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Necessary Conditions for Stability of Vehicle Formations

by

Pablo Enrique Baldivieso Blanco

A dissertation submitted in partial fulfillment of the
requirements for the degree of

Doctor of Philosophy
in
Mathematical Sciences

Dissertation Committee:
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Abstract

Necessary conditions for stability of coupled autonomous vehicles in $\mathbb{R}$ are established in this thesis. The focus is on linear arrays with decentralized vehicles, where each vehicle interacts with only a few of its neighbors. Decentralized means that there is no central authority governing the motion. Instead, each vehicle registers only velocity and position *relative to itself* and bases its acceleration only on those data. Explicit expressions are obtained for necessary conditions for asymptotic stability in the cases that a system consists of a periodic arrangement of two or three different types of vehicles, i.e. configurations as follows: ...2-1-2-1 or ...3-2-1-3-2-1. Previous literature indicated that the (necessary) condition for stability in the case of a single vehicle type (...1-1-1) held that the first moment of certain coefficients of the interactions between vehicles has to be zero. Here, we show that that does not generalize. Instead, the (necessary) condition in the cases considered is that the first moment *plus a nonlinear correction term* must be zero.

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Chapter 1

Introduction

Linear arrays of agents, or particles have been studied in many areas such as flock formations, see [22, 31] and vehicular platooning, see [3, 9, 19, 27, 4]. In this thesis, we direct our attention to autonomous vehicular formation in $\mathbb{R}$, namely N vehicles driving on a one-lane road. By autonomous vehicles, we mean that each vehicle does not have any human assistance other than its own set of initial values and a pre-specified set of interaction parameters between its neighbors.

The analysis of stability of systems of identical interconnected vehicles with symmetrical interactions has been presented in the literature of string stability, see [27, 7, 24]. The criteria of string stability are based on definitions of either time or frequency-domain. From the time-domain viewpoint, the interconnected system must have some form of a Lyapunov function. For our systems, it would be almost impossible to find such a Lyapunov function, [26], hence impossible to establish a string stability criteria. And studies from the frequency-domain point of view were focused in most cases, in symmetrical interactions and identical vehicle systems. Others [2, 19] have proposed the idea of coherence vehicular formation by local and global feedback and the analysis of *consensus* dynamics, see also [14, 25]. Equations of motion in the consensus type of systems, in most cases, are first order ordinary differential equations.

In this thesis, the symbol $\mathbf{z}$ is used for the N relative positions of vehicles on the

line, see Figure 2.1. The equations of motion have the following general form

$$\frac{d}{dt} \begin{pmatrix} \mathbf{z} \\ \dot{\mathbf{z}} \end{pmatrix} = \begin{pmatrix} 0 & I \\ L_x & L_v \end{pmatrix} \begin{pmatrix} \mathbf{z} \\ \dot{\mathbf{z}} \end{pmatrix}, \quad (1.1)$$

where I is the $N \times N$ identity, L_x and L_v are $N \times N$ so-called Laplacian matrices. This equation is meant to express the idea that the acceleration of the k th vehicle depends on the *positions relative to it* of some of his neighbors — this is expressed through the matrix L_x — and on the *velocities relative to it* — expressed through L_v . Vehicles whose response depends only on positions and velocities *relative to them* are called *decentralized*. The fact they are decentralized implies that L_x and L_v have row-sum zero. Hence they share many characteristics with the usual Laplacian operator (for details, see [18] and [29]). Ultimately, what we want to know is the behavior of the flock when the following happens. For $t \leq 0$ the formation is in equilibrium, that is: $\dot{z}_i = 0$ and $\ddot{z}_i$ is constant. For $t \geq 0$, the first vehicle changes its velocity, and the others “try” to follow.

Figure 1.1: *Periodic arrangement of flock with two types of vehicles, labeled by 1 and 2. Each agent uses information from four others; the arrows indicate information flow. At time $t = 0$, the first agent start moving to the right.*

In this thesis, we wish to consider more general (linear) systems than those studied with the notion(s) of string stability. In studying string stability, one usually makes several of the following assumptions: the number of agents is infinite [27, 7, 21, 8], the interactions are symmetric or are forward-looking only [23], interactions are small

[2], or L_x and L_v are identical, see [7, 19, 27]. In this thesis based on work done in [30] none of those assumptions are necessary.

But now, a double complication arises. First, L_x and L_v do not (generally) commute, and thus we have no analytical means of solving these equations, and second, there may be non-trivial boundary conditions at the beginning and end of the flock.

This problem was partially overcome in [6] and [5]. In those papers a series of conjectures was proposed that relate solutions of the system on the real line (with non-trivial boundary conditions) to solutions of system on the circle (i.e. periodic boundary conditions). The reason this simplifies the equations is that for systems on the circle, the Laplacians L_x and L_v become *circulant* matrices. Since circulant matrices can be simultaneously diagonalized ([17]), this renders the system on the circle, at least in principle, soluble by analytical means. Note that this takes care of *both* problems just mentioned, because on the circle there is no boundary, and, hence, no dependence on boundary conditions. That is: any quantitative outcome of the theory will be independent of boundary conditions. Naturally, flocks with few agents may be substantially influenced by boundary conditions. So, the theory that results from using the circular flocks to understand flocks on the line is asymptotic in N , the number of agents in the system. That is to say, it gives a prediction for the trajectories of the individual agents; and the relative difference between predicted and actual trajectories should go to zero as the number of agents, N , tends to infinity.

Thus we can solve these systems on the circle. The delicate part in this, of course, is to find out how exactly to transition from solutions in the circular flock to solution of the flock on the line. This is described in the conjectures formulated in [5]. These conjectures are quite detailed, but in spirit they are akin to the traditional “periodic boundary” approach commonly used in physical systems [1]. However, physical systems such as crystals have symmetric interactions, whereas the equations we consider

(generally) do not. Indeed, it is quite reasonable to allow for the possibility to react differently to a trajectory of car behind than to a car in front. As a result, the validity of the “periodic boundary” approach commonly used in physical systems does not imply validity of the conjectures in [5]. However, fairly extensive numerical testing has been performed by [5], [16], and [15], to the effect that in all simulations, the theory appears to have been confirmed.

The theory developed in [5] and [6] can also be used to develop a necessary condition for stability of the flock. Let P be the parameter space, then such a condition typically has the form $f(p) = 0$ where $f : P \rightarrow \mathbb{R}$. Let us take as example the systems studied in [5] and [6].

$$\ddot{z}_k = g_x \left(z_k + \sum_{j \neq 0} \rho_{x,j} z_{k+j} \right) + g_v \left(\dot{z}_k + \sum_{j \neq 0} \rho_{v,j} \dot{z}_{k+j} \right). \quad (1.2)$$

where $g_x, g_v, \rho_{x,j}$, and $\rho_{v,j}$ are real numbers. Here, the assumption is that all agents are equal, and so each agent interacts the same way with the k th agent in front (or behind) it. Due to the Laplacian property of L_x and L_v , we have $\sum_{j \neq 0} \rho_{x,j} = \sum_{j \neq 0} \rho_{v,j} = -1$. What was proved in [6] is that *if $\sum_{j \neq 0} \rho_{x,j} j \neq 0$, then for large N the system on the circle is unstable*. The conjectures in [5] then imply that if that condition holds, then for large N the system on the line has some form of instability. This means that either the system on the line is unstable (Definition 3.2.1), or it is stable but has a transient that grows exponentially in N , the number of agents (Definition 2.0.2). This was called *flock unstable* in [5]. Both types of instabilities are undesirable if we want to have large efficient traffic flow. Thus $\sum_{j \neq 0} \rho_{x,j} j = 0$ is a necessary condition for stability (though generally not sufficient).

Thus, it seemed that there was a very general principle that *first moment of the coefficients of the spatial Laplacian L_x* to the stability of the system. This was confirmed

by [16] and [15] in more detail and accompanied by extensive simulations. In looking to prove such a far-reaching statement, we, very unexpectedly, found that for more complicated systems — presented in this work — that statement is generally false. In what follows, we will show that for certain systems where we allow more than 1 type of agent, a necessary condition for stability may still be derived, but its form is more complicated than the previous papers led to expect. Corollaries 4.1.1 and 5.1.1 show that in the cases at hand, a nonlinear correction to the first moment needs to be taken into account. We also present numerical simulations to show that, in spite of this, the predictions to which the theory developed leads us, are still asymptotically (for large N) accurate.

This is of considerable importance if one studies the effect of non-symmetric interactions in these systems. Indeed, these formulas show that, surprisingly, *stability is a co-dimension one phenomenon!* Thus, without the help of these formulae, it would be nigh impossible to find stable flocks with non-symmetric interactions (in the spatial Laplacian) by experiment, and one might be tempted to conclude that there are none. On the other hand, the non-symmetric stable interactions are important, because they allow us to further optimize these systems for applications. In addition, they provide qualitatively new types of solutions (see [15]).

In the remainder of this thesis we give a detailed description of the response of a formation of vehicles to a perturbation in the trajectory of the lead vehicle. This is based on work done in [30]. The formations we consider are periodic formations of two types of vehicles (see Figure 3.1) and with decentralized interaction of up to two vehicles in front and two vehicles in the rear (see Figure 5.1). We also look at formations with three types of vehicles, but here we consider only nearest neighbor interaction (see Figure 4.1). The essential tool we use is a generalization of the periodic boundary condition hypothesis. For a more detailed description of how non-

trivial boundary conditions may influence the dynamics of a large system, we refer the reader to [28].

Chapter 2

Preliminaries Examples

In this chapter we present some background of the study of oscillators in one-dimensional array from coupled oscillator systems classical theory, see [1], and we introduce some general notation used through out this thesis. Then the following chapters expand on necessary conditions for stable systems. Some initial work and results were studied by Cantos et al. [5], and [6].

This thesis considers a finite number of vehicles in the vehicular formation. One of the many problems studying finite linear arrays of size N of oscillating agents is that the array has boundaries in both ends, which force us to set non-trivial boundary conditions. Boundary conditions in the system complicate the mathematical analysis because the Laplacian matrices L_x and L_v not necessarily commute nor are symmetric. There has been two general approaches to deal with this problem. Some assume that the array consists of infinite number of agents in the array. Others have set periodic boundary conditions and assumed the array is a circular array, i.e. $z_0 = z_N$, $z_{N+1} = z_1$, see [10].

In consequence of the fact that L_v and L_x are Laplacians, we see that for arbitrary constant x_0 and v_0 (1.1) has a solution $z_i = x_0 + v_0 t$. This is desirable for a flock. It *does* mean, however, that the matrix associated with this linear system must have a Jordan block of dimension 2 associated to the eigenvalue 0. In this thesis, we will call a flock stable if all *other* eigenvalues have strictly negative real part. For future

reference, we need a definition of stability.

Definition 2.0.1. *The system (1.1) is linearly stable if it has one eigenvalue zero with geometric multiplicity one and algebraic multiplicity two, and all other eigenvalues have real part less than zero. The system is unstable if at least one eigenvalue has positive real part.*

Definition 2.0.2. *The system (1.1) is flock-stable if it is linearly stable and if transients grow less than exponentially fast in the number of agents. It is called flock-unstable if the growth is exponential.*

2.1 One-Dimensional One-Mass Array

The following schematic figure shows a formation of particles and the deviation z_j of the element j from its equilibrium position.

Figure 2.1: Periodic arrangement of formations with a single type of mass.

The assumption here is that masses or particles interact with their closest neighbors only.

The harmonic potential energy of the system is

$$U = \frac{1}{2}K \sum_j (z_j - z_{j-1})^2, \quad j = 1, \dots, N \quad (2.1)$$

Where K is the interaction energy of two particles at a distance Δ apart, (without of lost of generality, we assume $\Delta = 1$, see [1], [12]) and M is the mass of the particle. For $j = 1, \dots, N$, the equations of motion are given by

$$M\ddot{z}_j = -\frac{\partial U}{\partial z_j} = -K \left(z_j - \frac{1}{2}z_{j-1} - \frac{1}{2}z_{j+1} \right) \quad (2.2)$$

Let $g = -K/M$, this will simplify our notation.

Assume solutions will have the form

$$z_j(t) = ce^{i(\phi j - \omega t)} \quad (2.3)$$

with Born-von Karman boundary condition, i.e. $z_0 = z_N$, $z_{N+1} = z_1$, so ϕ has to be $\phi = 2\pi m/N$ for $m = 0, 1, \dots, N-1$.

We know of no proof that this hypothesis — called periodic boundary conditions — leads to an approximation of the exact solution of the system with non-trivial boundary conditions. Generally, an intuitive reason is given for this fact, see e.g. [10]. This question: when exactly does the periodic boundary hypothesis lead to good approximation, has been addressed in [30]

Definition 2.1.1. *From now on, we set $\phi_m = \frac{2\pi m}{N}$, $m = 0, 1, \dots, N-1$. When there is no ambiguity, we will often drop the subscript m from ϕ_m .*

Differentiating equation (2.3), we obtain

$$\begin{aligned} \dot{z}_j(t) &= -ci\omega e^{i(\phi j - \omega t)} \\ \ddot{z}_j(t) &= -c\omega^2 e^{i(\phi j - \omega t)} \end{aligned} \quad (2.4)$$

Substituting this into the differential equation, we obtain

$$\begin{aligned}
-c\omega^2 e^{i(\phi j - \omega t)} &= g \left(c e^{i(\phi j - \omega t)} - \frac{c}{2} e^{i(\phi(j-1) - \omega t)} - \frac{c}{2} e^{i(\phi(j+1) - \omega t)} \right) \\
\omega^2 e^{i(\phi j - \omega t)} &= -g(1 - \cos \phi) e^{i(\phi j - \omega t)} \\
\omega^2 &= -g(1 - \cos \phi)
\end{aligned}$$

so

$$\omega(\phi) = \sqrt{2|g|} \left| \sin \frac{\phi}{2} \right| \quad (2.5)$$

Figure 2.2 shows the plot of the dispersion relation ω against ϕ on $[-\pi, \pi]$.

The equations of motion of each mass (2.2) on the circle can be thought as a system of first order ordinary linear system of ordinary differential equations. The system has the form shown in equation (2.7). Notice the matrix $\mathbf{L}$ is a tridiagonal matrix which eigenvalues can be obtained for different boundary conditions, see [28].

We now solve the same problem as an example to show some of the methodology used later in this thesis.

$$\frac{d}{dt} \begin{pmatrix} \mathbf{z} \\ \dot{\mathbf{z}} \end{pmatrix} = \mathbf{M} \begin{pmatrix} \mathbf{z} \\ \dot{\mathbf{z}} \end{pmatrix} \quad (2.6)$$

or

$$\frac{d}{dt} \begin{pmatrix} \mathbf{z} \\ \dot{\mathbf{z}} \end{pmatrix} = \left(\begin{array}{c|c} \mathbf{0} & \mathbf{I} \\ \hline g\mathbf{L} & \mathbf{0} \end{array} \right) \begin{pmatrix} \mathbf{z} \\ \dot{\mathbf{z}} \end{pmatrix} \quad (2.7)$$

where $\mathbf{L}$ is given by

$$\mathbf{L} = \begin{pmatrix} 1 & -\frac{1}{2} & 0 & 0 & 0 & \cdots & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & & & \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & & & \vdots \\ 0 & & & \ddots & & & \\ \vdots & & & & & & 0 \\ & & & & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & & & & 0 & -\frac{1}{2} & 1 \end{pmatrix}$$

We can show that the eigenvectors of $\mathbf{L}$ are given by

$$\mathbf{v}_m = \begin{pmatrix} 1 \\ e^{\phi i} \\ e^{2\phi i} \\ \vdots \\ e^{(N-1)\phi i} \end{pmatrix} \quad (2.8)$$

If $(\mathbf{v}_m, \lambda_m)$ is an eigenpair of $\mathbf{L}$, then $\mathbf{L}\mathbf{v}_m = \lambda_m\mathbf{v}_m$. Multiplying out the left and right hand side of this equation we can see that the j^{th} row is

$$\begin{aligned} \lambda_m(\mathbf{v}_m)_j &= (\mathbf{L}\mathbf{v}_m)_j \\ \lambda_m e^{i\phi j} &= -\frac{1}{2}e^{(j-1)i\phi} + e^{ji\phi} - \frac{1}{2}e^{i\phi(j+1)} \end{aligned}$$

Multiplying both sides by $e^{-i\phi j}$ we obtain

$$\begin{aligned}\lambda_m &= -\frac{1}{2}e^{-i\phi} - \frac{1}{2}e^{i\phi} + 1 \\ &= 2\sin^2 \frac{\phi}{2}\end{aligned}\tag{2.9}$$

Figure 2.2: Dispersion relation - Monatomic linear chain with $g = -10/2$.

There are N eigenpairs $(\lambda_m, \mathbf{v}_m)$ of $\mathbf{L}$, we find now the eigenpairs of the block matrix $\mathbf{M}$. Let $(\nu_m, (\mathbf{v}_m, \nu_m \mathbf{v}_m)^T)$ be an eigenpair of $\mathbf{M}$, then

$$\left(\begin{array}{c|c} \mathbf{0} & \mathbf{I} \\ \hline g\mathbf{L} & \mathbf{0} \end{array} \right) \begin{pmatrix} \mathbf{v}_m \\ \nu_m \mathbf{v}_m \end{pmatrix} = \nu_m \begin{pmatrix} \mathbf{v}_m \\ \nu_m \mathbf{v}_m \end{pmatrix}\tag{2.10}$$

$$\Rightarrow \mathbf{L}\mathbf{v}_m = (\nu_m^2/g)\mathbf{v}_m\tag{2.11}$$

From this we can see that ν_m^2/g is an eigenvalue of $\mathbf{L}$, so $\nu_m = \pm i\sqrt{\lambda_m|g|}$ (recall that $g < 0$); the matrix $\mathbf{M}$ has complex eigenvalues ν_m . We observe that for each

eigenvalue λ_m there are two eigenvalues ν_m , namely

$$\begin{aligned}\nu_{m,\pm} &= \pm i \sqrt{2|g| \sin^2 \frac{\phi}{2}} \\ &= \pm i \sqrt{2|g|} \left| \sin \frac{\phi}{2} \right|\end{aligned}\tag{2.12}$$

Notice that the dispersion relation (2.5) is the imaginary part of $\nu_{m,\pm}$, i.e. $\nu_{m,\pm} = i\omega_m$.

Figure 2.2 shows the plot of the dispersion relation against ϕ on $[-\pi, \pi]$.

For some constants a_m and b_m determined by an initial condition, the solution of the system (2.7) can be written as

$$\begin{pmatrix} \mathbf{z}(t) \\ \dot{\mathbf{z}}(t) \end{pmatrix} = \sum_m a_m e^{\nu_{m,-}t} \begin{pmatrix} \mathbf{v}_m \\ \nu_- \mathbf{v}_m \end{pmatrix} + \sum_m b_m e^{\nu_{m,+}t} \begin{pmatrix} \mathbf{v}_m \\ \nu_+ \mathbf{v}_m \end{pmatrix}\tag{2.13}$$

where each solution looks like

$$z_j(t) = \sum_{m=0}^{N-1} a_m e^{\nu_{m,-}t} e^{i\phi j} + \sum_{m=0}^{N-1} b_m e^{\nu_{m,+}t} e^{i\phi j}, \quad j = 1 \dots N$$

with phase velocity

$$v_m = \frac{\omega_m}{\phi}, \quad m = 1 \dots N-1$$

When $j = 0$, we have the particular case that the solution z_0 has the temporal period ($\tau = 2\pi/\omega_m$), but not the spatial period.

