x)}{Y(x)} \right] dx \quad (37)$$

which is then solved by graphical methods. The results they got were applied to calculating the resonant wavelengths of various types of tapers.

Others were quick to pick up on this development and begin applying the Walker-Wax technique to finding the geometry of a tapered matching section with the minimum overall reflection coefficient for a given bandwidth. Various authors, such as Bolinder (1950) [15], noted that, if the q_v^2 term was assumed very small, the Walker-Wax equation could be handled more easily and applied to a number of taper types.

This observation provided an opportunity for the use of another result. Since the

mid-1940's stepped lines had been studied and used as matching sections. These lines were built with short cascaded sections of uniform lines whose characteristic impedances gradually stepped between two given terminal values. The trick in designing this sort of line is to get the correct step size and line length which give both a low reflection coefficient and wide bandwidth. Cohn had observed [16] that one could optimize this sort of design by forcing step size selection to yield a Chebyshev-type reflection coefficient response. His design technique used lines which are a quarter-wavelength at the middle frequency of the expected useful range. He also included a compensation factor to minimize the effect of the step discontinuities, derived from a first-order approximation from circuit theory.

The problem of optimizing a matching section designed as a continuous taper was attacked from a couple of angles. S.I. Orlov (1955) [17] took an arbitrary taper, divided it into a number of discrete sections, and applied the definition of reflection coefficient to each section. He then summed the effect from each section and let the number of discrete sections go to infinity, thus obtaining an integral expression for the overall reflection coefficient. A second approach, taken by Klopfenstein (1956) [18], returned to the Walker and Wax equation (36) and applied the Chebyshev response characteristic. In doing so, two approximations were made: first, that the q_v^2 term is very small and can be eliminated, and second, that the line is lossless. He then equated the input reflection coefficient, ρ , with the limiting form of the Chebyshev polynomial:

$$\rho e^{j\beta l} = \rho_0 \frac{\cos \sqrt{bl^2 - A^2}}{\cosh(A)} \quad (39)$$

where β is the phase factor, l represents length, and A is a design parameter which determines the maximum reflection coefficient in the passband. From this, he was able to derive a function for the profile of the taper.

R.E. Collin, in an article published the same year [19], took essentially the same approach as Klopfenstein, but provided significantly more detailed derivations.

APPLICATION OF RICCATI EQUATION SOLUTIONS

Although no general solution has ever been found, a number of line profiles can be solved by transforming the transmission line equations into a single Riccati equation and specifying relationships among the various terms. In the early 1960's, Iwao Sugai, of ITT, wrote of a number of techniques to derive solutions. He worked both with the standard transmission line equations and with the equations written in terms of reflection coefficient.

Sugai was the first, apparently, to recognize that the reflection coefficient equation derived by Walker and Wax is a Riccati equation, whose general form is:

$$\frac{dr}{dx} + P(x)r + Q(x)r^2 = R(x) \quad (40)$$

where, in the transmission line case:

$$P(x) = -2\sqrt{Y(x)Z(x)} \quad (41)$$

$$Q(x) = \frac{1}{2Z(x)} \frac{dZ(x)}{dx} \quad (42)$$

$$\text{and} \quad R(x) = Q(x). \quad (43)$$

One approach he used was to transform this equation into a second order linear equation, then set each coefficient to zero. He provided two transformations which accomplish this [20]. The result is a formula for the required relationship between the various line parameters. A second, related approach used a change of variables :

$$r(x) = s(x) + T(x) \quad (44)$$

where $T(x)$ is related to P , Q , and R of the standard Riccati equation by:

$$R(x) = \frac{dT}{dx} + P(x)T(x) + Q(x)T(x)^2 \quad (45)$$

The resulting equation, in terms of the new variable $s(x)$, is in the form of a Bernoulli equation:

$$\frac{ds}{dx} + (P(x) + Q(x)T(x))s + Q(x)s^2 = 0 \quad (46)$$

for which a general solution exists [21].

S.C. Dutta Roy used the transform-and-constrain method of Sugai, applying it to an RC line, such as might be used in thin film semiconductor work [22]. The second order voltage equation for RC lines is:

$$\frac{dV}{dx} - \frac{1}{f} \frac{df}{dx} \frac{dV}{dx} - jafgV = 0 \quad (47)$$

where $R = r_0 f(x)$ and $C = c_0 g(x)$. With the transformation:

$$V(x) = w(x) \sqrt{f(x)}, \quad (48)$$

the following equation is derived:

$$\frac{dw}{dx} + \left[\frac{1}{2} \frac{d}{dx} \left(\frac{1}{f} \frac{df}{dx} \right) - \frac{1}{4} \left(\frac{1}{f} \frac{df}{dx} \right)^2 - jafg \right] w = 0 \quad (49)$$

If the restrictions:

$$fg = 1 \quad (50)$$

$$\text{and} \quad 2 \left(\frac{1}{f} \frac{df}{dx} \right) - \left(\frac{1}{f} \frac{df}{dx} \right)^2 = K^2 \text{ (a constant)} \quad (51)$$

are applied, the equation can be solved in general as:

$$f(x) = K_1 \sec^2(K_2 x + K_3) \quad (52)$$

where K_1 and K_2 are integration constants, and $K_3 = \frac{1}{2} K$.

Again, using the technique of transformation and restriction, Swamy and Bhattacharyya (1966)[23] were able to obtain a solution for the RC line equation in terms of Hermite functions. In this approach, two transformation steps take place: the first to transform the RC line equation into a second order linear differential equation, and the second (after the restriction is applied) to take the equation into Hermite form. Once again, the restrictions specify the relationship between the line parameters, in this case R and C.

GENERALIZED SOLUTIONS

Another approach to the solution of the transmission line for tapered structures consisted of generalizing already existing solutions. In the early and mid-sixties a number of authors used this approach.

Schwartz (1964)[24] worked out a set of rules for generalizing exponential, Bessel, and Legendre equation solutions. His method is fairly straightforward and is described here for the case of the exponential line. The goal is to find a rule for which the second order voltage differential equation:

$$\frac{d^2V}{dx^2} - \frac{1}{Z} \frac{dZ}{dx} \frac{dV}{dx} - \gamma^2 V = 0 \quad (53)$$

reduces to a second order equation with constant coefficients of the form:

$$\frac{d^2V}{du^2} - k \frac{dV}{du} - \Gamma^2 V = 0 \quad (54)$$

With some manipulation he arrived at a rule which determines the selection of k and Γ :

$$\frac{1}{\gamma} \frac{d}{dx} \ln(Y(x)Z(x)) = \pm \frac{2k}{\Gamma} \quad (55)$$

If a transmission line which is nonuniform with respect to x is to be transformed into an

exponential line with the distance variable u , the left hand side of the above expression must be a real or complex constant. Rules for transforming various tapered lines into Bessel or Legendre lines are similarly derived.

Berger (1966) [25] took a slightly different approach to the problem. He started with a nonuniform line characterized by $Z=A(x)$ and $Y=B(x)$ which has a known solution:

$$V(x) = K_1 V_1(x) + K_2 V_2(x) \quad (56)$$

where K_1 and K_2 are constants and V_1 and V_2 are linearly independent. Then he defined a new line for which:

$$Z(x) = f'(x) A[f(x)] \quad (57)$$

$$\text{and} \quad Y(x) = f'(x) B[f(x)]. \quad (58)$$

The transmission line equations now become:

$$\frac{dV}{df} = -I(f) A(f) \quad (59)$$

$$\frac{dI}{df} = -V(f) B(f) \quad (60)$$

with the solution:

$$V(x) = K_1 (V_1(f(x))) + K_2 (V_2(f(x))) \quad (61)$$

The generalized exponential line is described by:

$$\frac{dV}{dx} = -I(x) z_0 f'(x) e^{af(x)} \quad (62)$$

and

$$\frac{dI}{dx} = -V(x) y_0 f'(x) e^{-af(x)} \quad (63)$$

The solution in terms of voltage is:

$$V(x) = K_1 e^{\left(\frac{a}{2} - jb\right)f(x)} + K_2 e^{\left(\frac{a}{2} + jb\right)f(x)} \quad (64)$$

where $a = \sqrt{z_0 y_0}$ and $b = \sqrt{C^2 - \frac{a^2}{4}}$. Similar solutions are given for the uniform line and the Bessel line.

