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THESIS APPROVAL

The abstract and thesis of Mohammad Azadeh for the Master of Science in Electrical and Computer Engineering were presented October 4, 1994, and accepted by the thesis committee and the department.

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ABSTRACT

An abstract of the thesis of Mohammad Azadeh for the Master of Science in Electrical and Computer Engineering presented October 4, 1994.

Title: Reflection and Refraction of Light from Nonlinear Boundaries.

This thesis deals with the topic of reflection and refraction of light from the boundary of nonlinear materials in general, and saturating amplifiers in particular. We first study some of the basic properties of the light waves in nonlinear materials. We then develop a general formalism to model the reflection and refraction of light with an arbitrary angle of incidence from the boundary of a nonlinear medium. This general formalism is then applied to the case of reflection and refraction from the boundary of linear dielectrics. It will be shown that in this limit, it reduces to the well known Fresnel and Snell's formulas. We also study the interface of a saturating amplifier. The wave equation we use for this purpose is approximate, in the sense that it assumes the amplitude of the wave does not vary significantly in a distance of a wave length. The limits and implications of this approximation are also investigated. We derive expressions for electric field and intensity reflection and transmission coefficients for such materials. In doing so, we make sure that the above mentioned approximation is not violated. These results are compared with the case of reflection and refraction from the interface of a linear dielectric.

REFLECTION AND REFRACTION OF LIGHT FROM NONLINEAR
BOUNDARIES

by
MOHAMMAD AZADEH

A thesis submitted in partial fulfillment of the
requirements for the degree of

MASTER OF SCIENCE
in
ELECTRICAL AND COMPUTER ENGINEERING

Portland State University

1994

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NOTATION AND TERMINOLOGY

E_I	Complex amplitude of the incident electric field
E_R	Complex amplitude of the reflected electric field
E_T	Complex amplitude of the transmitted (refracted) electric field
I_I	Intensity of the incident wave
I_R	Intensity of the reflected wave
I_T	Intensity of the transmitted wave
θ_I	Angle of incidence
θ_R	Angle of reflection
θ_T	Angle of refraction
R_E	Reflection coefficient for electric field, the ratio of the amplitude of the reflected wave to that of the incident wave
T_E	Transmission coefficient for electric field, the ratio of the amplitude of the transmitted wave to that of the incident wave
R	Reflectance, defined as the ratio of the reflected intensity to the incident intensity
T	Transmittance, defined as the ratio of the transmitted intensity to the incident intensity

1. INTRODUCTION

The problem of reflection and refraction of light from dielectric boundaries has been studied for a long time. Most text books on optics include a chapter on the reflection and refraction (or transmission) of light from dielectric boundaries[1]. These classical studies can be summarized in two sets of formulas: The first, known as Snell's law of reflection and refraction, states that the angle of incidence of light is equal to the angle of reflection. Moreover, the ratio of the sine of the angle of incidence to the sine of the angle of refraction is equal to the ratio of the index of refraction of the first medium to that of the second medium. The second set, usually called Fresnel's law, relates the angle of incidence and the amplitude of the incident wave with the amplitudes of the reflected and refracted waves. It should also be mentioned that there are two versions for Fresnel's formulas depending on whether the electric component of the incident field is parallel with or perpendicular to the plane of incidence. Therefore, given an arbitrarily polarized wave incident on an interface with an arbitrary angle of incidence, one can decompose the incident wave into parallel and perpendicular polarized components, and use Fresnel's and Snell's laws to find the amplitude and direction of propagation of the reflected and refracted waves accordingly.

In all these studies, the two dielectrics are assumed to be linear. However, all materials start to behave nonlinearly when subjected to sufficiently large fields. Therefore, some possible deviations from the classical theory of reflection and refraction should be expected when the fields themselves become strong enough to modify the physical properties of the materials involved.

As is usually the case, the nonlinearity causes the problem to become much more difficult in many aspects. It is important to distinguish between two types of phenomena that might

occur in the nonlinear case. The first type, is the reflection and refraction due to the discontinuity *at* the boundary. This type of phenomenon can be viewed as the generalization of the phenomena described by the Fresnel and Snell's formulas in the linear case. However, reflection and refraction might also occur *inside* the nonlinear material, once the light has passed the interface separating the two media. This usually happens when the propagation inside a material is highly nonlinear, for example, in very high gain laser amplifiers[2], or in high loss materials[3].

In this study we are primarily concerned with the first type of phenomenon, however, it should be mentioned that in some cases both phenomena might be present. Studying the second type, or “self-reflection” of light inside the nonlinear materials requires solving the corresponding nonlinear differential equations inside the material. Usually this can not be done analytically, and one must resort to numerical solutions. However, for the first type, where one is considering the reflection and transmission of light at the interface, it is possible to obtain analytical solutions for a larger number of cases. In fact, it will be shown later that the nonlinear steady state behavior of light at a boundary is a function of the first space derivative(s) of the fields at the boundary. This means that if we can express the propagation of the wave inside the material with a first order differential equation, we would be able to relate the magnitudes of the fields at the two sides of the boundary by algebraic equations.

One of the first people who noticed that some interesting phenomena might happen at the boundary of a nonlinear material was Kaplan[4]. He predicted that for certain types of nonlinear materials for which the index of refraction is a linear function of intensity, and for plane waves incident on the boundary of such materials at almost glancing angles, hysteresis effects (or bistability) should be observed. In other words, by changing the angle of inci-

dence or the intensity of the incident light, jumps should be observed from the transmission regime to the regime of total internal reflection and back. Since then many studies of the effect have been carried out. Most of these studies have been on the subject of bistability in the case of Kerr-like materials for which the index of refraction is a linear function of intensity([5]–[10]). These studies involve seeking numerical solutions to the differential equations involved as well as doing experimental studies and measurements for practical setups.

In spite of all these studies, the problem of reflection and transmission at a nonlinear boundary is still controversial, even when it comes to such basic properties as bistable behavior and hysteresis[10]. Part of the reason for this lies in the fact that the reflection of light at the interface of a Kerr-like medium, especially at glancing angles (in which case switching behavior is most likely to occur), involves phenomena occurring at and inside the nonlinear medium. For example, it looks in some cases as if one branch of the beam bends back and “reflects ” out again or even splits into two beams, once entered into the nonlinear medium([5],[6],[9]).

It was not until recently that people started studying other types of nonlinearities. In Ref. [11] the authors claim to have done the first experimental study of the interface of a linear material and a saturable absorber. They have used a solution of HIDCI (hexamethylindodicarbocyanine iodine) in ethylene glycol with a dye concentration of 5.6 mM/L as the saturable medium. There is not much theoretical analysis of the interface of a saturable absorber found in the literature. Many peculiarities appear in the case of a dielectric–absorber interface and the problem of calculating reflection and absorption at such boundaries is far from being understood[12]. In Ref. [12] the authors have done some numerical calculations on

this problem. They have not made the assumption of low gain per wave length, and therefore they have used a second order differential equation for propagation of light inside the nonlinear medium. Numerical solution of the differential equation inside the boundary and application of the boundary conditions at the interface then results in nonlinear reflection coefficients at the boundary, and self-reflection of light inside the nonlinear absorber.

This work is ordered as follows. First some necessary concepts are developed which will be needed later in this study. These concepts include some mathematical relations and expressions which relate the electric and magnetic complex amplitudes of the wave inside the nonlinear medium. Also an expression relating the intensity of the wave with the complex amplitude of the electric field in the general nonlinear case is derived. These results will prove to be useful later, because usually, one is interested in what happens to the intensity of the wave upon reflection or transmission from the boundary. The complex amplitudes of the fields which are not directly measurable quantities are usually of secondary importance. Also these expressions become handy in finding the magnetic component of the field once the electric component is found.

Next, starting from Maxwell's equations and boundary conditions, a rather general formalism to handle the reflection and transmission of light from the interface between two arbitrary nonlinear media is worked out. It is also shown that all the interesting physics involved in the problem can be described by the space derivative(s) of the electric field at the two sides of the boundary.

In the next two chapters some special cases of this general formalism are considered. As the simplest special case, and also as a check on the theory, the limit where both media are linear dielectrics will be discussed. In this way, the familiar Fresnel and Snell's laws will

be derived as special cases of the previous expressions. Then the case where one of the materials is a linear dielectric and the other one is a saturable absorber or amplifier is studied. Here a low gain (loss) per wave length is assumed, which allows for using first order differential equations for the propagation of the waves inside the nonlinear material. While making this assumption will probably result in the neglecting of some interesting phenomena, it enables the expression of the space derivatives of the fields in terms of the fields themselves. Thus it would be possible to obtain algebraic equations for the amplitudes of the fields at the two sides of the boundary.

2. REFLECTION, REFRACTION, AND BOUNDARY CONDITIONS

2.1. PRELIMINARIES

One of the important aspects of propagation of electromagnetic waves in nonlinear media is that unlike the linear case, there is not a simple relationship between the electric and magnetic components of the field. For example, in a very high gain laser amplifier the electric and magnetic components of the wave might become partially decoupled. Moreover, each might have a different “local” wavelength[2] or a different “instantaneous” frequency[13]. In the context of this study, we might be able to find the reflection coefficient for the electric component of the wave, but we might also be interested to find the same coefficient for the magnetic component of the wave. If we have an expression to relate the electric and magnetic components of the waves in the general nonlinear case, our task would become simpler. Once we find such an expression, we would also be able to express the intensity of the wave in terms of the electric field. This is not obvious in the nonlinear case, and is of importance to us because the intensity is the primary measurable quantity.

Usually the electric or magnetic field of an electromagnetic wave is represented as the product of a slowly varying function of time and space and an exponential function which represents the rapid periodic changes in time and space. Often times it becomes necessary to integrate such expressions either with respect to time or space. Let us assume two arbitrary complex functions $F(z,t)$ and $f(z,t)$ defined as:

$$F(z,t) = f(z,t) \exp(ikz - i\omega t) \quad (2.1)$$

In other words, we have assumed that f is the complex amplitude of F . It is possible to express

the integral of F in terms of the derivatives of f and the constants k and ω . Application of the generalized formula of integration by parts[14] results in the following formulas:

$$\int_{-\infty}^t F(z, t') dt' = -\exp(ikz - i\omega t) \sum_{n=0}^N \left(\frac{-i}{\omega}\right)^{n+1} \frac{\partial^n f}{\partial t^n} + \left(\frac{-i}{\omega}\right)^{N+1} \int_{-\infty}^t \exp(ikz - i\omega t') \frac{\partial^{N+1} f}{\partial t'^{N+1}} dt' \quad (2.2)$$

$$\int_{-\infty}^z F(z', t) dz' = -\exp(ikz - i\omega t) \sum_{n=0}^N \left(\frac{i}{k}\right)^{n+1} \frac{\partial^n f}{\partial z^n} + \left(\frac{i}{k}\right)^{N+1} \int_{-\infty}^z \exp(ikz' - i\omega t) \frac{\partial^{N+1} f}{\partial z'^{N+1}} dz' \quad (2.3)$$

where N can be any natural number and the assumption has been made that f is differentiable up to $N+1$ times. A proof of these expressions based on induction is presented in Appendix.