2.2 One-Dimensional Two-Mass Linear Array

We now consider the case of the linear array with $2N$ (N each) alternating particles having masses M_1 and M_2 respectively. Again, in this section we bring some background from the classical theory presented by Ashcroft & Mermin [1].

Figure 2.3: Sketch of a one-dimensional 2-mass array

Each particle interacts with its nearest neighbors, so the equations of motions are found similarly as the monatomic case, in particular, the equations of motions for each type are

$$\begin{aligned} M_1 \ddot{z}_j^{(1)} &= -K_1(z_j^{(1)} - z_j^{(2)}) - K_2(z_j^{(1)} - z_{j-1}^{(2)}) \\ M_2 \ddot{z}_j^{(2)} &= -K_1(z_j^{(2)} - z_j^{(1)}) - K_2(z_j^{(2)} - z_{j+1}^{(1)}) \end{aligned}$$

Let

$$g_x^{(1)} = -K_1/M_1, \quad g_x^{(2)} = -K_2/M_2, \quad \rho_{x,1}^{(1)} = \frac{1}{g_x^{(1)}} \left(\frac{K_1}{M_1} \right), \quad \text{and} \quad \rho_{x,-1}^{(1)} = \frac{1}{g_x^{(1)}} \left(\frac{K_2}{M_1} \right) \quad (2.14)$$

Now we can write the system as

$$\begin{aligned}\ddot{z}_j^{(1)} &= g_x^{(1)} \left(z_j^{(1)} + \rho_{x,1}^{(1)} z_j^{(2)} + \rho_{x,-1}^{(1)} z_{j-1}^{(2)} \right) \\ \ddot{z}_j^{(2)} &= g_x^{(2)} \left(z_j^{(2)} + \rho_{x,1}^{(2)} z_{j+1}^{(1)} + \rho_{x,-1}^{(2)} z_j^{(1)} \right)\end{aligned}\tag{2.15}$$

We want solutions of the form

$$\begin{aligned}z_j^{(1)}(t) &= c_1 e^{i(\phi j - \omega t)} \\ z_j^{(2)}(t) &= c_2 e^{i(\phi j - \omega t)}\end{aligned}\tag{2.16}$$

Differentiating twice, we have

$$\begin{aligned}\ddot{z}_j^{(1)}(t) &= -c_1 \omega^2 e^{i(\phi j - \omega t)} \\ \ddot{z}_j^{(2)}(t) &= -c_2 \omega^2 e^{i(\phi j - \omega t)}\end{aligned}\tag{2.17}$$

Substituting these into equations (2.15) we get

$$\begin{aligned}-c_1 \omega^2 e^{i(\phi j - \omega t)} &= g_x^{(1)} (c_1 e^{i(\phi j - \omega t)} - \rho_{x,1}^{(1)} c_2 e^{i(\phi j - \omega t)} - \rho_{x,-1}^{(1)} c_2 e^{i(\phi(j+1) - \omega t)}) \\ -c_2 \omega^2 e^{i(\phi j - \omega t)} &= g_x^{(2)} (c_2 e^{i(\phi j - \omega t)} - \rho_{x,1}^{(2)} c_1 e^{i(\phi(j-1) - \omega t)} - \rho_{x,-1}^{(2)} c_1 e^{i(\phi j - \omega t)})\end{aligned}$$

then simplifying

$$\begin{aligned}-c_1 \omega^2 &= g_x^{(1)} (2c_1 - c_2 - c_2 e^{i\phi}) \\ -c_2 \omega^2 &= g_x^{(2)} (2c_2 - c_1 e^{-i\phi} - c_1)\end{aligned}$$

Rearranging these equations, we obtain

$$\begin{aligned}(-\omega^2 - 2g_x^{(1)})c_1 + g_x^{(1)}(1 + e^{i\phi})c_2 &= 0 \\ (-\omega^2 - 2g_x^{(2)})c_2 + g_x^{(2)}(e^{-i\phi} + 1)c_1 &= 0\end{aligned}$$

or in matrix form

$$\begin{bmatrix} -\omega^2 - 2g_x^{(1)} & g_x^{(1)}(1 + e^{i\phi}) \\ g_x^{(2)}(e^{-i\phi} + 1) & -\omega^2 - 2g_x^{(2)} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This is an homogeneous system of the form $A\mathbf{x} = \mathbf{0}$, which will have nontrivial solutions if $\det(A) = 0$. If we set the determinant of A equal to zero, we will obtain the dispersion relation, so

$$\begin{aligned}(-\omega^2 - 2g_x^{(1)})(-\omega^2 - 2g_x^{(2)}) - g_x^{(1)}(1 + e^{i\phi})g_x^{(2)}(e^{-i\phi} + 1) &= 0 \\ \omega^4 + 2(g_x^{(1)} + g_x^{(2)})\omega^2 + 2g_x^{(1)}g_x^{(2)}(1 - \cos\phi) &= 0\end{aligned}$$

Solving this equation as a quadratic in terms of ω^2 , we get

$$\omega^2 = -(g_x^{(1)} + g_x^{(2)}) \pm \sqrt{[g_x^{(1)}]^2 + [g_x^{(2)}]^2 + 2g_x^{(1)}g_x^{(2)}\cos\phi} \quad (2.18)$$

We verify now that the one-dimensional monatomic and diatomic array dispersion relations are comparable, that is, if we were to assume that the particles have the same mass, i.e. $g_x^{(1)} = g_x^{(2)} = g$, we obtain the monatomic dispersion relation with $2N$ entities.

$$\nu^2 = 2g \pm \sqrt{2g^2 + 2g^2\cos\phi}$$

Figure 2.4: Dispersion relation of the diatomic linear chain with the acoustic branch (solid line) and the optical branch (dashed line), and $g_x^{(1)} = -5$, and $g_x^{(2)} = -10/3$.

For each case, plus and minus sign of the square root, and for $m \in \{0, 1, \dots, N-1\}$ the previous equation reduces to

$$\nu = \begin{cases} \pm i \sqrt{4|g|} \sin \frac{\phi}{4} \\ \pm i \sqrt{4|g|} \cos \frac{\phi}{4} \end{cases} \quad (2.19)$$

The previous equations produce $4N$ eigenvalues which correspond to the monatomic case with $2N$ entities; For the cosine case, which is the optical branch, we use the identity $\cos \phi/4 = \sin(\phi/4 + \pi/2)$. The plot below shows that the optical branch on $[\pi, 2\pi]$ and $[-2\pi, -\pi]$ is equivalent to the acoustic branch plotted in figure 1.

Figure 2.5: Dispersion relation of the diatomic linear chain when the masses are equal, $g = -5$, with the acoustic branch (solid line) and the optical branch (dashed line).

Chapter 3

Two-Vehicle Linear Arrays with Nearest Neighbors Interactions

We now consider the case of two-vehicle linear arrays with $2N$ (N each) alternating vehicles or agents with “damping” coefficients, see Figure 3.1. Each vehicle interacts with its nearest neighbors, which are of the other type. The quantities $z_j^{(i)}$, and $\dot{z}_j^{(i)}$, $i = 1, 2$, and $j = 1 \dots N$ are relative positions and velocities respectively to the local observer j . These type of systems are called *decentralized*.

The equations of motions for each type of vehicle are

$$\begin{aligned}\ddot{z}_j^{(1)} &= g_x^{(1)} \left(z_j^{(1)} + \rho_{x,1}^{(1)} z_j^{(2)} + \rho_{x,-1}^{(1)} z_{j-1}^{(2)} \right) + g_v^{(1)} \left(\dot{z}_j^{(1)} + \rho_{v,1}^{(1)} \dot{z}_j^{(2)} + \rho_{v,-1}^{(1)} \dot{z}_{j-1}^{(2)} \right) \\ \ddot{z}_j^{(2)} &= g_x^{(2)} \left(z_j^{(2)} + \rho_{x,1}^{(2)} z_{j+1}^{(1)} + \rho_{x,-1}^{(2)} z_j^{(1)} \right) + g_v^{(2)} \left(\dot{z}_j^{(2)} + \rho_{v,1}^{(2)} \dot{z}_{j+1}^{(1)} + \rho_{v,-1}^{(2)} \dot{z}_j^{(1)} \right)\end{aligned}\tag{3.1}$$

subject to the constraints:

$$\rho_{x,1}^{(1)} + \rho_{x,-1}^{(1)} = -1, \quad \rho_{x,1}^{(2)} + \rho_{x,-1}^{(2)} = -1, \quad \rho_{v,1}^{(1)} + \rho_{v,-1}^{(1)} = -1, \quad \rho_{v,1}^{(2)} + \rho_{v,-1}^{(2)} = -1 \tag{3.2}$$

Figure 3.1: *Periodic arrangement of formations with two types of vehicles, labeled by 1, and 2. At time $t = 0$, the first vehicle start moving to the right.*

where $g_x^{(1)}, g_x^{(2)}, g_v^{(1)}$, and $g_v^{(2)}$ are real numbers. For the physical analogy, these coefficients can be thought as:

$$M_1 \ddot{z}_j^{(1)} = -K_1(z_j^{(1)} - z_j^{(2)}) - K_2(z_j^{(1)} - z_{j-1}^{(2)}) - D_1(\dot{z}_j^{(1)} - \dot{z}_j^{(2)}) - D_2(\dot{z}_j^{(1)} - \dot{z}_{j-1}^{(2)})$$

and

$$M_2 \ddot{z}_j^{(2)} = -K_1(z_j^{(2)} - z_j^{(1)}) - K_2(z_j^{(2)} - z_{j+1}^{(1)}) - D_1(\dot{z}_j^{(2)} - \dot{z}_j^{(1)}) - D_2(\dot{z}_j^{(2)} - \dot{z}_{j+1}^{(1)})$$

Let

$$g_x^{(1)} = -\left(\frac{K_1}{M_1} + \frac{K_2}{M_1}\right), \quad \rho_{x,1}^{(1)} = \frac{1}{g_x^{(1)}} \left(\frac{K_1}{M_1}\right), \quad \rho_{x,-1}^{(1)} = \frac{1}{g_x^{(1)}} \left(\frac{K_2}{M_1}\right)$$

and

$$g_v^{(1)} = -\left(\frac{D_1}{M_1} + \frac{D_2}{M_1}\right), \quad \rho_{v,1}^{(1)} = \frac{1}{g_v^{(1)}} \left(\frac{D_1}{M_1}\right), \quad \rho_{v,-1}^{(1)} = \frac{1}{g_v^{(1)}} \left(\frac{D_2}{M_1}\right) \quad (3.3)$$

Similarly, let

$$g_x^{(2)} = -\left(\frac{K_1}{M_2} + \frac{K_2}{M_2}\right), \quad \rho_{x,-1}^{(2)} = \frac{1}{g_x^{(2)}} \left(\frac{K_1}{M_2}\right), \quad \rho_{x,1}^{(2)} = \frac{1}{g_x^{(2)}} \left(\frac{K_2}{M_2}\right)$$

and

$$g_v^{(2)} = -\left(\frac{D_1}{M_2} + \frac{D_2}{M_2}\right), \quad \rho_{v,-1}^{(2)} = \frac{1}{g_v^{(2)}} \left(\frac{D_1}{M_2}\right), \quad \rho_{v,1}^{(2)} = \frac{1}{g_v^{(2)}} \left(\frac{D_2}{M_2}\right) \quad (3.4)$$

Notice that $\rho_{x,0}^{(1)} = \rho_{x,0}^{(2)} = \rho_{v,0}^{(1)} = \rho_{v,0}^{(2)} = 1$, and

$$\sum_{k=-1}^{k=1} \rho_{x,k}^{(1)} = \sum_{k=-1}^{k=1} \rho_{v,k}^{(1)} = \sum_{k=-1}^{k=1} \rho_{x,k}^{(2)} = \sum_{k=-1}^{k=1} \rho_{v,k}^{(2)} = 0$$

A circular boundary condition on these equations means that the first vehicle or agent sees the last vehicle (or agent). The system in matrix form of the circular array written as a first-order system of equations is given by

$$\frac{d}{dt} \begin{pmatrix} \mathbf{z}^{(1)} \\ \mathbf{z}^{(2)} \\ \dot{\mathbf{z}}^{(1)} \\ \dot{\mathbf{z}}^{(2)} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \\ g_x^{(1)} \mathbf{I} & g_x^{(1)} \mathbf{A}_x^{(1)} & g_v^{(1)} \mathbf{I} & g_v^{(1)} \mathbf{A}_v^{(1)} \\ g_x^{(2)} \mathbf{A}_x^{(2)} & g_x^{(2)} \mathbf{I} & g_v^{(2)} \mathbf{A}_v^{(2)} & g_v^{(2)} \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{z}^{(1)} \\ \mathbf{z}^{(2)} \\ \dot{\mathbf{z}}^{(1)} \\ \dot{\mathbf{z}}^{(2)} \end{pmatrix} \quad (3.5)$$

where

$$\begin{aligned} \mathbf{A}_x^{(1)} &= (-1 - \rho_{x,1}^{(1)}) \mathbf{P}_+ + \rho_{x,1}^{(1)} \mathbf{I} \\ \mathbf{A}_v^{(1)} &= (-1 - \rho_{v,1}^{(1)}) \mathbf{P}_+ + \rho_{v,1}^{(1)} \mathbf{I} \\ \mathbf{A}_x^{(2)} &= \rho_{x,1}^{(2)} \mathbf{P}_- + (-1 - \rho_{x,1}^{(2)}) \mathbf{I} \\ \mathbf{A}_v^{(2)} &= \rho_{v,1}^{(2)} \mathbf{P}_- + (-1 - \rho_{v,1}^{(2)}) \mathbf{I} \end{aligned} \quad (3.6)$$

and $\mathbf{P}_+$ and $\mathbf{P}_-$ are $N \times N$ permutations matrices defined as follows respectively

$$\mathbf{P}_+ = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \ddots & 0 \\ 0 & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}, \quad \mathbf{P}_- = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix} \quad (3.7)$$

Very often, we will write equation (3.5) simply as

$$\frac{d}{dt} \begin{pmatrix} \mathbf{z} \\ \dot{\mathbf{z}} \end{pmatrix} = \mathbf{M} \begin{pmatrix} \mathbf{z} \\ \dot{\mathbf{z}} \end{pmatrix} \quad (3.8)$$

Each $\mathbf{A}_x^{(1)}$, $\mathbf{A}_x^{(2)}$, $\mathbf{A}_v^{(1)}$, and $\mathbf{A}_v^{(2)}$ is an $N \times N$ matrix, and $\mathbf{I}$ is the identity matrix of dimension N . Each matrix $\mathbf{A}$ is circulant, therefore their eigenvectors are given by the discrete Fourier transform (see [6], [17]). So, for $m \in \{0, 1, \dots, N-1\}$, and $\phi = 2\pi m/N$, the eigenvectors of each matrix $\mathbf{A}$ can be shown to be

$$\mathbf{v}_m(\phi) = \begin{pmatrix} 1 \\ e^{\phi i} \\ e^{2\phi i} \\ \vdots \\ e^{(N-1)\phi i} \end{pmatrix} \quad (3.9)$$

and with corresponding eigenvalues

$$\begin{aligned}
\lambda_{x,m}^{(1)}(\phi) &= \rho_{x,1}^{(1)} + \rho_{x,-1}^{(1)} e^{-\phi i} \\
\lambda_{x,m}^{(2)}(\phi) &= \rho_{x,-1}^{(2)} + \rho_{x,1}^{(2)} e^{\phi i} \\
\lambda_{v,m}^{(1)}(\phi) &= \rho_{v,1}^{(1)} + \rho_{v,-1}^{(1)} e^{-\phi i} \\
\lambda_{v,m}^{(2)}(\phi) &= \rho_{v,-1}^{(2)} + \rho_{v,1}^{(2)} e^{\phi i}
\end{aligned} \tag{3.10}$$

With some abuse and for ease of notation, we drop the dependence of ϕ and the $\pm$. From now on we write $\mathbf{v}_m$, ν , $\mathbf{u}_\nu$, $\lambda_{x,m}^{(\cdot)}$, and $\lambda_{v,m}^{(\cdot)}$ instead of $\mathbf{v}_m(\phi)$, $\nu_\pm(\phi)$, $\mathbf{u}_\pm(\phi)$, $\lambda_{x,m}^{(\cdot)}(\phi)$, and $\lambda_{v,m}^{(\cdot)}(\phi)$ respectively.

We want to determine the eigenvalues and eigenvectors of the matrix $\mathbf{M}$ defined in (3.5).

Proposition 3.0.1. *The eigenvalues ν and associated eigenvectors $\mathbf{u}_\nu$ of $\mathbf{M}$ satisfy*

$$\mathbf{u}_\nu = \begin{pmatrix} \epsilon_1 \mathbf{v}_m \\ \epsilon_2 \mathbf{v}_m \\ \nu \epsilon_1 \mathbf{v}_m \\ \nu \epsilon_2 \mathbf{v}_m \end{pmatrix} \tag{3.11}$$

For each $m \in \{0, \dots, N-1\}$ given, there are four eigenpairs (counting multiplicity) determined by solving the following equation for ν and ϵ_i :

$$\begin{pmatrix} g_x^{(1)} + \nu g_v^{(1)} - \nu^2 & g_x^{(1)} \lambda_{x,m}^{(1)} + \nu g_v^{(1)} \lambda_{v,m}^{(1)} \\ g_x^{(2)} \lambda_{x,m}^{(2)} + \nu g_v^{(2)} \lambda_{v,m}^{(2)} & g_x^{(2)} + \nu g_v^{(2)} - \nu^2 \end{pmatrix} \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Proof. From equations (3.5) and (3.8), we see that an eigenvector $\begin{pmatrix} \mathbf{z} \\ \dot{\mathbf{z}} \end{pmatrix}$ associated to the eigenvalue ν satisfies $\dot{\mathbf{z}} = \nu\mathbf{z}$. Now (3.10) and $\mathbf{v}_m$ are the eigenvalues and eigenvectors of (3.6) respectively. Then by substituting $\mathbf{u}_\nu$ into (4.4), one sees that these are the eigenvectors of $\mathbf{M}$.

For the second part, by definition, we can write

$$\mathbf{M}\mathbf{u}_\nu = \nu\mathbf{u}_\nu \quad (3.12)$$

Let

$$\mathbf{u}_\nu = \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{u}_3 \\ \mathbf{u}_4 \end{pmatrix} \quad (3.13)$$

where each $\mathbf{u}_i$, $i = 1 \dots 4$ is an $N \times 1$ vector.