More comprehensive work on generalization and classification of solutions was done by Holt and Ahmed (1968) [26]. Their approach is similar to the one taken by Schwartz. They take the standard second order transmission line equation:

$$\frac{d^2 V}{dx^2} - \frac{1}{f} \frac{df}{dx} \frac{dV}{dx} - z_0 y_0 f g V = 0 \quad (65)$$

where the shunt and series parameters Z and Y are represented as,

$$Z(x) = z_0 f(x) \quad (66)$$

$$Y(x) = y_0 g(x) \quad (67)$$

and change the independent variable to get:

$$\frac{d^2 V}{dw^2} - \frac{1}{f} \frac{d}{dx} \left(\frac{f}{w'} \right) \frac{dV}{dw} - z_0 y_0 \frac{fg}{(w')^2} = 0 \quad (68)$$

where the new variable, w , is some arbitrary function of x . They then take the $\frac{dV}{dx}$ and V terms and determine how to transform the x variable into w . In the particular case of a generalized Bessel line,

$$\frac{1}{f} \frac{d}{dx} \left(\frac{f}{w'} \right) = -\frac{1}{w} \quad (69)$$

$$\text{and} \quad \frac{fg}{(w')^2} = k^2 - \frac{n^2}{w^2} \quad (70)$$

The first equation is solved for w in terms of $f(x)$. As noted before, $f(x)$ determines the taper profile.

$$w = k_2 e^{\frac{1}{k_1} \int f(x) dx} \quad (71)$$

This result is the desired transformation which takes the standard form of the

transmission line equation into a Bessel's equation.

The authors work out cases for a number of standard second order differential equation types and present it in tabular form with each transformation and solution. Equation families included are: exponential, Euler, Bessel, hypergeometric, and Hermite. This approach loses a bit of generality in that, with each given transformation, $g(x)$ becomes dependent on $f(x)$.

A slightly different method of generalization was introduced by Gough and Gould (1966) [27] and later developed significantly by Wescott (1969) [28]. Gough and Gould started by transforming the independent variable from x to z , using the formula:

$$z = \int_0^x \sqrt{Z(x)Y(x)} dx \quad (72)$$

and introducing the new dependent variable $U(x)$ defined by:

$$V(z) = U(z) \sqrt[4]{\frac{Z(x)}{Y(x)}} \quad (73)$$

to get:

$$\frac{d^2 U}{dz^2} + [F(z) - k]U(z) = 0 \quad (74)$$

where:

$$F(z) = \frac{1}{2} \frac{d^2}{dx^2} \ln \sqrt{\frac{Z(x)}{Y(x)}} - \frac{1}{4} \left(\frac{d}{dz} \ln \sqrt{\frac{Z(x)}{Y(x)}} \right)^2 \quad (75)$$

Further transformation of this last equation by:

$$\Phi(z) = \sqrt[4]{\frac{Y(x)}{Z(x)}} \quad (76)$$

yields:

$$\frac{d^2 \Phi}{dz^2} + F(z)\Phi(z) = 0 \quad (77)$$

These last two equations must both be satisfied in order to get the solution for voltage

(or current).

B. Wescott (1969) [28] expanded on the work of Gough and Gould by performing more detailed mathematical analysis of their method and showing how it can be used for hypergeometric equation solutions. Once again, general methods for obtaining transformations which take the transmission line equations into a particular form with known solutions are presented. Wescott also indicated how previous methods, such as those of Berger, Holt and Ahmed, and Dutta Roy, are particular cases of his work.

SUMMARY

Over the last hundred years, numerous practical and theoretical studies of non-uniform radio frequency transmission lines have been done. The comprehensive review presented in this chapter is the only one of its kind known to the author. The following table summarizes the main contributions to the classes of solutions discussed above. References to numerous articles written prior to 1955 and not directly cited here can be found in Kaufman's well-researched bibliography on nonuniform transmission lines [29].

TABLE I
SOLUTIONS FOR THE TRANSMISSION LINE EQUATIONS FOR TAPERED
LINES

Author	Ref.	Taper Profile	Solution Form
Heaviside	1	Reciprocal linear	Modified Bessel functions
Ravut	2	Differentiable contour	MacLaurin series
Carson	3	Integrable contour	Integral series
Arnold & Bechberger	4	Independent linear	Power series
Starr	6	Independent power of distance	Bessel functions
Burrows	8	Independent exponential	Exponential
Walker & Wax	14	Arbitrary (reflection-based)	Graphical
Orlov	17	Integrable contour (reflection-based)	Integral series
Sugai	19, 20	Restricted coefficients of Riccati Equation	Various closed-form
Dutta Roy	21	RC lines with restrictions	Elementary functions
Swamy & Bhattacharyya	22	RC lines with restrictions	Hermite polynomials
Schwartz	23	Generalized restrictions	Bessel and Legendre functions
Berger	24	Arbitrary, given known solution	Exponential
Holt & Ahmed	25	Generalized restrictions	Euler, Bessel, hypergeometric, Hermite

CHAPTER III

THE TRANSMISSION LINE EQUATIONS IN "HILL" EQUATION FORM

INTRODUCTION

The review of literature on solutions to the transmission line problem indicates that there can be significant advantages in transforming the second-order voltage or current equations into another form for which there are known solutions. This is especially the case when considering a tapered transmission line, since the equation used to model its behavior no longer has constant coefficients. In this chapter, a particular transformation which has been used to help solve the beam parameter equation for tapered fiber optics will be applied to the telegrapher's equation for nonuniform transmission lines. The material which follows breaks new ground in two areas. First, the particular transformation being considered has not been applied directly to the transmission line equations in any previous work. Secondly, the transformation presents an opportunity to analyze taper geometries which, up until now, have not been given any attention in the literature.

The idea of the transformation considered here is to take a second order ordinary differential equation with non-constant coefficients (such as the transmission line equations for a tapered line), and eliminate the first derivative term. The result is a second order equation of the form:

$$\frac{d^2V}{dx^2} + f(x)V(x) = 0 \quad (1)$$

This is sometimes referred to as the "Hill" form [30], though it is recognized here that equations which are derived with this transformation are not the classic Hill equation which is given as:

$$\frac{d^2y}{dx^2} + [a_0 + 2a_1 \cos(2x) + 2a_2 2 \cos(4x) + 2a_3 3 \cos(6x) + \dots]y = 0 \quad (2)$$

A number of solutions to equations of this form have been worked out in the past decade for use in dealing with the beam parameter equation of fiber optics [31]-[33]. In addition, a number of techniques have been developed for generating new solutions [34]-[37]. It is the ultimate aim of this section to demonstrate how one can take advantage of these solutions to solve the telegrapher's equation for transmission lines with particular classes of tapers.

TRANSFORMATION TO THE "HILL" FORM

The transformation under consideration has been described by Yamamoto and Makimoto (1971) [38], used by Hilbert and Courant [39], and cited by Zwillinger [40]. The description given here follows Casperson [41]. An equation in self-adjoint form:

$$\frac{d}{dr} [k_0(r) \frac{dR(r)}{dr}] + k_2(r)R(r) = 0 \quad (3)$$

can be expanded to the form:

$$\frac{d^2R(r)}{dr^2} + \frac{1}{k_0(r)} \frac{dk_0}{dr} \frac{dR(r)}{dr} + \frac{k_2(r)}{k_0(r)} R(r) = 0 \quad (4)$$

which later can be matched, coefficient by coefficient, with the transmission line equation. This expression can be transformed to another equation with new dependent and independent variables:

$$\frac{d}{dr'} [k_0'(r') \frac{dR'(r')}{dr'}] + k_2'(r') R'(r') = 0 \quad (5)$$

if the following relationships are defined:

$$k_0'(r') = k_0[r(r')] \frac{dr'}{dr(r')} \quad (6)$$

$$k_2'(r') = k_2[r(r')] \frac{dr(r')}{dr'} \quad (7)$$

$$\text{and} \quad R'(r') = R[r(r')] \quad (8)$$

In order that the transformed equation is of the desired "Hill" form, the $k_0'(r')$ term must be a constant. This is accomplished by constraining equation (6) with the relation:

$$\frac{dr'}{dr} = \frac{k_0'}{k_0(r)} \quad (9)$$

where k_0' is some arbitrarily specified constant. Integrated this gives:

$$r' = k_0' \int \frac{dr}{k_0(r)} \quad (10)$$

which is the required relationship between the new and old independent variables.

Equation (7) then becomes:

$$k_2'(r') = \frac{k_2(r)k_0(r)}{k_0'} \quad (11)$$

Since $k_0'(r')$ is a constant, equation (5), the transformed second order differential equation, can now be written as:

$$\frac{d^2 R'(r')}{dr'^2} + \frac{k_2'(r')}{k_0'} R(r') = 0 \quad (12)$$

Happily, the second order transmission line equation for nonuniform lines,

$$\frac{d^2V}{dx^2} - \frac{1}{Z} \frac{dZ}{dx} \frac{dV}{dx} - ZYV = 0 \quad (13)$$

is close to the form of (5), which is the starting point for the described transformation technique. With a slight modification, it can be made to correspond exactly to (5). Let

$$A(x) = \frac{1}{Z(x)} = \frac{1}{z_0 f(x)} \quad (14)$$

$$B(x) = -Y(x) = -y_0 g(x) \quad (15)$$

Substituting these into (13) one obtains:

$$\frac{d^2V}{dx^2} - \frac{1}{1/A} \frac{d(1/A)}{dx} \frac{dV}{dx} + \frac{1}{A} BV = 0 \quad (16)$$

which simplifies to:

$$\frac{d^2V}{dx^2} + \frac{1}{A} \frac{dA}{dx} \frac{dV}{dx} + \frac{B}{A} V = 0 \quad (17)$$

The transformed equation will be of the form:

$$\frac{d^2V'(x')}{dx'^2} + A(x')B(x')V'(x') = 0 \quad (18)$$

Note that the term k_0 from (12) in the description of the transformation has been arbitrarily chosen to be unity. The solution of this equation $V'(x')$ is transformed back into the previous domain by a simple change of variables using:

$$x' = \int \frac{1}{A(x)} dx = \int z_0 f(x) dx \quad (19)$$

This technique can be tested by checking a case where solutions in both domains are known. Let

$$Z(x) = \frac{z_0}{x} \quad (20)$$

$$\text{and} \quad Y(x) = y_0 x \quad (21)$$

which defines the linear taper studied by Heaviside and generalized by Starr and others.

