Pattern Recognition and ERP Waveform Analysis Using Wavelet Transform

Hong Qi
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AN ABSTRACT OF THE THESIS OF Hong Qi for the Master of Science in Electrical Engineering presented November 19, 1993.

Title: Pattern Recognition and ERP Waveform Analysis Using Wavelet Transform

APPROVED BY THE MEMBERS OF THE THESIS COMMITTEE:

[Signatures]

Fu Li, Chair
Andrew M. Fraser
Bradford R. Crain

Wavelet transform provides an alternative to the classical Short-Time Fourier Transform (STFT). In contrast to the STFT, which uses a single analysis window, the Wavelet Transform uses shorter windows at higher frequencies and longer windows at lower frequencies. For some particular wavelet functions, the local maxima of the wavelet transform correspond to the sharp variation points of the signal.

As an application, wavelet transform is introduced to the character recognition. Local maximum of wavelet transform is used as a local feature to describe character boundary. The wavelet method performs well in the presence of noise.
The maximum of wavelet transform is also an important feature for analyzing the properties of brain wave. In our study, we found the maximum of wavelet transform was related to the $P_{300}$ latency. It provides an easy and efficient way to measure $P_{300}$ latency.
PATTERN RECOGNITION AND ERP WAVEFORM ANALYSIS USING WAVELET TRANSFORM

by

HONG QI

A thesis submitted in partial fulfillment of the requirements for the degree of

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in
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CHAPTER I

INTRODUCTION

Singularities and irregular structures often carry the most important information in signals. Until recently, the Fourier transform was the main mathematical tool for analyzing singularities. The Fourier transform is global and provides a description of the overall regularity of signals, but it is not well adapted for finding the location and the spatial distribution of singularities. This was a major motivation for studying the wavelet transform in mathematics and in applied domains. By decomposing signals into elementary building blocks that are well localized both in space and frequency, the wavelet transform can detect sharp variation points [3].

The sharp variation points along an object boundary are rich in information content and are sufficient to characterize the shape of the objects. As an application, wavelet transform is introduced to the character recognition. Local maximum of wavelet transform is used as a local feature to describe character boundary. Wavelet method performs well in the presence of noise.

The maximum of wavelet transform is also an important feature for analyzing the properties of brain wave. In our study, we found the maximum of wavelet transform was related to the $P_{300}$ latency. It provides an easy and efficient way to describe the ERP waveform.

As a matter of fact, the wavelet theory covers quite a large area. It treats both the continuous and the discrete time cases. It provides general techniques that can be applied to many tasks in signal processing, and has many potential applications.
I.1 THESIS OUTLINE

This thesis is organized as follows:

Chapter I: General introduction.

Chapter II: We review the Short-Time Fourier Transform first, then describe a multiresolution analysis method—Wavelet Transform. Several types of wavelet transform are discussed. A particular class wavelet functions and a fast wavelet algorithm are presented.

Chapter III: As a local method, wavelet transform is introduced to character recognition. The local maximum of wavelet transform of boundary data is used as shape feature of character. Simulation results show that this wavelet method is robust in noisy environment.

Chapter IV: A general ERP waveform is presented. Local maximum of wavelet transform of ERP data is used to describe the ERP waveform. The experiment result shows that maximum of wavelet transform is related to $P_{300}$ latency.

I.2 NOTATION

$\mathbb{Z}$ denotes the set of integers.

$L^2$ denotes the Hilbert space of measurable, square-integrable one dimensional functions such that

$$\int_{-\infty}^{+\infty} |f(x)|^2 dx < +\infty$$

We denote the convolution of two functions $f(x) \in L^2$ and $g(x) \in L^2$ as

$$f(x) * g(x) = \int_{-\infty}^{+\infty} f(u) g(x-u) du$$

The Fourier transform of $f(x)$ is written by $\hat{f}(\omega)$ and is defined by

$$\hat{f}(\omega) = \int_{-\infty}^{+\infty} f(x) e^{-i\omega x} dx$$
The inner product of $f(x) \in L^2$ with $g(x) \in L^2$ is written by

$$<g(x), f(x)> = \int_{-\infty}^{\infty} f(x)g(x)dx$$

For any function $f(x)$, $f_s(x)$ denotes the dilation of $f(x)$ by the scale factor $s$

$$f_s(x) = \frac{1}{s} f\left(\frac{x}{s}\right)$$
CHAPTER II

WAVELET AND WAVELET TRANSFORM

II.1 INTRODUCTION

Wavelet Transform (WT) is a linear operation that decomposes a signal into components that appear at different scales. It provides an alternative to the classical Short-Time Fourier Transform (STFT). In contrast to the STFT, which uses a single analysis window, the Wavelet Transform uses shorter windows at higher frequencies and longer window at lower frequencies. The notion of scale in WT is introduced as an alternative to frequency, leading to a so-called time-scale representation.

There are several types of WT. For a continuous input signal, the time and scale parameters can be continuous, leading to the Continuous Wavelet Transform (CWT). The scale may as well be discrete, leading to a wavelet series expansion. One of the useful wavelet series expansion is called Dyadic Wavelet Transform, of which the scale varies only along a dyadic sequence. Finally, the wavelet transform can be defined for the discrete-time signal, leading to a Discrete Wavelet Transform.

For some particular wavelet functions, the local maxima of the wavelet transform correspond to the sharp variation points of the signal. Points of sharp variations are often among the most important features for analyzing the properties of signal and image.

In this chapter, we review the Short-Time Fourier Transform first, and then we describe an advanced method, which is Wavelet Transform. Several types of wavelet Transform will be discussed. A fast wavelet algorithm and a family of B-spline wavelets will also be introduced at the end of this chapter.
II.2 THE SHORT-TIME FOURIER TRANSFORM
---ANALYSIS WITH FIXED RESOLUTION

II.2.1 Fourier Transform

For a regular signal, the well-know Fourier Transform (FT) is defined as:

\[ \hat{f}(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx \]  

(1)

The coefficients define the notion of global frequency \(\omega\) in a signal. Analysis works well if the signal is composed of a few stationary components. However, any abrupt change in time in an irregular signal is spread out over the whole frequency axis in \(\hat{f}(\omega)\). Therefore, an analysis adapted to irregular signal requires more than the Fourier Transform.

The usual approach is to introduce time dependency in the Fourier analysis while preserving linearity. The idea is to introduce a "local frequency" parameter (local in time) so that "local" Fourier Transform looks at the signal through a window over which the signal is approximately stationary.

II.2.2 Short-Time Fourier Transform

Consider a signal \(f(x)\), and assume it is regular when seen through a window \(g(x)\) of limited extent, centered at time location \(\tau\). The Fourier transform of window signals \(f(x)g^*(x-\tau)\) yields the Short-Time Fourier Transform:

\[ STFT(\tau, \omega) = \int_{-\infty}^{\infty} f(x)g^*(x-\tau)e^{-i\omega x} dx \]  

(2)

which maps the signal into a two-dimensional function in a time-frequency plane \((\tau,\omega)\).

The parameter \(\omega\) in Equation (2) is similar to the Fourier frequency and many properties of FT carry over to STFT. However, the analysis here depends critically on the choice of the window \(g(x)\).

Figure 1 from [1] shows vertical stripes in the time-frequency plane, illustrating
Given a version of signal windowed around time \( x \), one computes the STFT for all "frequencies". At a given frequency \( \omega \), Equation (2) amounts to filtering the signal "at all time" with a bandpass filter having as impulse response the window function modulated to that frequency.

\[
\text{STFT}(\tau, \omega) = \int_{-\infty}^{\infty} g(x) x^\tau e^{-j\omega x} dx
\]

\[
\text{Sliding Window} \ g(x)
\]

Figure 1. Time-frequency plane corresponding to the Short-Time Fourier Transform.