Notice that $\mathbf{u}_3 = \nu\mathbf{u}_1$, and $\mathbf{u}_4 = \nu\mathbf{u}_2$, so basically (3.12) can be re-written as

$$\left(\begin{array}{c|c|c|c} \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \\ \hline g_x^{(1)}\mathbf{I} & g_x^{(1)}\mathbf{A}_x^{(1)} & g_v^{(1)}\mathbf{I} & g_v^{(1)}\mathbf{A}_v^{(1)} \\ \hline g_x^{(2)}\mathbf{A}_x^{(2)} & g_x^{(2)}\mathbf{I} & g_v^{(2)}\mathbf{A}_v^{(2)} & g_v^{(2)}\mathbf{I} \end{array} \right) \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \nu\mathbf{u}_1 \\ \nu\mathbf{u}_2 \end{pmatrix} = \nu \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \nu\mathbf{u}_1 \\ \nu\mathbf{u}_2 \end{pmatrix} \quad (3.14)$$

We can define $\mathbf{u}_1 = \epsilon_1 \mathbf{v}_m$, and $\mathbf{u}_2 = \epsilon_2 \mathbf{v}_m$ for some constants ϵ_1 and ϵ_2 . Then

$$\begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \\ g_x^{(1)} \mathbf{I} & g_x^{(1)} \mathbf{A}_x^{(1)} & g_v^{(1)} \mathbf{I} & g_v^{(1)} \mathbf{A}_v^{(1)} \\ g_x^{(2)} \mathbf{A}_x^{(2)} & g_x^{(2)} \mathbf{I} & g_v^{(2)} \mathbf{A}_v^{(2)} & g_v^{(2)} \mathbf{I} \end{pmatrix} \begin{pmatrix} \epsilon_1 \mathbf{v}_m \\ \epsilon_2 \mathbf{v}_m \\ \nu \epsilon_1 \mathbf{v}_m \\ \nu \epsilon_2 \mathbf{v}_m \end{pmatrix} = \nu \begin{pmatrix} \epsilon_1 \mathbf{v}_m \\ \epsilon_2 \mathbf{v}_m \\ \nu \epsilon_1 \mathbf{v}_m \\ \nu \epsilon_2 \mathbf{v}_m \end{pmatrix} \quad (3.15)$$

and expanding

$$\begin{aligned} \epsilon_1 g_x^{(1)} \mathbf{I} \mathbf{v}_m + \epsilon_2 g_x^{(1)} \mathbf{A}_x^{(1)} \mathbf{v}_m + \epsilon_1 \nu g_v^{(1)} \mathbf{I} \mathbf{v}_m + \epsilon_2 \nu g_v^{(1)} \mathbf{A}_v^{(1)} \mathbf{v}_m &= \nu^2 \epsilon_1 \mathbf{v}_m \\ \epsilon_1 g_x^{(2)} \mathbf{A}_x^{(2)} \mathbf{v}_m + \epsilon_2 g_x^{(2)} \mathbf{I} \mathbf{v}_m + \epsilon_1 \nu g_v^{(2)} \mathbf{A}_v^{(2)} \mathbf{v}_m + \epsilon_2 \nu g_v^{(2)} \mathbf{I} \mathbf{v}_m &= \nu^2 \epsilon_2 \mathbf{v}_m \end{aligned} \quad (3.16)$$

We know that $\mathbf{A}_x^{(i)} \mathbf{v}_m = \lambda_{x,m}^{(i)} \mathbf{v}_m$ and $\mathbf{A}_v^{(i)} \mathbf{v}_m = \lambda_{v,m}^{(i)} \mathbf{v}_m$ for $i = 1, 2$, then dropping $\mathbf{v}_m$ and rearranging

$$\begin{pmatrix} g_x^{(1)} + \nu g_v^{(1)} - \nu^2 & g_x^{(1)} \lambda_{x,m}^{(1)} + \nu g_v^{(1)} \lambda_{v,m}^{(1)} \\ g_x^{(2)} \lambda_{x,m}^{(2)} + \nu g_v^{(2)} \lambda_{v,m}^{(2)} & g_x^{(2)} + \nu g_v^{(2)} - \nu^2 \end{pmatrix} \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (3.17)$$

A nontrivial solution of this equation exists if and only if the determinant of the matrix on the left hand side is zero. ■

It is very clear that the determinant of the matrix in the previous proposition is given by

$$\begin{aligned} Q(\nu) = & \nu^4 + (-g_v^{(1)} - g_v^{(2)}) \nu^3 + (-g_x^{(1)} - g_x^{(2)} + g_v^{(1)} g_v^{(2)} (1 - \lambda_{v,m}^{(1)} \lambda_{v,m}^{(2)})) \nu^2 \\ & + (g_x^{(1)} g_v^{(2)} (1 - \lambda_{x,m}^{(1)} \lambda_{v,m}^{(2)}) + g_x^{(2)} g_v^{(1)} (1 - \lambda_{x,m}^{(2)} \lambda_{v,m}^{(1)})) \nu + g_x^{(1)} g_x^{(2)} (1 - \lambda_{x,m}^{(1)} \lambda_{x,m}^{(2)}) \end{aligned} \quad (3.18)$$

Notice that for each $m (= 0, 1, \dots, N-1)$ this equation produces four roots. A method

to find the roots is described in Appendix B. For convenience it is written below more compactly as

$$\nu^4 + b\nu^3 + c_m\nu^2 + d_m\nu + e_m = 0 \quad (3.19)$$

where b, c_m, d_m , and e_m are

$$\begin{aligned} b &= -g_v^{(1)} - g_v^{(2)} \\ c_m &= -g_x^{(1)} - g_x^{(2)} + g_v^{(1)}g_v^{(2)}(1 - \lambda_{v,m}^{(1)}\lambda_{v,m}^{(2)}) \\ d_m &= g_x^{(1)}g_v^{(2)}(1 - \lambda_{x,m}^{(1)}\lambda_{v,m}^{(2)}) + g_x^{(2)}g_v^{(1)}(1 - \lambda_{x,m}^{(2)}\lambda_{v,m}^{(1)}) \\ e_m &= g_x^{(1)}g_x^{(2)}(1 - \lambda_{x,m}^{(1)}\lambda_{x,m}^{(2)}) \end{aligned} \quad (3.20)$$

From (3.17), we are interested in when ϵ_1 and ϵ_2 are not independent, therefore, from (3.17), the relationship between ϵ_1 and ϵ_2 is

$$(g_x^{(1)} + \nu g_v^{(1)} - \nu^2) \epsilon_1 = - (g_x^{(1)}\lambda_{x,m}^{(1)} + \nu g_v^{(1)}\lambda_{v,m}^{(1)}) \epsilon_2 \quad (3.21)$$

3.1 Nearest Neighbor Interaction without Friction

An interesting results is when we assume that there is no friction in the system (by no friction we mean in the general sense that there is no damping term in the equation of motion). This case has been studied extensively, and one of the classical textbook in this topic is [1]. It is presented here to show that our more general approach is consistent with the classical theory. All friction factors and coefficients in equation (3.18) are zero:

$$g_v^{(1)} = g_v^{(2)} = 0$$

So, equation (3.18) reduces to

$$\nu^4 - (g_x^{(1)} + g_x^{(2)}) \nu^2 + g_x^{(1)} g_x^{(2)} (1 - \lambda_{x,m}^{(1)} \lambda_{x,m}^{(2)}) = 0 \quad (3.22)$$

Clearly this is a quadratic equation in ν^2 . So

$$\begin{aligned} \nu^2 &= 2^{-1} \left(g_x^{(1)} + g_x^{(2)} \pm \sqrt{(g_x^{(1)} + g_x^{(2)})^2 - 4g_x^{(1)} g_x^{(2)} (1 - \lambda_{x,m}^{(1)} \lambda_{x,m}^{(2)})} \right) \\ &= 2^{-1} \left(g_x^{(1)} + g_x^{(2)} \pm \sqrt{(g_x^{(1)} + g_x^{(2)})^2 - 4g_x^{(1)} g_x^{(2)} \left\{ 1 - (\rho_{x,1}^{(1)} + \rho_{x,-1}^{(1)} e^{-i\phi}) (\rho_{x,-1}^{(2)} + \rho_{x,1}^{(2)} e^{i\phi}) \right\}} \right) \end{aligned} \quad (3.23)$$

We will be able to show later that for stable systems, the equation above can be reduced to

$$\nu^2 = 2^{-1} \left(g_x^{(1)} + g_x^{(2)} \pm \sqrt{(g_x^{(1)} + g_x^{(2)})^2 - 8g_x^{(1)} g_x^{(2)} \rho_{x,1}^{(1)} \rho_{x,1}^{(2)} (1 - \cos \phi)} \right) \quad (3.24)$$

For example, if we take $\rho_x^{(1)} = \rho_x^{(2)} = -1/2$, then

$$\nu^2 = g_x^{(1)} + g_x^{(2)} \pm \sqrt{(g_x^{(1)})^2 + (g_x^{(2)})^2 + 2g_x^{(1)} g_x^{(2)} \cos \phi} \quad (3.25)$$

where $\phi = 2\pi m/N$ for $m = 0, 1, \dots, N-1$. This result is consistent with (2.18) (Recall that $\nu = i\omega_m$).

3.2 Nearest Neighbor Interaction with Friction, $\phi = 0$ and $\phi = \pi$.

Other important cases are when $\phi = 0$ and $\phi = \pi$. From the definition of ϕ , these cases occur when $m = 0$ and $m = N/2$. If $\phi = 0$, then each λ in (3.10) reduces to -1

by (3.2), that is:

$$\lambda_{x,0}^{(1)} = \lambda_{x,0}^{(2)} = \lambda_{v,0}^{(1)} = \lambda_{v,0}^{(2)} = -1$$

equation (3.18) becomes

$$\nu^2 \left[\nu^2 + (-g_v^{(1)} - g_v^{(2)}) \nu + (-g_x^{(1)} - g_x^{(2)}) \right] = 0 \quad (3.26)$$

Clearly, there is a zero eigenvalue with multiplicity two and two distinct eigenvalues (also it is possible to have a zero eigenvalue with multiplicity four). The two distinct non-zero eigenvalues, $\nu_{\pm}$, are obtained by solving the quadratic equation given in the second factor:

$$\nu_{\pm} = \frac{1}{2} \left(g_v^{(1)} + g_v^{(2)} \pm \sqrt{\left(g_v^{(1)} + g_v^{(2)} \right)^2 + 4 \left(g_x^{(1)} + g_x^{(2)} \right)} \right) \quad (3.27)$$

It is assumed through out this paper that $g_x^{(\cdot)}, g_v^{(\cdot)} \in \Re$. From (3.27) is easy to see one of the first necessary conditions for stability of (3.1).

Definition 3.2.1. *The system (3.5) is linearly stable if it has one eigenvalue zero with geometric multiplicity one and algebraic multiplicity two, and all other eigenvalues have real part less than zero. The system is unstable if at least one eigenvalue has positive real part.*

For ease in notation, we define the following (using the same notation as [15]):

Definition 3.2.2. *For $j > 0$, and $k = 1, 2, \dots$, we define*

$$\alpha_{x,j}^{(k)} = \rho_{x,j}^{(k)} + \rho_{x,-j}^{(k)}, \quad \text{and} \quad \beta_{x,j}^{(k)} = \rho_{x,j}^{(k)} - \rho_{x,-j}^{(k)} \quad (3.28)$$

Lemma 3.2.1. *Assume each $g_x^{(1)}, g_x^{(2)}, g_v^{(1)}$, and $g_v^{(2)}$ are real values. Then the necessary conditions for stability of the system (3.1) are $g_v^{(1)} + g_v^{(2)} < 0$ and $g_x^{(1)} + g_x^{(2)} < 0$.*

Proof. Since stability requires that $\nu_{\pm}$ both have negative real part, we must have that $\nu_- + \nu_+ = g_v^{(1)} + g_v^{(2)}$ has negative real part. This implies the first statement. Now, if this is satisfied and $g_x^{(1)} + g_x^{(2)} \geq 0$, then we see that ν_+ must be greater than or equal to 0. This implies the second statement. ■

From now on we will assume that $g_x^{(1)} + g_x^{(2)} < 0$, and $g_v^{(1)} + g_v^{(2)} < 0$.

Another important result that we see from $\phi = 0$ is the relationships between ϵ_1 and ϵ_2 given in (3.21). Substitute ν from (3.26) into (3.21), we obtain the relationship:

$$(g_x^{(2)} + \nu g_v^{(2)}) \epsilon_1 = - (g_x^{(1)} + \nu g_v^{(1)}) \epsilon_2 \quad (3.29)$$

If $\phi = \pi$ (or $m = N/2$), each eigenvalue in (3.10) is real. Using Definition 3.2.2, we can write

$$\begin{aligned} \beta_{x,1}^{(1)} &= \rho_{x,1}^{(1)} - \rho_{x,-1}^{(1)} \\ \beta_{x,1}^{(2)} &= \rho_{x,1}^{(2)} - \rho_{x,-1}^{(2)} \\ \beta_{v,1}^{(1)} &= \rho_{v,1}^{(1)} - \rho_{v,-1}^{(1)} \\ \beta_{v,1}^{(2)} &= \rho_{v,1}^{(2)} - \rho_{v,-1}^{(2)} \end{aligned} \quad (3.30)$$

The quartic equation (3.18) contains now real coefficients only:

$$\begin{aligned} \nu^4 + (-g_v^{(1)} - g_v^{(2)}) \nu^3 + (-g_x^{(1)} - g_x^{(2)} + g_v^{(1)} g_v^{(2)} (1 + \beta_v^{(1)} \beta_v^{(2)})) \nu^2 \\ + (g_x^{(1)} g_v^{(2)} (1 + \beta_x^{(1)} \beta_v^{(2)}) + g_x^{(2)} g_v^{(1)} (1 + \beta_x^{(2)} \beta_v^{(1)})) \nu + g_x^{(1)} g_x^{(2)} (1 + \beta_x^{(1)} \beta_x^{(2)}) = 0 \end{aligned} \quad (3.31)$$

From (3.20), we see that

$$\begin{aligned} b &= -g_v^{(1)} - g_v^{(2)} \\ c_{N/2} &= -g_x^{(1)} - g_x^{(2)} + g_v^{(1)} g_v^{(2)} (1 + \beta_v^{(1)} \beta_v^{(2)}) \\ d_{N/2} &= g_x^{(1)} g_v^{(2)} (1 + \beta_x^{(1)} \beta_v^{(2)}) + g_x^{(2)} g_v^{(1)} (1 + \beta_x^{(2)} \beta_v^{(1)}) \\ e_{N/2} &= g_x^{(1)} g_x^{(2)} (1 + \beta_x^{(1)} \beta_x^{(2)}) \end{aligned} \quad (3.32)$$

For convenience equation (3.31) is written below more compactly as

$$\nu^4 + b\nu^3 + c_{N/2}\nu^2 + d_{N/2}\nu + e_{N/2} = 0 \quad (3.33)$$

Here $b, c_{N/2}, d_{N/2}$, and $e_{N/2}$ are real numbers. Therefore using Routh-Hurwitz criterion (see [11]), we found the following necessary conditions for stability:

Lemma 3.2.2. *The necessary conditions for stability of (3.1) are $b > 0$, $d_{N/2} > 0$, $e_{N/2} > 0$, and $bc_{N/2}d_{N/2} > d_{N/2}^2 + b_{N/2}^2 e_{N/2}$. In particular, $e_{N/2} > 0$ implies that $g_x^{(i)} \neq 0$.*

Notice that from Lemma 3.2.1 also $b > 0$.

3.3 Necessary Condition for Linear Stability

In this section, necessary conditions for stability are established. We are interested in stable systems only, so from lemmas 3.2.1 and 3.2.2, we define the following.

Definition 3.3.1. Let $g_x^{(1)}, g_x^{(2)}, g_v^{(1)}$, and $g_v^{(2)}$ be real numbers such that $g_v^{(1)} + g_v^{(2)} < 0$, $g_x^{(1)} \neq 0$, $g_x^{(2)} \neq 0$ and $g_x^{(1)} + g_x^{(2)} < 0$.

Theorem 3.3.1. Let $g_x^{(1)}$ and $g_x^{(2)}$ be real numbers. If any of the following conditions are violated, then for large N , the system given by (3.1) on the circle is not stable:

- (i) $g_x^{(1)} \neq 0$, and $g_x^{(2)} \neq 0$, and
- (ii) $\beta_{x,1}^{(1)} + \beta_{x,1}^{(2)} = 0$

Proof. Consider again the quartic equation (3.18). For convenience it is written below more compactly as

$$\nu^4 + b\nu^3 + c_m\nu^2 + d_m\nu + e_m = 0 \quad (3.34)$$

where b, c_m, d_m , and e_m are

$$\begin{aligned} b &= -g_v^{(1)} - g_v^{(2)} \\ c_m &= -g_x^{(1)} - g_x^{(2)} + g_v^{(1)}g_v^{(2)}(1 - \lambda_{v,m}^{(1)}\lambda_{v,m}^{(2)}) \\ d_m &= g_x^{(1)}g_v^{(2)}(1 - \lambda_{x,m}^{(1)}\lambda_{v,m}^{(2)}) + g_x^{(2)}g_v^{(1)}(1 - \lambda_{x,m}^{(2)}\lambda_{v,m}^{(1)}) \\ e_m &= g_x^{(1)}g_x^{(2)}(1 - \lambda_{x,m}^{(1)}\lambda_{x,m}^{(2)}) \end{aligned} \quad (3.35)$$

The eigenvalues (3.10) of each block matrix A in (3.5) can be expanded in Taylor series, and using (3.2), we obtain

$$\begin{aligned} \lambda_{x,m}^{(1)} &= -1 + \left(-1 - \rho_{x,1}^{(1)}\right) \left(-i\phi - \frac{\phi^2}{2} + i\frac{\phi^3}{3!}\right) + \mathcal{O}(\phi^4) \\ \lambda_{x,m}^{(2)} &= -1 + \rho_{x,1}^{(2)} \left(i\phi - \frac{\phi^2}{2} - i\frac{\phi^3}{3!}\right) + \mathcal{O}(\phi^4) \\ \lambda_{v,m}^{(1)} &= -1 + \left(-1 - \rho_{v,1}^{(1)}\right) \left(-i\phi - \frac{\phi^2}{2} + i\frac{\phi^3}{3!}\right) + \mathcal{O}(\phi^4) \\ \lambda_{v,m}^{(2)} &= -1 + \rho_{v,1}^{(2)} \left(i\phi - \frac{\phi^2}{2} - i\frac{\phi^3}{3!}\right) + \mathcal{O}(\phi^4) \end{aligned} \quad (3.36)$$

Since there are several terms and factors in the quartic equation, we will work on each term and factor separately. Substitute the corresponding eigenvalues (3.36) in c_m . The term c_m becomes

$$c_m = -g_x^{(1)} - g_x^{(2)} + g_v^{(1)} g_v^{(2)} \left\{ \left(1 + \rho_{v,1}^{(1)} + \rho_{v,1}^{(2)} \right) i\phi + \left(1 + \rho_{v,1}^{(1)} + \rho_{v,1}^{(2)} + 2\rho_{v,1}^{(1)}\rho_{v,1}^{(2)} \right) \frac{\phi^2}{2} + \left(1 + \rho_{v,1}^{(1)} + \rho_{v,1}^{(2)} \right) \frac{i\phi^3}{3!} \right\} + \mathcal{O}(\phi^4)$$

Similarly the coefficients d_m , and e_m become

$$\begin{aligned} d_m &= g_x^{(1)} g_v^{(2)} \left\{ \left(1 + \rho_{x,1}^{(1)} + \rho_{v,1}^{(2)} \right) i\phi + \left(1 + \rho_{x,1}^{(1)} + \rho_{v,1}^{(2)} + 2\rho_{x,1}^{(1)}\rho_{v,1}^{(2)} \right) \frac{\phi^2}{2} + \left(1 + \rho_{x,1}^{(1)} + \rho_{v,1}^{(2)} \right) \frac{i\phi^3}{3!} \right\} \\ &\quad + g_x^{(2)} g_v^{(1)} \left\{ \left(1 + \rho_{v,1}^{(1)} + \rho_{x,1}^{(2)} \right) i\phi + \left(1 + \rho_{v,1}^{(1)} + \rho_{x,1}^{(2)} + 2\rho_{v,1}^{(1)}\rho_{x,1}^{(2)} \right) \frac{\phi^2}{2} + \left(1 + \rho_{v,1}^{(1)} + \rho_{x,1}^{(2)} \right) \frac{i\phi^3}{3!} \right\} + \mathcal{O}(\phi^4) \\ e_m &= g_x^{(1)} g_x^{(2)} \left\{ \left(1 + \rho_{x,1}^{(1)} + \rho_{x,1}^{(2)} \right) i\phi + \left(1 + \rho_{x,1}^{(1)} + \rho_{x,1}^{(2)} + 2\rho_{x,1}^{(1)}\rho_{x,1}^{(2)} \right) \frac{\phi^2}{2} + \left(1 + \rho_{x,1}^{(1)} + \rho_{x,1}^{(2)} \right) \frac{i\phi^3}{3!} \right\} + \mathcal{O}(\phi^4) \end{aligned}$$

