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# Mathematical Silences

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by

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## Abstract

Previous studies have shown that mathematicians employ lengthy silences in their collaborations, contrary to the norms that govern every-day collaborations (e.g. Sacks, Schegloff, & Jefferson, 1974). Such results shows the relevance of an investigation of silence in mathematical collaboration. This dissertation builds off these results in a series of three papers. The first paper describes a methodology that can be employed in the investigation of silence in mathematical collaboration. The second paper analyses silence in the mathematical collaboration of students in junior-level introductory proof classes. It identifies two forms of mathematical activity, reading and ruminating, that students regularly engage in, and which violate the norms of every-day conversation. Furthermore, it shows that these activities interact with each other in complicated ways. The final paper explores the norms that govern silence in mathematical collaboration. It finds that in their conversations, mathematicians display ongoing thought to conversationally relevant mathematics, and that this display of continued engagement with conversationally relevant mathematics allows lengthy thinking silences to come in the middle of conversation.

## Dedication

I dedicate this dissertation to my wife, Josie.

## Acknowledgements

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# 1

# Introduction

The foundation for my dissertation was laid in a series of smaller studies which identified and began to analyze an unstudied aspect of mathematicians' and students' mathematical activities. In the first of these smaller studies (Petersen, 2015), I identified an episode in the collaboration of three calculus students which suggests that, in their collaboration, students may need to employ silences, but doing so can be an interactive challenge. Second, I showed that when collaborating face-to-face, mathematicians employ silence far more extensively than is normal in every-day conversation (Petersen, 2017a). This result indicates that, during their apprenticeship into the mathematics discipline, students need to learn a novel form of interaction in which silence is utilized differently than in their prior interactions. Finally, in a theoretical paper (Petersen, 2018a) I laid the groundwork for examining silence as an interactive, embodied aspect of mathematical activity.

These studies laid the groundwork for—and established the possibility of—my dissertation, which follows a "three small papers" model to take three different perspectives on silence in collaborative mathematical activity. The paper presented in Chapter 1 (Petersen, 2020a) develops a methodology that can be employed to study silence. The first goal of this paper is to lay out a methodology for studying silence, and show how it can be used to differentiate between different sorts of silent activities (thinking, listening, disengaged). It further motivates the study of silence by pointing to different aspects of mathematical education research that may be aided by attending to how various silent activities are caught up with silence (for instance, a mathematics tutor may need to recognize that silence following their question is good). Finally, it demonstrates the viability of this methodology by showing it can be used to show that the meaning of particular silences may be the subject of a miscommunication that disrupts some group members' mathematical work.

In the second chapter (Petersen, 2020c), I examine the silent practices of math majors in junior-level introduction to proof classes when they collaborate on their homework. Little is known about how students employ silence in their collaboration, though since mathematicians employ it extensively (Petersen, 2020b), there is reason to think it is increasingly an aspect of their work as they pursue their studies. This study found that students use two different sorts of silent activities during their collaboration: First, the same sort of silent thinking activities as mathematicians employ (though for far shorter times), and second, reading activities. Furthermore, these activities interact with spoken mathematical work in surprising ways. Students will pursue both reading and thinking activities at the same time as a peer works to talk through a proof—sometimes to the detriment of the silent activity and other times to the detriment of the spoken activity. Furthermore, the pursuit of parallel activities seems to sometimes be a normally sanctioned situation. Finally, I show that thinking activities, even in their most cohesive form, can have a centrifugal effect on group cohesion.

In the paper presented in the final chapter (Petersen, 2020b), I attend to the connection between silent thinking and face-to-face collaboration in mathematicians' work, and specifically to the norms and physical practices that make extensive silent thinking interactionally possible. In conversation, some utterances make particular sorts of responses relevant, so that the participants in the conversation treat even the absence of a response as a particular sort of response. For instance, following an invitation to dinner, silence may be heard as a declension. If, however, this sort of utterance may normatively initiate a lengthy period of mutual silent thought (as my work shows it does sometimes for mathematicians), then very different norms must govern mathematicians' face-to-face collaboration than govern every-day conversation. Furthermore, particular material practices may help foster those norms. In this final paper, I demonstrate that, in order to think silently while collaborating and simultaneously maintain the sequence of their conversation, mathematicians employ distinct, board-oriented practices that display continued thought about conversationally relevant mathematics. Furthermore, silent thinking activities are pursued in tandem, and are mathematicians' work to achieve times of lengthy mutual silence. These two results raise the prospect that thinking may not only be an internal individual activity, but may be a communal activity of thinking together about a specific mathematical problem.