Although (2.2) and (2.3) are purely mathematical formulas, they can be given a physical meaning as well. Physically, if f is a slowly varying function of time and space, the derivative terms inside the integrals in the right hand side of (2.2) and (2.3) for a large enough value of N can be neglected:

$$\int_{-\infty}^t F(z, t') dt' \cong -\exp(ikz - i\omega t) \sum_{n=0}^N \left(\frac{-i}{\omega}\right)^{n+1} \frac{\partial^n f}{\partial t^n} \quad (2.4)$$

$$\int_{-\infty}^z F(z', t) dz' \cong -\exp(ikz - i\omega t) \sum_{n=0}^N \left(\frac{i}{k}\right)^{n+1} \frac{\partial^n f}{\partial z^n} \quad (2.5)$$

More specifically, if the series on the right hand side of these two expressions converge, N can be substituted with infinity. Then these expressions might be considered as expansions

of the integral of a physical quantity (electric or magnetic field) in terms of the derivatives of the complex amplitude of that quantity.

Next we concentrate on electromagnetic waves. For the time being we assume plane waves, and we choose a coordinate system in which the electric field is polarized in the x direction, and the wave is propagating in the positive z direction. With these assumptions, the electric component of the wave can be represented as:

$$E(z, t) = \frac{1}{2} E'(z, t) \exp(ikz - i\omega t) + c.c. \quad (2.6)$$

where $E(z, t)$ is meant to represent the magnitude of the electric field vector in the x direction and the notation $c.c.$ represents complex conjugate. Moreover, the harmonic space and time variations of the field are factored out and $E'(z, t)$ represents the complex amplitude of the electric field. In the case of pure harmonic waves, such as in lossless dielectrics, the complex amplitude E' can be assumed to be a constant. In a steady state analysis, E' can be assumed to be a function of z , and in the most general case, where no assumption on time or space variations are made, it can be a function of both z and t . To be able to derive an expression for the magnetic component of the field, we also assume that the dielectric involved is magnetically linear. With this assumption Ampere's law would be:

$$\frac{\partial E(z, t)}{\partial z} = -\mu \frac{\partial H(z, t)}{\partial t} \quad (2.7)$$

where $H(z, t)$ is meant to represent the magnitude of the magnetic field vector in the y direction and μ is the permeability of the material. This equation relates the electric and magnetic components of the electromagnetic wave together. So in principle, once the electric field is known, by applying (2.7), the magnetic field could also be found. We now assume that

the electric field is given by (2.6). Application of (2.7) requires first differentiating the electric field with respect to z :

$$\frac{\partial H(z, t)}{\partial t} = -\frac{1}{\mu} \frac{\partial E(z, t)}{\partial z} = -\frac{1}{2\mu} \left[\frac{\partial E'(z, t)}{\partial z} + ikE'(z, t) \right] \exp(ikz - i\omega t) + c.c. \quad (2.8)$$

To find H , we should integrate (2.8) with respect to time and we use (2.4) to do so. Application of (2.4) results in:

$$H(z, t) \cong \exp(ikz - i\omega t) \times \frac{1}{2\mu} \sum_{n=0}^N \left(\frac{-i}{\omega} \right)^{n+1} \frac{\partial^n}{\partial t^n} \left[\frac{\partial E'(z, t)}{\partial z} + ikE'(z, t) \right] + c.c. \quad (2.9)$$

Lets assume that H is also decomposed to a harmonic variation and a complex amplitude:

$$H(z, t) = \frac{1}{2} H'(z, t) \exp(ikz - i\omega t) + c.c. \quad (2.10)$$

Comparing (2.10) and (2.9) results in:

$$H'(z, t) \cong \frac{1}{\mu} \sum_{n=0}^N \left(\frac{-i}{\omega} \right)^{n+1} \frac{\partial^n}{\partial t^n} \left[\frac{\partial E'(z, t)}{\partial z} + ikE'(z, t) \right] \quad (2.11)$$

This expression relates the complex amplitude of the electric and magnetic fields. It should be emphasized that at this point k and ω are arbitrary, and hence in deriving (2.11) no assumptions on the form of the fields is made yet. The only assumptions we have made so far are that the electric field is polarized in the x direction, and that the material is magnetically simple.

An important special case of (2.11) is the steady state case, when E' and H' are not functions of time. In this limiting case all the terms of the series in the right hand side of (2.11), except for the first one, will be zero and (2.11) simplifies to the exact expression:

$$H'(z) = \frac{1}{\mu\omega} \left[kE'(z) - i \frac{dE'(z)}{dz} \right] \quad (2.12)$$

where ω is the angular frequency of the wave.

In a linear lossless dielectric, the derivative term in (2.12) will be zero and the familiar relation between the magnetic and electric fields for electromagnetic waves is obtained. It should again be emphasized that in a way (2.12) does not include any approximations, since there are no assumptions on the z dependence of the field. Equation (2.12) is the first of a series of expressions we derive in this study, all of which containing a first order z derivative in them. In all these expressions, depending on the case and the model used, one can substitute the derivative term with the field variable using the differential equation governing the propagation of light for that particular case. If the model involves first order differential equations for propagation of the wave, these expressions will reduce to algebraic equations involving the quantities of interest.

Now we draw our attention to the intensity of the wave. Usually, electric and magnetic fields can not be measured directly or at least easily, and one is primarily interested in the intensity of the wave. For a linear material, the intensity is simply proportional to the square of the electric (magnetic) field. In a nonlinear material, however, the situation might not be as simple[2]. In this study, we use the definitions of the intensity and energy as defined and worked out in Ref. [13] to develop a relationship between the intensity of the wave with the electric field amplitude. These results will be briefly reviewed in the following.

The Poynting vector is defined as the cross product of the electric and magnetic field vectors. With the assumptions made earlier about the electric and magnetic fields being perpendicular, the cross product will be simplified to normal product and one can assume:

$$\begin{aligned}
S(z, t) &= E(z, t)H(z, t) \\
&= \left[\frac{1}{2} E'(z, t) \exp(ikz - i\omega t) + c.c. \right] \left[\frac{1}{2} H'(z, t) \exp(ikz - i\omega t) + c.c. \right] \\
&= \frac{1}{4} [E'(z, t)H'^*(z, t) + E'^*(z, t)H'(z, t) \exp(2ikz - 2i\omega t) + c.c.]
\end{aligned} \tag{2.13}$$

where S is the magnitude of the Poynting vector. Equation (2.13) by itself is complicated and not easy to use. However, averaging over a time of one-half optical period, the complex exponentials can be eliminated and for intensity one obtains:

$$\begin{aligned}
I(z) &= \langle S(z, t) \rangle \\
&= \frac{1}{4} [E'(z)H'^*(z) + E'^*(z)H'(z)]
\end{aligned} \tag{2.14}$$

where E'^* and H'^* are the complex conjugates of the electric and magnetic fields respectively. Now using (2.11), we can substitute for H' in terms of E' to get an expression for the intensity of the electromagnetic field in terms of the amplitude of the electric field. However, to simplify the equations, we assume the steady state case, where (2.12) can be used in stead of (2.11). In this case, we will obtain:

$$I(z) = \frac{k}{2\omega\mu} |E'(z)|^2 + \frac{i}{4\omega\mu} \left[E'(z) \frac{\partial E'^*(z)}{\partial z} - E'^*(z) \frac{\partial E'(z)}{\partial z} \right] \tag{2.15}$$

Equation (2.15) relates the intensity of the electromagnetic wave to the complex amplitude of the electric field in the general nonlinear case. It is possible to write (2.15) in a slightly different way in the case of a dielectric, where the conductivity is assumed to be negligible. In such a material, the propagation constant can be written as: $k = \omega(\mu\epsilon)^{1/2}$. With this as-

sumption (2.15) can be rewritten as:

$$I(z) = \frac{1}{2} \left(\frac{\varepsilon}{\mu} \right)^{1/2} |E'(z)|^2 + \frac{i}{4\omega\mu} \left[E'(z) \frac{\partial E'^*(z)}{\partial z} - E'^*(z) \frac{\partial E'(z)}{\partial z} \right] \quad (2.16)$$

It is easy to check that in the limiting case of a linear lossless dielectric, the terms involving the derivatives will drop and then one is left with the familiar expression for intensity in terms of the square of the electric field. The results obtained in this section are going to be used later in this study.

In the next section, we start from boundary conditions for electric and magnetic fields and develop a general formalism to handle the reflection and refraction (transmission) of light from the interface separating two arbitrary nonlinear media.

2.2. THE WAVE REPRESENTATION AND BOUNDARY CONDITIONS

The reflection and transmission of electromagnetic waves at boundaries with no surface current (or with conductivity) is determined by two boundary conditions; i.e. the continuity of the tangential component of the electric and magnetic fields at the two sides of the boundary:

$$\vec{E}_1(x, y, z, t)|_{z=0} = \vec{E}_2(x, y, z, t)|_{z=0} \quad (2.17)$$

$$\vec{H}_1(x, y, z, t)|_{z=0} = \vec{H}_2(x, y, z, t)|_{z=0} \quad (2.18)$$

where we have assumed that z is the direction normal to the interface and the subscript t represents tangential component. Furthermore, the subscripts 1 and 2 refer to the first and second medium, respectively. If we assume that the materials involved are magnetically linear, one

can write Ampere's law for the two media as:

$$\nabla \times \vec{E}_1 = -\mu_1 \frac{\partial \vec{H}_1}{\partial t} \quad (2.19)$$

$$\nabla \times \vec{E}_2 = -\mu_2 \frac{\partial \vec{H}_2}{\partial t} \quad (2.20)$$

These two equations hold everywhere inside the materials. Therefore, the tangential components of the two sides of these two equations (at the boundary) should also be equal:

$$(\nabla \times \vec{E}_1)_t = -\mu_1 \frac{\partial \vec{H}_{1t}}{\partial t} \quad (2.21)$$

$$(\nabla \times \vec{E}_2)_t = -\mu_2 \frac{\partial \vec{H}_{2t}}{\partial t} \quad (2.22)$$

Noticing that (2.18) should hold for all t , it follows that the time derivative of the tangential component of the magnetic field should also be continuous. However, using (2.21) and (2.22), the time derivative of the tangential component of the magnetic field is proportional to the tangential component of the curl of the electric field. Therefore, one can in general express the boundary conditions in terms of the electric fields at the two sides of the interface as follows:

$$\vec{E}_{1t}(x, y, z, t)|_{z=0} = \vec{E}_{2t}(x, y, z, t)|_{z=0} \quad (2.23)$$

$$\frac{1}{\mu_1} (\nabla \times \vec{E}_1(x, y, z, t))|_{z=0} = \frac{1}{\mu_2} (\nabla \times \vec{E}_2(x, y, z, t))|_{z=0} \quad (2.24)$$

where E_1 and E_2 are vector functions, evaluated at the boundary. Each of the functions repre-

sents the total electric field inside each of the materials and the subscript t represents tangential component.

In a way (2.23) and (2.24) are the most general form of the boundary conditions in terms of the electric fields. Nevertheless, we are primarily interested in waves. In particular, we want to consider a beam of light incident on the interface from the first medium, resulting a reflected beam in the first medium and a transmitted beam in the second medium. To accomplish this we should adopt a more specific form for the electric field.