Using only the lowest terms in ϕ , we can write the quartic equation as

$$\begin{aligned} \nu^4 &+ (-g_v^{(1)} - g_v^{(2)}) \nu^3 + \left(-g_x^{(1)} - g_x^{(2)} + i g_v^{(1)} g_v^{(2)} (1 + \rho_{v,1}^{(1)} + \rho_{v,1}^{(2)}) \phi \right) \nu^2 \\ &+ \left(g_x^{(1)} g_v^{(2)} (1 + \rho_{x,1}^{(1)} + \rho_{v,1}^{(2)}) + g_x^{(2)} g_v^{(1)} (1 + \rho_{x,1}^{(2)} + \rho_{v,1}^{(1)}) \right) i\phi \nu \\ &+ i g_x^{(1)} g_x^{(2)} \left(1 + \rho_{x,1}^{(1)} + \rho_{x,1}^{(2)} \right) \phi + \mathcal{O}(\phi^2) = 0 \end{aligned} \tag{3.37}$$

If we take a look at the expanded quartic equation (3.37), for $\phi = 0$, the expanded quartic reduces to (3.26) and we see that one of the roots is zero. Roots of polynomials are continuous functions of their coefficients, so for $|\phi|$ small, $|\nu|$ is small. Now terms

with higher orders in $\nu^i \phi^j$ become negligible, and the expanded form (3.37) can be written as

$$(-g_x^{(1)} - g_x^{(2)}) \nu^2 + i g_x^{(1)} g_x^{(2)} (1 + \rho_{x,1}^{(1)} + \rho_{x,1}^{(2)}) \phi = 0 \quad (3.38)$$

Suppose that the factor $(1 + \rho_{x,1}^{(1)} + \rho_{x,1}^{(2)})$ is not equal to zero. Then, by Definition 3.3.1, $g_x^{(1)} g_x^{(2)} (1 + \rho_{x,1}^{(1)} + \rho_{x,1}^{(2)})$ is not equal to zero. For $|\phi|$ small, equation (3.38) has four branches of roots in the complex plane. At the origin, these branches are tangent to each of the four half-lines $(\pm 1 \pm i)t$ for t in the positive reals. Since two of these branches are in the right half plane, we must choose $1 + \rho_{x,1}^{(1)} + \rho_{x,1}^{(2)} = 0$ which can be written in terms of Definition 3.2.2. ■

3.4 Numerical Results

Our main focus in this thesis are linear arrays of vehicles. The boundary conditions which correspond to the interactions of the first vehicle (or agent), and the last vehicle must be treated carefully. We want to preserve the sum of coefficients in each row in (3.5) equal to zero. That is:

$$\sum_{j=-1}^1 \rho_{x,j}^{(i)} = 0, \text{ and } \sum_{j=-1}^1 \rho_{x,j}^{(i)} = 0 \quad (3.39)$$

where $i = 1, 2$, $\rho_{x,0}^{(i)} = 1$, and $\rho_{v,0}^{(i)} = 1$. However, on the boundaries, (3.39) is not equal to zero. This forces us to consider what happen on the boundaries and how we set proper boundary conditions which may depend on the application. We consider two sets of boundary conditions. For easy, let's call them Type I BC and Type II BC.

Type I BC adjusts the central coefficients $\rho_{x,0}^{(i)}$, and $\rho_{v,0}^{(i)}$ on the boundaries as follows:

$$\begin{aligned}\ddot{z}_1^{(1)} &= 0 \\ \ddot{z}_N^{(2)} &= g_x^{(2)} \left(-\rho_{x,-1}^{(2)} z_N^{(2)} + \rho_{x,-1}^{(2)} z_N^{(1)} \right) + g_v^{(2)} \left(-\rho_{v,-1}^{(2)} \dot{z}_N^{(2)} + \rho_{v,-1}^{(2)} \dot{z}_N^{(1)} \right)\end{aligned}\tag{3.40}$$

And for Type II BC, we keep the central coefficients $\rho_{x,0}^{(i)}$, and $\rho_{v,0}^{(i)}$ equal to 1 and we adjust the remaining coefficients accordingly such that the sum of coefficients is zero as follows:

$$\begin{aligned}\ddot{z}_1^{(1)} &= 0 \\ \ddot{z}_N^{(2)} &= g_x^{(2)} \left(z_N^{(2)} - z_N^{(1)} \right) + g_v^{(2)} \left(\dot{z}_N^{(2)} - \dot{z}_N^{(1)} \right)\end{aligned}\tag{3.41}$$

Figures 3.2 and 3.3 show the dynamics of a stable system with nearest neighbor interactions and boundary conditions Type I and Type II respectively. The maximum amplitude of each simulation is shown and the time at which this occurs. The time can be interpreted as the maximum delayed reaction time of the last vehicle in the array. The coefficients are chosen such that the stability condition of Theorem 3.3.1 is satisfied with the following values:

$$\begin{aligned}N &= 100 \text{ (of each type),} \\ g_x^{(1)} &= -1.4, \quad g_x^{(2)} = -1.2, \quad g_v^{(1)} = -1.5, \quad g_v^{(2)} = -1.1 \\ \rho_{x,1}^{(1)} &= -0.6, \quad \rho_{x,1}^{(2)} = -0.4, \quad \rho_{v,1}^{(1)} = -0.15, \quad \rho_{v,1}^{(2)} = -0.55.\end{aligned}$$

Figure 3.4 shows the dynamics of an unstable systems which occur when Theorem 3.3.1 is not satisfied. Figure 3.4 has exactly the same parameters as in Figure 3.3a, except that $\rho_{x,1}^{(1)} = -0.611$. Figure 3.4c shows the eigenvalues around zero which clearly one can see some eigenvalues with positive real part. Positive real parts

Figure 3.2: *Boundary Condition Type I. (a) Eigenvalues of a stable 2-vehicle linear array system (b) Dynamics of the systems. Maximum amplitude of -183.9045 at $t = 195.3337$*
(b) *Boundary Condition Type II. Maximum amplitude of -183.4982 at $t = 194.8469$*

Figure 3.3: *Boundary Condition Type II. (a) Eigenvalues of a stable 2-vehicle linear array system (b) Dynamics of the systems. Maximum amplitude of -183.4982 at $t = 194.8469$*

resulting in an unstable system.

Figure 3.4: *Unstable system. (a) Eigenvalues of the system. (b) Zoomed in eigenvalues around zero. (c) Unstable dynamics of the system.*

3.5 Approximating ν

For $\phi \neq 0$, the eigenvalues ν can be approximated in terms of power expansions of ϕ as

$$\nu = \nu_1 \phi + \frac{1}{2} \nu_2 \phi^2 + \dots \quad (3.42)$$

We now use this form of ν to re-express equation (3.18) in orders of ϕ and expand around $\phi = 0$. The constant term plus higher orders of the expansion is given by

$$g_x^{(1)} g_x^{(2)} \left(1 - (\rho_{x,1}^{(1)} + \rho_{x,-1}^{(1)}) (\rho_{x,1}^{(2)} + \rho_{x,-1}^{(2)}) \right) + \mathcal{O}(\phi) = 0 \quad (3.43)$$

By the constraints (3.2), the constant term vanishes. Similarly we look at the next order expansion:

$$\begin{aligned} & \left\{ i \left(g_x^{(1)} g_v^{(2)} (1 - (\rho_{x,1}^{(1)} + \rho_{x,-1}^{(1)}) (\rho_{v,-1}^{(2)} + \rho_{v,1}^{(2)})) \right. \right. \\ & \quad \left. \left. + g_x^{(2)} g_v^{(1)} (1 - (\rho_{x,-1}^{(2)} + \rho_{x,1}^{(2)}) (\rho_{v,1}^{(1)} + \rho_{v,-1}^{(1)})) \right) \nu_1 \right. \\ & \quad \left. + g_x^{(1)} g_x^{(2)} \left(-i(\rho_{x,1}^{(1)} + \rho_{x,-1}^{(1)}) \rho_{x,1}^{(2)} + i \rho_{x,-1}^{(1)} (\rho_{x,-1}^{(2)} + \rho_{x,1}^{(2)}) \right) \right\} \phi + \mathcal{O}(\phi^2) = 0 \end{aligned} \quad (3.44)$$

Again, by constraints (3.2), this expansion reduces to

$$i g_x^{(1)} g_x^{(2)} \left(1 + \rho_{x,1}^{(1)} + \rho_{x,1}^{(2)} \right) \phi + \mathcal{O}(\phi^2) = 0 \quad (3.45)$$

which by Theorem 3.3.1, the ϕ term is zero. Next, we show terms of orders ϕ^2 and ϕ^3 , these will be used in the proofs below.

The second order expansion of (3.34) around $\phi = 0$ has the following form:

$$(A \nu_1^2 - B \nu_1 + C) \phi^2 + \mathcal{O}(\phi^3) = 0 \quad (3.46)$$

where

$$\begin{aligned} A &= g_x^{(1)} + g_x^{(2)} \\ B &= g_x^{(1)} g_v^{(2)} \left(1 + \rho_{v,1}^{(2)} + \rho_{x,1}^{(1)}\right) + g_x^{(2)} g_v^{(1)} \left(1 + \rho_{v,1}^{(1)} + \rho_{x,1}^{(2)}\right) \\ C &= \frac{1}{2} g_x^{(1)} g_x^{(2)} \left(1 + \rho_{x,1}^{(1)} + \rho_{x,1}^{(2)} + 2\rho_{x,1}^{(1)} \rho_{x,1}^{(2)}\right) = g_x^{(1)} g_x^{(2)} \rho_{x,1}^{(1)} \rho_{x,1}^{(2)} \end{aligned}$$

The last equality for C follows from Theorem 3.3.1.

And the third order expansion of (3.34) around $\phi = 0$ has the following form:

$$\begin{aligned} &\left\{ \left(\frac{1}{2} g_x^{(2)} g_v^{(1)} (1 + \rho_{x,1}^{(2)} + \rho_{v,1}^{(1)}) + \frac{1}{2} g_x^{(1)} g_v^{(2)} (1 + \rho_{x,1}^{(1)} + \rho_{v,1}^{(2)}) - (g_x^{(1)} + g_x^{(2)}) \nu_1 \right) i \nu_2 + \right. \\ &i(g_v^{(1)} + g_v^{(2)}) \nu_1^3 - \frac{1}{6} i g_x^{(1)} g_x^{(2)} (1 + \rho_{x,1}^{(1)} + \rho_{x,1}^{(2)}) - i g_v^{(1)} g_v^{(2)} (1 + \rho_{v,1}^{(1)} + \rho_{v,1}^{(2)}) \nu_1^2 + \\ &\left. \frac{1}{2} i \left(g_x^{(2)} g_v^{(1)} (1 + \rho_{v,1}^{(1)} + 2\rho_{x,1}^{(2)} \rho_{v,1}^{(1)} + \rho_{x,1}^{(2)}) \right. \right. \\ &\left. \left. + g_x^{(1)} g_v^{(2)} (\rho_{v,1}^{(2)} + 2\rho_{v,1}^{(2)} \rho_{x,1}^{(1)} + 1 + \rho_{x,1}^{(1)}) \right) \nu_1 \right\} \phi^3 + \mathcal{O}(\phi^4) \end{aligned} \quad (3.47)$$

$$\begin{aligned} &\frac{1}{2} i \left(g_x^{(2)} g_v^{(1)} (1 + \rho_{v,1}^{(1)} + 2\rho_{x,1}^{(2)} \rho_{v,1}^{(1)} + \rho_{x,1}^{(2)}) \right. \\ &\left. + g_x^{(1)} g_v^{(2)} (\rho_{v,1}^{(2)} + 2\rho_{v,1}^{(2)} \rho_{x,1}^{(1)} + 1 + \rho_{x,1}^{(1)}) \right) \nu_1 \Big\} \phi^3 + \mathcal{O}(\phi^4) = 0 \end{aligned} \quad (3.48)$$

Proposition 3.5.1. *The factor ν_1 in (3.42) is given by*

$$\begin{aligned} \nu_1 &= \frac{1}{2} (g_x^{(1)} + g_x^{(2)})^{-1} \left(\left(g_x^{(1)} g_v^{(2)} (1 + \rho_{v,1}^{(2)} + \rho_{x,1}^{(1)}) + g_x^{(2)} g_v^{(1)} (1 + \rho_{v,1}^{(1)} + \rho_{x,1}^{(2)}) \right) \pm \right. \\ &\left[\left(g_x^{(1)} g_v^{(2)} (1 + \rho_{v,1}^{(2)} + \rho_{x,1}^{(1)}) + g_x^{(2)} g_v^{(1)} (1 + \rho_{v,1}^{(1)} + \rho_{x,1}^{(2)}) \right)^2 \right. \\ &\left. \left. - 4 g_x^{(1)} g_x^{(2)} \rho_{x,1}^{(1)} \rho_{x,1}^{(2)} (g_x^{(1)} + g_x^{(2)}) \right]^{1/2} \right) \end{aligned} \quad (3.49)$$

Proof. Clearly the second order expansion (3.46) around $\phi = 0$ is a quadratic equation in ν_1 , the result follows. $\blacksquare$

We have assumed that ν has the form given in (3.42). Proposition 3.5.1 gives an expression for ν_1 , the next proposition gives an expression for ν_2 .

Proposition 3.5.2. *The factor ν_2 in (3.42) is given by*

$$\begin{aligned} \nu_2 = & \left[-2(g_v^{(1)} + g_v^{(2)})\nu_1^3 + 2g_v^{(1)}g_v^{(2)}(1 + \rho_{v,1}^{(1)} + \rho_{v,1}^{(2)})\nu_1^2 \right. \\ & - \left\{ g_x^{(2)}g_v^{(1)}(1 + \rho_{v,1}^{(1)} + 2\rho_{x,1}^{(2)}\rho_{v,1}^{(1)} + \rho_{x,1}^{(2)}) + g_x^{(1)}g_v^{(2)}(1 + \rho_{v,1}^{(2)} + 2\rho_{v,1}^{(2)}\rho_{x,1}^{(1)} + \rho_{x,1}^{(1)}) \right\} \nu_1 \Big] \\ & \left(g_x^{(2)}g_v^{(1)}(1 + \rho_{x,1}^{(2)} + \rho_{v,1}^{(1)}) + g_x^{(1)}g_v^{(2)}(1 + \rho_{x,1}^{(1)} + \rho_{v,1}^{(2)}) - 2(g_x^{(1)} + g_x^{(2)})\nu_1 \right)^{-1} \end{aligned} \quad (3.50)$$

Proof. Assuming higher orders of ϕ are negligible, then the factor in curly braces $\{\}$ in (3.47) must be zero. Now, substituting the expression for ν_1 given by (3.49) in (3.47), we obtain the result. $\blacksquare$

Notice that the equation of ν_1 produces pairs of values since the $\pm$ signs, so for each sign, there is a ν_2 value.

Corollary 3.5.1. *For lowest orders of ϕ , and $|\phi|$ small, the roots of (3.18) can be written as*

$$\nu = \nu_1 \phi i + \frac{1}{2} \nu_2 \phi^2 + \mathcal{O}(\phi^3) \quad (3.51)$$

Figure 3.5: MATLAB and approximate values of ν given by (3.42) in the complex plane of 200 agents, (100 of each type). (a) Approximate values of ν when coefficients are chosen following the stability criteria given in this section (b) The approximation zoomed in near zero

where both, ν_1 and ν_2 are real numbers and are given by

$$\begin{aligned} \nu_1 &= \frac{1}{2}(g_x^{(1)} + g_x^{(2)})^{-1} \left(\left(g_x^{(1)} g_v^{(2)} (1 + \rho_{v,1}^{(2)} + \rho_{x,1}^{(1)}) + g_x^{(2)} g_v^{(1)} (1 + \rho_{v,1}^{(1)} + \rho_{x,1}^{(2)}) \right) \pm \right. \\ &\quad \left[\left(g_x^{(1)} g_v^{(2)} (1 + \rho_{v,1}^{(2)} + \rho_{x,1}^{(1)}) + g_x^{(2)} g_v^{(1)} (1 + \rho_{v,1}^{(1)} + \rho_{x,1}^{(2)}) \right)^2 \right. \\ &\quad \left. \left. - 4g_x^{(1)} g_x^{(2)} \rho_{x,1}^{(1)} \rho_{x,1}^{(2)} (g_x^{(1)} + g_x^{(2)}) \right]^{1/2} \right) \\ \nu_2 &= \left[-2(g_v^{(1)} + g_v^{(2)})\nu_1^3 + 2g_v^{(1)} g_v^{(2)} (1 + \rho_{v,1}^{(1)} + \rho_{v,1}^{(2)})\nu_1^2 \right. \\ &\quad \left. - \left\{ g_x^{(2)} g_v^{(1)} (1 + \rho_{v,1}^{(1)} + 2\rho_{x,1}^{(2)} \rho_{v,1}^{(1)} + \rho_{x,1}^{(2)}) + g_x^{(1)} g_v^{(2)} (1 + \rho_{v,1}^{(2)} + 2\rho_{v,1}^{(2)} \rho_{x,1}^{(1)} + \rho_{x,1}^{(1)}) \right\} \nu_1 \right] \\ &\quad \left(g_x^{(2)} g_v^{(1)} (1 + \rho_{x,1}^{(2)} + \rho_{v,1}^{(1)}) + g_x^{(1)} g_v^{(2)} (1 + \rho_{x,1}^{(1)} + \rho_{v,1}^{(2)}) - 2(g_x^{(1)} + g_x^{(2)})\nu_1 \right)^{-1} \end{aligned}$$

With these results we have found explicit expressions of the eigenvalues of the system (3.1) with their corresponding eigenvectors $\mathbf{u}_{\pm}(\phi)$ which are defined in equations (3.13)–(3.15). So for each m ($= 0, 1, \dots, N-1$) we obtain four eigenvalues and eigenvectors. For easy in notation, let us relabel the eigenvalues and eigenvectors as

ν_{mr} and $\mathbf{u}_{mr}$ respectively, where $r = 1, \dots, 4$. Then for some constants c_{mr} depending on the initial condition, the general solution of the system can be written as

$$\mathbf{z}(t) = \sum_{m=0}^{N-1} \sum_{r=1}^4 c_{mr} e^{\nu_{mr} t} \mathbf{u}_{mr} \quad (3.52)$$

or, for $j = 1, \dots, N$

$$\begin{aligned} z_j^{(1)}(t) &= \sum_{m=0}^{N-1} \sum_{r=1}^4 c_{mr} e^{\nu_{mr} t} \epsilon_1 e^{j\phi i} \\ z_j^{(2)}(t) &= \sum_{m=0}^{N-1} \sum_{r=1}^4 c_{mr} e^{\nu_{mr} t} \epsilon_2 e^{j\phi i} \end{aligned} \quad (3.53)$$

where c_1 and c_2 are defined in (3.21).