II.2.3 The Disadvantage of STFT

Consider the ability of the STFT to discriminate between two pure sinusoids. Given a window function \( g(x) \) and Fourier Transform \( \hat{g}(\omega) \) define the "Bandwidth" \( \Delta \omega \) of the filter as:

\[
\Delta \omega^2 = \frac{\int_{-\infty}^{\infty} \omega^2 |\hat{g}(\omega)|^2 d\omega}{\int_{-\infty}^{\infty} |\hat{g}(\omega)|^2 d\omega}
\]

(3)

Two sinusoids will be discriminated only if their frequencies are more than \( \Delta \omega \) apart. Thus, the resolution in frequency of STFT analysis is given by \( \Delta \omega \).
Similarly, the resolution in the time is given by $\Delta x$ as

$$\Delta x^2 = \frac{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^2 |g(x)|^2 dx}{\int_{-\infty}^{\infty} |\hat{g}(x)|^2 dx}$$

(4)

Two pulses in time can be discriminated only if they are more than $\Delta x$ apart.

Now, resolution in the time and frequency cannot be arbitrarily small, because their product is lower bounded.

$$\Delta x \Delta \omega \geq \frac{1}{2}$$

(5)

More important is that once a window has been chosen for the STFT, the time-frequency resolution given by Equations (3) and (4) is fixed over the entire time-frequency plane (since the same window is used at all frequencies). For example, if the signal is composed of small bursts associated with long quasi-stationary components, then each type of component can be analyzed with good time resolution or frequency resolution, but not both.

II.3 THE CONTINUOUS WAVELET TRANSFORM
----A MULTiresolution ANALYSIS

To overcome the resolution limitation of the STFT, one can imagine letting the resolution $\Delta x$ and $\Delta \omega$ vary in the time-frequency plane in order to obtain a multi-resolution analysis. When the analysis is viewed as a filter bank, the time resolution must increase with the central frequency of the analysis filter. We therefore impose that $\Delta \omega$ is proportional to $\omega$ or

$$\frac{\Delta \omega}{\omega} = c$$

(6)

where $c$ is a constant.

When Equation (6) is satisfied, we see that $\Delta \omega$ and therefore also $\Delta x$ changes with the center frequency. It seems that we use the shorter windows at higher frequencies
and the longer windows at lower frequencies. Of course, they still satisfy the Heisenberg inequality (5), but now, the time resolution becomes arbitrarily good at higher frequencies, while the frequency resolution becomes arbitrarily good at lower frequencies.

This kind of analysis of course looks best if the signal is composed of higher frequency components of shorter durations plus lower frequency components of longer durations, which is often the case with signals encountered in practice.

The Continuous Wavelet Transform exactly follows the above ideas while adding a simplification: all impulse responses of the filter are defined as scaled version of the same prototype \( \psi(x) \)

\[
\psi_s(x) = \frac{1}{s} \psi\left(\frac{x}{s}\right)
\]  

(7)

where \( s \) is a scale factor.

This results in the definition of Continuous Wavelet Transform. The Wavelet Transform of a function \( f(x) \) at scale \( s \) and time \( x \) (or position) is given by the convolution product

\[
W_s f(x) = f(x) \ast \psi_s(x)
\]  

(8)

If \( \psi(x) \) satisfies

\[
\int_{-\infty}^{\infty} \psi(x) dx = 0
\]  

(9)

\( \psi(x) \) is called the wavelet. \( \psi(x) \) is of finite energy and bandpass which implies that it oscillates in time like a short wave, hence the name "wavelet". The dilation of \( \psi(x) \) by a factor \( s \) is called wavelets \( \psi_s(x) \).

As the scale increases, wavelets (the filter impulse responses) \( \psi_s(x) = \frac{1}{s} \psi\left(\frac{x}{s}\right) \) spreads out in time, takes only long-time behavior into account, and \( W_s f(x) \) detects the local lower frequency components of signal \( f(x) \). When the scale \( s \) decreases, the support of \( \psi_s(x) \) decreases. The wavelet transform \( W_s f(x) \) detects the local higher frequency components of the signal \( f(x) \) and it is sensitive to finer details. The scale \( s \)
characterizes the size and regularity of the signal features extracted by the wavelet transform.

Figure 2(a) from [1] shows that the influence of the signal’s behavior around $x = x_0$ in the analysis is limited to a cone in the time-scale plane; It is therefore very "localized" around $x_0$ for small scale. In the STFT case, the corresponding region of influence is as large as the extent of the analysis window over all frequencies, as shown in Figure 2(b).

![Diagram of CWT and STFT](image)

(a) CWT  
(b) STFT

Figure 2. Regions of influence of a Dirac pulse at $x = x_0$, (a) for the CWT and (b) for the STFT.

The wavelet Transform $W_s f(x)$ can also be written as an inner product in $L^2$

\[
W_s f(x) = f(x) * \psi_s(x) \\
= \int_{-\infty}^{+\infty} f(\tau) \psi_s(x-\tau) d\tau \\
= \int_{-\infty}^{+\infty} f(\tau) \tilde{\psi}_s(\tau-x) d\tau \\
= \langle f(\tau), \tilde{\psi}_s(\tau-x) \rangle
\]

where $\tilde{\psi}_s(x) = \psi_s(-\tau)$. 

\[ (10) \]
II.4 DYADIC WAVELET TRANSFORM

The Continuous Wavelet Transform depends on two parameters $s$ and $x$ that vary continuously over the set of real numbers. For practical applications these parameters must be discreted. For a particular class of wavelets, the scale parameter can be sampled along the dyadic sequence, without modifying the overall properties of the transform [2]. The principle of such a dyadic decomposition was studied in mathematics by Little Wood and Paley in the 1930's.

II.4.1 Infinite-Scale Dyadic Wavelet Transform

We impose the scale which varies only along the dyadic sequence $(2^j)_{j \in \mathbb{Z}}$. The dilation of $\psi(x)$ by a factor $2^j$ is given by

$$\psi_{2^j}(x) = \frac{1}{2^j} \psi\left(\frac{x}{2^j}\right) \quad (11)$$

Its Fourier Transform $\hat{\psi}_{2^j}(\omega)$ is given by

$$\hat{\psi}_{2^j}(\omega) = \hat{\psi}(2^j \omega) \quad (12)$$

by imposing that

$$\sum_{j=\infty}^{\infty} |\hat{\psi}(2^j \omega)|^2 = 1 \quad (13)$$

we ensure that the whole frequency is covered by a dilation of $\hat{\psi}(\omega)$ by the scale factor $(2^j)_{j \in \mathbb{Z}}$.

Any wavelet satisfying Equation (13) is called a dyadic wavelet. The wavelet transform at scale $2^j$ and position $x$ is given by

$$W_{2^j} f(x) = f(x) * \psi_{2^j}(x) \quad (14)$$

At each scale $2^j$, the function $W_{2^j} f(x)$ is continuous since it is equal to the convolution of two functions in $L^2$. The Fourier transform of $W_{2^j} f(x)$ is given by

$$\hat{W}_{2^j} f(\omega) = \hat{f}(\omega) \hat{\psi}(2^j \omega) \quad (15)$$

Because $j$ varies between $-\infty$ and $+\infty$, we call the sequence of function
(W_2^j f(x))_{j \in \mathbb{Z}} \text{ Infinite-Scale Dyadic Wavelet Transform.}

From Equations (13), (15), and by apply the Parseval theorem, we obtain an energy conservation equation

\[ \int_{-\infty}^{\infty} |f(x)|^2 dx = \sum_{j=-\infty}^{\infty} \int_{-\infty}^{\infty} |W_2^j f(x)|^2 dx \]  \hspace{1cm} (16)

Let \( \tilde{\psi}_2^j(x) = \psi_2^j(-x) \). The function \( f(x) \) can be reconstructed from its dyadic wavelet transform [7].

\[ f(x) = \sum_{j=-\infty}^{\infty} W_2^j f(x) * \tilde{\psi}_2^j(x) \]  \hspace{1cm} (17)

II.4.2 Finite-Scale Dyadic Wavelet Transform

In practice we can not compute the wavelet transform at all scales \( 2^j \) for \( j \) varying from \(-\infty\) to \(+\infty\). We are limited by a finite larger scale and a nonzero finer scale. Let us suppose for normalization purposes that the finer scale is equal to 1 and that \( 2^J \) is the largest scale.