The first problem now rises from the fact that the electric field is a vector quantity, and only its tangential component is continuous at the two sides of the boundary. Therefore the polarization of the field becomes important. Since we are dealing with nonlinear media, it is not enough to decompose the field to parallel and perpendicular components and treat each component separately. We only consider perpendicular polarization, i.e., we assume that the electric field is only polarized in the direction normal to the plane of incidence (and parallel with the interface). In this case, (2.23) and (2.24) will simplify to:

$$E_1(x, y, z, t)|_{z=0} = E_2(x, y, z, t)|_{z=0} \quad (2.25)$$

$$\frac{1}{\mu_1} \frac{\partial}{\partial z} E_1(x, y, z, t) \Big|_{z=0} = \frac{1}{\mu_2} \frac{\partial}{\partial z} E_2(x, y, z, t) \Big|_{z=0} \quad (2.26)$$

These equations represent the most general form of boundary conditions for perpendicularly polarized electromagnetic waves, where the assumption of linearity in magnetic properties had allowed us to omit magnetic variables.

The next thing to do is to adopt a particular mathematical form for the electric field. The purpose here is to mathematically separate different beam characteristics. These characteristics include the beam transverse and longitudinal profile, direction of propagation, intensity,

etc. Once a particular mathematical representation for the beam is adopted, substituting it into (2.25) and (2.26) results in the necessary relations to find the characteristics of the reflected and transmitted beams.

First, we consider an arbitrary beam of light, propagating in an arbitrary direction. We can represent this beam in the following form:

$$E(x, y, z, t) = \frac{1}{2}A(x, y')B(z') \exp i(kz' - \omega t) + c.c. \quad (2.27)$$

where A and B are complex functions and represent the transverse and longitudinal profile of the beam respectively.

In the above equation, a new coordinate system (x, y', z') has been introduced. This coordinate system is tilted in the direction of propagation of the beam such that the beam is propagating along the positive z' axis. The two coordinate systems are related to each other via the following transformation :

$$z' = z \cos \theta - y \sin \theta \quad (2.28)$$

$$y' = z \sin \theta + y \cos \theta \quad (2.29)$$

where θ is the angle between the two coordinate systems as defined by the following figure:

Fig.(2.1) Definition of the reference coordinate system and the primed coordinate system attached to the beam with respect to the interface

The introduction of the new coordinate system is useful, since the beam can now be represented in the new coordinate system in such a way that the different characteristics of the beam are decoupled as suggested by (2.27). It should further be noted that in assuming (2.27), we have also assumed that the beam's transverse profile is not changing along the direction of propagation. That is, A is a function of x and y' only. In our study here, this can be justified because we are only interested in the behavior of the beam in the vicinity of the boundary. For that matter, we can assume that the beam profile changes negligibly in the vicinity of the interface. We have also assumed the steady state case in which A and B in (2.27) are not functions of time. All these assumptions can be removed if we want to have a more accurate model.

Now by assuming (2.27) as the general form for the beam, the incident, reflected, and transmitted beams can be represented as:

$$E_I(x, y, z, t) = \frac{1}{2} A_I(x, y') B_I(z') \exp i(k_1 z' - \omega t) + c.c. \quad (2.30)$$

$$E_R(x, y, z, t) = \frac{1}{2} A_R(x, y') B_R(z') \exp i(k_1 z' - \omega t) + c.c. \quad (2.31)$$

$$E_T(x, y, z, t) = \frac{1}{2} A_T(x, y') B_T(z') \exp i(k_2 z' - \omega t) + c.c. \quad (2.32)$$

where the subscripts I , R , and T represent the incident, reflected, and transmitted beams respectively. It should be mentioned that in each of these equations, the primed coordinate system is with reference to the beam itself. Hence, each primed coordinate system is different from the primed coordinate system used to represent another beam. Therefore, for each of the equations (2.30), (2.31), and (2.32), the coordinate system (x, y', z') is related to the main coordinate system (x, y, z) according to the transformations (2.28) and (2.29). In above equation θ for the incident beam is $\pi + \theta_I$, for the reflected beam it is $-\theta_R$, and for the transmitted beam it is $\pi + \theta_T$. The following figure depicts the situation for each case:

Fig.(2.2) Definition of the angles of incidence, reflection, and refraction

Also it should be noted that at this point we have not necessarily assumed that the angle of incidence is equal to the angle of reflection.

The next step is to apply the boundary conditions (2.25) and (2.26) using the beam representation just introduced. Using (2.25), one will obtain:

$$E_I(x, y, z, t)|_{z=0} + E_R(x, y, z, t)|_{z=0} = E_T(x, y, z, t)|_{z=0} \quad (2.33)$$

And substituting from (2.30)–(2.32) and cancelling the $\exp(i\omega t)$ terms from both sides of the equation, one obtains:

$$A_I(x, y')B_I(z')\exp i(k_1 z')|_{z=0} + A_R(x, y')B_R(z')\exp i(k_1 z')|_{z=0} = A_T(x, y')B_T(z')\exp i(k_2 z')|_{z=0} \quad (2.34)$$

However, one should remember that the primed coordinates used in (2.34) are relative to the beams themselves. In each case the beam is propagating along the positive z' direction of that coordinate system, furthermore, these coordinates are not the same in the incident, reflected, and transmitted beams. Therefore the next step is to write (2.34) with respect to the “absolute” coordinate system which is attached to the interface. In doing so, the transformations (2.28) and (2.29) can be used, replacing θ in each of the beams with the appropriate angle, i.e., the angle between the beam and the normal to the interface:

$$\begin{aligned} A_I(x, -y \cos \theta_I)B_I(y \sin \theta_I)\exp i(k_1 y \sin \theta_I)|_{z=0} + A_R(x, y \cos \theta_R)B_R(y \sin \theta_R)\exp i(k_1 y \sin \theta_R)|_{z=0} \\ = A_T(x, -y \cos \theta_T)B_T(y \sin \theta_T)\exp i(k_2 y \sin \theta_T)|_{z=0} \end{aligned} \quad (2.35)$$

where the substitution $z=0$ was made wherever z had appeared explicitly.

This equation relates the characteristics of the incident beam with those of the reflected and transmitted beams. In the linear theory of reflection of plane waves from dielectric media, all the A and B functions in the above equation are simply constants. Therefore, one is left with the exponential terms and the only way to satisfy (2.35) for all x and y is to require:

$$k_1 y \sin \theta_I = k_1 y \sin \theta_R = k_2 y \sin \theta_T \quad (2.36)$$

which is Snell's law of reflection and refraction stating the fact that the angle of incidence is equal to the angle of reflection, and the ratio of the angle of incidence to the angle of refraction is equal to the ratio of the indices of refraction of the two media.

It is evident from (2.35) that in the general nonlinear case the above condition might not be the case. In fact, the only thing one can say at this point is:

$$B_I(y \sin \theta_I) \exp i(k_1 y \sin \theta_I) + B_R(y \sin \theta_R) \exp i(k_1 y \sin \theta_R) = B_T(y \sin \theta_T) \exp i(k_2 y \sin \theta_T) \quad (2.37)$$

In other words, the phase expressions which contain the angles are now coupled with the gain expressions. It is worth asking that under what conditions it is possible to deduce (2.36) from (2.37). One possible answer might be related to how fast the amplitude terms in (2.37) vary with distance. The exponential terms in (2.37) are periodic functions changing in distances of the order of the wavelength. On the other hand, the B functions in (2.37) which represent the amplitude of the wave along the surface are usually slowly varying functions. Therefore, it seems one can reach at the following conclusion : If the gain or loss is small enough so that the amplitude of the wave changes negligibly in a wavelength, it is possible to deduce (2.36) from (2.37) ,i.e., Snell's law of reflection and refraction holds. If this assumption is not true,

(2.37) is not conclusive by itself. Hence, one needs to consider the other equation resulting from (2.26) as well.

So far one of the boundary conditions has been used. We saw in certain cases (certainly in the limiting case of linear dielectrics) one can find the angles of the reflected and transmitted beams from this first equation. However, the amplitudes of the reflected and transmitted waves are also unknown. Therefore, we should consider the second equation (2.26) as well.

To do this, we again choose (2.30)–(2.32) as the form of the beams, and substitute these in (2.26). We then obtain:

$$\frac{1}{\mu_1} \frac{\partial}{\partial z} (E_I + E_R) \Big|_{z=0} = \frac{1}{\mu_2} \frac{\partial}{\partial z} E_T \Big|_{z=0} \quad (2.38)$$

or equivalently:

$$\begin{aligned} & \frac{1}{\mu_1} \left(B_I \frac{\partial A_I}{\partial z} + A_I \frac{\partial B_I}{\partial z} + ik_1 A_I B_I \frac{\partial z'}{\partial z} \right) \exp i(k_1 z' - \omega t) \Big|_{z=0} \\ & + \frac{1}{\mu_1} \left(B_R \frac{\partial A_R}{\partial z} + A_R \frac{\partial B_R}{\partial z} + ik_1 A_R B_R \frac{\partial z'}{\partial z} \right) \exp i(k_1 z' - \omega t) \Big|_{z=0} \\ & = \frac{1}{\mu_2} \left(B_I \frac{\partial A_I}{\partial z} + A_I \frac{\partial B_I}{\partial z} + ik_2 A_T B_T \frac{\partial z'}{\partial z} \right) \exp i(k_2 z' - \omega t) \Big|_{z=0} \end{aligned} \quad (2.39)$$

Noticing that E_I , E_R , and E_T are primarily expressed in primed coordinates, and using the chain rule, we obtain:

$$\begin{aligned}
& \frac{1}{\mu_1} \left(\frac{\partial A_I}{\partial y'} \frac{\partial y'}{\partial z} B_I + A_I \frac{\partial B_I}{\partial z'} \frac{\partial z'}{\partial z} + ik_1 A_I B_I \frac{\partial z'}{\partial z} \right) \exp i(k_1 z') \Big|_{z=0} + \\
& \frac{1}{\mu_1} \left(\frac{\partial A_R}{\partial y'} \frac{\partial y'}{\partial z} B_R + A_R \frac{\partial B_R}{\partial z'} \frac{\partial z'}{\partial z} + ik_1 A_R B_R \frac{\partial z'}{\partial z} \right) \exp i(k_1 z') \Big|_{z=0} \\
& = \frac{1}{\mu_2} \left(\frac{\partial A_T}{\partial y'} \frac{\partial y'}{\partial z} B_T + A_T \frac{\partial B_T}{\partial z'} \frac{\partial z'}{\partial z} + ik_1 A_T B_T \frac{\partial z'}{\partial z} \right) \exp i(k_2 z') \Big|_{z=0} \quad (2.40)
\end{aligned}$$

where the $\exp(i\omega)$ terms have been cancelled from both sides. Again, remembering that the primed coordinates are different for each beam, and using the transformations (2.28) and (2.29) for each, we obtain:

$$\begin{aligned}
& \frac{1}{\mu_1} \left(- \frac{\partial A_I}{\partial y'} B_I \sin \theta_I - A_I \frac{\partial B_I}{\partial z'} \cos \theta_I - ik_1 A_I B_I \cos \theta_I \right) \exp i(k_1 y \sin \theta_I) \Big|_{z=0} + \\
& \frac{1}{\mu_1} \left(- \frac{\partial A_R}{\partial y'} B_R \sin \theta_R + A_R \frac{\partial B_R}{\partial z'} \cos \theta_R + ik_1 A_R B_R \cos \theta_R \right) \exp i(k_1 y \sin \theta_R) \Big|_{z=0} \\
& = \frac{1}{\mu_2} \left(- \frac{\partial A_T}{\partial y'} B_T \sin \theta_T - A_T \frac{\partial B_T}{\partial z'} \cos \theta_T - ik_2 A_T B_T \cos \theta_T \right) \exp i(k_2 y \sin \theta_T) \Big|_{z=0} \quad (2.41)
\end{aligned}$$

It should be noted that both sides of (2.41) are to be evaluated at the boundary, and the substitution $z=0$ has been made wherever z had appeared explicitly. Therefore, one should be careful that all the functions in (2.41), including the derivative terms, should be evaluated at $z=0$.