In the standard domain, the telegrapher's equation for this line is written:

$$\frac{d^2V}{dx^2} - \frac{x}{z_0} \frac{d}{dx} \left(\frac{z_0}{x} \right) \frac{dV}{dx} - z_0 y_0 V = 0 \quad (22)$$

which simplifies to:

$$\frac{d^2V}{dx^2} + \frac{1}{x} \frac{dV}{dx} - z_0 y_0 V = 0 \quad (23)$$

This is recognized as a form of Bessel's equation which, following Starr [42], has the general solution:

$$V(x) = c_1 J_0(j\sqrt{z_0 y_0} x) + c_2 Y_0(j\sqrt{z_0 y_0} x) \quad (24)$$

The same result can be obtained using the transformation described above. The first step is to redefine the coefficients so that the problem is posed in the correct form. Thus,

$$A(x) = \frac{x}{z_0} \quad (25)$$

$$\text{and} \quad B(x) = -y_0 x \quad (26)$$

Then, to get the coefficient of the non-derivative term, combine $A(x)$ and $B(x)$

$$A(x)B(x) = -\frac{y_0 x^2}{z_0} \quad (27)$$

The new independent variable is:

$$x' = \int \frac{dx}{A(x)} = \int \frac{z_0}{x} dx \quad (28)$$

$$\text{or} \quad x' = z_0 \ln(x) \quad (29)$$

Solving for x and substituting into (27), we get:

$$A(x')B(x') = -\frac{y_0 e^{2x'/z_0}}{z_0} \quad (30)$$

The equation to be solved in the transformed domain is, therefore:

$$\frac{d^2 V'}{dx'^2} - \frac{y_0 e^{2x'/z_0}}{z_0} = 0 \quad (31)$$

The solution of this equation, from Abramowitz and Stegun [43] is:

$$V'(x') = c_1 J_0(j\sqrt{y_0 z_0} e^{x'/z_0}) + c_2 Y_0(j\sqrt{y_0 z_0} e^{x'/z_0}) \quad (32)$$

When the variable x' is changed back to x , using (29), the result is:

$$V(x) = c_1 J_0(j\sqrt{y_0 z_0} x) + c_2 Y_0(j\sqrt{y_0 z_0} x) \quad (33)$$

which agrees exactly with (24) and completes the demonstration.

OBTAINING SOLUTIONS FOR AN ARBITRARY TAPER

It is clear that taking any arbitrary tapered line in the standard domain does not necessarily mean that it can be solved in the transformed domain. Equations in "Hill" form do not have general solutions. Only a handful of special solutions exist, although recently developed construction techniques have now demonstrated that there are several larger classes of solutions. Still, for a given taper geometry with specified $Z(x)$ and $Y(x)$ it is not usual (in fact it may be quite rare) that a solution may be obtained by transforming the telegrapher's equation into the "Hill" form. The most immediate benefit of using this transformation for obtaining solutions to the telegrapher's equation is to find a method for using the known solutions to the "Hill" type equation and seeing what sorts of tapers can be solved.

Toward this goal, let us generalize the form of the telegrapher's equation as it appears in the transformed domain. We begin by defining:

$$Z(x) = z_0 f(x) \quad (34)$$

and $Y(x) = y_0 g(x)$ (35)

The telegrapher's equation is then written:

$$\frac{d^2 V}{dx^2} - \frac{1}{z_0 f(x)} \frac{dz_0 f(x)}{dx} \frac{dV}{dx} - z_0 f(x) y_0 g(x) V = 0 \quad (36)$$

To get this into the proper form (i.e., to get the signs correct), the z_0 term is inverted:

$$\frac{d^2 V}{dx^2} + z_0 f(x) \frac{d}{dx} \left(\frac{1}{f(x)} \right) \frac{dV}{dx} - \frac{y_0 g(x)}{z_0 f(x)} V = 0 \quad (37)$$

Again, recalling the procedure outlined above, the transformed equation is written:

$$\frac{d^2 V'}{dx'^2} - \frac{y_0 g(x)}{z_0 f(x)} V' = 0 \quad (38)$$

To complete the transformation, the variable, x , in the second term must be changed.

This is accomplished by applying the definition from (19):

$$x' = \int z_0 f(x) dx \quad (39)$$

the transformed equation becomes:

$$\frac{d^2 V'}{dx'^2} - \frac{y_0 g(x')}{z_0 f(x')} V' = 0 \quad (40)$$

The telegrapher's equation when transformed into a "Hill" type equation will always be of this form. Therefore, given a "Hill" type equation with a known solution, one can always find a tapered transmission line profile which can be described by the known solution. The process is simply to define the function $f(x)$ and apply this definition to equation (39), thus giving the relationship between the independent variables in each domain. This relationship can then be used to solve for the $g(x)$ profile term and to rewrite the solution $V'(x')$ in the standard domain as $V(x)$.

An example will show how this procedure works. In general, if the non-derivative term is written:

$$F(x') = -\frac{y_0 g(x')}{z_0 f(x')} \quad (41)$$

and the definition for $x'(x)$, equation (39), is applied, the $g(x)$ profile term can be written in terms of the given $F(x')$ and the arbitrarily selected $f(x)$, as:

$$g(x) = g(x'(x)) = -\frac{z_0}{y_0} F\left(\int z_0 f(x) dx\right) f\left(\int z_0 f(x) dx\right) \quad (42)$$

If a simple taper profile such as $f(x)=x$ is chosen and $F(x')$ is selected from a "Hill" type equation with a known solution such as Casperson's [32] :

$$F(x') = \frac{\gamma^2 G \cos(\gamma x')}{1 + G \cos(\gamma x')} \quad (43)$$

the resulting $g(x)$ is found to be:

$$g(x) = -\left(\frac{z_0^2 x^2}{2 y_0}\right) \frac{\gamma^2 G \cos\left(\gamma \frac{z_0 x^2}{2}\right)}{1 + G \cos\left(\gamma \frac{z_0 x^2}{2}\right)} \quad (44)$$

The solution for this line is:

$$V(x) = c_1 \left(\frac{1 + G \cos\left(\gamma \frac{z_0 x^2}{2}\right)}{1 + G} \right) + c_2 \left(\frac{1 + G \cos\left(\gamma \frac{z_0 x^2}{2}\right)}{1 + G} \right) \left[\frac{1}{\gamma(1 - G^2)} \right] \\ \times \left\{ \frac{G \sin\left(\gamma \frac{z_0 x^2}{2}\right)}{1 + G \cos\left(\gamma \frac{z_0 x^2}{2}\right)} - \frac{2}{(1 - G^2)^{\frac{1}{2}}} \tan^{-1} \left[\frac{(1 - G^2)^{\frac{1}{2}}}{1 + G} \tan\left(\gamma \frac{z_0 x^2}{2}\right) \right] \right\} \quad (45)$$

It is important to note that when using this procedure, the $f(x)$ profile function must define a realizable taper geometry. Similarly, the resulting $g(x)$ must be checked so that it, too, defines a geometry which can actually be constructed. So, while (44) is a cumbersome expression, it is a realizable taper, since if G is properly defined, the

function will never change sign and will never equal zero, except at the origin.

Simple "starter" profiles such as $f(x)=kx$ or $f(x)=k$, where k is some constant, can be used to develop solvable transmission lines from any of the known solutions for "Hill" type equations. Particularly in the cases where the $F(x)$ term consists of trigonometric functions, these solutions obtained with the method above are new. In the other cases, the novelty of the solution will depend on the selection of the constant terms.

The known closed-form solutions to the basic "Hill" type equation mentioned at the beginning of this chapter have been indexed by Tovar [30]. This list is given in Table II.

Taper profiles derived in this fashion are difficult to construct, but certainly realizable. In all cases where the series and shunt profiles, $f(x)$ and $g(x)$, are not proportional, one must vary the material constants of the dielectric material to obtain the desired taper. Techniques such as this were used for the tapering of some of the early transatlantic cables [7]. Because of the difficulty in controlling the materials involved, construction of this type of line was not attempted for this project.

OBTAINING SOLUTIONS FOR A RECIPROCAL LINE

There is a major class of realizable taper geometries for which the profile functions, $f(x)$ and $g(x)$ are reciprocal up to a constant. The most familiar of these are the coaxial line and the two-wire (twin-lead) line whose series and shunt parameters are shown here in Table III. The reciprocal relation is also approximately valid for microstrip lines.