Let us introduce a function \( \phi(x) \) whose Fourier transform is given by

\[ |\hat{\phi}(\omega)|^2 = \sum_{j=1}^{\infty} |\hat{\psi}(2^j \omega)|^2 \]  \hspace{1cm} (18)

From Equations (13) and (18), we have equation

\[ \lim_{\omega \to 0} |\hat{\phi}(\omega)| = 1 \]  \hspace{1cm} (19)

So that the energy of the Fourier transform \( \hat{\phi}(\omega) \) is concentrated in low frequency and \( \phi(x) \) is a smoothing function. The smoothing operator \( S_2^j \) is defined by

\[ S_2^j f(x) = f(x) * \phi_2^j(x) \]  \hspace{1cm} (20)

with

\[ \phi_2^j(x) = \frac{1}{2^j} \phi\left(\frac{x}{2^j}\right) \]  \hspace{1cm} (21)

The Fourier transform of \( \phi_2^j(x) \) is given by

\[ \hat{\phi}_2^j(\omega) = \hat{\phi}(2^j \omega) \]  \hspace{1cm} (22)
The larger scale $2^j$ we used, the more detail of $f(x)$ are removed by the smoothing operator $S_{2^j}$.

The Fourier transform of $S_{1f}(x)$, $S_{2^j}f(x)$ and $W_{2^j}f(x)$ are respectively given by

\begin{align}
\hat{S}_{1f}(\omega) &= \hat{\phi}(\omega) \hat{f}(\omega) \\
\hat{S}_{2^j}f(\omega) &= \hat{\phi}(2^j \omega) \hat{f}(\omega)
\end{align}

Equation (18) yield

\begin{align}
|\phi(\omega)|^2 &= \sum_{j=1}^{\infty} |\psi(2^j \omega)|^2 \\
&= \sum_{j=1}^{\infty} |\hat{\psi}(2^j \omega)|^2 + \sum_{j=1}^{\infty} |\hat{\psi}(2j \omega)|^2 \\
&= \sum_{j=1}^{\infty} |\hat{\psi}(2^j \omega)|^2 + |\hat{\phi}(2^j \omega)|^2
\end{align}

Using Parseval's theorem, we get the following energy conservation equation

\begin{align}
\int_{-\infty}^{\infty}|S_{1f}(x)|^2dx &= \sum_{j=1}^{\infty} \int_{-\infty}^{\infty}|W_{2^j}f(x)|^2dx + \int_{-\infty}^{\infty}|S_{2^j}f(x)|^2dx
\end{align}

This equation proves that the dyadic wavelet transform $(W_{2^j}f(x))_{1 \leq j \leq f}$ between the scale $1$ and $2^j$ provides the details available in $S_{1f}(x)$, but not in $S_{2^j}f(x)$. The higher frequencies of $S_{1f}(x)$ that have disappeared in $S_{2^j}f(x)$ can be found in the dyadic wavelet transform $(W_{2^j}f(x))_{1 \leq j \leq f}$ between the scale $1$ and $2^j$. The signal $S_{1f}(x)$ can be reconstructed from $\left\{S_{2^j}f(x), (W_{2^j}f(x))_{1 \leq j \leq f}\right\}$ and we called $\left\{S_{2^j}f(x), (W_{2^j}f(x))_{1 \leq j \leq f}\right\}$ Finite-Scale Dyadic Wavelet Transform of signal $S_{1f}(x)$.

**II.4.3 Discrete Dyadic Wavelet Transform**

In practice, the signal we process is given by a discrete sequence of value $(d_n)$. Any discrete signal of finite energy can be interpreted as uniform sampling of some func-
tion smoothed at scale 1 \cite{2}. This means there exists a (nonunique) function $f(x) \in L^2$. For any $n \in \mathbb{Z}$, we have

$$S_1 f (n) = d_n$$

The input signal can thus be rewritten by

$$D = (S_1 f (n))_{n \in \mathbb{Z}}$$

For any coarse scale $2^J$, the sequence of discrete signals $\left\{S_{2^j} f, \{W_{2^j} f \}_{i \leq j \leq J} \right\}$ is called the Discrete Dyadic Wavelet Transform of $D = (S_1 f (n))_{n \in \mathbb{Z}}$.

In practice, the original discrete signal $D$ has a finite number $N$ of nonzero value:

$$D = (d_n)_{1 \leq n \leq N}$$

For the class of wavelets we used in this paper, one can prove that when the scale is as large as $2^j=2N$, $S_{2^j} f$ is constant and equal to the mean value of the original signal $D$ \cite{4}. We thus decompose any signal of $N$ samples over $J = \log_2(N)+1$ scale.

Figure 3 (a) is the plot of a discrete signal of 256 samples. Figure 3 (b) shows its discrete dyadic wavelet transform computed on nine scales.

II.5 FAST WAVELET ALGORITHMS

Here, we define the class of wavelets used for implementation of discrete algorithms. We first define the function $\phi(x)$ and then we build the wavelet $\psi(x)$ which is associated with $\phi(x)$.

We impose that the Fourier transform of the smoothing function $\phi(x)$ defined by Equation (18) can be written as an infinite product

$$\mathcal{F}(\omega) = \prod_{p=1}^{\infty} H(2^p \omega)$$

where, $H(\omega)$ is a $2\pi$ periodic differentiable function such that

$$|H(\omega)|^2 + |H(\omega+\pi)|^2 \leq 1$$ \hspace{1cm} (28a)

$$|H(0)| = 1$$ \hspace{1cm} (28b)
Figure 3. Signal of 256 sample and its Dyadic WT.
The function $H(\omega)$ can be interpreted as the transfer function of a discrete low-pass filter. Equation (27) implies that

$$\hat{\phi}(2\omega) = H(\omega)\hat{\phi}(\omega)$$

(29)

Let us now characterize the corresponding wavelet $\psi(x)$. As a consequence of Equation (18), we have

$$|\hat{\psi}(2\omega)|^2 = |\hat{\phi}(\omega)|^2 - |\hat{\phi}(2\omega)|^2$$

(30)

Substitute Equation (30) by (29), we obtain

$$\hat{\psi}(2\omega) = G(\omega)\hat{\phi}(\omega)$$

(31)

where, $G(\omega)$ is also a $2\pi$ periodic function, and

$$|G(\omega)|^2 + |H(\omega)|^2 = 1$$

(32)

$G(\omega)$ can be interpreted as the transfer function of a discrete high-pass filter.

From Equations (23), (24), (25), (29) and (31), we obtain

$$W_{2^j}f(\omega) = S_{2^j}f(\omega)G(2^j\omega)$$

(33)

$$S_{2^j-1}f(\omega) = S_{2^j}f(\omega)H(2^j\omega)$$

(34)

where, $j$ is between 0 and $J$. In time domain, equations (33) and (34) are equivalent to the following equations

$$W_{2^j}f = S_{2^j}f*G_j$$

(35)

$$S_{2^j-1}f = S_{2^j}f*H_j$$

(36)

where, $G_j$ and $H_j$ are inverse Fourier transform of $G(2^j\omega)$ and $H(2^j\omega)$.

Equation (35) and (36) can be interpreted by Figure 4. At each scale $2^j$, it decomposes $S_{2^j}f$ into $S_{2^j-1}f$ and $W_{2^j}f$.