Equation (2.41) is the second equation which relates, in general, the characteristics of the incident beam with those of the reflected and transmitted beams. Therefore, (2.41) along with (2.35) form a complete set which enables us to calculate the amplitudes and angles of the reflected and transmitted beams.

Considering (2.41) carefully reveals that the reflection and transmission characteristics of the beam are related to:

- 1) The amplitudes of the beams at the boundary;
- 2) The derivative of the transverse profile of the beam along the direction perpendicular; to the direction of the propagation at the boundary. Obviously, this is a consequent of the beam (transverse) profile *inside* the boundary;
- 3) The derivative of the longitudinal profile of the beam along the direction of the propagation at the boundary. Of course, this is again the propagation characteristic of the beam *inside* either medium, and is a result of the solution of the nonlinear wave equation inside either medium;
- 4) The angle of propagation of the beam with respect to the interface.

Again it should be emphasized that in general (2.41) and (2.35) should be considered together. Using them, one should be able to calculate the amplitudes and angles of propagation of the reflected and transmitted beams, given the amplitude and angle of propagation of the input beam *and* the propagation characteristics for beams inside both media close to the interface. It is clear that (2.41) and (2.35) are in general coupled, and one can not simply deduce Snell's law for the angles of propagation of the beams, as would be the case with the limiting case of linear dielectrics.

In conclusion, in this section we obtained two general equations. In this way, given the amplitude and angle of the incident beam, one should be able to calculate those of the reflected and transmitted beams. For further reference we rewrite these two equations together again.

$$A_I B_I \exp i(k_1 z') + A_R B_R \exp i(k_1 z') = A_T B_T \exp i(k_2 z') \quad (2.42)$$

$$\begin{aligned} & \frac{1}{\mu_1} \left(-\frac{\partial A_I}{\partial y'} B_I \sin \theta_I - A_I \frac{\partial B_I}{\partial z'} \cos \theta_I - i k_1 A_I B_I \cos \theta_I \right) \exp i(k_1 z') + \\ & \frac{1}{\mu_1} \left(-\frac{\partial A_R}{\partial y'} B_R \sin \theta_R + A_R \frac{\partial B_R}{\partial z'} \cos \theta_R + i k_1 A_R B_R \cos \theta_R \right) \exp i(k_1 z') \\ & = \frac{1}{\mu_2} \left(-\frac{\partial A_T}{\partial y'} B_T \sin \theta_T - A_T \frac{\partial B_T}{\partial z'} \cos \theta_T - i k_2 A_T B_T \cos \theta_T \right) \exp i(k_2 z') \end{aligned} \quad (2.43)$$

where the A and B functions are primarily expressed in terms of the primed coordinates evaluated at $z=0$. The angles are all assumed to be absolute values and are defined in Fig.(2.2). Alternatively, we can explicitly express the arguments of some of these functions in terms of the absolute coordinates right at this moment, but this will make the notation much more difficult, and it doesn't work for the derivative terms anyway.

2.3. PLANE WAVES

Up until now we were focusing on the general case of the beams. From now on, however, we focus on the important special case of plane waves. We postulate at this moment that if the incident wave is a plane wave, the transmitted and reflected waves will also be plane waves. With this assumption, we can assume that the A functions in (2.42) and (2.43), representing the transverse profile of the waves, are constants. Therefore (2.42) and (2.43) can be rewritten:

$$A_I B_I(z') \exp i(k_1 z') + A_R B_R(z') \exp i(k_1 z') = A_T B_T(z') \exp i(k_2 z') \quad (2.44)$$

$$\begin{aligned} & \frac{1}{\mu_1} \left(-A_I \frac{\partial B_I(z')}{\partial z'} \cos \theta_I - ik_1 A_I B_I(z') \cos \theta_I \right) \exp i(k_1 z') + \\ & \frac{1}{\mu_1} \left(A_R \frac{\partial B_R(z')}{\partial z'} \cos \theta_R + ik_1 A_R B_R(z') \cos \theta_R \right) \exp i(k_1 z') \\ & = \frac{1}{\mu_2} \left(-A_T \frac{\partial B_T(z')}{\partial z'} \cos \theta_T - ik_2 A_T B_T(z') \cos \theta_T \right) \exp i(k_2 z') \end{aligned} \quad (2.45)$$

The above equations, not being with reference to one single coordinate system, are not very useful. Therefore, we should express these equations in terms of the absolute coordinate system. Using the transformations (2.28) and (2.29), and with appropriate choice of θ for each case, we will obtain:

$$A_I B_I(y \sin \theta_I) \exp i(k_1 y \sin \theta_I) + A_R B_R(y \sin \theta_R) \exp i(k_1 y \sin \theta_R) = A_T B_T(y \sin \theta_T) \exp i(k_2 y \sin \theta_T) \quad (2.46)$$

$$\begin{aligned} & \frac{1}{\mu_1} \cos \theta_I \left(-A_I \frac{\partial B_I(z')}{\partial z'} - ik_1 A_I B_I(y \sin \theta_I) \right) \exp i(k_1 y \sin \theta_I) + \\ & \frac{1}{\mu_1} \cos \theta_R \left(A_R \frac{\partial B_R(z')}{\partial z'} + ik_1 A_R B_R(y \sin \theta_R) \right) \exp i(k_1 y \sin \theta_R) \\ & = \frac{1}{\mu_2} \cos \theta_T \left(-A_T \frac{\partial B_T(z')}{\partial z'} - ik_2 A_T B_T(y \sin \theta_T) \right) \exp i(k_2 y \sin \theta_T) \end{aligned} \quad (2.47)$$

These expressions couple different characteristics of the incident, reflected, and transmitted plane waves together for an arbitrary angle of incidence. As is clear from these equations, the derivative of the fields at the boundary also plays a role in the problem. These derivatives model the nonlinearities involved: In order to solve the problem of reflection and refraction, one should also know the propagation characteristics of the wave inside the medium close

to the boundary. In general this requires solving the nonlinear wave equation in either medium. So in a way the problem of nonlinear reflection and refraction of light is reduced to nonlinear propagation of light. However, in some cases these derivatives can be expressed without having to solve the nonlinear wave equation. This can be done when the wave equation governing the propagation of light in the medium is of first order.

It should also be noticed that the form we had chosen for the wave has been useful. It has allowed us to separate the transverse profile of the wave (which in this case is constant) from the longitudinal profile of the wave. Another point to emphasize is that unlike the linear case, Snell's law of reflection and refraction does not necessarily follow from these equations. In the linear case, the only way for (2.46) to hold for all x and y is to require $\theta_I = \theta_R$ and $k_I \sin(\theta_I) = k_2 \sin(\theta_R)$. This is due to the fact that B_I , B_R , and B_T are constants in this case. In the nonlinear case, this may not necessarily be true in general and the angle and amplitude unknowns are coupled. However, one might ask under what conditions Snell's law follows from (2.46)? The exponential terms in (2.46) are fast functions of distance. Therefore, if the functions B_I , B_R , and B_T are slow functions of distance (vary negligibly in a wavelength), one can decouple the amplitude and phase terms in (2.46) and require:

$$\exp i(k_1 y \sin \theta_I) = \exp i(k_1 y \sin \theta_R) = \exp i(k_2 y \sin \theta_T) \quad (2.48)$$

$$A_I B_I(y \sin \theta_I) + A_R B_R(y \sin \theta_R) = A_T B_T(y \sin \theta_T) \quad (2.49)$$

The situation is then similar to the linear case, Snell's laws follow from (2.48), and (2.49) should be solved together with (2.47) in order to obtain a relationship between the amplitudes of the fields.

A very important special case of (2.46) and (2.47) is the case of normal incidence. In this case, it is assumed that the incident, reflected, and refracted waves are all propagating normal to the interface. The primed coordinates will thus be the same as the absolute coordinates. If one assumes that $\theta_I = \theta_R = \theta_T = 0$; then (2.46) and (2.47) will simplify to:

$$A_I B_I + A_R B_R = A_T B_T \quad (2.50)$$

$$\frac{1}{\mu_1} \left(-A_I \frac{\partial B_I(z)}{\partial z} - ik_1 A_I B_I \right) + \frac{1}{\mu_1} \left(A_R \frac{\partial B_R(z')}{\partial z'} + ik_1 A_R B_R \right) = \frac{1}{\mu_2} \left(-A_T \frac{\partial B_T(z')}{\partial z'} - ik_2 A_T B_T \right) \quad (2.51)$$

We will later use these equations to examine the case of normal reflection and refraction.

In the next chapters we use the formalism developed here to study some specific problems. We first start with the linear case. We show that in this limit, the formalism developed here will reduce to the familiar Fresnel and Snell's formulas. Then we consider the case of internal and external reflection from a saturating material, which might have gain or loss.

3. REFLECTION AND REFRACTION FROM LINEAR DIELECTRICS

In the previous chapter we derived some expressions ((2.44) and (2.45)) governing the reflection and refraction of plane waves from a boundary of a nonlinear medium. In this chapter, our goal is to study the simplest special case of those expressions, i.e., the linear case. As a check on (2.44) and (2.45), we expect to obtain Snell's law for angles of incidence, reflection, and refraction, and Fresnel equations for the amplitudes of reflected and transmitted waves.