This reciprocal relationship is valid only for high frequency TEM mode propagation on a lossless line. It is assumed that the series resistance (R) and shunt conductance (G) terms are very small and that external inductance of each conductor is

TABLE II
CLOSED FORM SOLUTIONS TO "HILL" TYPE EQUATIONS

Non-constant coefficient term	Solution Form	Reference
$F(1-\gamma x')^2$	Trigonometric functions	Casperson, [32]
$F(1-\gamma x'/2)^4$	Trigonometric functions	Casperson, [32]
$\frac{F}{(1+G \cos \gamma x')^4} + \frac{\gamma^2 G \cos \gamma x'}{1+G \cos \gamma x'}$	Trigonometric functions	Casperson, [31]
$\frac{g_0}{1-(x'/L)^2}$	Trigonometric functions	Gomez-Reino and Linares, [33]
$\frac{g_0}{1-(x'/L)}$	Trigonometric functions	Gomez-Reino and Linares, [33]
$1 + \frac{F+G^2-1}{[1+G \cos(2x')]^2}$	Trigonometric Functions	Wu and Shih, [34]
$(V^2 \operatorname{sech}(x'/a) - B^2)/a^2$	Trigonometric functions	Love and Ghatak, [49]
$\lambda^2 - \frac{v^2 - 1/4}{x'^2}$	Bessel functions	Abramowitz and Stegun, [43]
$\frac{\lambda^2}{4x'} - \frac{v^2 - 1}{4x'^2}$	Bessel functions	Abramowitz and Stegun, [43]
$\lambda^2 x'^{p-2}$	Bessel functions	Abramowitz and Stegun, [43]
$\lambda^2 e^{2x'} - v^2$	Bessel functions	Abramowitz and Stegun, [43]
$F(1+2\gamma x')$	Airy functions	Casperson, [32]
$-1/4 + \kappa/x' + (1/4 - \mu^2)/x'^2$	Whittaker functions	Abramowitz and Stegun, [43]
$ax'^2 + bx' + c$	Hypergeometric functions	Abramowitz and Stegun, [43]
$a - 2q \cos(2x')$	Mathieu functions	Abramowitz and Stegun, [43]
$\frac{1}{(a+bx'^2)^2}$	Trigonometric functions	Tovar, [30]

TABLE III
PARAMETER FORMULAE FOR COAXIAL AND TWO-WIRE LINES

Parameter	Coaxial Line	Two-wire Line
Series Inductance per meter	$\frac{\mu}{2\pi} \ln(r_o / r_i)$	$\frac{\mu}{\pi} \cosh^{-1}(s / d)$
Shunt Capacitance per meter	$\frac{2\pi\epsilon}{\ln(r_o / r_i)}$	$\frac{\pi\epsilon}{\cosh^{-1}(s / d)}$
r_o = outer diameter of the inside conductor r_i = inner diameter of the outside conductor d = conductor diameter s = center-to-center spacing between conductors ϵ = permittivity of the dielectric μ = permeability of the dielectric		

much larger than its internal inductance, all of which happens at higher (greater than 1 MHz) frequencies. The frequency of operation must be low enough that higher order modes do not propagate and that skin effect does not introduce serious losses.

Depending on conductor geometry, spacing, and dielectric composition, a couple of gigahertz might be considered an upper limit. One should consult Chipman's handbook (1968)[44] or Grives' work on high frequency lines (1970) [45] for a further discussion of the validity of the approximations made above.

Given the above assumptions, let us work toward a procedure for taking a solution to a "Hill" type equation and seeing what type of transmission line profile might be solved. Right away, since we are now working with the restricted case in which $f(x) = 1/g(x)$, the general transformed transmission line equation (40) can be simplified to:

$$\frac{d^2 V'}{dx'^2} - \frac{y_0}{z_0 f(x')^2} V' = 0 \quad (46)$$

To solve for $f(x)$ and therefore $g(x)$, one must find the relationship between the independent variable of the "Hill" type equation and the independent variable of the standard equation. The coefficient of the non-derivative term of the given "Hill" type equation is of the form $A(x')B(x')$, which can be written as:

$$F(x') = -y_0 z_0 A(x')^2 \quad (47)$$

since $-y_0 z_0 A(x') = B(x)$ when the reciprocal condition is applied. This is convenient, since the definition of $A(x)$ (and therefore, of $A(x')$), equation (14), is written in terms of $f(x)$, the desired taper profile. Therefore, solving (47) for $A(x')$, we get:

$$A(x')^2 = -\frac{F(x')}{z_0 y_0} \quad (48)$$

Recalling the definition of $x'(x)$ and rewriting this equation in terms of x , we get:

$$A(x)^2 = \frac{1}{(z_0 f(x))^2} = \frac{F(\int f(x) dx)}{z_0 y_0} \quad (49)$$

This is the equation which must be solved to find the taper profile $f(x)$ and therefore its reciprocal, $g(x)$. The validity of this procedure can be checked by applying it to the earlier example of the linear taper studied by Heaviside, where the profile was defined by:

$$Z(x) = z_0 f(x) = \frac{z_0}{x} \quad (50)$$

The non-derivative term in this case, derived earlier in equation (30), is:

$$F(x') = -\frac{y_0 e^{2x'/z_0}}{z_0} \quad (51)$$

Now, apply the basic formula developed above (49), and insert the expression for $F(x')$:

$$-\frac{1}{-z_0 y_0} \frac{y_0 e^{2x'(x)/z_0}}{z_0} = -\frac{e^{2z_0 \int f(x) dx / z_0}}{z_0^2} = \frac{1}{(z_0 f(x))^2} \quad (52)$$

This is now solved for the profile term, $f(x)$.

$$e^{\int f(x) dx} = \frac{1}{f(x)} \quad (53)$$

$$\int f(x) dx = -\ln(f(x)) \quad (54)$$

$$f(x) = -\frac{1}{f(x)} \frac{df}{dx} \quad (55)$$

$$\frac{df}{dx} + f(x)^2 = 0 \quad (56)$$

This has the solution:

$$f(x) = \frac{1}{x + c} \quad (57)$$

which is the original taper profile (50).

The method just described has been applied to a number of the known solutions for "Hill" type equations given in Table II. Table IV presents the results of these calculations. The linear, exponential, power (squared) taper, and inverse cube root profiles have solutions found by other methods, as detailed in the second chapter. The sinusoidal is a new configuration which has not been studied up to this time.

Many constructed solutions for "Hill" type equations, found in the literature, have the non-constant coefficients written in terms of trigonometric functions. These cases are very difficult to deal with when trying to find a reciprocal line which they can solve. Future research might look more carefully at these cases. In addition, one might wish to use construction techniques to find "Hill" type equations with coefficient functions not

TABLE IV
RECIPROCAL LINE PROFILES FOR "HILL" TYPE EQUATIONS WITH KNOWN SOLUTIONS

Hill Equation Coefficient	Profile Term in Standard Form	Taper Profile Type	Solution of Hill-type Equation (in terms of x')
$F / (1 - \gamma x')^2$	$f(x) = e^{-\gamma x' \sqrt{-F/z_0 \gamma_0}}$	Exponential	$V'(x') = c_1(1 - \gamma x') \cos\left[-\left(F - \frac{\gamma^2}{4}\right)^{\frac{1}{2}} \gamma^{-1} \ln(1 - \gamma x')\right] \\ + c_2(1 - \gamma x') \sin\left[-\left(F - \frac{\gamma^2}{4}\right)^{\frac{1}{2}} \gamma^{-1} \ln(1 - \gamma x')\right]$
$F / (1 - \gamma x' / 2)^4$	$f(x) = \frac{4 \cdot \sqrt{-F z_0 / \gamma_0}}{(\gamma z_0 x')^2}$	Power (squared)	$V'(x') = c_1\left(1 - \frac{\gamma x'}{2}\right) \cos\left[F^{\frac{1}{2}}\left(1 - \frac{\gamma x'}{2}\right)^{-1} x'\right] \\ + c_2\left(1 - \frac{\gamma x'}{2}\right) \cos\left[F^{\frac{1}{2}}\left(1 - \frac{\gamma x'}{2}\right)^{-1} x'\right]$
$\frac{g_0}{1 - (x' / L)^2}$	$f(x) = c_1 e^{\sqrt{4Q}x} + c_2 e^{-\sqrt{4Q}x}$ $Q = \frac{\gamma_0 z_0}{g_0 L^2}$	Sinusoidal (exponential)	$V'(x') = c_1 \left\{ \frac{L}{b} \cos\left(1 - \frac{x'}{L}\right) \sin\left[b \ln\left(\sec\left(1 - \frac{x'}{L}\right) + \tan\left(1 - \frac{x'}{L}\right)\right)\right] \right\} \\ + c_2 \left\{ \cos\left(1 - \frac{x'}{L}\right) \cos\left[b \ln\left(\sec\left(1 - \frac{x'}{L}\right) + \tan\left(1 - \frac{x'}{L}\right)\right)\right] \right\}$ <p style="text-align: center;">where $b^2 = (g_0 L)^2 - 1$</p>