Throughout these papers, I adopt a cultural-historical perspective (Roth & Radford, 2011) to the mathematics discipline and profession. From this perspective, learning mathematics involves learning to participation in cultural-historical forms of labor. Or, if, as Arendt (1958), argues, labor is too specific a term, learning mathematics involves coming to participate in cultural-historical given forms of *vita activa*. Because the practitioner of mathematics acts in a cultural-historical situation, their individual goals and motivations are caught up in and contribute to the reproduction of Political and economic aspects (just and unjust) of our current world (Pais, 2014; Pais & Valero, 2012); and learning mathematics involves training students to engage in disciplinary practices (Foucault, 1977, 1988) of reading, speaking, and thinking, which are given by a particular, historically situated culture (Popkewitz, 2004). For instance, Radford (2010) argues that particular disciplined forms of attending to structures visible, to the disciplined eye, in patterns representing algebraic growth, are a critical aspect of mathematical practice.

Pais and Valero (2012) commend Radford's theoretical approach as one of the most sophisticated socio-cultural perspectives on mathematical learning, but also critique his work for focusing exclusively on mathematical aspects of learning, to the exclusion of Political aspects of the current division of labor—that is, on their analysis, to the ways schooling serves to reproduce capitalist systems (see also Pais, 2014; Pais & Valero, 2011). (On their reading, by focusing on classroom activity rather than the ways schools reproduce capitalist social systems, even high-quality research from a socio-political perspective, Gutiérrez, 2013, falls into these same traps.) Pais and Valero's primary focus is on the way education, and mathematics education in particular, reproduce economic, class-based inequalities. Nevertheless, by calling for attention to Political aspects of mathematical education Pais and Valero open up avenues for mathematics education research (though they do not suggest these avenues).

Following Pinxten and François's (2011), researchers can adopt the anthropological perspective that any forms of disciplined mathematical practice is ethnomathematical, including the mathematics of academic mathematicians. From this perspective, which I adopt in these papers, for all its very real success, beauty, and centrality in our current world, mathematicians' mathematics is not (or should not be) a master, hegemonic discipline Mathematics (Greer & Mukhopadhyay, 2012), to which various people's ethnomathematics compare, and toward which they reach. Rather, academic mathematics is a particular disciplinary practice, in which people pursue (and learn to pursue) historical-culturally given goals, which utilizes specific technologies, and which is fostered through specific social practices. Furthermore, this institution is itself deeply interconnected with other aspects of modern life (with epidemiology, national defense, advertising, sports, and the activities of corporations like Google and Amazon), and, finally, in spite the relatively high number of Asian and Eastern European mathematicians, academic mathematics is (inescapably) constructed as a Western institution with a hegemonic authority over other forms of mathematics (Bishop, 1990; Greer & Mukhopadhyay, 2012).

In this saying this, I am not claiming that, say, 2+2 only equals four in a particular discipline and that, outside that discipline $2+2$ may be whatever a particular people take it to be. Rather, for instance, the goal of proving a theorem only exists in for people trained in particular disciplines, and, furthermore, different mathematical disciplines can put these proofs to different ends. For instance, though proof exists in both academic mathematics and medieval Islamic mathematics, Islamic mathematics was part of a tradition of theological reflection on the way God (who is One) is manifested in creation through numbers, preeminently through the number one (Nasr, 1968), whereas in academic mathematics, a mathematician is not unlikely to search for a proof because of its inherent beauty (Lang, 1985), without any theological reference. Similarly, Urton and Llanos (1997) argue that, for many Quechua numbers have a social meaning that reflects their understanding of just familial structures. If Urton and Llanos's analysis is accurate, then, though both academic mathematicians and Quechuan farmers count, precisely because numbers have different significances in these two disciplinary communities, the act of counting differs (subtly) for people trained in these two communities. To say that all mathematics, including academic mathematics, is an ethnomathematics, is to call attention to these different goals and significances mathematics can have, without privileging academic mathematics as a default against which other mathematical disciplines are measured, or to whose significances and purposes other mathematical traditions may add non-mathematical "extras" like theology or familial structures.

These studies adopt this perspective by attending to some of the material means through which mathematicians (apprentice and expert) create novel mathematics. Because I am a member (to some degree) of the disciplinary community of academic mathematicians, teach students to be members of that community, and research how students are and can learn to be members of the disciplinary community of academic mathematics, the studies presented in this paper investigate the practices of academic mathematicians, and, are ultimately oriented to the successful incorporation of new members into that discipline. For this reason, in these papers, academic mathematics is not compared to or set alongside other mathematical practices. This, however, should not be taken as a claim that academic mathematics is normative for all forms of mathematics. Quite the contrary: One of my aims is to open up investigation into the peculiarities of academic mathematics, and the ways the practice of this particular discipline intersects with one particular historical cultural moment—ours.