The following algorithm computes the discrete wavelet transform of the discrete signal $S_{2^j}f$. 

...
Figure 4. Block diagram of the discrete wavelet transform implemented with discrete-time filters.

\[ j=0 \]
while \((j<J)\),

\[ W_{\frac{j}{2}} f = S_{\frac{j}{2}} f * G_j \]
\[ S_{\frac{j}{2}} f = S_{\frac{j}{2}} f * H_j \]
\[ j=j+1 \]
end of while.

The inverse wavelet transform algorithm [2] reconstructs \( S_{\frac{j}{2}} f \) from the discrete dyadic wavelet transform. At each scale \(2^j\), it reconstructs \( S_{\frac{j}{2}} f \) from \( S_{\frac{j}{2}} f \) and \( W_{\frac{j}{2}} f \).

\[ j=J \]
while \((j>0)\)

\[ S_{\frac{j}{2}} f = W_{\frac{j}{2}} f * \tilde{G}_{j-1} + S_{\frac{j}{2}} f * \tilde{H}_{j-1}, \]
\[ j=j-1 \]
end of while.

Where, \( \tilde{G}_j \) and \( \tilde{H}_j \) are the filters whose transfer functions are respectively \( \bar{H}(2^j \omega) \) and \( \bar{G}(2^j \omega) \) (complex conjugates of \( H(2^j \omega) \) and \( G(2^j \omega) \)).
II.6 LOCAL MAXIMUM OF WAVELET TRANSFORM

II.6.1 Local Maximum of WT and Sharp Variation Points

Points of sharp variation are often among the most important features for characterizing a signal. This section explains how sharp variation points are related to the wavelet transform.

A smoothing function $\theta(x)$ is such a function

$$\int_{-\infty}^{+\infty} \theta(x) \, dx = 1$$

and

$$\lim_{x \to +\infty} \theta(x) = 0$$
$$\lim_{x \to -\infty} \theta(x) = 0$$

We suppose that $\theta(x)$ is one differentiable, $\psi^1(x)$ is the first derivative of $\theta(x)$

$$\psi^1(x) = \frac{d \theta(x)}{dx} \tag{37}$$

By definition, the function $\psi^1(x)$ can be considered to be wavelet because its integral is equal to 0.

$$\int_{-\infty}^{+\infty} \psi^1(x) \, dx = 0$$

The wavelet transform of $f(x)$ at scale $s$ and position $x$, computed with respect to the wavelet $\psi^1(x)$, is defined by

$$W_s f(x) = f(x) \ast \psi^1_s(x) \tag{38}$$

with

$$\psi^1_s(x) = \frac{1}{s} \psi^1(x)$$

from Equation (37), we derive

$$\psi^1_s(x) = \frac{1}{s} \frac{d \theta \left( \frac{x}{s} \right)}{d \left( \frac{x}{s} \right)} = \frac{d \theta_s \left( \frac{x}{s} \right)}{d \left( \frac{x}{s} \right)} = s \frac{d \theta_s \left( \frac{x}{s} \right)}{dx} \tag{39}$$

from Equations (38) and (39), we obtain following equation
The wavelet transform $W_s f(x)$ is the first derivative of the signal smoothed at the scale $s$. The local maxima of $W_s f(x)$ thus correspond to the inflection points of $f(x) \ast \theta_s(x)$. The maxima of the absolute value of the first derivative are sharp variation points of $f(x) \ast \theta_s(x)$, whereas the minima correspond to slow variations.

We can easily select the sharp variation points by detecting only the local maxima of $|W_s f(x)|$. When the scale $s$ is large, the signal $f(x) \ast \theta_s(x)$ removes small signal fluctuations, we therefore only detect the sharp variations of large structure. When detecting local maxima locations, we can also record the values $|W_s f(x)|$ at the maxima locations, which measure the derivative at inflection points. One can prove that the extrema of the wavelet transform built from $\psi(x)$ is essentially equivalent to a Canny Edge Detection [7].

Local maxima of WT at nine scale of signal shown in Figure 3(a) are shown in Figure 5. From the local extrema, we can also reconstruct original signal [7].

II.6.2 Local Maxima of WT and Regularity of Signal

Signal sharp variations produce modulus maxima at different scales $2^j$. We know that the value of a modulus maximum at scale $2^j$ measures the derivative of the signal smoothed at scale $2^j$, but it is not clear how to combine these different values to characterize the signal variation. The wavelet theory gives an answer to this question by showing that the evolution across scales of wavelet transform depends on the local Lipschitz regularity of the signal [3].

Definition: Let $0 \leq \alpha \leq 1$. A function $f(x)$ is uniformly Lipschitz $\alpha$ over an interval $[a,b]$ if and if there exits a constant $K$ such that for any $(x_0,x_1) \in [a,b]$

$$|f(x_0)-f(x_1)| \leq K |x_0-x_1|^\alpha$$

We refer to the Lipschitz uniform regularity of $f(x)$ as the upper bound $\alpha_0$ of all $\alpha$ such
that \( f(x) \) is uniformly Lipschitz \( \alpha \).

If \( f(x) \) is differentiable at \( x_0 \), then it is Lipschitz \( \alpha = 1 \). If the uniform Lipschitz regularity \( \alpha_0 \) is larger, the singularity at \( x_0 \) will be more "regular". If \( f(x) \) is discontinuous but bounded in the neighborhood of \( x_0 \), its uniform Lipschitz regularity in the neighborhood of \( x_0 \) is 0. The following Theorem proves that the Lipschitz exponent of a function can be measured from the evolution across scales of the absolute value of the wavelet transform.

**Theorem:** Let \( 0<\alpha<1 \). A function \( f(x) \) is uniformly Lipschitz \( \alpha \) over \([a,b]\) if and only if there exists a constant \( K>0 \) such that for all \( x \in [a,b] \), the wavelet transform satisfies

\[
|W_{2^j} f(x)| \leq K (2^j)^\alpha
\]

So that wavelet transform can not only detect sharp variations but also characterize the local regularity of signal. The proof of this theorem can be found in [3].

### 11.7 B-SPLINE WAVELET

A family of wavelets used in this thesis is B-spline wavelet \( \psi^n(x) \). They come from the B-spline function \( \beta^n(x) \).

\[
\psi^n(x) = \frac{d}{dx} (\beta^n(x)) \quad (41)
\]

The function \( \beta^n(x) \) is the central B-spline of order \( n \) that can be generated by repeated convolution of a spline of order 0.

\[
\beta^n(x) = \beta^0(x)*\beta^{n-1}(x) \quad (42)
\]

where \( \beta^0(x) \) is the indicator function in the interval \([-1/2,1/2]\), as shown in Fig. 6(a).

First order B-spline function \( \beta^1(x) \), quadratic spline function \( \beta^2(x) \) and cubic spline function \( \beta^3(x) \) that can be obtained by Equation (42) are shown in Figure 6(b),(c) and (d). The spline wavelets \( \psi^n(x) \) that are related to spline function \( \beta^n(x) \) are shown in Figure 7.
Figure 5. Local maximum of the Dyadic WT of signal shown in Figure 3(a).
Figure 6. B-Spline (a) Zero order spline, (b) first order spline, (c) quadratic spline, (d) cubic spline.
Figure 7. B-Spline Wavelet. (a) Zero order spline wavelet, (b) first order spline wavelet, (c) quadratic spline wavelet, (d) cubic spline wavelet.
The corresponding $2\pi$ periodic function $H(\omega)$ that is defined by Equation (28) is given by [2]

$$H(\omega) = e^{\frac{i\omega}{2}} (\cos\frac{\omega}{2})^n.$$

(43)

The fundamental characteristic of B-spline wavelets is their compact support [26], the property that makes them useful in a variety of applications.