Remark: For each $m (= 0, 1, \dots, N-1)$, we have equations (3.53) since we defined $\phi = 2\pi m/N$ and ν_{mr} in equation (3.42).

For a stable system, we notice that $\text{Re}(\nu) \leq 0$, that means an exponential decay. To see the behavior of the oscillations of the stable system, the constants c_{mr} , ϵ_1 and ϵ_2 and the decay factor can be lumped together for a moment as C_{mr} . Then equations (3.53) have the following form:

$$\begin{aligned} z_j^{(1)}(t) &= \sum_{m=0}^{N-1} \sum_{r=1}^4 C_{mr}^{(1)} e^{(Im(\nu_{mr})t + j\phi)i} \\ z_j^{(2)}(t) &= \sum_{m=0}^{N-1} \sum_{r=1}^4 C_{mr}^{(2)} e^{(Im(\nu_{mr})t + j\phi)i} \end{aligned} \quad (3.54)$$

Chapter 4

Three-Vehicle Linear Arrays with Nearest Neighbor Interactions

Linear flocks in $\mathbb{R}$ of type ...1-1-1 with nearest neighbor interactions have been thoroughly studied ([5], [16]). The necessary condition for stability is that the first moment of the coefficients of the spatial Laplacian must be zero. For flocks of type ...2-1-2-1, the same is true. Details of the latter can be found in the previous chapter. Here we will look at the arrangement ...3-2-1-3-2-1. Some results of this chapter have been submitted to [30]. Thus we consider linear arrays with $3N$ (N of each type) agents in which each agent interacts with its nearest neighbors. The quantities $z_j^{(i)}$ are the deviations from the equilibrium position at a fixed distance from the leader (or “positions”, for short). The quantities $z_j^{(i)}$, and $\dot{z}_j^{(i)}$, $i = 1, 2, 3$, and $j = 1 \dots N$ are relative positions and velocities respectively to the local observer j . These type of systems are called *decentralized*.

Figure 4.1: *Periodic arrangement of flock with three types of agents, labeled by 1, 2, and 3. At time $t = 0$, the first agent start moving to the right.*

The equations of motions for each vehicle type are

$$\begin{aligned}
\ddot{z}_j^{(1)} &= g_x^{(1)} \left(z_j^{(1)} + \rho_{x,1}^{(1)} z_j^{(2)} + \rho_{x,-1}^{(1)} z_{j-1}^{(3)} \right) + g_v^{(1)} \left(\dot{z}_j^{(1)} + \rho_{v,1}^{(1)} \dot{z}_j^{(2)} + \rho_{v,-1}^{(1)} \dot{z}_{j-1}^{(3)} \right) \\
\ddot{z}_j^{(2)} &= g_x^{(2)} \left(z_j^{(2)} + \rho_{x,1}^{(2)} z_j^{(3)} + \rho_{x,-1}^{(2)} z_j^{(1)} \right) + g_v^{(2)} \left(\dot{z}_j^{(2)} + \rho_{v,1}^{(2)} \dot{z}_j^{(3)} + \rho_{v,-1}^{(2)} \dot{z}_j^{(1)} \right) \\
\ddot{z}_j^{(3)} &= g_x^{(3)} \left(z_j^{(3)} + \rho_{x,1}^{(3)} z_{j+1}^{(1)} + \rho_{x,-1}^{(3)} z_j^{(2)} \right) + g_v^{(3)} \left(\dot{z}_j^{(3)} + \rho_{v,1}^{(3)} \dot{z}_{j+1}^{(1)} + \rho_{v,-1}^{(3)} \dot{z}_j^{(2)} \right)
\end{aligned} \tag{4.1}$$

subject to the constraints:

$$\begin{aligned}
\rho_{x,1}^{(1)} + \rho_{x,-1}^{(1)} &= -1, & \rho_{v,1}^{(1)} + \rho_{v,-1}^{(1)} &= -1 \\
\rho_{x,1}^{(2)} + \rho_{x,-1}^{(2)} &= -1, & \rho_{v,1}^{(2)} + \rho_{v,-1}^{(2)} &= -1 \\
\rho_{x,1}^{(3)} + \rho_{x,-1}^{(3)} &= -1, & \rho_{v,1}^{(3)} + \rho_{v,-1}^{(3)} &= -1
\end{aligned} \tag{4.2}$$

where $g_x^{(1)}, g_x^{(2)}, g_x^{(3)}, g_v^{(1)}, g_v^{(2)}$, and $g_v^{(3)}$ are real numbers.

Again we impose periodic boundary conditions meaning that the first particle or agent sees the last particle (or agent) and viseversa. The system in matrix form of the circular array written as a first-order system of equations is given by

$$\begin{aligned}
\frac{d}{dt} \begin{pmatrix} \mathbf{z}^{(1)} \\ \mathbf{z}^{(2)} \\ \mathbf{z}^{(3)} \\ \dot{\mathbf{z}}^{(1)} \\ \dot{\mathbf{z}}^{(2)} \\ \dot{\mathbf{z}}^{(3)} \end{pmatrix} &= \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ g_x^{(1)} \mathbf{I} & g_x^{(1)} \rho_{x,1}^{(1)} \mathbf{I} & g_x^{(1)} \rho_{x,-1}^{(1)} \mathbf{P}_- \\ g_x^{(2)} \rho_{x,-1}^{(2)} \mathbf{I} & g_x^{(2)} \mathbf{I} & g_x^{(2)} \rho_{x,1}^{(2)} \mathbf{I} \\ g_x^{(3)} \rho_{x,1}^{(3)} \mathbf{P}_+ & g_x^{(3)} \rho_{x,-1}^{(3)} \mathbf{I} & g_x^{(3)} \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \\ g_v^{(1)} \mathbf{I} & g_v^{(1)} \rho_{v,1}^{(1)} \mathbf{I} & g_v^{(1)} \rho_{v,-1}^{(1)} \mathbf{P}_- \\ g_v^{(2)} \rho_{v,-1}^{(2)} \mathbf{I} & g_v^{(2)} \mathbf{I} & g_v^{(2)} \rho_{v,1}^{(2)} \mathbf{I} \\ g_v^{(3)} \rho_{v,1}^{(3)} \mathbf{P}_+ & g_v^{(3)} \rho_{v,-1}^{(3)} \mathbf{I} & g_v^{(3)} \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{z}^{(1)} \\ \mathbf{z}^{(2)} \\ \mathbf{z}^{(3)} \\ \dot{\mathbf{z}}^{(1)} \\ \dot{\mathbf{z}}^{(2)} \\ \dot{\mathbf{z}}^{(3)} \end{pmatrix}, \tag{4.3}
\end{aligned}$$

where $\mathbf{P}_+$ and $\mathbf{P}_-$ are $N \times N$ permutations matrices defined in (3.7)

or just

$$\frac{d}{dt} \begin{pmatrix} \mathbf{z} \\ \dot{\mathbf{z}} \end{pmatrix} = \mathbf{M} \begin{pmatrix} \mathbf{z} \\ \dot{\mathbf{z}} \end{pmatrix} \tag{4.4}$$

Clearly $\rho_{x,-1}^{(1)} \mathbf{P}_+$, $\rho_{v,-1}^{(1)} \mathbf{P}_+$, $\rho_{x,1}^{(3)} \mathbf{P}_-$, and $\rho_{v,1}^{(3)} \mathbf{P}_-$ are circulant matrices and respectively, their eigenvalues are given by

$$\begin{aligned}
\lambda_x^{(1)}(\phi) &= \rho_{x,-1}^{(1)} e^{-\phi i} \\
\lambda_v^{(1)}(\phi) &= \rho_{v,-1}^{(1)} e^{-\phi i} \\
\lambda_x^{(3)}(\phi) &= \rho_{x,1}^{(3)} e^{\phi i} \\
\lambda_v^{(3)}(\phi) &= \rho_{v,1}^{(3)} e^{\phi i}
\end{aligned} \tag{4.5}$$

where $\phi = 2\pi m/N$, and $m \in \{0, 1, \dots, N-1\}$. So for each m , we can find the corresponding eigenvector, call it $\mathbf{v}_m$. Now the eigenvectors of diagonal matrices can adopt the same form of $\mathbf{v}_m$ for reasons that will help in simplifying the characteristic equation of the larger system.

We want to determine the eigenvalues and eigenvectors of the matrix $\mathbf{M}$ defined in (4.4).

Proposition 4.0.1. *The eigenvalues ν and associated eigenvectors $\mathbf{u}_\nu(\phi_m)$ of M satisfy*

$$\mathbf{u}_\nu(\phi_m) = \begin{pmatrix} \epsilon_1 \mathbf{v}_m \\ \epsilon_2 \mathbf{v}_m \\ \epsilon_3 \mathbf{v}_m \\ \nu \epsilon_1 \mathbf{v}_m \\ \nu \epsilon_2 \mathbf{v}_m \\ \nu \epsilon_3 \mathbf{v}_m \end{pmatrix}.$$

For each $m \in \{0, \dots, N-1\}$ given, there are six eigenpairs (counting multiplicity)

determined by solving the following equation for ν and ϵ_i :

$$\begin{pmatrix} g_x^{(1)} + \nu g_v^{(1)} - \nu^2 & g_x^{(1)} \rho_{x,1}^{(1)} + \nu g_v^{(1)} \rho_{v,1}^{(1)} & g_x^{(1)} \lambda_{x,m}^{(1)} + \nu g_v^{(1)} \lambda_{v,m}^{(1)} \\ g_x^{(2)} \rho_{x,-1}^{(2)} + \nu g_v^{(2)} \rho_{v,-1}^{(2)} & g_x^{(2)} + \nu g_v^{(2)} - \nu^2 & g_x^{(2)} \rho_{x,1}^{(2)} + \nu g_v^{(2)} \rho_{v,1}^{(2)} \\ g_x^{(3)} \lambda_{x,m}^{(3)} + \nu g_v^{(3)} \lambda_{v,m}^{(3)} & g_x^{(3)} \rho_{x,-1}^{(3)} + \nu g_v^{(3)} \rho_{v,-1}^{(3)} & g_x^{(3)} + \nu g_v^{(3)} - \nu^2 \end{pmatrix} \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Proof. From equations (4.4) and (4.3), we see that an eigenvector $\begin{pmatrix} \mathbf{z} \\ \dot{\mathbf{z}} \end{pmatrix}$ associated to the eigenvalue ν satisfies $\dot{\mathbf{z}} = \nu \mathbf{z}$. Now $P_+^n = I$, and so $e^{i\phi_m}$ and $\mathbf{v}_m$ are the eigenvalues and eigenvectors of P_+ , and $e^{-i\phi_m}$ and $\mathbf{v}_m$ of P_- . Then by substituting $\mathbf{u}_\nu$ into (4.4), one sees that these are the eigenvectors of $\mathbf{M}$.

For the second part, let the eigenvector of $\mathbf{M}$ be

$$\mathbf{u}_\pm(\phi) = \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{u}_3 \\ \mathbf{u}_4 \\ \mathbf{u}_5 \\ \mathbf{u}_6 \end{pmatrix} \quad (4.6)$$

where each $\mathbf{u}_i$, $i = 1, \dots, 6$ is an $N \times 1$ vector.

With some abuse and for ease of notation, we drop the dependence of ϕ and the $\pm$. From now on we write ν , $\mathbf{u}_\nu$, $\lambda_{x,m}^{(\cdot)}$, and $\lambda_{v,m}^{(\cdot)}$ instead of $\nu_\pm(\phi)$, $\mathbf{u}_\pm(\phi)$, $\lambda_x^{(\cdot)}(\phi)$, and $\lambda_v^{(\cdot)}(\phi)$ respectively.

By definition, we can write

$$\mathbf{M}\mathbf{u} = \nu\mathbf{u} \quad (4.7)$$

Notice that $\mathbf{u}_4 = \nu\mathbf{u}_1$, $\mathbf{u}_5 = \nu\mathbf{u}_2$, and $\mathbf{u}_6 = \nu\mathbf{u}_3$, so basically (4.7) can be re-written as

$$\mathbf{M} \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{u}_3 \\ \nu\mathbf{u}_1 \\ \nu\mathbf{u}_2 \\ \nu\mathbf{u}_3 \end{pmatrix} = \nu \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{u}_3 \\ \nu\mathbf{u}_1 \\ \nu\mathbf{u}_2 \\ \nu\mathbf{u}_3 \end{pmatrix} \quad (4.8)$$

Using the eigenvalues of each circulant matrix above, and their corresponding eigenvectors $\mathbf{v}_m$, then we can define $\mathbf{u}_1 = \epsilon_1\mathbf{v}_m$, $\mathbf{u}_2 = \epsilon_2\mathbf{v}_m$, and $\mathbf{u}_3 = \epsilon_3\mathbf{v}_m$ for some constants ϵ_1 , ϵ_2 and ϵ_3 . The vector $\mathbf{u}$ defined above is an eigenvector of the matrix $\mathbf{M}$, that is

$$\mathbf{M} \begin{pmatrix} \epsilon_1\mathbf{v}_m \\ \epsilon_2\mathbf{v}_m \\ \epsilon_3\mathbf{v}_m \\ \nu\epsilon_1\mathbf{v}_m \\ \nu\epsilon_2\mathbf{v}_m \\ \nu\epsilon_3\mathbf{v}_m \end{pmatrix} = \nu \begin{pmatrix} \epsilon_1\mathbf{v}_m \\ \epsilon_2\mathbf{v}_m \\ \epsilon_3\mathbf{v}_m \\ \nu\epsilon_1\mathbf{v}_m \\ \nu\epsilon_2\mathbf{v}_m \\ \nu\epsilon_3\mathbf{v}_m \end{pmatrix} \quad (4.9)$$

Since $\mathbf{P}_\pm\mathbf{v}_m = \lambda_{(\cdot,m)}^{(i)}\mathbf{v}_m$ for $i = 1, 2, 3$. The equations above can be simplified and

rearranged as:

$$\begin{pmatrix} g_x^{(1)} + \nu g_v^{(1)} - \nu^2 & g_x^{(1)} \rho_{x,1}^{(1)} + \nu g_v^{(1)} \rho_{v,1}^{(1)} & g_x^{(1)} \lambda_{x,m}^{(1)} + \nu g_v^{(1)} \lambda_{v,m}^{(1)} \\ g_x^{(2)} \rho_{x,-1}^{(2)} + \nu g_v^{(2)} \rho_{v,-1}^{(2)} & g_x^{(2)} + \nu g_v^{(2)} - \nu^2 & g_x^{(2)} \rho_{x,1}^{(2)} + \nu g_v^{(2)} \rho_{v,1}^{(2)} \\ g_x^{(3)} \lambda_{x,m}^{(3)} + \nu g_v^{(3)} \lambda_{v,m}^{(3)} & g_x^{(3)} \rho_{x,-1}^{(3)} + \nu g_v^{(3)} \rho_{v,-1}^{(3)} & g_x^{(3)} + \nu g_v^{(3)} - \nu^2 \end{pmatrix} \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (4.10)$$

The determinant of the matrix above gives the characteristic equation of the matrix in (4.4). Obviously the characteristic equation in its full form will be somehow cumbersome, however it will have the following form in terms of ν :

$$Q(\nu) = \nu^6 + b_5 \nu^5 + b_4 \nu^4 + b_3 \nu^3 + b_2 \nu^2 + b_1 \nu + b_0 \quad (4.11)$$

where the coefficients b_0 and b_1 are as follows

$$\begin{aligned} b_0 &= g_x^{(1)} g_x^{(2)} g_x^{(3)} \left(\rho_{x,1}^{(1)} \rho_{x,1}^{(2)} \lambda_{x,m}^{(3)} + \rho_{x,-1}^{(2)} \rho_{x,-1}^{(3)} \lambda_{x,m}^{(1)} - \rho_{x,1}^{(1)} \rho_{x,-1}^{(2)} - \rho_{x,1}^{(2)} \rho_{x,-1}^{(3)} - \lambda_{x,m}^{(1)} \lambda_{x,m}^{(3)} + 1 \right) \\ b_1 &= g_v^{(1)} g_x^{(2)} g_x^{(3)} \left(\rho_{v,1}^{(1)} \rho_{x,1}^{(2)} \lambda_{x,m}^{(3)} + \rho_{x,-1}^{(2)} \rho_{x,-1}^{(3)} \lambda_{v,m}^{(1)} - \rho_{v,1}^{(1)} \rho_{x,-1}^{(2)} - \rho_{x,1}^{(2)} \rho_{x,-1}^{(3)} - \lambda_{v,m}^{(1)} \lambda_{x,m}^{(3)} + 1 \right) \\ &\quad + g_x^{(1)} g_v^{(2)} g_x^{(3)} \left(\rho_{x,1}^{(1)} \rho_{v,1}^{(2)} \lambda_{x,m}^{(3)} + \rho_{v,-1}^{(2)} \rho_{x,-1}^{(3)} \lambda_{x,m}^{(1)} - \rho_{x,1}^{(1)} \rho_{v,-1}^{(2)} - \rho_{v,1}^{(2)} \rho_{x,-1}^{(3)} - \lambda_{x,m}^{(1)} \lambda_{x,m}^{(3)} + 1 \right) \\ &\quad + g_x^{(1)} g_x^{(2)} g_v^{(3)} \left(\rho_{x,1}^{(1)} \rho_{x,1}^{(2)} \lambda_{v,m}^{(3)} + \rho_{x,-1}^{(2)} \rho_{v,-1}^{(3)} \lambda_{x,m}^{(1)} - \rho_{x,1}^{(1)} \rho_{x,-1}^{(2)} - \rho_{x,1}^{(2)} \rho_{v,-1}^{(3)} - \lambda_{x,m}^{(1)} \lambda_{v,m}^{(3)} + 1 \right) \end{aligned}$$

The rest of the coefficients of (4.11) are not given here for lack of space and they are not necessary for our purpose. ■

4.1 Necessary Condition for Linear Stability

In order to be able to manage the amount of terms and factors in (4.11), let's introduce the following notation:

Definition 4.1.1. Let r , s , and t be real numbers, define

$$C(r, s, t) = 2rst + rs + rt + st + r + s + t + 1 \quad (4.12)$$

Theorem 4.1.1. Let $g_x^{(1)}$, $g_x^{(2)}$, and $g_x^{(3)}$ be real numbers, then necessary conditions for stability of (4.1) on the circle are:

$$(i) \ g_x^{(1)} \neq 0, \ g_x^{(2)} \neq 0, \text{ and } g_x^{(3)} \neq 0$$

$$(ii) \ b_2 \neq 0 \text{ and}$$

$$(iii) \ \prod_{i=1}^3 \beta_{x,1}^{(i)} + \sum_{i=1}^3 \beta_{x,1}^{(i)} = 0$$

Proof. Part (i), suppose either $g_x^{(1)} = 0$, $g_x^{(2)} = 0$, or $g_x^{(3)} = 0$, then $b_0 = 0$ for each m , which implies that the zero eigenvalue has multiplicity of at least N , contradicting Definition (3.2.1).

Suppose (ii) is false. Then for $\phi = 0$, we get multiplicity 3 for the eigenvalue zero. This violates Definition 3.2.1.

For part (iii), first, notice in (4.11) that when $\phi = 0$ then $b_0 = 0$. This is easily seen it by substituting (4.5) and the constraints given in (4.2). The characteristic equation (4.11) has a zero root when $\phi = 0$. So for ϕ small, also ν is small since roots of polynomials are continuous functions of their coefficients.