In this case we are dealing with linear lossless media, therefore, the longitudinal profile of the wave, i.e., B_I , B_R , and B_T are constants. Hence their derivatives are zero. With these simplifications, (2.44) and (2.45) will reduce to:

$$A_I B_I \exp i(k_1 z') + A_R B_R \exp i(k_1 z') = A_T B_T \exp i(k_2 z') \quad (3.1)$$

$$\begin{aligned} \frac{1}{\mu_1} (-ik_1 A_I B_I \cos \theta_I) \exp i(k_1 y \sin \theta_I) + \frac{1}{\mu_1} (ik_1 A_R B_R \cos \theta_R) \exp i(k_1 y \sin \theta_R) \\ = \frac{1}{\mu_2} (-ik_2 A_T B_T \cos \theta_T) \exp i(k_2 y \sin \theta_T) \end{aligned} \quad (3.2)$$

As expected in these equations, A and B appear together as products, which is the amplitude of the field. Also we should express z' in these equations in terms of the absolute coordinates. The results will be:

$$E_I \exp i(k_1 y \sin \theta_I) + E_R \exp i(k_1 y \sin \theta_R) = E_T \exp i(k_2 y \sin \theta_T) \quad (3.3)$$

$$\begin{aligned} \frac{1}{\mu_1} (-ik_1 E_I \cos \theta_I) \exp i(k_1 y \sin \theta_I) + \frac{1}{\mu_1} (ik_1 E_R \cos \theta_R) \exp i(k_1 y \sin \theta_R) \\ = \frac{1}{\mu_2} (-ik_2 E_T \cos \theta_T) \exp i(k_2 y \sin \theta_T) \end{aligned} \quad (3.4)$$

The first expression should be valid for all values of y . The only way for this to happen is require:

$$\exp i(k_1 y \sin \theta_I) = \exp i(k_1 y \sin \theta_R) = \exp i(k_2 y \sin \theta_T) \quad (3.5)$$

$$E_I + E_R = E_T \quad (3.6)$$

Expression (3.5) results in Snell's law of reflection and refraction:

$$k_1 \sin \theta_I = k_1 \sin \theta_R = k_2 \sin \theta_T$$

$$\theta_I = \theta_R, \quad \frac{\sin \theta_I}{\sin \theta_T} = \frac{k_2}{k_1} \quad (3.7)$$

Expression (3.6) relates the amplitudes of the waves, but is not conclusive by itself. We should consider (3.4) now, which will further simplify if we use (3.7):

$$\frac{1}{\mu_1} (k_1 E_I \cos \theta_I) - \frac{1}{\mu_1} (k_1 E_R \cos \theta_R) = \frac{1}{\mu_2} (k_2 E_T \cos \theta_T) \quad (3.8)$$

Combining (3.8) with (3.6) results in the familiar Fresnel formulas[1]:

$$T_E \equiv \frac{E_T}{E_I} = \frac{2k_1\mu_2 \cos \theta_I}{k_1\mu_2 \cos \theta_I + k_2\mu_1 \cos \theta_T} \quad (3.9)$$

$$R_E \equiv \frac{E_R}{E_I} = \frac{k_1\mu_2 \cos \theta_I - k_2\mu_1 \cos \theta_T}{k_1\mu_2 \cos \theta_I + k_2\mu_1 \cos \theta_T} \quad (3.10)$$

where T_E and R_E are defined as the *transmission* and *reflection coefficients* for the electric field. Expressions (3.9) and (3.10) are Fresnel equations for the transmission and reflection coefficients for the case of perpendicularly polarized electric fields. Usually the materials involved are magnetically similar, that is, $\mu_1 = \mu_2 = \mu_0$. Also, the materials are assumed to be pure dielectrics which means one can assume $k_1 = \omega n_1 / c$ and $k_2 = \omega n_2 / c$. Substituting these in (3.9) and (3.10), we get:

$$T_E \equiv \frac{E_T}{E_I} = \frac{2n_1 \cos \theta_I}{n_1 \cos \theta_I + n_2 \cos \theta_T} \quad (3.11)$$

$$R_E \equiv \frac{E_R}{E_I} = \frac{n_1 \cos \theta_I - n_2 \cos \theta_T}{n_1 \cos \theta_I + n_2 \cos \theta_T} \quad (3.12)$$

These two equations are the more familiar forms of Fresnel's equations for the reflection and refraction of a perpendicularly polarized plane wave from the boundary of a dielectric.

It is also useful to derive the reflection and transmission coefficients for the intensity of the waves. The *transmittance* of the surface is defined as the ratio of the transmitted power to the incident power. Remembering that the power crossing a certain area is proportional to the cosine of the angle of incidence, one can write:

$$T \equiv \frac{\text{Transmitted power}}{\text{Incident power}} = \frac{I_T \cos \theta_T}{I_I \cos \theta_I} = \frac{n_2 \cos \theta_T}{n_1 \cos \theta_I} \left(\frac{E_T}{E_I} \right)^2 = \frac{n_2 \cos \theta_T}{n_1 \cos \theta_I} (T_E)^2 \quad (3.13)$$

So by knowing the transmission coefficient for the electric field, one can find the transmittance of the surface using (3.13).

The *reflectance* of an interface is defined as the ratio of the reflected power to the incident power. So the reflectance is related to the reflection coefficient for the electric field by:

$$R \equiv \frac{\text{Reflected power}}{\text{Incident power}} = \frac{I_R \cos \theta_R}{I_I \cos \theta_I} = \frac{I_R}{I_I} = \left(\frac{E_R}{E_I} \right)^2 = (R_E)^2 \quad (3.14)$$

In chapter 4, we will generalize these formulas for the case of a nonlinear saturating medium.

We do not continue the subject of linear reflection and refraction any further as complete discussion on this topic can be found in any standard text book[1]. Our purpose, however, was to check the formalism developed earlier in the easiest special case and to make sure that the reduction will result in the familiar Fresnel and Snell's formulas. Also it would be useful to compare the expressions we derive in chapter 4 for a nonlinear interface with these standard expressions for a linear interface. In the next chapter, we study the case of a saturable material.

4. REFLECTION AND REFRACTION AT A SATURABLE INTERFACE

4.1. LIGHT PROPAGATION IN SATURABLE AMPLIFIERS

As was discussed earlier, reflection and refraction from a nonlinear boundary is a direct consequence of nonlinear propagation characteristics inside the materials at the two sides of the interface. Our purpose now is to examine the case of saturable materials. These materials might act as amplifiers, or as absorbers. The formalism for the two types is not very different. In this study we consider the case of amplifiers.

To find the governing equation for the propagation of the light waves in a saturating amplifier, one might start from Maxwell's equations. Using these equations and adopting a propagating wave form for the electric field and the polarization, one would obtain a second order differential equation for the complex amplitude of the electric field with the polarization as the driving factor. For convenience, we rewrite equation (7) in ref. [2] which is the result of the procedure just mentioned :

$$c^2 \frac{d^2 E'(z)}{dz^2} + 2i\Omega c \frac{dE'(z)}{dz} + [\omega^2 - \Omega^2 + i\frac{\sigma}{\epsilon}\omega]E'(z) = -\frac{\omega^2}{\epsilon} \int_0^\infty \int_{-\infty}^\infty P'(\nu, \omega_a, z) d\nu d\omega_a \quad (4.1)$$

where $E'(z)$ is the complex amplitude of the electric field vector in x direction, $c=(\mu\epsilon)^{-1/2}$ and $\Omega=k(\mu\epsilon)^{-1/2}$, and P' is the complex amplitude of the polarization. It is also assumed that the field is propagating in the z direction. This equation is a second order differential equation, and it is usual to eliminate the second derivative with respect to the first derivative, by arguing that the field envelope varies negligibly in a wave length. The resulting equation (equation (8) in ref. [2]) is then of first order.

Our starting point is the equations (60) and (61) in ref. [2], which describe the behavior of a laser amplifier in the limit of low gain per wavelength. For convenience, we repeat those two equations here:

$$c \frac{dA_r(z)}{dz} = -\gamma_c \left[A_r(z) + \frac{yA_i(z) - A_r(z)}{1 + y^2 + A_r(z)^2 + A_i(z)^2} D_0(z) \right] \quad (4.2)$$

$$c \frac{dA_i(z)}{dz} = -\gamma_c \left[A_i(z) - \frac{yA_r(z) + A_i(z)}{1 + y^2 + A_r(z)^2 + A_i(z)^2} D_0(z) \right] \quad (4.3)$$

In the above equations, $\gamma_c = \sigma/2\varepsilon$ is the field decay rate, $y = (\omega - \omega_0)/\gamma$ is the normalized frequency with ω_0 the center frequency of transition, and $D_0(z)$ is the normalized unsaturated population difference. Moreover, $A_r(z)$ and $A_i(z)$ are the real and imaginary parts of $A(z)$, the normalized complex amplitude of the electric field inside the medium given by:

$$A(z) = \frac{\mu}{\hbar} \left[\frac{\gamma_a - \gamma_{ab} + \gamma_b}{2\gamma\gamma_a\gamma_b} \right]^{1/2} E'(z) \equiv mE'(z) \quad (4.4)$$

where the parameter m is introduced as a normalization constant (which is a function of the properties of the material). In this equation $E'(z)$ is the complex amplitude of the electric field, μ is the dipole moment of the transition, γ_a and γ_b are the total decay rates for the upper and lower laser levels, γ_{ab} is the rate of direct decays from the upper level to the lower level, and γ is the decay rate for the off-diagonal elements. The normalized unsaturated population difference used in (4.2) and (4.3) is given by:

$$D_0(U, z) = \frac{\gamma\omega_0\mu^2}{2k\varepsilon\gamma\hbar} D_0(\omega_a, z) \quad (4.5)$$

where μ^2 in this definition is the square of the magnitude of the dipole moment of the transition and $D_0(\omega_a, z)$ is the population difference between the lower and upper lasing levels.

Adding the two equations (4.2) and (4.3), a single equation for the normalized complex amplitude of the field can be obtained. It is also useful to work with the non-normalized fields since the normalization constant is a function of the parameters of the medium. In other words, it changes while going from one medium to another. Also to keep the notation simple, from now on we drop the primes from the field variables while maintaining their significance as complex amplitudes. The result of adding (4.2) and (4.3) would be:

$$c \frac{dE(z)}{dz} = -\gamma_c \left[1 - \frac{(1 + iy)D_0(z)}{1 + y^2 + m^2|E(z)|^2} \right] E(z) \quad (4.6)$$

where use has been made of (4.4) to represent the field in the non-normalized form. Finally, dividing both sides by c and noticing that the speed of light in the material can be written as the ratio of the speed of light in vacuum over the index of refraction, and also noticing that the definition of D_0 as given by (4.5) involves γ_c , (4.6) can be rewritten as:

$$\frac{dE(z)}{dz} = \left[-\frac{n\gamma_c}{c} + \frac{n}{c} \times \frac{(1 + iy)D_0(z)}{1 + y^2 + m^2|E(z)|^2} \right] E(z) \quad (4.7)$$

where n is the index of refraction of the material and c is now the speed of light in vacuum. Therefore, the term D_0 used in the above equation is now defined as:

$$D_0(U, z) = \frac{\gamma \omega_a \mu^2}{2k\epsilon\hbar} D_0(\omega_a, z) \quad (4.8)$$

Equation (4.7) is the general nonlinear wave equation for the propagation of light waves. The only approximation here is the assumption of slowly varying wave amplitudes. Howev-

er, in most of the practical problems, the conductivity of the dielectrics involved is negligible. This means that γ_c can be set to zero. This allows (4.7) to be written in a simpler way as:

$$\frac{dE(z)}{dz} = \frac{n}{c} \left[\frac{(1 + iy)D_0(z)}{1 + y^2 + m^2|E(z)|^2} \right] E(z) \quad (4.9)$$

This equation allows for expressing the derivative of the electric field inside the saturable material in terms of the field itself. This is what we need in order to use in expressions (2.46) and (2.47) to study the characteristics of reflection and refraction from a saturable medium.