II.8 CONCLUSION

We showed that Short-Time Fourier Transform and Wavelet Transform represent alternative ways to divide the time-frequency (or time-scale) plane. In contrast to STFT, which uses a single analysis window, the WT uses shorter windows at higher frequencies and longer windows at lower frequencies. For some particular wavelet functions, the local maxima of the wavelet transform correspond to the sharp variation points of the signal. When detecting locations of local maxima, we can also record the values of $W_s f(x)$ at these locations, which measure the derivative at inflection points. Finally, a fast wavelet algorithm and a family of B-Spline wavelets were also discussed. Because of B-Spline's compact support, it has many applications. In our study, we used cubic B-Spline wavelet as a basic wavelet.
CHAPTER III

CHARACTER RECOGNITION

III.1 INTRODUCTION

Character recognition can be broadly classified as template matching techniques and feature analysis techniques.

Template matching techniques directly compare an input character to a standard set of stored prototypes. The prototype that matches most closely provides recognition. This type of technique suffers from sensitivity to noise and is not adaptive to difference in writing style.

Feature analysis techniques are the most frequently used techniques for character recognition. In these methods, significant features are extracted from a character and compared to the feature descriptions of ideal character. In order to obtain a high success ratio in classification, features should satisfy the following three requirements:

*Small intraclass invariance.* Patterns with similar shapes and similar general characteristics should end up with numerically close numbers for the features. The features should be independent of translation, rotation, and scale of character.

*Large interclass separation.* The distinction between the features from different classes should be as large as possible.

*Small feature number.* The number of features used in classification and recognition should be as small as possible.
Many feature analysis techniques have been developed and applied to character recognition. Since the main information about a character can be found in its boundary, we utilize the boundary position data in making features. Present object recognition methods based on boundaries can be categorized as either global or local in nature.

Global methods are based on global feature of the boundary. Such techniques are the Fourier Descriptor [24], the invariant moments [10], etc. Global methods have the disadvantage that a small distortion in a section of a boundary of an object will result in changes to all global features.

Local methods use local features such as critical points. They perform extremely well in the presence of noise and distortion since such effects on an isolated region of the contour alter only the local features associated with that region, leaving all other local features unaffected. As a local method, wavelet transform is introduced to character recognition in this chapter.

In Chapter II, we proved that for some particular wavelet function, the extrema of wavelet transform correspond to the sharp variation points of signal at different scales, and we can also reconstruct original signal from its extrema of wavelet transform. Wavelet transform can compress the data size significantly. So that we choose the local maxima of wavelet transform of boundary character as shape features. After several success normalizations, they are not only invariant of character size, rotation and translation but also insensitive to noise.

After feature extraction, the second step is classification of the features. A class label is assigned to a test character by examining its extracted features and comparing them with the feature descriptions of ideal characters.

The nearest-neighbor rule [23] is a well-know statistical classifier. Decision is made based on the city block distance between the features of a test character and an ideal character.
In our study, we have used numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, and 0 as test characters. The digitized images of these numbers are provided by Dr. Z. Fan at Xerox Corporation. By the end of this chapter, we will have compared the performance of the wavelet method with Fourier Descriptor method. We also checked the performance when the noise effects are significant.

III.2 DATA COLLECTION

For a large pattern recognition, the use of the object boundary to provide information is attractive because of the reduction in quantity of information, compared to that of original 2D images. There are several steps to get normalized boundary data.

We assume that a character lies in each image plane. As an example, character "8" is showed in Figure 8(a). It is captured in 120x120 pixel with 2 gray level (dark is 0, bright is 1). The pixel data is stored as a 120x120 matrix. Starting at first point of first line of boundary and tracing once around it along the inverse clock direction, we obtain a sequence of outer boundary position data \([x_n, y_n]\) for \(1 \leq n \leq N\). \(N\) is the total points along the outer boundary.

![Character 8](image1.png)

![Character 8 rotated](image2.png)

Figure 8. Character "8" and \(\pi/2\) rotation of character "8".

The coordinates \((\bar{x}, \bar{y})\) of the centroid of the considered boundary are given by
\[ \bar{x} = \frac{1}{N} \sum_{n=1}^{N} x_n \]
\[ \bar{y} = \frac{1}{N} \sum_{n=1}^{N} y_n \]

The distances of the centroid from the points of the boundary is given by
\[ z_n = \sqrt{(x_n - \bar{x})^2 + (y_n - \bar{y})^2} \quad 1 \leq n \leq N \]

The sequence \( Z = [z_1, z_2, \cdots, z_N] \) contains all the distances of the centroid \((\bar{x}, \bar{y})\) from the points of the boundary of the considered character.

For any \( n \), we can find a point \( J \) such that
\[ z_J \leq z_n \quad 1 \leq n \leq N \]

Point \( J \) is therefore the point which has minimum distance from the centroid point. Starting at this point and tracing once around boundary along inverse clock direction, we obtain sequence \( D \) that is invariant under rotation and translation.

\[ D = [z_J, z_{J+1}, \cdots, z_N, z_1, z_2, \cdots, z_{J-1}] \]

Figure 9 shows sequences \( D \) associated to the character "8" shown in Figure 8(a) and the character shown in Figure 8(b).

III.3 FEATURE EXTRACTION

III.3.1 Wavelet Transform

For some particular wavelet functions, the extrema of the wavelet transform correspond to the sharp variation points of the signal. The wavelet theory states that evolution across scales of the wavelet transform modulus maximum can characterize the local Lipschitz regularity of input signal. Thus, the wavelet transform cannot only detect the signal's sharp variations but also characterize their local shapes. Hence, it is intuitive that the wavelet transform is a suitable candidate for extracting features. So we used the extrema of wavelet transform and its position as shape features to describe characters. Cubic spline wavelet is selected as basic wavelet \( \psi(x) \) because it preserves continuity of the boundary and compact support.
Figure 9. The normalized boundary data of characters "8". (a) for character shown in Figure 8(a), (b) for character shown in Figure 8(b).
Figure 10. Wavelet transform of boundary data shown in Figure 9.
Figure 11. Local maximum of wavelet transform of boundary data shown in Figure 9.
The wavelet transform of the normalized boundary data shown in Figure 9 is given by Figure 10. The local maxima of wavelet transform of the normalized boundary data are shown in Figure 11.

Figure 11 shows that the sharp variation points are moved when scale is changed. This is because the scale defines the size of the neighborhood where the signal changes are computed. For larger scale, the more neighborhood is considered. The wavelet transform is sensitive to finer scale.

Inherent in boundary extraction of a character are several sources of error. For instance, noise during image acquisition, quantization on digitization, and inaccuracy in edge detection. Usually, the wavelet transform at finer scale comes with noise, so that we are more interested in large scale. But if the scale is too large, we will lose some detail information which is used to identify character. The selection of scale is very flexible and it depends on different applications.

### III.3.2 Feature Extraction Procedure

In our study, we choose eight sharp variation points at some scale as feature descriptions \([v_j,w_j]_{j=1,2,...,8}\). \(v_j\) is the position of the sharp variation point and \(w_j\) is the wavelet transform value at one scale and position \(v_j\).

The selection of these eight sharp variation points obey the following rules:

1. If there are eight local maximum points of wavelet transform at scale \(s = 2^l\), we choose these 8 sharp variation points as shape features.

2. If there are less than eight local maximum points of wavelet transform at scale \(s = 2^l\), for example, there are \(k\) points \((k < 8\) ), we go to check wavelet transform at scale \(s = s^{l-1}\). If there are exactly eight local maximum points of wavelet transform at scale \(s = 2^{l-1}\), we take these eight sharp variation points as shape features. If there are less than eight local maximum points of wavelet
transform at scale $s=2^{l-1}$, we go down to the next smaller scale. If there are more than eight local maximum points of wavelet transform at scale $s=2^{l-1}$, we take the points that propagate to large scale $s=2^l$ and select other 8-k extrema points which have larger products of their wavelet transform value at scale $s=2^{l-1}$ and their distance to nearest extrema points that propagate to scale $s=2^l$.