Now, let's approximate the characteristic equation (4.11) using a Taylor expansion around $\phi = 0$. We see that b_0 has the following form

$$\begin{aligned} b_0 = & g_x^{(1)} g_x^{(2)} g_x^{(3)} \left[\left(\rho_{x,1}^{(1)} \rho_{x,1}^{(2)} \rho_{x,1}^{(3)} - \rho_{x,-1}^{(1)} \rho_{x,-1}^{(2)} \rho_{x,-1}^{(3)} \right) i\phi + \rho_{x,1}^{(1)} \rho_{x,1}^{(2)} \rho_{x,1}^{(3)} + \rho_{x,-1}^{(1)} \rho_{x,-1}^{(2)} \rho_{x,-1}^{(3)} \right. \\ & \left. - \rho_{x,1}^{(1)} \rho_{x,-1}^{(2)} - \rho_{x,-1}^{(1)} \rho_{x,1}^{(3)} - \rho_{x,1}^{(2)} \rho_{x,-1}^{(3)} + 1 \right] + \mathcal{O}(\phi^2) \end{aligned}$$

and using the constraints in (4.2), it reduces to

$$b_0 = g_x^{(1)} g_x^{(2)} g_x^{(3)} C \left(\rho_{x,1}^{(1)}, \rho_{x,1}^{(2)}, \rho_{x,1}^{(3)} \right) i\phi + \mathcal{O}(\phi^2)$$

In similar manner, the coefficient b_1 becomes

$$\begin{aligned} b_1 = & \left[g_v^{(1)} g_x^{(2)} g_x^{(3)} C \left(\rho_{v,1}^{(1)}, \rho_{x,1}^{(2)}, \rho_{x,1}^{(3)} \right) + g_x^{(1)} g_v^{(2)} g_x^{(3)} C \left(\rho_{x,1}^{(1)}, \rho_{v,1}^{(2)}, \rho_{x,1}^{(3)} \right) \right. \\ & \left. + g_x^{(1)} g_x^{(2)} g_v^{(3)} C \left(\rho_{x,1}^{(1)}, \rho_{x,1}^{(2)}, \rho_{v,1}^{(3)} \right) \right] i\phi + \mathcal{O}(\phi^2) \end{aligned}$$

The characteristic equation (4.11) becomes

$$\begin{aligned} b_2 \nu^2 + & \left[g_v^{(1)} g_x^{(2)} g_x^{(3)} C \left(\rho_{v,1}^{(1)}, \rho_{x,1}^{(2)}, \rho_{x,1}^{(3)} \right) + g_x^{(1)} g_v^{(2)} g_x^{(3)} C \left(\rho_{x,1}^{(1)}, \rho_{v,1}^{(2)}, \rho_{x,1}^{(3)} \right) \right. \\ & \left. + g_x^{(1)} g_x^{(2)} g_v^{(3)} C \left(\rho_{x,1}^{(1)}, \rho_{x,1}^{(2)}, \rho_{v,1}^{(3)} \right) \right] i\phi \nu + g_x^{(1)} g_x^{(2)} g_x^{(3)} C \left(\rho_{x,1}^{(1)}, \rho_{x,1}^{(2)}, \rho_{x,1}^{(3)} \right) i\phi \\ & + \mathcal{O}(\nu^2 \phi^2) + \mathcal{O}(\phi^2) = 0 \end{aligned} \quad (4.13)$$

Higher order terms in (4.13) are very small for ϕ small and ν small. Equation (4.13) can have solutions with positive real part. If we assume parts (i) and (ii), the equation satisfies all conditions of Proposition A.0.1 *except* the condition that $b'_0(0) \neq 0$. That proposition implies instability, and so to avoid the system from being unstable, we must have $b'_0(0) = 0$. Part (i) and (4.13) then imply

$$\left. \frac{\partial}{\partial \phi} \left(g_x^{(1)} g_x^{(2)} g_x^{(3)} C \left(\rho_{x,1}^{(1)}, \rho_{x,1}^{(2)}, \rho_{x,1}^{(3)} \right) i\phi \right) \right|_{\phi=0} = 0$$

The factor $C \left(\rho_{x,1}^{(1)}, \rho_{x,1}^{(2)}, \rho_{x,1}^{(3)} \right)$ must be zero.

By Definition 3.2.2 and the constraints (4.2), we have the following relationship:

$$\rho_{x,1}^{(i)} = \frac{1}{2} \left(\beta_{x,1}^{(i)} - 1 \right) \quad (4.14)$$

Substitute this into (4.13). Part (iii) follows by differentiating and setting $\phi = 0$. ■

Corollary 4.1.1. *The conjectures of [5] imply the following. If $g_x^{(1)} \neq 0$, $g_x^{(2)} \neq 0$, and $g_x^{(3)} \neq 0$ and*

$$\sum_{i=1}^3 \beta_{x,1}^{(i)} + \prod_{i=1}^3 \beta_{x,1}^{(i)} \neq 0 ,$$

then for large N , the system on the line given by 4.1 has some form of instability (Definitions 3.2.1 or 2.0.2).

4.2 Numerical Results

Now we want to pay attention to the first and last vehicles (or agents) of the array, and we want to preserve the sum of each row in (4.1) equals zero. That is the sum of coefficientts:

$$\sum_{j=-1}^1 \rho_{x,j}^{(i)} = 0, \text{ and } \sum_{j=-1}^1 \rho_{v,j}^{(i)} = 0 \quad (4.15)$$

where $i = 1, 2, 3$, $\rho_{x,0}^{(i)} = 1$, and $\rho_{v,0}^{(i)} = 1$. However, on the boundaries, (4.15) is not equal to zero. This forces us to consider what happen on the boundaries and how we set proper boundary conditions which may depend on the application. We consider two sets of boundary conditions. For easy, let's call them Type I BC and Type II BC. Type I BC adjusts the central coefficients $\rho_{x,0}^{(i)}$, and $\rho_{v,0}^{(i)}$ on the boundaries as follows:

$$\begin{aligned} \ddot{z}_1^{(1)} &= 0 \\ \ddot{z}_N^{(3)} &= g_x^{(3)} \left(-\rho_{x,-1}^{(3)} z_N^{(3)} + \rho_{x,-1}^{(3)} z_N^{(2)} \right) + g_v^{(3)} \left(-\rho_{v,-1}^{(3)} \dot{z}_N^{(3)} + \rho_{v,-1}^{(3)} \dot{z}_N^{(2)} \right) \end{aligned} \quad (4.16)$$

And for Type II BC, we keep the central coefficients $\rho_{x,0}^{(i)}$, and $\rho_{v,0}^{(i)}$ equal to 1 and

we adjust the remaining coefficients accordingly such that the sum of coefficients is zero as follows:

$$\begin{aligned}\ddot{z}_1^{(1)} &= 0 \\ \ddot{z}_N^{(3)} &= g_x^{(3)} \left(z_N^{(3)} - z_N^{(2)} \right) + g_v^{(3)} \left(\dot{z}_N^{(3)} - \dot{z}_N^{(2)} \right)\end{aligned}\tag{4.17}$$

Figure 4.2 and 4.3 are a numerical simulations with parameters satisfying Theorem 4.1.1 as follows:

$$\begin{aligned}N &= 60 \text{ (of each type), } g_x^{(1)} = g_x^{(2)} = g_x^{(3)} = -1, g_v^{(1)} = g_v^{(2)} = g_v^{(3)} = -1.4 \\ \rho_{x,1}^{(1)} &= -0.6, \rho_{x,1}^{(2)} = -0.8, \rho_{x,1}^{(3)} = -0.1429, \rho_{x,1}^{(1)} = \rho_{v,1}^{(2)} = \rho_{v,1}^{(3)} = -0.3\end{aligned}$$

We see clearly that the whole system oscillates for a period of time, then later, it becomes stable. The maximum amplitude of each simulation is shown and the time at which this occurs. The time can be interpreted as the maximum delayed reaction time of the last element in the array.

Figure 4.4 shows the dynamics of a *flock unstable* system. Notice the large magnitude of the oscillations ($\times 10^9$). In fact, the parameters are satisfying $\rho_{x,1}^{(1)} + \rho_{x,1}^{(2)} + \rho_{x,1}^{(3)} = -3/2$ which is referred to a conjecture called the method of moments described in [5], [15] and others as follows:

$$\begin{aligned}N &= 60 \text{ (of each type), } g_x^{(1)} = g_x^{(2)} = g_x^{(3)} = -1, g_v^{(1)} = g_v^{(2)} = g_v^{(3)} = -1.4 \\ \rho_{x,1}^{(1)} &= -0.6, \rho_{x,1}^{(2)} = -0.8, \rho_{x,1}^{(3)} = -0.1, \rho_{x,1}^{(1)} = \rho_{v,1}^{(2)} = \rho_{v,1}^{(3)} = -0.3\end{aligned}$$

The results show that that conjecture does not hold.

Figure 4.4 shows the dynamics of an unstable systems which occur when Theorem 4.1.1 is not satisfied. Figure 4.4 has exactly the same parameters as in Figure 4.2, except that $\rho_{x,1}^{(1)} = -0.4$, and $\rho_{x,1}^{(3)} = -0.3$.

(a)

(b)

Figure 4.2: *Boundary Condition Type I. (a) Eigenvalues of a stable 3-mass linear array system (b) Dynamics of the systems. Maximum amplitude of -215.1125 at $t = 239.5618$*

(a)

(b)

Figure 4.3: *Boundary Condition Type II. (a) Eigenvalues of a stable 3-mass linear array system (b) Dynamics of the systems. Maximum amplitude of -214.8957 at $t = 239.5187$*

Figure 4.4: *Unstable system. (a) Eigenvalues of the system. (b) Zoomed in eigenvalues around zero. (c) Unstable dynamics of the system.*

Chapter 5

Two-Vehicle Linear Arrays with Next Nearest Neighbors Interactions

In this section, we consider a coupled system of one-dimensional array of damped harmonic diatomic oscillators (or particles or agents). The oscillators are interacting (or interchanging information) from the nearest and next nearest neighbors. This model can be considered as one step further to the general case of linear array of damped oscillators and from which some models studied in this paper are particular cases.

Figure 5.1: *Periodic arrangement of flock with two types of agents, labeled by 1 and 2. Each agent uses information from four others; the arrows indicate information flow. At time $t = 0$, the first agent start moving to the right.*

Each $z_j^{(i)}$, and $\dot{z}_j^{(i)}$, $i = 1, 2$, and $k = 1 \dots N$ are relative positions and velocities respectively to the local observer j . See Figure 2.3.

$$\begin{aligned}
\ddot{z}_k^{(1)} &= g_x^{(1)}(z_k^{(1)} + \rho_{x,1}^{(1)}z_k^{(2)} + \rho_{x,-1}^{(1)}z_{k-1}^{(2)} + \rho_{x,2}^{(1)}z_{k+1}^{(1)} + \rho_{x,-2}^{(1)}z_{k-1}^{(1)}) \\
&\quad + g_v^{(1)}(\dot{z}_k^{(1)} + \rho_{v,1}^{(1)}\dot{z}_k^{(2)} + \rho_{v,-1}^{(1)}\dot{z}_{k-1}^{(2)} + \rho_{v,2}^{(1)}\dot{z}_{k+1}^{(1)} + \rho_{v,-2}^{(1)}\dot{z}_{k-1}^{(1)}) \\
\ddot{z}_k^{(2)} &= g_x^{(2)}(z_k^{(2)} + \rho_{x,-1}^{(2)}z_k^{(1)} + \rho_{x,1}^{(2)}z_{k+1}^{(1)} + \rho_{x,-2}^{(2)}z_{k-1}^{(2)} + \rho_{x,2}^{(2)}z_{k+1}^{(2)}) \\
&\quad + g_v^{(2)}(\dot{z}_k^{(2)} + \rho_{v,-1}^{(2)}\dot{z}_k^{(1)} + \rho_{v,1}^{(2)}\dot{z}_{k+1}^{(1)} + \rho_{v,-2}^{(2)}\dot{z}_{k-1}^{(2)} + \rho_{v,2}^{(2)}\dot{z}_{k+1}^{(2)})
\end{aligned} \tag{5.1}$$

subject to the constraints:

$$\begin{aligned}
\rho_{x,1}^{(1)} + \rho_{x,-1}^{(1)} + \rho_{x,2}^{(1)} + \rho_{x,-2}^{(1)} &= -1, & \rho_{x,1}^{(2)} + \rho_{x,-1}^{(2)} + \rho_{x,2}^{(2)} + \rho_{x,-2}^{(2)} &= -1 \\
\rho_{v,1}^{(1)} + \rho_{v,-1}^{(1)} + \rho_{v,2}^{(1)} + \rho_{v,-2}^{(1)} &= -1, & \rho_{v,1}^{(2)} + \rho_{v,-1}^{(2)} + \rho_{v,2}^{(2)} + \rho_{v,-2}^{(2)} &= -1
\end{aligned} \tag{5.2}$$

where $g_x^{(1)}, g_x^{(2)}, g_v^{(1)}$, and $g_v^{(2)}$ are real numbers. The system can be written more compactly in matrix form

$$\frac{d}{dt} \begin{pmatrix} \mathbf{z} \\ \dot{\mathbf{z}} \end{pmatrix} = \mathbf{M} \begin{pmatrix} \mathbf{z} \\ \dot{\mathbf{z}} \end{pmatrix} \tag{5.3}$$

or

$$\frac{d}{dt} \begin{pmatrix} \mathbf{z}^{(1)} \\ \mathbf{z}^{(2)} \\ \dot{\mathbf{z}}^{(1)} \\ \dot{\mathbf{z}}^{(2)} \end{pmatrix} = \left(\begin{array}{c|c|c|c} \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \\ \hline g_x^{(1)}\mathbf{B}_x^{(1)} & g_x^{(1)}\mathbf{A}_x^{(1)} & g_v^{(1)}\mathbf{B}_v^{(1)} & g_v^{(1)}\mathbf{A}_v^{(1)} \\ \hline g_x^{(2)}\mathbf{A}_x^{(2)} & g_x^{(2)}\mathbf{B}_x^{(2)} & g_v^{(2)}\mathbf{A}_v^{(2)} & g_v^{(2)}\mathbf{B}_v^{(2)} \end{array} \right) \begin{pmatrix} \mathbf{z}^{(1)} \\ \mathbf{z}^{(2)} \\ \dot{\mathbf{z}}^{(1)} \\ \dot{\mathbf{z}}^{(2)} \end{pmatrix} \tag{5.4}$$

where $\mathbf{P}_{+/-}$ are permutation matrices defined in (3.7). The eigenvalues of each $\mathbf{A}_x^{(1)}$, $\mathbf{A}_x^{(2)}$, $\mathbf{A}_v^{(1)}$, and $\mathbf{A}_v^{(2)}$ are given by (3.10). Similarly, we can find the eigenvalues of each matrix $\mathbf{B}_x^{(1)}$, $\mathbf{B}_x^{(2)}$, $\mathbf{B}_v^{(1)}$, and $\mathbf{B}_v^{(2)}$. Each matrix $\mathbf{B}$ is also circulant, therefore

their eigenvectors are given by the discrete Fourier transform (see [6], [17]). So, for $m = \{0, 1, \dots, N-1\}$, and $\phi = 2\pi m/N$, the eigenvectors of each matrix $\mathbf{A}$, and $\mathbf{B}$ can be shown to be

$$\mathbf{v}_m = \begin{pmatrix} 1 \\ e^{\phi i} \\ e^{2\phi i} \\ \vdots \\ e^{(N-1)\phi i} \end{pmatrix} \quad (5.5)$$

The list of all matrices and their eigenvalues are shown here.

$$\begin{aligned} \mathbf{A}_x^{(1)} &= \rho_{x,1}^{(1)} \mathbf{I} + \rho_{x,-1}^{(1)} \mathbf{P}_- ; & \lambda_x^{(1)}(\phi) &= \rho_{x,1}^{(1)} + \rho_{x,-1}^{(1)} e^{-i\phi} . \\ \mathbf{A}_x^{(2)} &= \rho_{x,-1}^{(2)} \mathbf{I} + \rho_{x,1}^{(2)} \mathbf{P}_+ ; & \lambda_x^{(2)}(\phi) &= \rho_{x,-1}^{(2)} + \rho_{x,1}^{(2)} e^{i\phi} . \\ \mathbf{A}_v^{(1)} &= \rho_{v,1}^{(1)} \mathbf{I} + \rho_{v,-1}^{(1)} \mathbf{P}_- ; & \lambda_v^{(1)}(\phi) &= \rho_{v,1}^{(1)} + \rho_{v,-1}^{(1)} e^{-i\phi} . \\ \mathbf{A}_v^{(2)} &= \rho_{v,-1}^{(2)} \mathbf{I} + \rho_{v,1}^{(2)} \mathbf{P}_+ ; & \lambda_v^{(2)}(\phi) &= \rho_{v,-1}^{(2)} + \rho_{v,1}^{(2)} e^{i\phi} . \\ \mathbf{B}_x^{(1)} &= \mathbf{I} + \rho_{x,-2}^{(1)} \mathbf{P}_- + \rho_{x,2}^{(1)} \mathbf{P}_+ ; & \mu_x^{(1)}(\phi) &= 1 + \rho_{x,2}^{(1)} e^{i\phi} + \rho_{x,-2}^{(1)} e^{-i\phi} . \\ \mathbf{B}_x^{(2)} &= \mathbf{I} + \rho_{x,-2}^{(2)} \mathbf{P}_- + \rho_{x,2}^{(2)} \mathbf{P}_+ ; & \mu_x^{(2)}(\phi) &= 1 + \rho_{x,2}^{(2)} e^{i\phi} + \rho_{x,-2}^{(2)} e^{-i\phi} . \\ \mathbf{B}_v^{(1)} &= \mathbf{I} + \rho_{v,-2}^{(1)} \mathbf{P}_- + \rho_{v,2}^{(1)} \mathbf{P}_+ ; & \mu_v^{(1)}(\phi) &= 1 + \rho_{v,2}^{(1)} e^{i\phi} + \rho_{v,-2}^{(1)} e^{-i\phi} . \\ \mathbf{B}_v^{(2)} &= \mathbf{I} + \rho_{v,-2}^{(2)} \mathbf{P}_- + \rho_{v,2}^{(2)} \mathbf{P}_+ ; & \mu_v^{(2)}(\phi) &= 1 + \rho_{v,2}^{(2)} e^{i\phi} + \rho_{v,-2}^{(2)} e^{-i\phi} . \end{aligned} \quad (5.6)$$

Again we are interested in knowing the eigenvalues ν and eigenvectors of the matrix of the system in (5.4). The following proposition is derived in the same way as the analogous proposition in the previous chapters.