It is now important to investigate the range of parameters for which (4.9) would be valid. In other words, one should be aware of the fact that by choosing some specific sets of values for the parameters in (4.9), the slowly varying amplitude assumption, according to which (4.9) is derived, might be violated. This translates roughly to an upper limit for the pump term in this equation.

It is possible to impose a more quantitative measure for the validity of the wave equation (4.9). One should remember that in deriving this equation, the second derivative of the field in (4.1) was assumed to be negligible compared to the first derivative. In other words, the following assumption was made:

$$c^2 \left| \frac{d^2 E'(z)}{dz^2} \right| \ll 2\Omega c \left| \frac{dE'(z)}{dz} \right| \quad (4.10)$$

To investigate the possible implications of this approximation, we calculate the second derivative of the complex amplitude of the field using the wave equation (4.9). For simplicity, we consider the case of line center ($y=0$) and constant pump. The result would be:

$$\frac{d^2 E}{dz^2} = \frac{D_0 n}{c} \times \frac{\frac{dE}{dz} - m^2 E^2 \frac{dE^*}{dz}}{(1 + m^2 E E^*)^2} \quad (4.11)$$

Now, we can again substitute for the first derivative terms in this equation, and obtain an expression for the second derivative of the complex amplitude of the electric field in terms of the field itself and the constant parameters:

$$\frac{d^2 E}{dz^2} = \left(\frac{D_0 n}{c} \right)^2 \times \frac{1 - m^2 E E^*}{(1 + m^2 E E^*)^3} E \quad (4.12)$$

Finally, substituting (4.12) and (4.9) in (4.10) one obtains:

$$G \equiv \frac{D_0 |1 - m^2 E E^*|}{2\omega (1 + m^2 E E^*)^2} \ll 1 \quad (4.13)$$

where ω is the frequency of the electromagnetic wave.

Also a new parameter G is introduced as a measure of the error. In other words, as long as the parameter G as defined by (4.13) is “small” compared to unity, the slowly varying amplitude approximation is valid. Equation (4.13) defines a “parameter space” in which the wave equation is valid. In other words, it is a general restriction for the application of the wave equation, and one should be aware that if it is not satisfied, the slowly varying amplitude approximation might be violated. In that case, one should use the second order wave equation from the beginning. It is clear from (4.13), as expected, that increasing the pump will generally reduce the accuracy of the approximation. It is also interesting to note that for one particular intensity, i.e., when $m^2 E E^* = 1$, the error completely vanishes. Moreover,

in general, the condition (4.13) is better satisfied with higher intensities. This can easily be described, since higher intensities cause a decrease in the effective gain.

In chapter 2, we derived an expression for the intensity of electromagnetic waves in a general nonlinear medium. It is convenient to use that expression and obtain an expression for intensity in terms of the amplitude of the electric wave in the case of a saturable material. If we take the complex conjugate of (4.7), we will obtain:

$$\frac{dE^*(z)}{dz} = \left[-\frac{m\gamma_c}{c} + \frac{n}{c} \times \frac{(1 - iy)D_0(z)}{1 + y^2 + m^2|E(z)|^2} \right] E^*(z) \quad (4.14)$$

Now (4.7) and (4.14) can be substituted in (2.16). With this substitution, all the complex quantities cancel out and we will get:

$$I(z) = \frac{1}{2} \left(\frac{\epsilon}{\mu} \right)^{1/2} |E(z)|^2 + \frac{nD_0(z)}{2\omega\mu c} \left[\frac{y|E(z)|^2}{1 + y^2 + m^2|E(z)|^2} \right] \quad (4.15)$$

This expression relates the intensity of a plane wave to the magnitude of the electric field in a saturating amplifier. It can be seen from (4.15) that the intensity is no longer directly proportional to the square of the amplitude of the electric field. It does, however, reduce to the familiar expression for intensity in the three limiting cases of: 1) at line center ($y = 0$), 2) far from line center ($y = \infty$), 3) no gain ($D_0 = 0$).

It is also useful to rewrite (4.15) in a slightly different way. With obvious manipulations one obtains:

$$2c\mu I(z) = \left[1 + \frac{yD_0(z)}{\omega(1 + y^2 + m^2|E(z)|^2)} \right] |E(z)|^2 \quad (4.16)$$

where $c=(\mu\epsilon)^{-1/2}$ is defined as the background speed of light in the material. The left hand side of (4.16) can be considered as the normalized intensity. The right hand side, however, consists of the sum of two parts: the square of the electric field and an extra term which is a measure of the nonlinearity. It should be noticed that if one does not consider (4.13), one might get a negative intensity with some choices of parameters in (4.16).

Examining (4.16) further reveals that below the line center, the intensity grows more slowly compared to the square of the electric field. On the other hand, above the line center, the intensity grows faster with respect to the square of the electric field.

So far we have derived some expressions for the intensity of the electromagnetic wave in terms of the electric field inside a saturating medium. These expressions are useful in deriving the reflection and refraction properties for the intensity once those properties are known for the electric field. Our goal now is to use the general formalism derived earlier to investigate the reflection and refraction properties of electromagnetic waves from the boundary of a saturating material. We can combine the terms representing the longitudinal and transverse wave profiles in (2.46) and (2.47) into a single term for the complex electric amplitude and get:

$$E_I(y \sin \theta_I) \exp i(k_1 y \sin \theta_I) + E_R(y \sin \theta_R) \exp i(k_1 y \sin \theta_R) = E_T(y \sin \theta_T) \exp i(k_2 y \sin \theta_T) \quad (4.17)$$

$$\begin{aligned} & \frac{1}{\mu_1} \cos \theta_I \left(-\frac{\partial E_I(z')}{\partial z'} - ik_1 E_I(y \sin \theta_I) \right) \exp i(k_1 y \sin \theta_I) + \\ & \frac{1}{\mu_1} \cos \theta_R \left(\frac{\partial E_R(z')}{\partial z'} + ik_1 E_R(y \sin \theta_R) \right) \exp i(k_1 y \sin \theta_R) \\ & = \frac{1}{\mu_2} \cos \theta_T \left(-\frac{\partial E_T(z')}{\partial z'} - ik_2 E_T(y \sin \theta_T) \right) \exp i(k_2 y \sin \theta_T) \end{aligned} \quad (4.18)$$

where z' is the direction of propagation for each wave, and all the functions are to be evaluated at the boundary. Combining (4.6) with (4.17) and (4.18) will give us a set of equations for the unknown quantities we want to find.

4.2. EXTERNAL REFLECTION AND REFRACTION

Fig.(3.1) shows the case of external reflection and refraction. The incident and the reflected waves are in the linear medium, and the transmitted wave is propagating in the non-linear medium.

Fig.(3.1) External reflection and refraction

When the incident wave is in the linear medium, (4.17) and (4.18) can be further simplified. This is due to the fact that the complex amplitudes of the incident and reflected fields inside the linear medium are constant, and therefore their derivatives vanish. The only derivative term remaining is thus in the saturating medium:

$$E_I \exp i(k_1 y \sin \theta_I) + E_R \exp i(k_1 y \sin \theta_R) = E_T(y \sin \theta_T) \exp i(k_2 y \sin \theta_T) \quad (4.19)$$

$$\begin{aligned} -\frac{ik_1}{\mu_1} E_I \cos \theta_I \exp i(k_1 y \sin \theta_I) + \frac{ik_1}{\mu_1} E_R \cos \theta_R \exp i(k_1 y \sin \theta_R) \\ = \frac{1}{\mu_2} \cos \theta_T \left(-\frac{\partial E_T(z')}{\partial z'} - ik_2 E_T(y \sin \theta_T) \right) \exp i(k_2 y \sin \theta_T) \end{aligned} \quad (4.20)$$

The next step is to combine (4.6) and the above equations. The result would be a set of two complex algebraic equations, coupling the various quantities of interest together. Particularly, the angle unknowns and the amplitude unknowns are coupled. However, one might make

some arguments to separate the angle and amplitude unknowns in (4.19). One, for instance, might argue that the amplitude term in the right hand side of (4.19) which represents the magnitude of the electric field inside the saturating medium along the boundary should be constant because for semi-infinite plane waves there should not be any variations along the boundary. In other words, the wave amplitudes have long reached a steady state value and are therefore constants. With this argument, the amplitude and angle terms in (4.19) decouple, and we get Snell's law of reflection and refraction, and an expression relating the amplitudes of the fields:

$$E_I + E_R = E_T \quad (4.21)$$

$$\theta_I = \theta_R \quad ; \quad k_1 \sin \theta_I = k_2 \sin \theta_T \quad (4.22)$$

Moreover, combining (4.9) and (4.20) will give us another expression:

$$\frac{k_1}{\mu_1} \cos \theta_I (E_I - E_R) = \frac{1}{\mu_2} \cos \theta_T \left[k_2 E_T + i \frac{n_2 \gamma_c}{c} E_T - i \frac{n_2 D_0(0)}{c} \left(\frac{1 + iy}{1 + y^2 + m^2 |E_T|^2} \right) E_T \right] \quad (4.23)$$

where n_2 and k_2 are defined for the nonlinear medium. The two expressions (4.21) and (4.23) can now be solved for either the reflected or transmitted amplitude in terms of the incident amplitude. The easiest thing to do is to omit E_R between (4.21) and (4.23). The result would be:

$$E_I = \left(\frac{1}{2} + \frac{\mu_1 \cos \theta_T}{2\mu_2 k_1 \cos \theta_I} \left[k_2 + i \frac{n_2 \gamma_c}{c} - i \frac{n_2 D_0(0)}{c} \left(\frac{1 + iy}{1 + y^2 + m^2 |E_T|^2} \right) \right] \right) E_T \quad (4.24)$$

where n_2 is the index of refraction of the nonlinear medium. Equation (4.24) is an algebraic equation which relates E_T and E_I . So for a given angle of incidence and incident field E_I ,

(4.24) can be solved for the transmitted field E_T . Once E_T is found, it can be substituted in (4.21) to find E_R . As can be seen from (4.24), to find E_T in terms of E_I explicitly, one should solve a rather difficult nonlinear complex algebraic equation. However, it is possible to treat the saturation term in the denominator as a parameter, and obtain:

$$T_E \equiv \frac{E_T}{E_I} = \frac{2\mu_2 k_1 \cos \theta_I}{\mu_2 k_1 \cos \theta_I + \mu_1 \cos \theta_T \left[k_2 + i \frac{n_2 \gamma_c}{c} - i \frac{n_2 D_0(0)}{c} \left(\frac{1 + iy}{1 + y^2 + m^2 |E_T|^2} \right) \right]} \quad (4.25)$$

$$R_E \equiv \frac{E_R}{E_I} = \frac{\mu_2 k_1 \cos \theta_I - \mu_1 \cos \theta_T \left[k_2 + i \frac{n_2 \gamma_c}{c} - i \frac{n_2 D_0(0)}{c} \left(\frac{1 + iy}{1 + y^2 + m^2 |E_T|^2} \right) \right]}{\mu_2 k_1 \cos \theta_I + \mu_1 \cos \theta_T \left[k_2 + i \frac{n_2 \gamma_c}{c} - i \frac{n_2 D_0(0)}{c} \left(\frac{1 + iy}{1 + y^2 + m^2 |E_T|^2} \right) \right]} \quad (4.26)$$

where T_E and R_E are defined as the transmission and reflection coefficients for the electric wave.