TABLE IV

RECIPROCAL LINE PROFILES FOR HILL-TYPE EQUATIONS WITH KNOWN SOLUTIONS
(continued)

$\frac{g_0}{1-(x'/L)}$	$f(x) = \frac{y_0}{2g_0L} x$	Linear	$V'(x') = c_1 \left[\frac{-L}{b} \left(1 - \frac{x'}{L}\right)^{\frac{1}{2}} \sin \left(b \ln \left(1 - \frac{x'}{L}\right)\right) \right] + c_2 \left[\left(1 - \frac{x'}{L}\right)^{\frac{1}{2}} \cos \left(b \ln \left(1 - \frac{x'}{L}\right)\right) \right] - \left(\frac{1}{2b} \right) \sin \left(b \ln \left(1 - \frac{x'}{L}\right)\right) \}$ <p style="text-align: center;">where $b^2 = (g_0L)^2 - \frac{1}{4}$</p>
$F(1+\gamma x')$	$f(x) = \sqrt[3]{\frac{-y_0}{3z_0^2 F \gamma x}}$	Cube root	$V'(x') = c_1 Ai \left(- \left[(2\gamma)^{\frac{2}{3}} F^{\frac{1}{3}} + (2\gamma F)^{\frac{1}{3}} x' \right] \right) + c_2 Bi \left(- \left[(2\gamma)^{\frac{2}{3}} F^{\frac{1}{3}} + (2\gamma F)^{\frac{1}{3}} x' \right] \right)$ <p style="text-align: center;">where Ai and Bi are the Airy functions</p>
$\lambda^2 e^{2x'} - v^2$	$f(x) = z_0 x$ valid only for $v=0$	Linear	$V'(x') = c_1 J_0(\lambda e^{x'}) + c_2 Y_0(\lambda e^{x'})$
$\lambda^2 x'^{p-2}$	$f(x) = e^{kx} \quad (p=0)$ $f(x) = \frac{1}{x} \quad (p=1)$ (profiles link to choice of p)	Linear, exponential, power	$V'(x') = \sqrt{x'} c_1 J_{1/p}(2\lambda x'^{p/2}) + \sqrt{x'} c_2 Y_{1/p}(2\lambda x'^{p/2})$

couched in terms of elementary functions. This study could add to the number of solvable tapered lines.

SUMMARY

A technique for transforming second-order differential equations in self-adjoint form into second-order differential equations which do not have a first derivative has been described and applied to the transmission line equations. Solutions which have been obtained in laser and optics studies can therefore be applied to radio frequency transmission lines. An example was presented, which showed that, given an arbitrary series impedance profile term, $f(x)$, a shunt conductance profile function, $g(x)$, can be found, given a "Hill" type equation with a known solution. This procedure enables one to find closed form solutions to a great number of new transmission line profiles that have never been studied before.

Finally, a procedure for finding taper profiles for reciprocal lines, i.e., lines for which the shunt and series profiles are reciprocal up to a constant, based on "Hill" type equations with a known solutions, has been developed and presented. Solutions which may be obtained in this way are presented in tabular form.

CHAPTER IV

DERIVATION OF THE TRANSMISSION LINE EQUATIONS

INTRODUCTION

The application of the length transformation to the solution of the transmission line equation has been demonstrated. It has been shown that the transformation opens up the possibility of analyzing the behavior of lines with tapers which have not yet been thoroughly studied. The purpose of this section is to investigate how well the transmission line equations, and therefore the solutions which have been derived by the method of the third chapter, model the actual performance of a tapered transmission line.

The derivation of the transmission line equations is discussed first. The equations can be developed from an argument based on circuit theory or from one based directly on electromagnetic theory. Both derivations are presented here. The validity of the lumped circuit parameters of the transmission line equations (R , L , G , and C) is also discussed. Finally, implications of reflections internal to a tapered line section and higher-order mode propagation are presented. From all this information, a clearer idea of how well the transmission line equations model the behavior of a tapered line can be gotten.

DEVELOPMENT OF THE EQUATIONS FROM CIRCUIT THEORY

To begin with, some discussion of the derivation of the transmission line equations is in order. As mentioned in the historical review in the first chapter, the

earliest transmission line models were based on circuit theory. A two conductor line right off, has the look and feel of a long capacitor. Similarly, one would suspect that each of the conductors contains a resistive component. Again, applying circuit theory, knowing that wire conductors contain an inductance, the addition of a series inductance term seems in order. And, knowing that no dielectric is perfectly insulating, a shunt conductance component might be added. In short, the lumped parameter model, with its series inductance and resistance and its shunt capacitance and conductance, falls out fairly easily from our knowledge of how conductors and dielectrics work when excited by voltages or currents.

Once this model is accepted, a bit of calculus can be coupled with a little more circuit theory to derive the transmission line equations. If the circuit elements are reduced to per-unit-length form and a voltage is applied to one port, the change in voltage over a small incremental distance along the line is the current flowing in the conductor times the series resistance plus the rate of change of the current times the series inductance:

$$-\frac{\partial v}{\partial x} \Delta x = (R \cdot \Delta x)i + (L \cdot \Delta x) \frac{\partial i}{\partial x} \quad (1)$$

Similarly, the difference in current between the input and output ports is the sum of the current caused by the voltage v across the shunt conductance and the displacement current through the capacitance caused by the rate of change of the voltage:

$$-\frac{\partial i}{\partial x} \Delta x = (G \cdot \Delta x)v + (C \cdot \Delta x) \frac{\partial v}{\partial x} \quad (2)$$

These expressions can become partial differential equations if the Δx terms are factored out:

$$-\frac{\partial v}{\partial x} = Ri + L \frac{\partial i}{\partial x} \quad (3)$$

$$-\frac{\partial i}{\partial x} = Gv + C \frac{\partial v}{\partial x} \quad (4)$$

If the current and voltage are restricted to be sinusoidal, they can be represented as

$$v = Ve^{(j\omega t + \theta)} \quad (5)$$

$$i = Ie^{(j\omega t + \phi)} \quad (6)$$

If these are substituted into (3) and (4), the time derivatives go away. The resulting expressions are the familiar pair of transmission line equations in the frequency domain

$$\frac{dV}{dx} = -(R + j\omega L)I \quad (7)$$

$$\frac{dI}{dx} = -(G + j\omega C)V \quad (8)$$

Deriving the transmission line equations from circuit theory and the physical structures which constitute various circuit elements works quite well. It predicts very accurately, the transient and steady state responses of a uniform line. The success of efforts to apply tapered lines as broadband terminations and impedance matching devices based on this model (as can be seen from the references in the first chapter), indicates that it can be extended, with appropriate caution, into the realm of nonuniform lines. However, studying a two conductor structure from an electromagnetic point of view gets a little closer to the fundamental basis of these equations.

DEVELOPMENT OF THE EQUATIONS FROM MAXWELL'S EQUATIONS

The starting point for this discussion is Maxwell's equations: the four equations and accompanying constitutive relationships which are the foundation of classical electromagnetic theory. Although there could be some argument as to how fundamental Maxwell's equations really are, for the purposes of this paper, they are considered given. They are listed below in differential point form, since this rendering is most useful in

showing the link between them and the transmission line equations.

$$\nabla \times \mathbf{H} = \left(\frac{\partial \mathbf{D}}{\partial t} + \mathbf{J} \right) \quad (9)$$

$$\nabla \times \mathbf{E} = - \frac{\partial \mathbf{B}}{\partial t} \quad (10)$$

$$\nabla \cdot \mathbf{D} = \rho \quad (11)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (12)$$

The constitutive relations are written as:

$$\mathbf{D} = \epsilon \mathbf{E} \quad (13)$$

$$\mathbf{B} = \mu \mathbf{H} \quad (14)$$

$$\mathbf{J} = \sigma \mathbf{E} \quad (15)$$

where μ , σ , and ϵ are defined as the usual permeability, conductivity and permittivity in SI units.

Maxwell's equations show that electromagnetic energy can propagate along various guiding structures in a variety of modes. Each mode is based on the geometry of the waveguide and on the relationship between the electric and magnetic fields traveling along the guide. While many modes can propagate along a two conductor line, the transmission line equations describe only the most fundamental mode, the "transverse electromagnetic" or "TEM" mode. In this mode, the electric and magnetic fields have components in the transverse direction, the direction normal to the axis of propagation. The components of the fields in the direction of propagation are zero. As is noted in every discussion of guided electromagnetic waves, the TEM mode is not possible in a single conductor guide.

Any multi-conductor structures could be used in the derivation of the transmission line equations. Because it is so common, a general coaxial structure with two conductors is considered here. However, the flow of the argument would be the

same for any other line geometry or with a greater number of conductors.