(3) If there are more than eight local maximum points of wavelet transform at scale $s=2^l$, we go to check to wavelet transform at scale $s=2^{l+1}$. If there are exactly eight local maximum points of wavelet transform at scale $s=2^{l+1}$, we take these eight sharp variation points as shape features. If there are more than eight local maximum points of wavelet transform at scale $s=2^{l+1}$, we go up to next larger scale. If there are less than eight local maximum points of wavelet transform at scale $s=2^{l+1}$, for example, there are $k$ extrema points, we go back to wavelet transform at scale $s=2^l$. We take those $k$ points that propagate to scale $s=2^{l+1}$, and select other 8-$k$ extrema points which have larger products of their wavelet transform value at scale $s=2^l$ and their distance to nearest extrema points that propagate to scale $s=2^{l+1}$.

III.3.3 Normalization

For the same character but different size, the total points $N$ along the boundary will be different. For normalization purpose, we define $p_j$ by

$$p_j = \frac{y_j}{N} \quad (48)$$

Because wavelet transform value $w_j$ may be from different scale $s$ for the different characters, the wavelet transform value will be increased as the scale $s$ is increased. In order to provide invariance under scaling, we need a method of normalization. We define $q_j$ by
\[ q_j = \frac{w_j - \bar{w}}{\sigma} \]

where \( j = 1, 2, \ldots, 8 \), and

\[
\bar{w} = \frac{1}{8} \sum_{j=1}^{8} w_j \\
\sigma = \sqrt{\frac{1}{8-1} \sum_{j=1}^{8} (w_j - \bar{w})^2}
\]

Therefore the feature \( [p_j, q_j]_{j=1,2,\ldots,8} \) are invariant under scaling, as well as rotation and translation. The features of ten characters "0,1,2,\ldots,9" shown in Figure 12 are shown in Table I and Table II.
### TABLE I

THE FEATURES OF CHARACTERS "1", "2", "3", "4", "5"

<table>
<thead>
<tr>
<th>Features</th>
<th>&quot;1&quot;</th>
<th>&quot;2&quot;</th>
<th>&quot;3&quot;</th>
<th>&quot;4&quot;</th>
<th>&quot;5&quot;</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q_1$</td>
<td>0.9132</td>
<td>1.5847</td>
<td>1.4871</td>
<td>0.2086</td>
<td>-0.4104</td>
</tr>
<tr>
<td>$q_2$</td>
<td>-0.2481</td>
<td>1.2373</td>
<td>-0.5637</td>
<td>-1.4347</td>
<td>-0.3976</td>
</tr>
<tr>
<td>$q_3$</td>
<td>0.8799</td>
<td>-1.4659</td>
<td>0.7182</td>
<td>0.2326</td>
<td>1.8984</td>
</tr>
<tr>
<td>$q_4$</td>
<td>-0.2998</td>
<td>-0.6133</td>
<td>0.5040</td>
<td>0.9612</td>
<td>-0.7591</td>
</tr>
<tr>
<td>$q_5$</td>
<td>-0.2723</td>
<td>0.4542</td>
<td>0.0742</td>
<td>1.0561</td>
<td>-0.4306</td>
</tr>
<tr>
<td>$q_6$</td>
<td>0.7783</td>
<td>-0.3548</td>
<td>0.0063</td>
<td>-0.0188</td>
<td>1.3029</td>
</tr>
<tr>
<td>$q_7$</td>
<td>-1.7541</td>
<td>-0.3167</td>
<td>-1.2389</td>
<td>-0.3020</td>
<td>-0.4113</td>
</tr>
<tr>
<td>$q_8$</td>
<td>-1.2181</td>
<td>-0.0130</td>
<td>-1.4700</td>
<td>-1.6471</td>
<td>-0.6637</td>
</tr>
<tr>
<td>$p_1$</td>
<td>0.0588</td>
<td>0.0711</td>
<td>0.0738</td>
<td>0.0519</td>
<td>0.0451</td>
</tr>
<tr>
<td>$p_2$</td>
<td>0.1272</td>
<td>0.1772</td>
<td>0.2047</td>
<td>0.1935</td>
<td>0.2102</td>
</tr>
<tr>
<td>$p_3$</td>
<td>0.1943</td>
<td>0.2407</td>
<td>0.2919</td>
<td>0.2777</td>
<td>0.2850</td>
</tr>
<tr>
<td>$p_4$</td>
<td>0.3410</td>
<td>0.3402</td>
<td>0.4284</td>
<td>0.4140</td>
<td>0.5048</td>
</tr>
<tr>
<td>$p_5$</td>
<td>0.4629</td>
<td>0.4836</td>
<td>0.5291</td>
<td>0.5143</td>
<td>0.5843</td>
</tr>
<tr>
<td>$p_6$</td>
<td>0.6095</td>
<td>0.7308</td>
<td>0.7080</td>
<td>0.6380</td>
<td>0.6841</td>
</tr>
<tr>
<td>$p_7$</td>
<td>0.7791</td>
<td>0.8304</td>
<td>0.8009</td>
<td>0.8244</td>
<td>0.8694</td>
</tr>
<tr>
<td>$p_8$</td>
<td>0.9382</td>
<td>0.9453</td>
<td>0.9150</td>
<td>0.9390</td>
<td>0.9584</td>
</tr>
</tbody>
</table>
TABLE II
THE FEATURES OF CHARACTERS "6", "7", "8", "9", "0"

<table>
<thead>
<tr>
<th>Features</th>
<th>&quot;6&quot;</th>
<th>&quot;7&quot;</th>
<th>&quot;8&quot;</th>
<th>&quot;9&quot;</th>
<th>&quot;0&quot;</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q_1$</td>
<td>0.6396</td>
<td>1.2640</td>
<td>1.1568</td>
<td>0.5607</td>
<td>1.4913</td>
</tr>
<tr>
<td>$q_2$</td>
<td>-0.3274</td>
<td>-0.3296</td>
<td>0.6517</td>
<td>-0.4690</td>
<td>-0.2201</td>
</tr>
<tr>
<td>$q_3$</td>
<td>0.6000</td>
<td>0.4275</td>
<td>-0.7928</td>
<td>0.5608</td>
<td>-0.8902</td>
</tr>
<tr>
<td>$q_4$</td>
<td>0.0104</td>
<td>-0.0749</td>
<td>-0.8410</td>
<td>0.0662</td>
<td>0.4517</td>
</tr>
<tr>
<td>$q_5$</td>
<td>-0.2289</td>
<td>-0.1220</td>
<td>0.9058</td>
<td>-0.6345</td>
<td>1.0619</td>
</tr>
<tr>
<td>$q_6$</td>
<td>1.1485</td>
<td>-0.7593</td>
<td>1.0610</td>
<td>-0.4771</td>
<td>-0.8200</td>
</tr>
<tr>
<td>$q_7$</td>
<td>-1.2852</td>
<td>-1.9905</td>
<td>-1.1675</td>
<td>1.4125</td>
<td>-1.2396</td>
</tr>
<tr>
<td>$q_8$</td>
<td>-1.7608</td>
<td>0.7427</td>
<td>-0.7694</td>
<td>-1.8777</td>
<td>-0.6282</td>
</tr>
<tr>
<td>$p_1$</td>
<td>0.0436</td>
<td>0.1111</td>
<td>0.0625</td>
<td>0.0419</td>
<td>0.0567</td>
</tr>
<tr>
<td>$p_2$</td>
<td>0.1424</td>
<td>0.2793</td>
<td>0.2048</td>
<td>0.1387</td>
<td>0.3214</td>
</tr>
<tr>
<td>$p_3$</td>
<td>0.2412</td>
<td>0.4571</td>
<td>0.3297</td>
<td>0.2486</td>
<td>0.4433</td>
</tr>
<tr>
<td>$p_4$</td>
<td>0.3328</td>
<td>0.5603</td>
<td>0.4688</td>
<td>0.3353</td>
<td>0.5231</td>
</tr>
<tr>
<td>$p_5$</td>
<td>0.5305</td>
<td>0.6682</td>
<td>0.5816</td>
<td>0.4711</td>
<td>0.6092</td>
</tr>
<tr>
<td>$p_6$</td>
<td>0.6860</td>
<td>0.8079</td>
<td>0.7101</td>
<td>0.5361</td>
<td>0.7542</td>
</tr>
<tr>
<td>$p_7$</td>
<td>0.7645</td>
<td>0.9016</td>
<td>0.8212</td>
<td>0.6792</td>
<td>0.8508</td>
</tr>
<tr>
<td>$p_8$</td>
<td>0.9433</td>
<td>0.9651</td>
<td>0.9479</td>
<td>0.9465</td>
<td>0.9391</td>
</tr>
</tbody>
</table>
III.4 CLASSIFICATION