This representation allows us, for instance, to plot the transmission and reflection coefficients versus E_T , which is related to the intensity of the incident wave. These two expressions can be viewed as the generalization of the Fresnel formulas, and it is in fact easy to check that they reduce to the Fresnel equations (3.9) and (3.10) in the limit of no initial population inversion ($D_0 = 0$) or in the limit of far away from the line center (y being infinity).

The two equations (4.25) and (4.26) are a bit too complicated to use, and we can make some simplifying assumptions to make them look simpler and easier, without losing much generality. In practical situations, usually the permeability of the materials involved is not different from that of vacuum, that is: $\mu_1 = \mu_2 = \mu_0$. Also it can be assumed that the conductivity of the materials involved is negligible. Consequently, the background propagation

constants of the materials k_1 and k_2 can be assumed to be: $k_1 = \omega(\mu_0\epsilon_1)^{1/2} = \omega/c_1 = n_1\omega/c$ and $k_2 = \omega(\mu_0\epsilon_2)^{1/2} = \omega/c_2 = n_2\omega/c$. With these assumptions, (4.25) and (4.26) can be rewritten as:

$$T_E \equiv \frac{E_T}{E_I} = \frac{2n_1 \cos \theta_I}{n_1 \cos \theta_I + n_2 \cos \theta_T \left[1 - i \frac{D(0)}{\omega} \left(\frac{1+iy}{1+y^2 + m^2|E_T|^2} \right) \right]} \quad (4.27)$$

$$R_E \equiv \frac{E_R}{E_I} = \frac{n_1 \cos \theta_I - n_2 \cos \theta_T \left[1 - i \frac{D(0)}{\omega} \left(\frac{1+iy}{1+y^2 + m^2|E_T|^2} \right) \right]}{n_1 \cos \theta_I + n_2 \cos \theta_T \left[1 - i \frac{D(0)}{\omega} \left(\frac{1+iy}{1+y^2 + m^2|E_T|^2} \right) \right]} \quad (4.28)$$

These equations express the external reflection and refraction coefficients for the electric field in terms of more familiar parameters such as the indices of refraction of the two materials.

It should be noticed that all these equations are based on the wave equation, which is valid as long as the condition (4.13) is satisfied. Therefore, in using these expressions, the condition (4.13) should also be remembered. The following figure shows a typical plot of the magnitude of the reflection and refraction coefficients at line center vs. the normalized magnitude of the transmitted electric field for a somewhat extreme choice of parameters. Also the parameter G which was defined by (4.13) is plotted. As it can be seen from this figure, increasing the intensity causes the transmission and reflection coefficients to approach their saturated values, which is what Fresnel's formulas predict as well. Also, it should be noticed that we have chosen the parameters in a way that the maximum of the parameter G is much less than 1 (less than 0.15). The parameters used in this figure are: $y=0$, $\omega=10$, $D_0=20$, $n_1=1$, and $n_2=1.5$, $\theta_I=\theta_T=0$.

Fig.(3.2) Transmission & reflection coefficients and the parameter G vs. the normalized amplitude of the transmitted electric field.

It is important to note that the parameter G , as defined by equation (4.13) and plotted in Fig.(3.2), has a zero at $m|E_T|=1$. Therefore, the results at that point are exact.

The two equations (4.27) and (4.28) are by themselves useful expressions. They describe the reflection and transmission properties of the nonlinear interface for the electric field. However, as mentioned earlier, it is also desirable to find such expressions for the intensity reflection and transmission coefficients, or the reflectance and transmittance of the interface, since it is the intensity and not the field which is of primary practical interest.

Earlier in this chapter, we derived an expression for the intensity of a plane wave in a saturating material (Eq. (4.15)). As one expects, in the linear limit, the intensity is proportional to the square of the electric field. In the problem we are considering now, the incident and reflected waves are both in the linear medium. Noticing that the incident power on the interface is proportional to the cosine of the angle of incidence, we obtain:

$$R \equiv \frac{\text{The reflected intensity}}{\text{The incident intensity}} = \frac{I_R \cos \theta_R}{I_I \cos \theta_I} = \frac{|E_R|^2}{|E_I|^2} = |R_E|^2 \quad (4.29)$$

where R is defined as the reflectance of the interface, and R_E is defined by (4.28) or more generally by (4.26). Therefore, the reflectance of the nonlinear interface is simply obtained by taking the square of the reflection coefficient for the electric field defined by (4.28). If we compare (4.29) which describes the nonlinear case with (3.14) which describes the linear case, we see that in both cases the reflectance of the surface is equal to the square of the reflection coefficient, although the reflection coefficient itself is different for the two cases.

To obtain the intensity transmission coefficient or the transmittance of the interface, we notice that the transmitted wave is propagating in the nonlinear medium, and hence its intensity is not simply the square of the intensity of the electric field anymore. Therefore, for the transmitted wave, we have to use (4.15) :

$$\begin{aligned} T \equiv \frac{\text{The transmitted intensity}}{\text{The incident intensity}} &= \frac{I_T \cos \theta_T}{I_I \cos \theta_I} = \frac{\cos \theta_T}{\cos \theta_I} \times \frac{\frac{1}{2} \left(\frac{\epsilon_2}{\mu_0} \right)^{1/2} |E_T|^2 + \frac{n_2 D_0(0)}{2 \mu_0 \omega c} \left[\frac{y |E_T|^2}{1 + y^2 + m^2 |E_T|^2} \right]}{\frac{1}{2} \left(\frac{\epsilon_1}{\mu_0} \right)^{1/2} |E_I|^2} \\ &= \frac{\cos \theta_T}{\cos \theta_I} \left(\frac{\epsilon_2}{\epsilon_1} \right)^{1/2} \frac{|E_T|^2}{|E_I|^2} + \frac{\cos \theta_T}{\cos \theta_I} \frac{n_2 D_0(z)}{n_1 \omega} \left[\frac{y}{1 + y^2 + m^2 |E_T|^2} \right] \frac{|E_T|^2}{|E_I|^2} \end{aligned} \quad (4.30)$$

where T is defined as the transmittance of the interface. Therefore, we obtain:

$$T = \frac{n_2 \cos \theta_T}{n_1 \cos \theta_I} \left(1 + \frac{D_0}{\omega} \left[\frac{y}{1 + y^2 + m^2 |E_T|^2} \right] \right) |T_E|^2 \quad (4.31)$$

where T_E is defined by (4.27) or more generally by (4.25).

It is again useful to compare (4.31) with its linear counterpart, (3.13). This comparison reveals that unlike reflectance, the relationship between the transmittance and the transmission coefficient in the nonlinear case is not as simple as its linear counterpart. Nevertheless, it does reduce to it in the limit of $D_0 = 0$. The two expressions (4.29) and (4.31) enable us to calculate the external reflectance and transmittance for a saturating material, once the external reflection and transmission coefficients are known. The following figure shows the plot of reflectance and transmittance vs. the magnitude of the normalized transmitted electric field using the same parameters used for the previous figure. Also, we have plotted the parameter G for comparison. Again, it should be noticed that at the point $m|E_T|=1$, the parameter G becomes zero and hence the results are exact at that point.

Fig.(3.3) External reflectance & transmittance and the parameter G vs. the normalized magnitude of the transmitted electric field.

Now, we may ask an important question: Do the transmittance and reflectance of the non-linear interface as defined by (4.29) and (4.31) add up to unity? We expect this to be the case, because *at* the two sides of the boundary, the waves have not had the chance to travel for any distance, and possibly gain or loose energy. Therefore, the principle of conservation of energy requires that the sum of the intensity of the incident and reflected waves be equal to the intensity of the transmitted wave. This by itself requires the reflectance and the transmittance of the interface to add up to unity. We have also plotted the sum of the reflectance and the transmittance of the interface in Fig.(3.3). It can be seen that they do add up to unity. In fact, a long and tedious arithmetic calculation (given in the appendix) shows that T and

R , as defined by (4.29) and (4.31), do add up identically to unity, which is a check for them as well.

5. CONCLUSION

In this study, we first developed an expression for the magnetic component of an electromagnetic plane wave propagating in a nonlinear material in terms of the electric component of the wave. This expression reduced to an exact expression in the steady state case, and allowed us to express the energy content or the intensity of the electromagnetic wave in terms of the electric field component.

In chapter 2, we worked out a general frame work for the reflection and refraction of electromagnetic beams from the interface separating two nonlinear materials for arbitrary angles of incidence. Then we discussed the special case of plane waves as beams with constant transverse profile. It was shown that in a steady state analysis, the problem of reflection and refraction from a nonlinear boundary can be reduced to the problem of expressing the first derivative of the electric field amplitude at the two sides of the boundary.

Next in chapter 3, we showed that in the case of linear lossless dielectrics, the formalism previously developed indeed reduces to the familiar Fresnel and Snell's formulas for reflection and refraction.

In chapter 4, we first discussed the propagation of light in a saturating amplifier. We further studied the validity of the slowly varying amplitude approximation. We derived an inequality for the parameters which had to be satisfied if the approximation was to remain valid. Based on this inequality, we defined a parameter which had to be small compared to one, if the validity of the aforementioned approximation was to be maintained.

Then we discussed the problem of external reflection and refraction from the boundary of a saturating amplifier as a special case of the general formalism developed earlier. We

derived some expressions for the reflection and transmission coefficients for the electric field component of the wave. These expressions could be considered as generalizations of Fresnel's formulas for linear dielectrics.

Since the primary quantity of interest is usually the intensity, we also derived the same expressions for the reflectance and transmittance of the interface using the results of the first section. We further showed that the sum of the two is unity—consistent with the principle of conservation of energy.

An important point in these expressions was that although the approximations for the most part grew larger as the interface became less saturated (or equivalently, as the gain was higher), there was one point at which they became exact: For one specific value of the transmitted electric field, the parameter which we had defined as a measure of error becomes zero and in that case, the expressions would be exact.

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APPENDIX

A. INTEGRATION OF A FUNCTION IN TERMS OF ITS COMPLEX AMPLITUDE

Here we introduce a proof of the two integration formulas we presented in the second chapter. We want to prove that for arbitrary complex valued functions $F(z,t)$ and $f(z,t)$ of the two real variables z and t , related to each other by:

$$F(z,t) = f(z,t) \exp(ikz - i\omega t) \quad (\text{A.1})$$

the following two relations hold:

$$\int_{-\infty}^t F(z,t') dt' = \exp(ikz - i\omega t) \sum_{n=0}^N - \left(\frac{-i}{\omega} \right)^{n+1} \frac{\partial^n f}{\partial t^n} + \left(\frac{-i}{\omega} \right)^{N+1} \int_{-\infty}^t \exp(ikz - i\omega t') \frac{\partial^{N+1} f}{\partial t'^{N+1}} dt' \quad (\text{A.2})$$

$$\int_{-\infty}^z F(z',t) dz' = \exp(ikz - i\omega t) \sum_{n=0}^N - \left(\frac{i}{k} \right)^{n+1} \frac{\partial^n f}{\partial z^n} + \left(\frac{i}{k} \right)^{N+1} \int_{-\infty}^z \exp(ikz' - i\omega t) \frac{\partial^{N+1} f}{\partial z'^{N+1}} dz' \quad (\text{A.3})$$

for any natural number N , and assuming f is differentiable up to $N+1$ times.