Proposition 5.0.1. *The eigenvalues ν and associated eigenvectors $\mathbf{u}_\nu(\phi_m)$ of M*

satisfy

$$\mathbf{u}_\nu(\phi_m) = \begin{pmatrix} \epsilon_1 \mathbf{V}_m \\ \epsilon_2 \mathbf{V}_m \\ \nu \epsilon_1 \mathbf{V}_m \\ \nu \epsilon_2 \mathbf{V}_m \end{pmatrix}.$$

For each $m \in \{0, \dots, N-1\}$ given, there are four eigenpairs (counting multiplicity) determined by solving the following equation for ν and ϵ_i (we dropped the argument ϕ):

$$\begin{pmatrix} g_x^{(1)} \mu_x^{(1)} + \nu g_v^{(1)} \mu_v^{(1)} - \nu^2 & g_x^{(1)} \lambda_x^{(1)} + \nu g_v^{(1)} \lambda_v^{(1)} \\ g_x^{(2)} \lambda_x^{(2)} + \nu g_v^{(2)} \lambda_v^{(2)} & g_x^{(2)} \mu_x^{(2)} + \nu g_v^{(2)} \mu_v^{(2)} - \nu^2 \end{pmatrix} \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Proof. From equations (5.3) and (5.4), we see that an eigenvector $\begin{pmatrix} \mathbf{z} \\ \dot{\mathbf{z}} \end{pmatrix}$ associated to the eigenvalue ν satisfies $\dot{\mathbf{z}} = \nu \mathbf{z}$. Now $P_+^n = I$, and so $e^{i\phi_m}$ and $\mathbf{v}_m$ are the eigenvalues and eigenvectors of P_+ , and $e^{-i\phi_m}$ and $\mathbf{v}_m$ of P_- . Then by substituting $\mathbf{u}_\nu$ into (5.4), one sees that these are the eigenvectors of $\mathbf{M}$.

For the second part, let the eigenvector of $\mathbf{M}$ be

$$\mathbf{u}_\pm(\phi) = \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{u}_3 \\ \mathbf{u}_4 \end{pmatrix} \tag{5.7}$$

where each $\mathbf{u}_i$, $i = 1, \dots, 4$ is an $N \times 1$ vector.

Following the same argument as in (3.12), and knowing the expressions for the eigenvalues of each matrix $\mathbf{A}$ and $\mathbf{B}$ respectively, and for some constant c_i , $i = 1, 2$, we can write

$$\begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \\ g_x^{(1)} \mathbf{B}_x^{(1)} & g_x^{(1)} \mathbf{A}_x^{(1)} & g_v^{(1)} \mathbf{B}_v^{(1)} & g_v^{(1)} \mathbf{A}_v^{(1)} \\ g_x^{(2)} \mathbf{A}_x^{(2)} & g_x^{(2)} \mathbf{B}_x^{(2)} & g_v^{(2)} \mathbf{A}_v^{(2)} & g_v^{(2)} \mathbf{B}_v^{(2)} \end{pmatrix} \begin{pmatrix} c_1 \mathbf{v}_m \\ c_2 \mathbf{v}_m \\ \nu c_1 \mathbf{v}_m \\ \nu c_2 \mathbf{v}_m \end{pmatrix} = \nu \begin{pmatrix} c_1 \mathbf{v}_m \\ c_2 \mathbf{v}_m \\ \nu c_1 \mathbf{v}_m \\ \nu c_2 \mathbf{v}_m \end{pmatrix} \quad (5.8)$$

Then

$$\begin{aligned} c_1 g_x^{(1)} \mathbf{B}_x^{(1)} \mathbf{v}_m + c_2 g_x^{(1)} \mathbf{A}_x^{(1)} \mathbf{v}_m + c_1 \nu g_v^{(1)} \mathbf{B}_v^{(1)} \mathbf{v}_m + c_2 \nu g_v^{(1)} \mathbf{A}_v^{(1)} \mathbf{v}_m &= \nu^2 c_1 \mathbf{v}_m \\ c_1 g_x^{(2)} \mathbf{A}_x^{(2)} \mathbf{v}_m + c_2 g_x^{(2)} \mathbf{B}_x^{(2)} \mathbf{v}_m + c_1 \nu g_v^{(2)} \mathbf{A}_v^{(2)} \mathbf{v}_m + c_2 \nu g_v^{(2)} \mathbf{B}_v^{(2)} \mathbf{v}_m &= \nu^2 c_2 \mathbf{v}_m \end{aligned} \quad (5.9)$$

We know that $\mathbf{A}_x^{(i)} \mathbf{v}_m = \lambda_{x,m}^{(i)} \mathbf{v}_m$, and $\mathbf{B}_x^{(i)} \mathbf{v}_m = \mu_{x,m}^{(i)} \mathbf{v}_m$ for each $i = 1, 2$, and then dropping $\mathbf{v}_m$ and rearranging

$$\begin{aligned} c_1 g_x^{(1)} \mu_{x,m}^{(1)} + c_2 g_x^{(1)} \lambda_{x,m}^{(1)} + c_1 \nu g_v^{(1)} \mu_{v,m}^{(1)} + c_2 \nu g_v^{(1)} \lambda_{v,m}^{(1)} &= \nu^2 c_1 \\ c_1 g_x^{(2)} \lambda_{x,m}^{(2)} + c_2 g_x^{(2)} \mu_{x,m}^{(2)} + c_1 \nu g_v^{(2)} \lambda_{v,m}^{(2)} + c_2 \nu g_v^{(2)} \mu_{v,m}^{(2)} &= \nu^2 c_2 \end{aligned} \quad (5.10)$$

$$\begin{pmatrix} g_x^{(1)} \mu_{x,m}^{(1)} + \nu g_v^{(1)} \mu_{v,m}^{(1)} - \nu^2 & g_x^{(1)} \lambda_{x,m}^{(1)} + \nu g_v^{(1)} \lambda_{v,m}^{(1)} \\ g_x^{(2)} \lambda_{x,m}^{(2)} + \nu g_v^{(2)} \lambda_{v,m}^{(2)} & g_x^{(2)} \mu_{x,m}^{(2)} + \nu g_v^{(2)} \mu_{v,m}^{(2)} - \nu^2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (5.11)$$

■

We want the determinant to be zero of the matrix on the lefthand side of the equation above.

Lemma 5.0.1. *When $\phi = 0$, the matrix of Proposition 5.0.1 has determinant $Q(\nu)$ equal to*

$$\nu^2 \left[\nu^2 + \nu \left(g_v^{(1)} \alpha_{v,1}^{(1)} + g_v^{(2)} \alpha_{v,1}^{(2)} \right) + \left(g_x^{(1)} \alpha_{x,1}^{(1)} + g_x^{(2)} \alpha_{x,1}^{(2)} \right) \right] .$$

The full expression of the constant term of $Q(\nu)$ is $a_0(\phi)$, where

$$a_0(\phi) = g_x^{(1)} g_x^{(2)} \left(\mu_x^{(1)}(\phi) \mu_x^{(2)}(\phi) - \lambda_x^{(1)}(\phi) \lambda_x^{(2)}(\phi) \right) .$$

Proof. The full determinant of the matrix in Proposition 5.0.1 is equal to

$$\begin{aligned} & \nu^4 + \left(-g_v^{(1)} \mu_{v,m}^{(1)} - g_v^{(2)} \mu_{v,m}^{(2)} \right) \nu^3 \\ & + \left(-g_x^{(1)} \mu_{x,m}^{(1)} - g_x^{(2)} \mu_{x,m}^{(2)} + g_v^{(1)} g_v^{(2)} \left(\mu_{v,m}^{(1)} \mu_{v,m}^{(2)} - \lambda_{v,m}^{(1)} \lambda_{v,m}^{(2)} \right) \right) \nu^2 \\ & + \left(g_x^{(1)} g_v^{(2)} \left(\mu_{x,m}^{(1)} \mu_{v,m}^{(2)} - \lambda_{x,m}^{(1)} \lambda_{v,m}^{(2)} \right) + g_v^{(1)} g_x^{(2)} \left(\mu_{v,m}^{(1)} \mu_{x,m}^{(2)} - \lambda_{v,m}^{(1)} \lambda_{x,m}^{(2)} \right) \right) \nu \\ & + g_x^{(1)} g_x^{(2)} \left(\mu_{x,m}^{(1)} \mu_{x,m}^{(2)} - \lambda_{x,m}^{(1)} \lambda_{x,m}^{(2)} \right) = Q(\nu) \end{aligned} \tag{5.12}$$

or in short

$$Q(\nu) = \nu^4 + b_3 \nu^3 + b_2 \nu^2 + b_1 \nu + b_0 \tag{5.13}$$

Now set $\phi = 0$. From (5.6) and recalling Definition 3.2.2, we see that for $r \in \{x, v\}$ and $i \in \{1, 2\}$:

$$\mu_r^{(i)}(0) = 1 + \alpha_{r,2}^{(i)} \quad \text{and} \quad \lambda_r^{(i)}(0) = \alpha_{r,1}^{(i)} .$$

Note that the constraint (5.2) gives for $r \in \{x, v\}$

$$1 + \alpha_{r,1}^{(i)} + \alpha_{r,2}^{(i)} = 0 \quad \implies \quad -\mu_r^{(i)}(0) = \lambda_r^{(i)}(0) = \alpha_{r,1}^{(i)} .$$

Substituting this, and some algebra, yields the Lemma.

■

5.1 Necessary Conditions for Linear Stability

The next results are entirely analogous to the ones in the previous chapters, and we mention them almost without comments or proof.

Theorem 5.1.1. *Let $g_x^{(1)}$ and $g_x^{(2)}$ be real numbers. Then necessary conditions for stability of (5.1) are*

- (i) $g_x^{(1)} \neq 0$, and $g_x^{(2)} \neq 0$
- (ii) $g_x^{(1)} \alpha_{x,1}^{(1)} + g_x^{(2)} \alpha_{x,1}^{(2)} > 0$, and $g_v^{(1)} \alpha_{v,1}^{(1)} + g_v^{(2)} \alpha_{v,1}^{(2)} > 0$ and
- (iii) $\alpha_{x,1}^{(2)} \left(\beta_{x,1}^{(1)} + 2\beta_{x,2}^{(1)} \right) + \alpha_{x,1}^{(1)} \left(\beta_{x,1}^{(2)} + 2\beta_{x,2}^{(2)} \right) = 0$.

Proof. Part (i), By and large, this proof is very similar to that of Theorem 4.1.1. Part (ii) is now more easily derived by explicitly solving for the zero in $Q(\nu)$ when $\phi = 0$ (see Lemma 5.0.1). In (iii), let us consider the eigenvalues (5.6) expanded in the Taylor series around $\phi = 0$ are

$$\begin{aligned}
 \lambda_{x,m}^{(1)} &= \rho_{x,1}^{(1)} + \rho_{x,-1}^{(1)} - i\rho_{x,-1}^{(1)}\phi + \mathcal{O}(\phi^2) \\
 \lambda_{x,m}^{(2)} &= \rho_{x,1}^{(2)} + \rho_{x,-1}^{(2)} + i\rho_{x,1}^{(2)}\phi + \mathcal{O}(\phi^2) \\
 \lambda_{v,m}^{(1)} &= \rho_{v,1}^{(1)} + \rho_{v,-1}^{(1)} - i\rho_{v,-1}^{(1)}\phi + \mathcal{O}(\phi^2) \\
 \lambda_{v,m}^{(2)} &= \rho_{v,1}^{(2)} + \rho_{v,-1}^{(2)} + i\rho_{v,1}^{(2)}\phi + \mathcal{O}(\phi^2) \\
 \mu_{x,m}^{(1)} &= 1 + \rho_{x,2}^{(1)} + \rho_{x,-2}^{(1)} + i(\rho_{x,2}^{(1)} - \rho_{x,-2}^{(1)})\phi + \mathcal{O}(\phi^2) \\
 \mu_{x,m}^{(2)} &= 1 + \rho_{x,2}^{(2)} + \rho_{x,-2}^{(2)} + i(\rho_{x,2}^{(2)} - \rho_{x,-2}^{(2)})\phi + \mathcal{O}(\phi^2) \\
 \mu_{v,m}^{(1)} &= 1 + \rho_{v,2}^{(1)} + \rho_{v,-2}^{(1)} + i(\rho_{v,2}^{(1)} - \rho_{v,-2}^{(1)})\phi + \mathcal{O}(\phi^2) \\
 \mu_{v,m}^{(2)} &= 1 + \rho_{v,2}^{(2)} + \rho_{v,-2}^{(2)} + i(\rho_{v,2}^{(2)} - \rho_{v,-2}^{(2)})\phi + \mathcal{O}(\phi^2)
 \end{aligned} \tag{5.14}$$

Then the characteristic polynomial (5.12) becomes

$$\begin{aligned}
Q(\nu) = & b_2 \nu^2 + g_v^{(1)} g_x^{(2)} \left(\rho_{v,2}^{(1)} - \rho_{v,-2}^{(1)} + \rho_{x,2}^{(2)} - \rho_{x,-2}^{(2)} - \rho_{v,1}^{(1)} \rho_{x,1}^{(2)} + \rho_{v,-1}^{(1)} \rho_{x,-1}^{(2)} \right. \\
& + 2 \left(\rho_{v,2}^{(1)} \rho_{x,2}^{(2)} - \rho_{v,-2}^{(1)} \rho_{x,-2}^{(2)} \right) \Big) i \phi \nu \\
& + g_x^{(1)} g_x^{(2)} \left(\rho_{x,2}^{(1)} - \rho_{x,-2}^{(1)} + \rho_{x,2}^{(2)} - \rho_{x,-2}^{(2)} - \rho_{x,1}^{(1)} \rho_{x,1}^{(2)} + \rho_{x,-1}^{(1)} \rho_{x,-1}^{(2)} \right. \\
& + 2 \left(\rho_{x,2}^{(1)} \rho_{x,2}^{(2)} - \rho_{x,-2}^{(1)} \rho_{x,-2}^{(2)} \right) \Big) i \phi + \mathcal{O}(\nu^2 \phi^2) + \mathcal{O}(\phi^2) + \mathcal{O}(\nu^3)
\end{aligned} \tag{5.15}$$

Higher order terms in (5.12) are very small. So for ϕ small, ν is small and (5.15) can have solutions with positive real part which we want to avoid. The equation satisfies all conditions of Proposition A.0.1 *except* the condition that $b'_0(0) \neq 0$. That proposition implies instability, and so to avoid the system from being unstable, we must have $b'_0(0) = 0$. It follows from part (i) of the theorem and differentiating (5.15), we must have to achieve stability:

$$\rho_{x,2}^{(1)} - \rho_{x,-2}^{(1)} + \rho_{x,2}^{(2)} - \rho_{x,-2}^{(2)} - \rho_{x,1}^{(1)} \rho_{x,1}^{(2)} + \rho_{x,-1}^{(1)} \rho_{x,-1}^{(2)} + 2 \left(\rho_{x,2}^{(1)} \rho_{x,2}^{(2)} - \rho_{x,-2}^{(1)} \rho_{x,-2}^{(2)} \right) = 0 \tag{5.16}$$

By Definition 3.2.2 and the constraints (5.2), we have that $\alpha_1^{(i)} + \alpha_2^{(i)} = -1$ and (5.16) is actually the conclusion of the Theorem. ■

Corollary 5.1.1. *The conjectures of [5] imply the following. If $g_x^{(1)} \neq 0$ and $g_x^{(2)} \neq 0$ and*

$$\alpha_{x,1}^{(2)} \left(\beta_{x,1}^{(1)} + 2\beta_{x,2}^{(1)} \right) + \alpha_{x,1}^{(1)} \left(\beta_{x,1}^{(2)} + 2\beta_{x,2}^{(2)} \right) \neq 0,$$

then for large N , the system on the line given by (5.1) has some form of instability (Definitions 3.2.1 or 2.0.2).

Denoting the first moment of the coefficients $\rho_{x,j}^{(i)}$ of vehicle of type i by $M^{(i)}$, we

can reformulate this condition as:

$$M^{(1)} + M^{(2)} - \frac{\alpha_{x,1}^{(1)}}{\alpha_{x,1}^{(1)} + \alpha_{x,2}^{(1)}} M^{(1)} - \frac{\alpha_{x,1}^{(2)}}{\alpha_{x,1}^{(1)} + \alpha_{x,2}^{(1)}} M^{(1)} \neq 0 .$$

Thus, we see that the first moment needs a quadratic correction.

5.2 Numerical Results

Again, as mentioned on the previous systems, we are studying linear arrays, and the boundary conditions must be adjusted, and preserve the sum of each row in (5.3) equals zero. That is the sum of coefficients:

$$\sum_{j=-2}^2 \rho_{x,j}^{(i)} = 0, \text{ and } \sum_{j=-2}^2 \rho_{v,j}^{(i)} = 0 \quad (5.17)$$

where $i = 1, 2$, $\rho_{x,0}^{(i)} = 1$, and $\rho_{v,0}^{(i)} = 1$. However, on the boundaries, (5.17) is not equal to zero. This forces us to consider what happen on the boundaries and how we set proper boundary conditions which may depend on the application. We consider two sets of boundary conditions. For easy, let's call them Type I BC and Type II BC.

Type I BC adjusts the central coefficients $\rho_{x,0}^{(i)}$, and $\rho_{v,0}^{(i)}$ on the boundaries as follows:

$$\begin{aligned}
\ddot{z}_1^{(1)} &= 0 \\
\ddot{z}_N^{(1)} &= g_x^{(1)} \left(-(\rho_{x,1}^{(1)} + \rho_{x,-1}^{(1)} + \rho_{x,-2}^{(1)})z_N^{(1)} + \rho_{x,1}^{(1)}z_N^{(2)} + \rho_{x,-1}^{(1)}z_{N-1}^{(2)} + \rho_{x,-2}^{(1)}z_{N-1}^{(1)} \right) \\
&\quad + g_v^{(1)} \left(-(\rho_{v,1}^{(1)} + \rho_{v,-1}^{(1)} + \rho_{v,-2}^{(1)})\dot{z}_N^{(1)} + \rho_{v,1}^{(1)}\dot{z}_N^{(2)} + \rho_{v,-1}^{(1)}\dot{z}_{N-1}^{(2)} + \rho_{v,-2}^{(1)}\dot{z}_{N-1}^{(1)} \right) \\
\ddot{z}_1^{(2)} &= g_x^{(2)} \left(-(\rho_{x,-1}^{(2)} + \rho_{x,1}^{(2)} + \rho_{x,2}^{(2)})z_1^{(2)} + \rho_{x,-1}^{(2)}z_1^{(1)} + \rho_{x,1}^{(2)}z_2^{(1)} + \rho_{x,2}^{(2)}z_2^{(2)} \right) \\
&\quad + g_v^{(2)} \left(-(\rho_{v,-1}^{(2)} + \rho_{v,1}^{(2)} + \rho_{v,2}^{(2)})\dot{z}_1^{(2)} + \rho_{v,-1}^{(2)}\dot{z}_1^{(1)} + \rho_{v,1}^{(2)}\dot{z}_2^{(1)} + \rho_{v,2}^{(2)}\dot{z}_2^{(2)} \right) \\
\ddot{z}_N^{(2)} &= g_x^{(2)} \left(-(\rho_{x,-1}^{(2)} + \rho_{x,-2}^{(2)})z_N^{(2)} + \rho_{x,-1}^{(2)}z_N^{(1)} + \rho_{x,-2}^{(2)}z_{N-1}^{(2)} \right) \\
&\quad + g_v^{(2)} \left(-(\rho_{v,-1}^{(2)} + \rho_{v,-2}^{(2)})\dot{z}_N^{(2)} + \rho_{v,-1}^{(2)}\dot{z}_N^{(1)} + \rho_{v,-2}^{(2)}\dot{z}_{N-1}^{(2)} \right)
\end{aligned} \tag{5.18}$$