The derivation, adapted from Adler, Chu, and Fano (1960) [46], begins with the equations for Faraday's law and Ampere's law:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\mu \frac{\partial \mathbf{H}}{\partial t} \quad (16)$$

$$\nabla \times \mathbf{H} = \left(\frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}\right) = \sigma \mathbf{E} + \epsilon \frac{\partial \mathbf{E}}{\partial t} \quad (17)$$

The assumption is made that the conductors are perfect, which leads to boundary conditions on the electric and magnetic fields. The component of $\mathbf{E}$ tangential to the conductors and the component of $\mathbf{H}$ normal to the conductors are zero. Recall, also, that the propagation mode considered here is TEM, which implies that the z-axis components of $\mathbf{E}$ and $\mathbf{H}$ are also zero. Using this last condition, Faraday's and Ampere's laws can be rewritten as:

$$\frac{\partial}{\partial z}(\mathbf{a}_z \times \mathbf{E}_T) = -\mu \frac{\partial \mathbf{H}_T}{\partial t} \quad (18)$$

$$\frac{\partial}{\partial z}(\mathbf{a}_z \times \mathbf{H}_T) = \sigma \mathbf{E}_T + \epsilon \frac{\partial \mathbf{E}_T}{\partial t} \quad (19)$$

where the T subscripts indicate that the field vectors have only transverse components and $\mathbf{a}_z$ represents the unit vector in the z-direction. The electric potential between the two conductors is defined as the line integral of the electric field from the surface of one conductor to the surface of the other:

$$V(z, t) = \int_{(1)}^{(2)} \mathbf{E}_t \cdot d\mathbf{s} \quad (20)$$

where any continuous path between the inner and outer conductor can be selected. The charge per unit length on each conductor can be written as:

$$q_1(z, t) = \epsilon \oint_{C_1} \mathbf{n}_1 \cdot \mathbf{E}_T \cdot d\mathbf{l} \quad (21)$$

where $d\mathbf{l}$ is an infinitesimal arc along the surface of one of the conductors and $\mathbf{n}_1$ is the unit vector normal to the surface of conductor 1. It can be shown that the charge on the other conductor is equal and opposite, by applying Gauss' law and assuming that the region between the conductors is source-free.

In circuit theory, capacitance is defined as charge per unit volt. Using the formulae obtained already, capacitance can be written as:

$$C \equiv \frac{q_1}{V} = \frac{\epsilon \oint_{C_1} \mathbf{n}_1 \cdot \mathbf{E}_T \cdot d\mathbf{l}}{\int_{(1)}^{(2)} \mathbf{E}_T \cdot d\mathbf{s}} \quad (22)$$

For the inner conductor considered here, the current can be defined as:

$$I(z, t) = \oint_{C_1} \mathbf{H}_T \cdot d\mathbf{l} \quad (23)$$

It can also be shown that the current on the inner and outer conductors is equal and opposite by applying Stokes' theorem and the given boundary conditions. The flux linkage per unit length can be defined as:

$$\Lambda(z, t) = \int_{(1)}^{(2)} \mathbf{H}_T \cdot (\mathbf{a}_z \times d\mathbf{s}) \quad (24)$$

Referring once again to circuit theory, inductance can be defined as flux linkage per unit current. Using the formulae obtained here, inductance per unit length can be written as

$$L \equiv \frac{\Lambda}{I_1} = \frac{\int_{(1)}^{(2)} \mathbf{H}_T \cdot (\mathbf{a}_z \times d\mathbf{s})}{\oint_{C_1} \mathbf{H}_T \cdot d\mathbf{l}} \quad (25)$$

To obtain the transmission line differential equations, the above definitions for voltage, current, inductance, and capacitance, given in terms of electric and magnetic fields, are applied to equation (18), Faraday's law. First, it is rewritten as:

$$\frac{\partial \mathbf{E}_T}{\partial z} = \mu \frac{\partial}{\partial t} (\mathbf{a}_z \times \mathbf{H}_T). \quad (26)$$

Integrating this expression from the surface of one conductor to the other (applying the definition of voltage), and manipulating the cross product expression gives:

$$\frac{\partial V}{\partial z} = -\frac{\partial}{\partial t} [\mu \int_{(1)}^{(2)} (\mathbf{a}_z \times \mathbf{H}_T) \cdot d\mathbf{s}] \quad (27)$$

But the term in brackets is the flux linkage term, $\Lambda = LI$ defined above (24). In terms of inductance and current, this becomes:

$$\frac{\partial V}{\partial t} = -L \frac{\partial I}{\partial t} \quad (28)$$

which is the basic transmission line equation for voltage.

By a similar argument, equation (19), the expression for Ampere's law, can be rewritten as:

$$\frac{\partial}{\partial z} \int_{C_1}^{C_2} (\mathbf{a}_z \times \mathbf{H}_T) \cdot d\mathbf{s} = \sigma V + \epsilon \frac{\partial V}{\partial t} \quad (29)$$

and, with some manipulation, becomes:

$$-\frac{\partial I}{\partial z} = \frac{\sigma \mu}{L} V + \frac{\mu \epsilon}{L} \frac{\partial V}{\partial t} \quad (30)$$

Conductance can be shown to be:

$$G \equiv \frac{\sigma \oint_{C_1} \mathbf{E}_T \cdot \mathbf{n}_1 d\mathbf{l}}{\int_{(1)}^{(2)} \mathbf{E}_T \cdot d\mathbf{s}} = \left(\frac{\sigma}{\epsilon}\right) C \quad (31)$$

Combining this with the fact that $LC = \epsilon\mu$, which can be derived from the equation for voltage (26) and the definition of capacitance, equation (30) can be rewritten as:

$$\frac{\partial I}{\partial z} = -(GV + C \frac{\partial V}{\partial t}) \quad (32)$$

the basic transmission line equation for current. Thus, the transmission line equations can be derived from basic electromagnetic theory, applying the definitions of inductance and capacitance.

Two further comments need to be made. First, the development above assumes a lossless line. If the conductors have some loss (finite conductivity), then a resistance term must be added to the voltage equation (28). Secondly, if sinusoidal waveforms are applied to the line, the time dependence can be taken away, leaving a somewhat simpler pair of equations:

$$\frac{dV}{dz} = -(R + j\omega L) \quad (33)$$

$$\frac{dI}{dz} = -(G + j\omega C) \quad (34)$$

The transmission line equations appear mainly in this form in the literature. The same convention has been used in this paper, as well.

ACCURACY OF THE LUMPED PARAMETER MODEL

The accuracy of the transmission line equations depends on how well the lumped parameter model conforms to a given line structure excited by a particular voltage and current. Certain precautions must be taken when trying to predict line performance. Some of these precautions deal directly with the model parameters. The two parameters which seem to require the most attention are the series resistance and inductance terms.

In even the highest quality line, conductors have some resistance. While this resistance may seem negligible in small electronics lab set-up's, longer runs of cable,

such as used in local area networks or cable TV distribution networks exhibit significant loss due to resistance. As an example, RG59/U, a 75 Ohm low-loss cable used in TV studios, reduces signal power by 1.1 dB every 100 feet for signals of 10 MHz. The resistance factor is compounded by skin effect, which is not dealt with directly in the lumped parameter model. Skin effect, of course, causes the higher frequency components of a signal to be attenuated more than the lower frequency ones. Again, considering RG59/U cable, a signal at 100 MHz sees 3.3 dB loss per hundred feet of cable, while a signal of 1 GHz will have a loss of 11.5 dB over the same distance. This has the effect of "drooping" the leading edge of square wave signals, for example. If the effect is pronounced enough, problems with detection at the receiving end could result. In the case of a timing signal, timing inaccuracies ("jitter") can result due to the longer risetime of the received signal.

Another problem factor is the inductance parameter. Throughout most of the frequency range of a transmission line, the inductance can be calculated from the line geometry and dimensions and conductor permeability using standard formulae given in most texts. However, the calculated inductance is the inductance external to the conductor only. The inductance internal to the conductor is not considered by these first-order formulae. At higher frequencies, it turns out that the external inductance is by far the dominant factor. But at lower frequencies, the internal inductance is an appreciable part of the total inductance and must be considered.

Much research has been done on how skin effect and the non linearity of the inductance parameter can be adjusted for given conditions. High frequency, mid-frequency, and low frequency formulae have been derived and are summarized by Chipman (1968) [44]. He also presents more complex expressions which give additional accuracy if the general formulae are insufficient.

Thus far, the limitations of the lumped parameter transmission line equations

discussed apply to all classes of lines, uniform and nonuniform. Still to be considered is what problems arise with the accuracy of the model when the line is not uniform. Two areas of concern arise: inaccuracies due to reflections and non-TEM mode wave propagation, both of which will be discussed here.