Proof:

We use induction to prove these formulas. We carry out the proof for the first expression (A.2). The proof for (A.3) can be carried out in the very same way.

First we have to show that (A.2) holds for the case of $N=0$. If we write (A.2) for $N=0$ we obtain:

$$\int_{-\infty}^t F(z, t') dt' = \frac{i}{\omega} f(z, t) \exp(ikz - i\omega t) - \frac{i}{\omega} \int_{-\infty}^t \exp(ikz - i\omega t') \frac{\partial f}{\partial t'} dt' \quad (\text{A.4})$$

To prove (A.4) we differentiate both sides of (A.4) with respect to t :

$$F(z, t) = \frac{i}{\omega} \frac{\partial}{\partial t} [f(z, t) \exp(ikz - i\omega t)] - \frac{i}{\omega} \exp(ikz - i\omega t) \frac{\partial f}{\partial t} \quad (\text{A.5})$$

Finally we have to differentiate the product on the right hand side of (A.5):

$$\begin{aligned} F(z, t) &= \frac{i}{\omega} \frac{\partial}{\partial t} [f(z, t) \exp(ikz - i\omega t)] - \frac{i}{\omega} \exp(ikz - i\omega t) \frac{\partial f}{\partial t} \\ &= \frac{i}{\omega} \exp(ikz - i\omega t) \frac{\partial f}{\partial t} + f \exp(ikz - i\omega t) - \frac{i}{\omega} \exp(ikz - i\omega t) \frac{\partial f}{\partial t} \\ &= f \exp(ikz - i\omega t) \end{aligned} \quad (\text{A.6})$$

which is true by assumption and this completes the proof for the case of $N=0$.

Now we want to show that if (A.2) is true for N , it is also true for $N+1$. To do this we first write (A.2) for N :

$$\int_{-\infty}^t F(z, t') dt' = \exp(ikz - i\omega t) \sum_{n=0}^N - \left(\frac{-i}{\omega} \right)^{n+1} \frac{\partial^n f}{\partial t^n} + \left(\frac{-i}{\omega} \right)^{N+1} \int_{-\infty}^t \exp(ikz - i\omega t') \frac{\partial^{N+1} f}{\partial t'^{N+1}} dt' \quad (\text{A.7})$$

The integral on the right hand side of (A.7) can be carried out using integration by parts:

$$\begin{aligned}
\int_{-\infty}^t F(z, t') dt' &= \exp(ikz - i\omega t) \sum_{n=0}^N - \left(\frac{-i}{\omega}\right)^{n+1} \frac{\partial^n f}{\partial t'^n} \\
&+ \left(\frac{-i}{\omega}\right)^{N+1} \left\{ -\frac{1}{i\omega} \exp(ikz - i\omega t') \frac{\partial^{N+1} f}{\partial t'^{N+1}} dt' - \frac{1}{-i\omega} \int_{-\infty}^t \exp(ikz - i\omega t') \frac{\partial^{N+2} f}{\partial t'^{N+2}} dt' \right\} \quad (A.8)
\end{aligned}$$

Simplifying the right hand side of (A.8), we obtain:

$$\begin{aligned}
\int_{-\infty}^t F(z, t') dt' &= \exp(ikz - i\omega t) \sum_{n=0}^N - \left(\frac{-i}{\omega}\right)^{n+1} \frac{\partial^n f}{\partial t'^n} \\
&- \left(\frac{-i}{\omega}\right)^{N+2} \exp(ikz - i\omega t') \frac{\partial^{N+1} f}{\partial t'^{N+1}} dt' + \left(\frac{-i}{\omega}\right)^{N+2} \int_{-\infty}^t \exp(ikz - i\omega t') \frac{\partial^{N+2} f}{\partial t'^{N+2}} dt' \quad (A.9)
\end{aligned}$$

Finally, the single term and the series in the right hand side of (A.9) can be combined together:

$$\int_{-\infty}^t F(z, t') dt' = \exp(ikz - i\omega t) \sum_{n=0}^{N+1} - \left(\frac{-i}{\omega}\right)^{n+1} \frac{\partial^n f}{\partial t'^n} + \left(\frac{-i}{\omega}\right)^{N+2} \int_{-\infty}^t \exp(ikz - i\omega t') \frac{\partial^{N+2} f}{\partial t'^{N+2}} dt' \quad (A.10)$$

which is the same as (A.2) for the case of $N+1$ and this completes the proof. Briefly, we showed that (A.2) holds for the case of $N=0$, furthermore, if it holds for the case of N , it also holds for the case of $N+1$. Therefore, the induction is complete and we proved that (A.2) is true for any natural number N .

B. CONSERVATION OF ENERGY IN EXTERNAL REFLECTION AND REFRACTION

In this appendix, we want to prove that the expressions obtained in the previous chapter describing the external reflection and refraction also satisfy the principle of conservation of energy. In particular, we want to show that:

$$T + R = 1 \quad (\text{B.1})$$

This is the direct consequence of the principle of conservation of energy, assuming that there is no energy source or sink *at* the boundary. In (B.1), T and R are the external transmissivity and reflectivity of the saturating interface, defined by:

$$T = \frac{\cos \theta_T n_2}{\cos \theta_I n_1} \left(1 + \frac{D_0(0)}{\omega} \left[\frac{y}{1 + y^2 + m^2 |E_T|^2} \right] \right) |T_E|^2 \quad (\text{B.2})$$

$$R = |R_E|^2 \quad (\text{B.3})$$

and the electric field reflection and refraction coefficients are defined by:

$$T_E \equiv \frac{E_T}{E_I} = \frac{2n_1 \cos \theta_I}{n_1 \cos \theta_I + n_2 \cos \theta_T \left[1 + i \frac{\gamma_c}{\omega} - i \frac{D(0)}{\omega} \left(\frac{1 + iy}{1 + y^2 + m^2 |E_T|^2} \right) \right]} \quad (\text{B.4})$$

$$R_E \equiv \frac{E_R}{E_I} = \frac{n_1 \cos \theta_I - n_2 \cos \theta_T \left[1 + i \frac{\gamma_c}{\omega} - i \frac{D(0)}{\omega} \left(\frac{1 + iy}{1 + y^2 + m^2 |E_T|^2} \right) \right]}{n_1 \cos \theta_I + n_2 \cos \theta_T \left[1 + i \frac{\gamma_c}{\omega} - i \frac{D(0)}{\omega} \left(\frac{1 + iy}{1 + y^2 + m^2 |E_T|^2} \right) \right]} \quad (\text{B.5})$$

To prove (B.1), first we introduce the following notational abbreviations:

$$N_1 \equiv n_1 \cos \theta_I$$

$$N_2 \equiv n_2 \cos \theta_T$$

$$\xi \equiv 1 + y^2 + m^2 |E_T|^2$$

With these definitions, the previous equations can be rewritten as:

$$T = \frac{N_2}{N_1} \left(1 + \frac{D_0(0)y}{\omega \xi} \right) |T_E|^2 \quad (\text{B.6})$$

$$R = |R_E|^2 \quad (\text{B.7})$$

$$T_E \equiv \frac{E_T}{E_I} = \frac{N_1}{N_1 + N_2 \left[1 + i \frac{\gamma_c}{\omega} - i \frac{D_0(0)(1+iy)}{\omega \xi} \right]} \quad (\text{B.8})$$

$$R_E \equiv \frac{E_R}{E_I} = \frac{N_1 - N_2 \left[1 + i \frac{\gamma_c}{\omega} - i \frac{D_0(0)(1+iy)}{\omega \xi} \right]}{N_1 + N_2 \left[1 + i \frac{\gamma_c}{\omega} - i \frac{D_0(0)(1+iy)}{\omega \xi} \right]} \quad (\text{B.9})$$

Now, we substitute (B.8) and (B.9) into (B.6) and (B.7). The result is:

$$T = 4 \left(\frac{N_2}{N_1} + \frac{N_2 D_0 y}{N_1 \omega \xi} \right) \left| \frac{N_1}{N_1 + N_2 + i \frac{N_2}{\omega} \left(\gamma_c - \frac{D_0(1+iy)}{\xi} \right)} \right|^2 \quad (\text{B.10})$$

$$R = \left| \frac{2N_1}{N_1 + N_2 + i \frac{N_2}{\omega} \left(\gamma_c - \frac{D_0(1+iy)}{\xi} \right)} - 1 \right|^2 \quad (\text{B.11})$$

Adding (B.10) and (B.11) results:

$$T + R =$$

$$4 \left(\frac{N_2}{N_1} + \frac{N_2 D_0 y}{N_1 \omega \xi} \right) \left[\frac{N_1^2 \left(N_1 + N_2 + \frac{N_2 D_0 y}{\omega \xi} \right)^2}{K^2} + \frac{N_1 L^2}{K^2} \right] + \left[2 \frac{N_1 \left(N_1 + N_2 + \frac{N_2 D_0 y}{\omega \xi} \right)}{K} - 1 \right]^2 + 4 \frac{N_1^2 L^2}{K^2} \quad (\text{B.12})$$

where the two new parameters K and L are defined as:

$$L \equiv \frac{N_2}{\omega} \left(\gamma_c - \frac{D_0}{\xi} \right) \quad (\text{B.13})$$

$$K = \left(N_1 + N_2 + \frac{N_2 D_0 y}{\omega \xi} \right)^2 + L^2 \quad (\text{B.14})$$

Next step is to simplify (B.12). If one expands (B.12) in the form of a numerator and denominator, after simplification one obtains:

$$\begin{aligned} \text{Numerator} = & \omega^3 \xi^3 (N_1^2 \omega^2 \xi^2 + 2N_1 N_2 \omega^2 \xi^2 + 2N_1 N_2 D_0 \xi \omega y + N_2^2 \omega^2 \xi^2 + 2N_2^2 D_0 y \omega \xi + N_2^2 D_0^2 y^2 + \\ & + N_2^2 \gamma^2 \xi^2 - 2N_2^2 \gamma D_0 \xi + N_2^2 D_0) \end{aligned}$$

$$\begin{aligned} \text{Denominator} = & \omega^3 \xi^3 (N_1^2 \omega^2 \xi^2 + 2N_1 N_2 \omega^2 \xi^2 + 2N_1 N_2 D_0 \xi \omega y + N_2^2 \omega^2 \xi^2 + 2N_2^2 D_0 y \omega \xi + N_2^2 D_0^2 y^2 + \\ & + N_2^2 \gamma^2 \xi^2 - 2N_2^2 \gamma D_0 \xi + N_2^2 D_0) \end{aligned}$$

The numerator and denominator are equal, and hence the right hand side of (B.12) is identically one, and the proof is complete.