And for Type II BC, we keep the central coefficients $\rho_{x,0}^{(i)}$, and $\rho_{v,0}^{(i)}$ equal to 1 and we adjust the remaining coefficients accordingly such that the sum of coefficients is zero as follows:

$$\begin{aligned}
\ddot{z}_1^{(1)} &= 0 \\
\ddot{z}_N^{(1)} &= g_x^{(1)} \left(z_N^{(1)} + \rho_{x,1}^{(1)}z_N^{(2)} + \rho_{x,-1}^{(1)}z_{N-1}^{(2)} - (1 + \rho_{x,1}^{(1)} + \rho_{x,-1}^{(1)})z_{N-1}^{(1)} \right) \\
&\quad + g_v^{(1)} \left(\dot{z}_N^{(1)} + \rho_{v,1}^{(1)}\dot{z}_N^{(2)} + \rho_{v,-1}^{(1)}\dot{z}_{N-1}^{(2)} - (1 + \rho_{v,1}^{(1)} + \rho_{v,-1}^{(1)})\dot{z}_{N-1}^{(1)} \right) \\
\ddot{z}_1^{(2)} &= g_x^{(2)} \left(z_1^{(2)} + \rho_{x,-1}^{(2)}z_1^{(1)} + \rho_{x,1}^{(2)}z_2^{(1)} - (1 + \rho_{x,1}^{(2)} + \rho_{x,-1}^{(2)})z_2^{(2)} \right) \\
&\quad + g_v^{(2)} \left(\dot{z}_1^{(2)} + \rho_{v,-1}^{(2)}\dot{z}_1^{(1)} + \rho_{v,1}^{(2)}\dot{z}_2^{(1)} - (1 + \rho_{v,1}^{(2)} + \rho_{v,-1}^{(2)})\dot{z}_2^{(2)} \right) \\
\ddot{z}_N^{(2)} &= g_x^{(2)} \left(z_N^{(2)} + (\rho_{x,1}^{(2)} + \rho_{x,-1}^{(2)})z_N^{(1)} + (\rho_{x,2}^{(2)} + \rho_{x,-2}^{(2)})z_{N-1}^{(2)} \right) \\
&\quad + g_v^{(2)} \left(\dot{z}_N^{(2)} + (\rho_{v,1}^{(2)} + \rho_{v,-1}^{(2)})\dot{z}_N^{(1)} + (\rho_{v,2}^{(2)} + \rho_{v,-2}^{(2)})\dot{z}_{N-1}^{(2)} \right)
\end{aligned} \tag{5.19}$$

We run simulations of the dynamics of the system considering these two boundary conditions with initial condition:

$$z_k^{(i)}(0) = 0, \dot{z}_1^{(1)}(0) = v_0, \dot{z}_{k+1}^{(1)}(0) = 0, \text{ and } \dot{z}_k^{(2)}(0) = 0$$

Figure 5.2 and 5.3 shows the dynamics of a system with next nearest neighbor interactions with boundary conditions Type I and Type II respectively. The parameters were chosen to satisfy Theorem 5.1.1 as follows:

$$\begin{aligned} \rho_{x,1}^{(1)} &= -1/12, \rho_{x,-1}^{(1)} = -1/4, \rho_{x,2}^{(1)} = -1/3, \rho_{x,-2}^{(1)} = -1/3 \\ \rho_{x,1}^{(2)} &= -9/20, \rho_{x,-1}^{(2)} = -3/20, \rho_{x,2}^{(2)} = -1/5, \rho_{x,-2}^{(2)} = -1/5 \\ \rho_{v,1}^{(1)} &= -3/10, \rho_{v,-1}^{(1)} = -7/10, \rho_{v,2}^{(1)} = 0, \rho_{v,-2}^{(1)} = 0 \\ \rho_{v,1}^{(2)} &= -3/10, \rho_{v,-1}^{(2)} = -7/10, \rho_{v,2}^{(2)} = 0, \rho_{v,-2}^{(2)} = 0 \\ N &= 200 \text{ (of each type)}, g_x^{(1)} = g_x^{(2)} = g_v^{(1)} = g_v^{(2)} = -1 \end{aligned} \tag{5.20}$$

Figure 5.4 shows the dynamics of an unstable system with next nearest neighbor interactions. The maximum amplitude of each simulation is shown and the time at which this occurs. The time can be interpreted as the maximum delayed reaction time of the last element in the array. The parameters were chosen such that they do not satisfy the condition of stability of Theorem 5.1.1. We see clearly the expanding

Figure 5.2: *Boundary Condition Type I.*

(a) *Eigenvalues of a stable 2-vehicle system with next nearest neighbors interaction.*

(b) *Dynamics of the system, maximum amplitude of -303.9 at $t = 314.7$.*

amplitude as time increases.

$$\begin{aligned}
 \rho_{x,1}^{(1)} &= -0.30, \rho_{x,-1}^{(1)} = -0.25, \rho_{x,2}^{(1)} = -0.25, \rho_{x,-2}^{(1)} = -0.20 \\
 \rho_{x,1}^{(2)} &= -0.30, \rho_{x,-1}^{(2)} = -0.55, \rho_{x,2}^{(2)} = -0.10, \rho_{x,-2}^{(2)} = -0.05 \\
 \rho_{v,1}^{(1)} &= -3/10, \rho_{v,-1}^{(1)} = -7/10, \rho_{v,2}^{(1)} = 0, \rho_{v,-2}^{(1)} = 0 \\
 \rho_{v,1}^{(2)} &= -3/10, \rho_{v,-1}^{(2)} = -7/10, \rho_{v,2}^{(2)} = 0, \rho_{v,-2}^{(2)} = 0 \\
 N &= 200 \text{ (of each type)}, g_x^{(1)} = g_x^{(2)} = g_v^{(1)} = g_v^{(2)} = -1
 \end{aligned} \tag{5.21}$$

Figure 5.3: *Boundary Condition Type II.*
 (a) *Eigenvalues of a stable 2-vehicle system with next nearest neighbors interaction.*
 (b) *Dynamics of the system, maximum amplitude of -303.1 at $t = 314.3$.*

(a)

(b)

(c)

Figure 5.4: *Unstable system. (a) Eigenvalues of the system. (b) Zoomed in eigenvalues around zero. (c) Unstable dynamics of the system.*

Chapter 6

Conclusions

Necessary conditions were defined in terms of nearest and next nearest neighbors interaction between vehicles in linear array systems for various vehicle-type configurations. These interactions were expressed as the coefficients of the systems.

One of the many problems studying finite linear arrays of size N agents is that the array has boundaries in both ends, which force us to set non-trivial boundary conditions. Boundary conditions in the system complicate the mathematical analysis because the Laplacian matrices L_x and L_v not necessarily commute nor are symmetric. The problem can be overcome by following the conjectures established in [5], [6] by setting periodic boundary conditions assuming the array is a circular array. Then relating solutions on the real line with non-trivial boundary conditions and solutions on the circle which has periodic boundary conditions. Non-trivial boundary conditions may influence the dynamics of a large system, see [28].

Contrasting the results presented in this thesis with string stability theory which has have some acceptance among the public, we see that none of the assumptions required in string stability are necessary to establish necessary conditions for stability of linear array systems. String stability usually makes several of the following assumptions: the number of agents is infinite, the interactions are symmetric or are forward-looking only, interactions are small, or the Laplacian matrices L_x and L_v are identical.

In addition, some studies suggested that the (necessary) condition for stability was that the first moment of certain coefficients of the interactions between vehicles has to be zero. This thesis shows that that does not generalize to systems consisting of various vehicle type configurations. Instead, the (necessary) condition in the cases considered in chapter 3 and 4 show that the first moment plus a nonlinear correction term must be zero (see [30]).

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Appendix A

Roots Near the Origin

Proposition A.0.1. *For $n \geq 2$, define Q_n as follows:*

$$Q_n(z) = \sum_{i=2}^n a_i(t)z^i + 2a_1(t)z + a_0(t) ,$$

where the a_i are analytic functions on $\mathbb{R}$ modulo 2π into $\mathbb{C}$, see [20]. Assume further that

$$a_0(0) = a_1(0) = 0 \quad \text{and} \quad a_2(0) \neq 0 \quad \text{and} \quad a'_0(0) \neq 0 .$$

Then there is a neighborhood N of the origin and an $\epsilon > 0$ in which the zeros of $\{Q_n(t)\}_{t \in (-\epsilon, \epsilon)}$ form two differentiable curves intersecting orthogonally at the origin.

In particular, it follows that near the origin, the solutions form a perpendicular cross and thus at least one on the arms of the cross extends into the right half-plane.

Proof. We start with $n = 2$. In this case, we can write out the solutions:

$$z_{\pm}(t) = \frac{-a_1 \pm \sqrt{-a_0a_2 + a_1^2}}{a_2} = \pm \sqrt{-\frac{a_0}{a_2}} \sqrt{1 - \frac{a_1^2}{a_0a_2}} - \frac{a_1}{a_2} .$$

Let us define a curve $\delta(t)$ to be tangent to a curve $\eta(t)$ at the origin for $t = 0$ if $\delta(0) = \eta(0) = 0$ and

$$\lim_{t \searrow 0} \frac{|\delta(t) - \eta(t)|}{|\eta(t)|} = 0 . \tag{A.1}$$

One checks that we need all the assumptions on the coefficients a_i , $i \in \{0, 1, 2\}$, to

show that $z_{\pm}(t)$ is tangent to $\pm \sqrt{-\frac{a'_0(0)}{a_2(0)}} t$.

We proceed by doing $n-2$ induction steps. Given Q_n , we form all the intermediate polynomials $\{Q_k\}_{k=2}^n$. Consider $t \in N_{\epsilon} = (-\epsilon, \epsilon)$ for ϵ small. We wish to prove that $t \in N_{\epsilon}$, the solutions of Q_k form two curves $z_{k,\pm}(t)$ tangent (in the sense of equation

A.1) at the origin to $\pm \sqrt{-\frac{a'_0(0)}{a_2(0)}} t$ which we will from now on denote by $\pm \sqrt{ct}$. See

Figure A.1.

Figure A.1: The curve γ_L around $z_{k,+}(t)$ (solid) which itself is on a curve tangent to $\sqrt{ct}$ (dashed).

We proved the statement holds for $n = 2$. The induction hypothesis is that the above statement holds for some fixed $k \in \{2, \dots, n-1\}$. Fix an arbitrarily large L , (and at least as large as n).. Then fix $\epsilon > 0$ small enough, so that the conditions in the following hold for all $t \in N_\epsilon$. Without loss of generality, take $t \geq 0$ and specialize to one branch, namely $z_{k,+}(t)$.

Q_k has no other zeros in an $2\sqrt{|c\epsilon|}$ neighborhood of the origin. By continuity, for $|z| < \sqrt{|c\epsilon|}$, we can write Q_k as $(z - z_{k,+})(z - z_{k,-})\tilde{Q}_k(t, z)$, where $|\tilde{Q}_k(t, z)| \geq \frac{1}{2}|\tilde{Q}_k(0, 0)| \neq 0$. Similarly, we may assume that $|a_{k+1}(t)| \leq 2|a_{k+1}(0)|$. Let $\gamma_L(s)$ be the curve $z_{k,+}(t) + \frac{|z_{k,+}(t)|}{L}e^{is}$. Then γ_L contains no zeroes. By the induction hypothesis, $z_{k,+}(t)$ is tangent to $\sqrt{ct}$ or:

$$|z_{k,+}(t) - \sqrt{|ct|}| \leq \frac{1}{L}\sqrt{|ct|} \iff (1 - L^{-1})\sqrt{|ct|} \leq |z_{k,+}(t)| \leq (1 + L^{-1})\sqrt{|ct|}.$$

$$\begin{aligned} |a_{k+1}(t)\gamma_L^{k+1}| &\leq |a_{k+1}(t)| |z_{k,+}(t)|^{k+1} |1 + L^{-1}|^{k+1} \\ &\leq 2|a_{k+1}(0)| \left|1 + \frac{1}{k+1}\right|^{k+1} |ct|^{\frac{k+1}{2}} \left|1 + \frac{1}{k+1}\right|^{k+1} = 2e^2 |a_{k+1}(0)| |ct|^{\frac{k+1}{2}}. \\ |Q_k(\gamma_L)| &= |\gamma_L - z_{k,+}| |\gamma_L - z_{k,-}| |\tilde{Q}_k(t, \gamma_L)| \quad \text{where } \tilde{Q}_k(0, 0) \neq 0 \\ &= \frac{|z_{k,+}(t)|}{L} |z_{k,+}(t) + \frac{|z_{k,+}(t)|}{L}e^{is} - z_{k,-}(t)| |\tilde{Q}_k(t, z)| \\ &\geq (L^{-1} - L^{-2})\sqrt{|ct|} \frac{\sqrt{|ct|}}{2} \frac{|\tilde{Q}_k(0, 0)|}{2}. \end{aligned}$$

Thus we can choose t small enough so that, on γ_L , $|a_{k+1}(t)z^{k+1}|$ is smaller than $|Q_k(z)|$. Since neither function has poles, Rouché's theorem [20] implies that $a_{k+1}(t)z^{k+1} + Q_k(z)$ has the same number of zeros inside γ_L as does $Q_k(z)$, namely one. Thus $Q_{k+1}(z)$ has a unique zero within γ_L . Since we can do this for any value of L (at the price of making ϵ small enough), it follows that $z_{k+1,+}(t)$ is tangent to $z_{k,+}(t)$ and hence to $\sqrt{ct}$. Since we need only finitely many induction steps to get to $z_{n,+}(t)$, the statement of the proposition follows. $\blacksquare$

Appendix B

Solutions of the Quartic Equation

In this section, a method to obtain solutions of the quartic equation (3.18) is shown by following Ferrari's method ([13]).

B.1 Exact Solutions

We can write this quartic equation more compactly as

$$\nu^4 + b\nu^3 + c\nu^2 + d\nu + e = 0 \quad (\text{B.1})$$

where each coefficient b, c, d , and e is defined in (5.12) in terms of the original parameters, that is

$$\begin{aligned} b &= -g_v^{(1)}\mu_{v,m}^{(1)} - g_v^{(2)}\mu_{v,m}^{(2)} \\ c &= -g_x^{(1)}\mu_{x,m}^{(1)} - g_x^{(2)}\mu_{x,m}^{(2)} + g_v^{(1)}g_v^{(2)}(\mu_{v,m}^{(1)}\mu_{v,m}^{(2)} - \lambda_{v,m}^{(1)}\lambda_{v,m}^{(2)}) \\ d &= g_x^{(1)}g_v^{(2)}(\mu_{x,m}^{(1)}\mu_{v,m}^{(2)} - \lambda_{x,m}^{(1)}\lambda_{v,m}^{(2)}) + g_v^{(1)}g_x^{(2)}(\mu_{v,m}^{(1)}\mu_{x,m}^{(2)} - \lambda_{v,m}^{(1)}\lambda_{x,m}^{(2)}) \\ e &= g_x^{(1)}g_x^{(2)}(\mu_{x,m}^{(1)}\mu_{x,m}^{(2)} - \lambda_{x,m}^{(1)}\lambda_{x,m}^{(2)}) \end{aligned} \quad (\text{B.2})$$

Let ν be $\nu = y - b/4$. We make this change of variable to obtain the *depressed quartic* equation which now is in terms of the new variable y :

$$y^4 + \left(-\frac{3b^2}{8} + c\right)y^2 + \left(\frac{b^3}{8} - \frac{cb}{2} + d\right)y - \frac{3b^4}{256} + \frac{cb^2}{16} - \frac{db}{4} + e = 0 \quad (\text{B.3})$$

Let us denote the coefficients of the depressed quartic as

$$\begin{aligned} p &= \frac{-3b^2 + 8c}{8} \\ q &= \frac{b^3 - 4cb + 8d}{8} \\ r &= \frac{-3b^4 + 16cb^2 - 64db + 256e}{256} \end{aligned}$$

So, the depressed quartic can be written more compactly as

$$y^4 + py^2 + qy + r = 0 \quad (\text{B.4})$$

Rearranging and completing the squares we have

$$\left(y^2 + \frac{p}{2}\right)^2 = -qy - r - \frac{p^2}{4} \quad (\text{B.5})$$

If we introduce a quantity s inside the parenthesis on the left hand side, we need to compensate the right hand side of the equation by $2sy^2 + ps + s^2$ to balance the equation, so

$$\left(y^2 + \frac{p}{2} + s\right)^2 = 2sy^2 - qy + ps + s^2 - r + \frac{p^2}{4} \quad (\text{B.6})$$

The right hand of this equation is quadratic in y and we wish that to become a perfect square, we will be able to do that if the discriminant of the right hand is zero, that is

$$(-q)^2 - 4(2s) \left(ps + s^2 - r + \frac{p^2}{4}\right) = 0 \quad (\text{B.7})$$

Simplifying and collecting terms in s , we obtain what is called the *resolvent cubic* of the quartic equation.

$$8s^3 + 8ps^2 + (2p^2 - 8r)s - q^2 = 0 \quad (\text{B.8})$$

If s is a root of the resolvent cubic, the discriminant of the quadratic expression in y is zero, so the right hand side of (B.6) is the square of

$$\left(\sqrt{2s} y - \frac{q}{2\sqrt{2s}}\right)^2 \quad (\text{B.9})$$

Obviously, we avoid solutions of $s = 0$ of the resolvent cubic. Therefore (B.6) can be written as

$$\left(y^2 + \frac{p}{2} + s\right)^2 = \left(\sqrt{2s} y - \frac{q}{2\sqrt{2s}}\right)^2 \quad (\text{B.10})$$

or

$$\left(y^2 + \frac{p}{2} + s\right)^2 - \left(\sqrt{2s} y - \frac{q}{2\sqrt{2s}}\right)^2 = 0 \quad (\text{B.11})$$

We can factor this different of squares

$$\left[\left(y^2 + \frac{p}{2} + s \right) + \left(\sqrt{2s} y - \frac{q}{2\sqrt{2s}} \right) \right] \left[\left(y^2 + \frac{p}{2} + s \right) - \left(\sqrt{2s} y - \frac{q}{2\sqrt{2s}} \right) \right] = 0 \quad (\text{B.12})$$

$$\left[y^2 + \sqrt{2s} y + \frac{p}{2} + s - \frac{q}{2\sqrt{2s}} \right] \left[y^2 - \sqrt{2s} y + \frac{p}{2} + s + \frac{q}{2\sqrt{2s}} \right] = 0 \quad (\text{B.13})$$

Each of these factors is quadratic in y and they can be solved separately, so

$$y = \frac{-\sqrt{2s} \pm \sqrt{-2s - 2p + \sqrt{\frac{2}{s}} q}}{2}, \text{ and } y = \frac{\sqrt{2s} \pm \sqrt{-2s - 2p - \sqrt{\frac{2}{s}} q}}{2} \quad (\text{B.14})$$

Therefore the roots of the original quartic equation are given by

$$\nu = -\frac{b}{4} + \frac{-\sqrt{2s} \pm \sqrt{-2s - 2p + \sqrt{\frac{2}{s}} q}}{2}, \text{ and } \nu = -\frac{b}{4} + \frac{\sqrt{2s} \pm \sqrt{-2s - 2p - \sqrt{\frac{2}{s}} q}}{2} \quad (\text{B.15})$$

where s is a real root of the resolvent cubic of the quartic equation. Notice that there are 4 solutions (since we have a quartic equation). Four solutions for each $\phi = 2\pi m/N$, $m = 0, 1, \dots, N-1$.

The whole problem has been reduced to solve a cubic equation, that is, finding the right s that will reduce the right hand side of (B.6) to a perfect square. The good thing is that we need just one real solution of the resolvent cubic. We can proceed and find an algebraic expression for s , one solution of the cubic equation, or we can get a numerical approximation. Now the advantage of a numerical approximation is that any cubic polynomial will always have at least one real root.