As an electromagnetic wave propagates along a guiding structure, reflections of some of the energy will occur whenever a change in the structure is encountered. A change in physical dimensions is one case which could occur. A change in the conducting medium will also cause reflections. By definition, a tapered transmission line is constantly changing its physical parameters. Therefore, some reflection will always occur. As was seen in the second chapter, a great deal of effort, particularly in the 1940's and 1950's, went into developing taper geometries which minimized reflections. It was found that smoother profiles, such as a hyperbolic tangent, provide significantly better performance than, say, a linear taper, when designing a tapered matching section. This is discussed, for instance, in the article by Scott (1953) [47]. It is important to note that while the results presented in the articles on minimizing reflections are ultimately based on the lumped circuit transmission line model, simply solving the transmission line equations for voltage or current will not give any indication of the reflections which might occur within a tapered line. The solutions for voltage and current on a tapered line, obtained from:

$$\frac{d^2V}{dx^2} - \frac{1}{Z} \frac{dZ}{dx} \frac{dV}{dx} - ZYV = 0 \quad (35)$$

assume that there are no reflections due to the changing characteristic impedance. When designing a tapered line or analyzing the performance of such a line, the additional work of checking for reflections must be done. The solution techniques presented in the articles discussed in the second chapter provide a reasonable starting point for such a check. Most, however, were accomplished graphically. An opportunity exists here for

application of computerized numerical techniques to the problem of predicting reflections for a given taper. The literature, to date, contains no such effort. The literature does suggest a rule of thumb for designing a taper. Based primarily on the graphical data presented by Klopfenstein [18] and Scott [47], a taper which transitions between lines having a 3:1 characteristic impedance ratio would have an overall reflection coefficient of $< 5\%$ if the tapered section is > 5 wavelengths at the lowest frequency of interest. Some tapers have better performance than others. The hyperbolic line described by Scott has less than 5% reflection at two wavelengths, but does not perform well for shorter lengths. The Chebyshev line discussed by Klopfenstein has less than 5% reflection at half a wavelength, but has characteristic Chebyshev "ripples" at half-wavelength intervals which contribute reflections greater than 5% up to taper lengths of 5 wavelengths.

As has been indicated, the validity of the transmission line equations is based on the assumption that the mode of wave propagation is transverse electromagnetic (TEM). There are definite conditions which must prevail if one wants assurance that no higher order modes exist. For uniform coaxial lines, formulae have been worked out to determine cut-off frequencies of various modes. Ramo, Whinnery, and VanDuzer (1965) [48] give an approximate formula for the cutoff frequency for the n^{th} order TE mode in a coaxial line:

$$\lambda_c \approx \frac{2\pi}{n} \left(\frac{r_o + r_i}{2} \right) \quad (36)$$

where r_o is the inside diameter of the outer conductor and r_i is the outside diameter of the inner conductor. This formula is derived from an approximate solution to the wave equation for a coaxial waveguide. Unfortunately in this case, the wave equation has been set up for a uniform line. One would hope that operating a tapered line at frequencies significantly below the lowest order non-TEM mode cut-off frequency for a uniform line

of the same maximum dimensions would assure TEM mode propagation exclusively. Future research might want to revisit this issue, developing a closed form or numerical solution to the wave equation for a tapered coaxial line to determine at what rate of taper higher order modes begin to propagate.

SUMMARY

In this chapter, the connection between circuit theory, electromagnetic theory and the transmission line equations has been reviewed and explained. In fact, with proper definitions applied, it has been demonstrated that the transmission line equations can be derived from Maxwell's equations. Still, caution must be exercised when applying the transmission line equations to a given line operating with a given signal. The resistance and inductance terms, in particular, need to be adjusted, depending on the frequency of the applied signal. In addition, one must take care that the taper is not so steep as to create large reflected signals. Finally, the line dimensions must be specified in such a way to assure that higher order (non-TEM) modes will not propagate. While some work has been done to predict when higher order modes propagate in a uniform line, the problem has not been solved for a tapered line. With some work, one could solve Maxwell's equations for a tapered structure and predict when higher-order modes will propagate, based on the rate of taper. This and further investigation of the problem of characterizing reflections present opportunities for future research.

CHAPTER V

EXPERIMENT WITH A TAPERED TRANSMISSION LINE

INTRODUCTION

A tapered transmission line was designed and built, using one of the taper profiles derived from a solution to the "Hill" type equation worked out in connection with fiber optics. This chapter gives details on how the line was designed: how its physical parameters were determined and how it is driven. Considerable work went into the measurement of the performance of the line. The development and verification of the measurement method are discussed. Finally, measurement results are presented and correlated with values predicted by the derived solution to the transmission line equation.

DESIGN OF THE LINE

A number of considerations went into the design of the transmission line used for this experiment. Two particular areas received special attention: 1) ease of fabrication and 2) ease of measurement. Based on these guidelines, a two-conductor "parallel" wire line geometry was chosen. First, from the various types of lines available, this line is easiest to build with an accurate taper -- easier than a microstrip line, which has the drawback of small dimensions, and certainly easier than a coaxial line. Use of standard brass rod stock assured high quality conductors with well-controlled diameter. Conductor spacing remained the only other critical design feature. This was easily controlled with thin acrylic spacers whose width was easily verified.

The "parallel" wire geometry makes several measurement options available, since both conductors as well as the space between and around them can be accessed by a variety of probe devices. Once again, the coaxial and microstrip lines are somewhat limited in this area.

Selection of the "parallel" wire line is not without its drawbacks. Since it is a balanced geometry, it must be driven by a signal source with a balanced output. Such devices are not readily available at the frequencies of interest (hundreds of megahertz). Therefore, a balun had to be designed to convert the single-ended 50 Ohm output of a standard signal generator to a balanced output with the appropriate impedance. The choice of this line geometry also introduces a greater chance for losses due to radiation, since it is not as well shielded as a coaxial line or even as a microstrip line. The taper profile chosen for the design of this line is the $\sqrt[3]{1/3x}$ profile which was obtained in chapter three from a solution to the "Hill" type equation worked out by Casperson [32]. From Table III, the general form of the profile is:

$$f(x) = \sqrt[3]{\frac{-y_0}{3z_0^2 F \gamma x}} \quad (1)$$

For simplicity of design, the constants from the original equation can be set as follows:

$$F = 1 \quad (2)$$

$$\text{and, } \gamma = \frac{-y_0}{z_0^2} \quad (3)$$

The taper profile thus becomes:

$$f(x) = \sqrt[3]{1/3x} \quad (4)$$

As noted in the third chapter, the parallel wire line (as well as the coaxial line) are physical forms in which the series impedance term and the shunt conductance term vary inversely with respect to each other by a constant. In the case of the line used for this

experiment, the series impedance term, $Z(x) = z_0 f(x) = z_0 \sqrt[3]{1/3x}$, where z_0 is a constant. The shunt conductance term is similarly, $Y(x) = y_0 / f(x) = y_0 / \sqrt[3]{1/3x}$. The solution to the transmission line equation in the transformed domain, from Table IV is:

$$V'(x') = c_1 Ai(-[(2\gamma)^{\frac{2}{3}} F^{\frac{1}{3}} + (2\gamma F)^{\frac{1}{3}} x']) + c_2 Bi(-[(2\gamma)^{\frac{2}{3}} F^{\frac{1}{3}} + (2\gamma F)^{\frac{1}{3}} x']) \quad (5)$$

where Ai and Bi are the Airy functions. This is transformed back to the standard domain by applying the definition of the x' variable introduced in the third chapter:

$$x' = \int z_0 f(x) dx = z_0 \int \sqrt[3]{1/3x} dx \quad (6)$$

For this experiment, the line was assumed to be lossless, which means that the series and shunt impedance terms in the transmission line equations simplify to $j\omega L(x)$ and $j\omega C(x)$. With both these assumptions the solution, in terms of voltage can be written:

$$V'(x') = c_1 Ai(-[(2\gamma)^{\frac{2}{3}} F^{\frac{1}{3}} + (2\gamma F)^{\frac{1}{3}} x']) + c_2 Bi(-[(2\gamma)^{\frac{2}{3}} F^{\frac{1}{3}} + (2\gamma F)^{\frac{1}{3}} x']) \quad (7)$$

To determine the values of z_0 and y_0 , the formulae for series inductance and shunt capacitance for parallel wire lines was applied. The overall initial conditions for the solution were calculated based on the assumption that the line was lossless. No loss of power implies that the product of voltage and current at the input be the same as that on the output. Since characteristic impedance is the ratio of voltage to current, $V_{out} = \sqrt{k} V_{in}$ where k is the ratio of the input characteristic impedance to the terminating characteristic impedance.

Some care was exercised in the selection of the range of the variable x , the distance variable. If the values for x are too small, the rate of taper becomes very steep, which can cause excessive reflection and could excite higher-order modes. On the other hand, if the values of x are generally large, the rate of taper approaches that of a uniform

line and it becomes impossible for the experiment to yield any information about the performance of a tapered transmission line. A design range was selected, roughly centered about $x = 1/3$, and appropriately scaled to make the line interesting over its eight foot length. A table showing the conductor spacing and characteristic impedance in 10 inch increments is given below.