We denote by \([p_j,q_j]_{j=1,2,...,8}\) the features of ideal character \(i\), where \(i = 0,1,2,...,9\). The features of the test character \(X\) is defined by \([p_j,q_j]_{j=1,2,...,8}\). The distance between a test character \(X\) and ideal character \(i\) is measured by

\[
D(X,i) = \sqrt{\sum_{j=1}^{8} [(p_j-p_j)^2+(q_j-q_j)^2]}
\]  

With the nearest-neighbor rule, the character \(I\) is assigned to \(X\) if the features of character \(I\) has minimum distance from \(X\) among all the training patterns. This means that there is a character \(I, D(X,I) \leq D(X,i)\), where \(i = 0,1,...,9\). \(I\) is one of \(i\).

III.5 SIMULATION RESULTS

In this section, we compare the performance of the wavelet method with Fourier Descriptor. We will also check the performance when the noise effects are significant. Let us review a little about Fourier Descriptor before we show the simulation result.

III.5.1 Fourier Descriptor

Suppose a character boundary position data is \([x_n,y_n]\) for \(1 \leq n \leq N\). Taking the discrete Fourier transform of the data, \(x_n,y_n,1 \leq n \leq N\), we obtain the Fourier coefficients for \(0 \leq m \leq N-1\):

\[
a_m = \frac{1}{N} \sum_{n=1}^{N} x_n e^{-i n \frac{2\pi m}{N}}, \quad 0 \leq m \leq N-1.
\]  

\[
b_m = \frac{1}{N} \sum_{n=1}^{N} y_n e^{-i n \frac{2\pi m}{N}}, \quad 0 \leq m \leq N-1.
\]

We discard the DC components, \(a_0\) and \(b_0\), since they carry information only about the position of the image center. To obtain rotation invariance, we use the energy spectrum \(r_m\) defined by

\[
r_m = \sqrt{|a_m|^2+|b_m|^2}, \quad 1 \leq m \leq N-1
\]
where \(|a_m|\) and \(|b_m|\) denote the absolute value of the complex numbers \(a_m\) and \(b_m\), and signify the energy spectra of \(x_n\) and \(y_n\). In order to provide invariance under scaling,

\[ s_m = \frac{r_m}{r_1}, \quad 1 \leq m \leq N-1 \]  

(53)

Then \(s_m\) is invariant under scaling, as well as rotation and translation. In order to compare with wavelet method, we select first \(8 \times 2 = 16\) components, \((s_1, s_2, \ldots, s_{16})\) as a feature vector.

The nearest-neighbor rule is also selected as a classifier. The distance between a test character \(X = [s_{m1}^x]_{m=1,2,\ldots,16}\) and a ideal character \(I = [s_{m1}^i]_{m=1,2,\ldots,16}\) is measured by

\[ D(X, I) = \sqrt{\sum_{m=1}^{16} (s_{m}^x - s_{m}^i)^2} \]  

(54)

For the performance test of the ten characters we made about 100 images (10 images for each character) by varying position, rotation, and size. Table III shows the recognition results of the ten characters. The rate of accurate recognition of the wavelet method and Fourier Descriptor are almost same. From Table III, we know that wavelet method could not recognize character "0" very well. This is because the distances between its boundary points and its centroid do not vary very much. It is difficult to extract suitable feature from character "0" using wavelet method.

III.5.2 Recognition of Noisy Characters

Generally, the wavelet method is robust in a noisy environment. To check the performance of the wavelet method in such an environment, we made noisy characters by adding Gaussian random noise to the boundary data of the original images. We performed the experiments with 8 different test image for each character for the case where the noise variance are \(\sigma^2 = 9\), \(\sigma^2 = 25\), \(\sigma^2 = 36\), and \(\sigma^2 = 64\). The recognition results are shown in Table IV, in which it is clear that when the noise effect is significant the wavelet method outperforms Fourier Descriptor. In the case where \(\sigma^2 = 64\), the accuracy
rate of the wavelet method is 8% higher than the Fourier Descriptor.

III.5.3 Conclusion

As a local method, the wavelet method has been used to feature extraction. After several success normalizations the features described by wavelet transform are invariant under character size, rotation, and translation. The simulation results show that the performance of wavelet method is the same as Fourier Descriptor in the noise-free environment and much better than FD in noisy environment.

TABLE III

THE ACCURACIES OF RECOGNITION FOR WAVELET METHOD AND FOURIER DESCRIPTOR

<table>
<thead>
<tr>
<th>Character</th>
<th>Wavelet method</th>
<th>Fourier Descriptor</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>80%</td>
<td>100%</td>
</tr>
<tr>
<td>1</td>
<td>100%</td>
<td>100%</td>
</tr>
<tr>
<td>2</td>
<td>100%</td>
<td>100%</td>
</tr>
<tr>
<td>3</td>
<td>100%</td>
<td>100%</td>
</tr>
<tr>
<td>4</td>
<td>100%</td>
<td>100%</td>
</tr>
<tr>
<td>5</td>
<td>100%</td>
<td>100%</td>
</tr>
<tr>
<td>6</td>
<td>100%</td>
<td>90%</td>
</tr>
<tr>
<td>7</td>
<td>100%</td>
<td>100%</td>
</tr>
<tr>
<td>8</td>
<td>100%</td>
<td>100%</td>
</tr>
<tr>
<td>9</td>
<td>100%</td>
<td>90%</td>
</tr>
</tbody>
</table>
TABLE IV
RECOGNITION ACCURACIES UNDER NOISE ENVIRONMENT

<table>
<thead>
<tr>
<th>Variance($\sigma^2$)</th>
<th>Wavelet method</th>
<th>Fourier Descriptor</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>100%</td>
<td>100%</td>
</tr>
<tr>
<td>25</td>
<td>100%</td>
<td>97.25%</td>
</tr>
<tr>
<td>36</td>
<td>98.5%</td>
<td>92.5%</td>
</tr>
<tr>
<td>64</td>
<td>95.25%</td>
<td>87.5%</td>
</tr>
</tbody>
</table>
CHAPTER IV

EVENT RELATED POTENTIALS ANALYSIS

IV.1 INTRODUCTION

Extracting interesting and possibly useful information from the Event Related Potentials (ERP) waveform is an important issue in brain research. The ERP is the time-locked component of brain electrical activity measured as a scalp potential following a stimulus. For instance, a word presented on a computer screen can produce a characteristic ERP waveform.

Peak analysis and area measures are two traditional techniques of ERP data analysis.