TABLE V
LINE DIMENSIONS AND CHARACTERISTIC IMPEDANCES

Distance	Spacing	Characteristic Impedance
0"	0.20"	138 Ohms
10"	0.20"	138 Ohms
20"	0.21"	143 Ohms
30"	0.24"	148 Ohms
40"	0.27"	154 Ohms
50"	0.30"	162 Ohms
60"	0.35"	171 Ohms
70"	0.42"	183 Ohms
80"	0.55"	199 Ohms
90"	0.75"	222 Ohms
100"	1.45"	264 Ohms

The operating frequency was selected to be approximately 450 MHz. A couple of considerations drove this choice. It was desirable that the frequency be high in order that the taper be several wavelengths long. This allowed the rate of impedance change per wavelength to be small enough to keep reflections to a minimum. The rule of thumb

discussed in the fourth chapter -- that the characteristic impedance change by 2-to-1 in a minimum of five wavelengths -- was followed. Material availability suggested that the overall length of the line be kept to around nine feet. When all these factors were combined, a minimum operating frequency of about 350 MHz was determined.

An upper limit to the operating frequency was dictated by the available range of conductor spacing and the selection of a measurement device with a nominal 400 MHz bandwidth. To achieve a good range of characteristic impedance variation, conductor spacings had to run from about 0.200" to 1.30". In order to prevent higher-order mode propagation, another rule of thumb from the fourth chapter needed to be followed -- that the maximum conductor spacing not exceed 1/10th of a wavelength. With the effect of the acrylic spacing and support material taken into account, 1.30" is 1/10th of a wavelength at approximately 450 MHz. Since there was no advantage in selecting a lower frequency of operation, 450 MHz became the choice.

A ground plane was used as part of the line in order to assure that the signal be properly balanced. This was necessary, since the line is being driven by a single-ended generator. A television impedance matching transformer was used to provide the transition from the single-ended to the balanced line. However, this balun used did not contain the extra winding required to force the transmission line to operate in balanced mode. The addition of the ground plane provided a means to force balance by allowing each leg of the line to be terminated in half the line's terminal characteristic impedance.

The balun built for this experiment was a modified 75 Ohm - 300 Ohm impedance matching transformer used for home television. A resistor was put in parallel with the 300 Ohm side so that the effective output impedance became 140 Ohms at 450 MHz. This was verified by measuring return loss, S_{11} , on a network analyzer with the 75 Ohm side terminated. No modifications were made to the 75 Ohm side of the transformer. Network analyzer measurements also underscored an unfortunate side

effect of using this type of balun: narrow bandwidth. It was observed that the output impedance varied up to ± 5 Ohms over a band of about 30 MHz. It was important, therefore, to operate the line very close to 450 MHz to keep the impedance match with the input side of the transmission line as close as possible.

The terminal impedance of the line was calculated with the normal formula for the characteristic impedance of a "parallel wire" line with dimensions the same as those at the end of the tapered line. This calculation was verified experimentally by checking the termination performance with a differential TDR (time domain reflectometry) measurement, made with a Tektronix 11802 Oscilloscope with an SD-24 sampling head. Minor adjustments were made until the TDR display representing the termination was smooth at that point.

The line conductors were two brass rods, 0.155" in diameter, laid on a 2" x 1/4" strip of acrylic plastic material, as shown in Figures 1 and 2. The plastic strip was laid over a 3/4" thick strip of particle board, under which was a long strip of 2 1/2" x 0.25" brass, used for the ground plane. Spacers made of acrylic plastic material similar to that used for the rod support were used at various intervals to separate the rods the appropriate distance. Strips 1/4" square were placed along the entire outside length of the rods, to hold them firmly against the spacers. Measurements were taken with the TDR apparatus used above, to determine the effect of plastic spacers on the characteristic impedance of the line. Readings were taken with and without spacers. It was found that the change in characteristic impedance of the line increased approximately 2 Ohms at the location where a spacer was used. Worst case, this represents a bit less than a 2% change in the characteristic impedance and is probably less than the voltage measurement error.

The following diagrams show the experimental setup and a view of the transmission line from one end.

Figure 1. Experimental Setup.

Figure 2. Tapered Transmission Line Viewed from One End.

MEASUREMENT TECHNIQUES

The transmission line equation solution is given in terms of voltage (or current) as a function of distance along the line. Verifying the solution to this line meant developing a technique for measuring the potential between the conductors at locations on the transmission line. This turned out to be a challenging task. The methods used to make this measurement are discussed here.

A first attempt was made using oscilloscope probes connected directly to the line at the location of interest. Although a voltage measurement can clearly be made this way, there was some concern that having the probe contact the line might alter the characteristic impedance at that point. This suspicion was verified by connecting a TDR instrument to the line and observing the display of the characteristic impedance, while

contacting the line with an oscilloscope probe. It was found that touching the line with a probe resulted in a change in characteristic impedance exceeding 20%. Further tests were made with large value resistors in series with the probe to increase the input impedance. These additions made very little difference to this problem and led to the conclusion that this technique was not adequate to make the measurement.

A less invasive measurement technique was attempted which used a small coil, which was connected to the oscilloscope input channel with a short length of semi-rigid coaxial cable. The objective of this test was to determine if the coil could accurately measure the current on the line by sending a signal to the oscilloscope proportional to the strength of the magnetic field. A voltage signal was observed, but experiments indicated that the coil was picking up some of the electric field. Orienting the coil so that it should pick up the maximum magnetic field did not produce the correct result. Hence, it was concluded that another coupling mechanism, probably involving the electric field, was also at work

These results led to the investigation of measuring the electric field. Two types of devices were used as electric field probes, the first being a set oscilloscope probes. The second was a simple parallel plate device made of two small rectangular pieces of brass shim material, one soldered to the center conductor of an SMA coaxial connector and the second soldered to the ground lead of the same conductor. These were then connected to an oscilloscope with a pair of short sections of semi-rigid coaxial cable. The oscilloscope, with attached probing devices, was set up at a number of locations along the line. One probing device was located next to the first conductor, while the second was placed the same distance from the second conductor. In this way, a measurement of voltage was made.

Measurements made with a pair scope probes (positioned the same way as the brass probes) gave similar results to those made with the brass probes, although with

considerably more difficulty. The probes were quite sensitive to position, harder to hold in place, and, when handled by the user, gave erratic readings. This was likely due to the fact that scope probes are high impedance devices (10 MOhm inputs) and therefore much more responsive to environmental changes.

MEASUREMENT RESULTS

The following graph shows the results of the measurements taken versus the expected values.

Figure 3. Voltage Measurements of a Tapered Transmission Line.

These measurements were taken with the parallel plate capacitive probes and the pair of oscilloscope probes described above. The numbers above are normalized to show the best fit between the calculated and actual values.

As mentioned above, the measurement readings were difficult to make and extremely sensitive to probe position and the location of the operator's hands and arms! This could account for some of the mismatch between actual and calculated results. Similarly, there could be standing waves on the line that are excited only when the

driving circuitry was attached to the line. It is fairly certain, because of the TDR measurements made from the source end with the line terminated, that the line itself is reasonably non-reflective. Some mismatch between the driving circuitry and the line is a more likely cause of standing waves.

SUMMARY

A "parallel wire" transmission line, designed with a profile obtained from one of the known solutions to the "Hill" form equation, was built and measured. The design is described in detail. Special note is made of design rules which were followed to ensure that reflections and higher-order modes were kept to a minimum. Various measurement methods were discussed. The results from the best measurements were graphed and analyzed. The fact that the measurements were difficult and error-prone points to an opportunity to find a better way to make them. This author would enjoy the chance to refine techniques for making accurate non-invasive measurements on transmission lines of all kinds.

CHAPTER VI

CONCLUSION

A mathematical connection has been established between the beam parameter equation of fiber optics and the transmission line equations. Solutions found in one domain can now be applied to the other, and vice versa.

A procedure has been developed to take solutions to the "Hill" form equation and derive profiles of tapered radio frequency transmission lines for which the voltage and current equations can be solved. This procedure has been further refined to apply to the special case of reciprocal transmission lines, those for which the series impedance and shunt conductance are reciprocal, up to a constant. Thus, given a solution to the "Hill" form equation it is often possible to derive a reciprocal transmission line to which the solution applies.

A comprehensive review of the closed form solutions to the transmission line equation and the methods for getting those solutions has also been presented. This review is summarized in a table listing the major contributors in the search for solutions to the transmission line equation. A thorough search of the literature on this subject shows that no other such review has ever been done.

A parallel wire transmission line was built to specifications derived from a "Hill" form equation whose solution was discovered in an optics application. Measurements were taken and compared to the results calculated based on the solution to the transmission line equations. The correlation between actual and expected results was

fair, which offers an opportunity for further research into measurement techniques in this area.

The new solutions to the transmission line equations made available with the techniques and existing solutions presented in this paper offer possibilities for future study. Further investigation of the closed form solutions may yield information about useful properties of transmission lines with exotic taper profiles. Selective impedance matching, filtering and signal synthesis are some of the applications which come to mind.

Finally, there is the challenge of accurately determining the voltage between the conductors along a nonuniform transmission line. The difficulties experienced in making these measurements present an exciting opportunity to engineers in the test and measurement business. Good measurement methods could open the door to further studies of tapered lines and additional applications.

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