Peak analysis is probably the simplest approach to quantify the differences in amplitudes of the ERP waveforms. A peak is defined as either the largest or the smallest voltage value in an interval representing the latency range of the waveform being measured. Because the measurements are based on a single point, moderate amounts of non-systematic variance can obscure real differences, also it reveals nothing about the wave shape.

Unlike peak measurement, area analysis is based on combining values measured at several time points. If an analyst chooses an appropriate interval, then it is likely that most of the contribution of the component to the total ERP will be included. But determination of integration limits is often difficult, or even arbitrary, because components cannot be accurately resolved by visual inspection. Finally, the procedure yields no information about ERP wave shape.
In this chapter, we will use local maximum of wavelet transform of ERP waveform at different scales to describe the ERP's. As mentioned in Chapter II, the local maximum of the wavelet transform at different scales can not only detect the waveform's sharp variations but also characterize its local shape. The values of $W_s f(x)$ at the maximum locations measure the derivative at inflection points. In our experiment, we found that the maximum of the wavelet transform at specific scale was associated with $P_{300}$ latency, and that using wavelet method to identify differences in waveform between two task conditions during a word-comparison experiment was more efficient than using $P_{300}$ peak measurements from the raw data.

IV.2 ERP WAVEFORM AND WAVELET TRANSFORM

IV.2.1 Event Related Potentials Waveform

A subject response to a stimulus can be divided into stimulus processing, encoding, decoding, and response selection, as shown in Figure 13.

In many studies, a typical ERP waveform can be divided into three main segments: (1) $P_1-N_1-P_2-N_2$ complex, (2) $P_{300}$ component and (3) a late potential. An average of the 20 target trials is shown in Figure 14. Because a peak value usually appears at about 300ms after stimulus, this peak point is called $P_{300}$. $P_{300}$ is an important feature in brain waveform analysis. $P_{300}$ latency varies with the subject's attention, alertness, age, stimulus processing speed and memory ability.

As mentioned in Chapter II, wavelet transform of signal can detect sharp variation points, and the values of $W_s f(x)$ at the sharp variation points measure the derivative at inflection points. The largest value of Wavelet transform of ERP waveform is related to $P_{300}$ latency. As a matter of fact, Wavelet method provides an efficient way to measure $P_{300}$ latency.
Figure 13. Block diagram of the response procedure.
Figure 14. Average of the 20 single ERP trials.

IV.2.2 WT of ERP waveform

The wavelet transform of ERP waveform shown in Figure 14 is shown in Figure 15. The local maxima of the wavelet transform are shown in Figure 16.

Figure 16 shows that the position of sharp variation points are shifted as the scale is changed. As mentioned in Chapter II, the scale defines the size of the neighborhood where the signal changes are computed. The larger scale, the more neighborhood is considered. The wavelet transform is sensitive to finer scale.

At the finer scale $2^1$, when SNR is small, the signal is dominated by the noise. Therefore we are more interested in a larger scale. But if the scale is too large, we will lose some detail information. In Figure 16, there are $2^8$ samples, and we found that the
Figure 15. Wavelet transform of ERP waveform shown in Figure 13.
Figure 16. Local maximum of wavelet transform of ERP waveform shown in Figure 13.
Figure 17. A practical brain waveform of 25000 samples.
positions of sharper variation points almost remain the same when scale varies between $2^3$ and $2^5$. Therefore our study focuses on scales between $2^3$ and $2^5$ when we have $2^8$ samples in one stimulus period. The largest value of $W_s f(x)$ at sharp variation points at these scales is related to $P_{300}$ latency of ERP waveform shown in Figure 14.

In next section, we will show how the wavelet transform allow the discrimination of the two types of task conditions in different type of ERP experiment.

IV.3 EXPERIMENTS AND RESULTS

A actual EEG sequence from a scalp electrode at central midline electrode (provided by Erickson Clinic) is shown in Figure 17. There are 25000 sample data points with 250 sample points each second and about 512 sampling points in each stimulus period. The stimulus were presented approximately every two seconds, and our task was to detect the relevant ERP signal within the background electrical noise.

In this experiment, several type of word stimulus shown in Figure 18 were presented. Stimulus 1 is background stimulus (a root word), which is given to each subject before each target stimulus (comparison word) is given. Stimulus 2 denotes a target stimulus that is identical to stimulus 1. Stimulus 3 denotes a target stimulus that is different from stimulus 1. For instance, if stimulus 1 is given by the word "rain", stimulus 2 would be identical, the same as "rain". In the next pair, if stimulus 1 is "hat", stimulus 3 would be different, here, "coat".

![Figure 18. The experiment pattern.](image-url)
Taking the wavelet transform of the signal shown in Figure 17 and finding local maximum of wavelet transform, we obtained the sharp variation points at scale $s = (2^j)_{1 \leq j \leq 10}$. In this case with 512 sampling points each stimulus period, we found that there is no shift for sharpest variation points when scale is between $2^4$ and $2^6$. So that we focused on scale $2^5$. The sharpest variation points in each stimulus period at scale $2^5$, which have the largest wavelet transform value in this stimulus period, are shown in Figure 19.

Subtracting the largest wavelet transform value $W_j^i$ at scale $s = 2^5$ in the $i$th stimulus 3 (different word) period and the largest wavelet transform value $W_{i-3}^j$ at scale $s = 2^5$ in its previous stimulus 1 period, we obtained

$$\Delta W_j^i = W_j^i - W_{i-3}^j$$

Using the same calculation for the stimulus 2 (same word) period, we obtained

$$\Delta W_j^i = W_j^i - W_{i-2}^i$$

Where $i$ means the $i$th stimulus 2, $W_j^i$ is the largest wavelet transform at scale $s = 2^5$ in $i$th stimulus 2 period and $W_{i-2}^i$ is the largest wavelet transform at scale $s = 2^5$ in its previous stimulus 1 period. $\Delta W_j^i$ and $\Delta W_j^i$ are shown in Figure 20.

Figure 20 shows that the value of $\Delta W_j^i$, where $i=1,2,\ldots,11$, is larger than the value of $\Delta W_j^i$, for a young, healthy, normal subject. The value of $W_{2sf}(x)$ at the sharpest variation point in a stimulus period is therefore associated with whether the comparison word is the same or different than the root word.

The largest peak values in each stimulus period are shown in Figure 21. The differences between the largest voltage in the target stimulus period and the largest voltage in its previous background stimulus period are shown in Figure 22.

But either Figure 21 or Figure 22 couldn't tell the distinction between stimulus 3 and stimulus 2.
The same method was then applied to data from an Alzheimer subject, in which verbal processing abilities are reduced. Unlike the young healthy normal subject, there was no distinction between stimulus 2 and stimulus 3. In addition, the largest value of wavelet transform were smaller for three kinds of stimulus than the young healthy normal subjects.

IV.4 CONCLUSION

In summary, not only the local maximum of wavelet transform at different scales can detect the ERP sharp variations but also characterize their local shape. The value of wavelet transform at sharpest variation point is associated with the P300 latency. The wavelet method provide a easy way to identify the differences in waveform between the task condition in a word-comparison experiment. We believe that this work on ERP analysis using the wavelet method is only a beginning. The wavelet method provides very general techniques that can be applied to many tasks in brain wave analysis and therefore has numerous potential applications.
Figure 19. The sharpest variation points in each stimulus period. "+" is related to stimulus 1, "o" is related stimulus 2, and "*" is related stimulus 3.
Figure 20. The differences between the largest WT values in target stimulus periods and the largest WT in its background stimulus 1 periods. "*" is related to stimulus 3, "o" is related stimulus 2.
Figure 21. The largest peak value in each stimulus period. "+" is related to stimulus 1, "o" is related stimulus 2, and "*" is related stimulus 3.
Figure 22. The differences between the largest peak values in target stimulus periods and largest peak values in its previous background stimulus periods. "*" is related to stimulus 3, "o" is related to stimulus 2.
REFERENCES