Nevertheless, it is not immediately clear how acknowledging that there are other mathematical disciplines makes a Political difference (indeed, see Pais, 2010). Though there are many avenues that ethnomathematical investigation into academic mathematics can take, these papers take inspiration from anthropological studies (Asad, 2003; Mahmood, 2005; Strhan, 2015), and Comparative Literature (Allan, 2016), are intended to allow for an investigation into the sedimented modes of bodily and of bodily ways relating to people and mathematics, fostered in the academic mathematics discipline. Allan's (2016) work analyzing the way the international community of letters functions as a particular disciplinary community, in Foucault's (1977) sense, is particularly instructive. According to Allan, the discipline of literature is not a universal discipline that trains readers to read texts from any particular nation (whether *Sense and Sensibility*, *al-Ayyām*, or *The Quran*), but a distinctive way of reading. This discipline, though geographically universal, practices forms of reading that level differences between texts other literary disciplines treat as central, and in the sort of disciplined subjectivity a reading aims to create in a reader—or rather, that a reader seeks to form in herself (Foucault, 1988). These practices differ from those utilized in other literary disciplines, for instance, in translating the text on the Rosetta Stone, scholars treated the languages as equivalent, even though, for those who produced the stone, there were important practical differences between the hieratic and demotic languages. Likewise, for many Muslims, memorization and recitation of the whole Quran plays important role in forming the *ummah* as a community formed through the faithful study and hearing of the Quran.

The differences between literary disciplines are important to Allan because a failure to recognize them, and relate neighborly to people trained in a different literary discipline creates real conflict, both in Egypt and Europe, between practitioners of different disciplines (see also Asad, Brown, Butler, & Mahmood, 2009). Similar considerations, however, do not inform these studies. Rather, my hope is that, by attending to non-mathematical aspects of mathematical activity, these studies will provide an impetus into the forms of sedimented disposition formed in the academic mathematics discipline, how those forms of subjectivity influence activity outside of specifically mathematical contexts, and the interaction between these forms of subjectivity and our broader Political projects. This attempt to link the disciplinary practices of academic mathematics (and those practiced in the classroom) and Political structures has the potential for research because, as Mahmood (2005) notes, different technologies of the self (Foucault, 1988) presuppose different political orders.

I am approaching this investigation of academic mathematical practices as something an academic mathematician myself. That is, though I am not a practicing mathematician, I have been employed teaching mathematics to undergraduates for the last ten years, including, for the last two years, teaching courses for Math Majors. Since teaching undergraduates involves apprenticing them into uses of mathematics in their profession, and teaching math majors involves apprenticing them into the mathematics discipline, I have worked as a member of the mathematics community. Nevertheless, I am not a practicing mathematician, and my training lies in mathematics education. Anthropology has often relegated its subjects to a "past", incapable of speaking with their own voice to present concerns, and so has been a tool of imperial power (Fabian, 1983). The danger, therefore, is that by investigating an anthropological investigation of mathematicians' mathematics, that I would initiate the colonization of the mathematics discipline by the mathematics education discipline. It is important, therefore, that these studies be conducted with a deference to the practices of the mathematics discipline, and specifically to the bodily techniques mathematicians employ to arrive at their conclusions.

# 2

# A Methodology for Studying Silence

> Studies of motor skill learning [e.g. in Martial Arts] have the advantage of being an easier research problem than mathematics learning, for within these domains of practice, expert and novice actions are overtly physical, affording researchers greater transparency and insight—when compared to domains like mathematics, where substantial stretches of activity take the form of Rodin's *Le Penseur* (Trninic, 2018, p. 137).

Originally intended to depict Dante working hard in thought over his *Comedia*, but later abstracted from that context, Rodin's statue *Le Penseur*, or *The Thinker*, depicts a male nude deep in silent thought, who, precisely because he is thinking, holds all his muscles in tension, as Trninic notes. Trninic's quotation, therefore suggests that specific silent postures, held with muscle tension, are an integral aspect of mathematical thinking, but that this embodied aspect of mathematics is neither "overt", nor readily amenable to research. For these reasons, Trninic's quote claims that the project of understanding the ways the body is utilized in mathematics and mathematics education requires the development of new methodologies adequate to the exploration of silence as an aspect of mathematical activity. This paper furthers this work by providing a methodology that can be used for qualitative research into forms of silent mathematical activity. The most inovative aspect of this methodology is a set of novel transcription conventions that can be used to focus analytic attention on silence, outline an episode, and systematically differentiate different kinds of silent activities by analyze the precise timing of gestures and postural changes.

Silence seems like an odd entry point for studying forms of embodied mathematical activity, because silence seems to be an absence of speech, and thus doubly unfitting for embodied research into mathematics education. First, inasmuch as silence seems to be an absence of *speech*, silence seems to be more akin to discourse than to embodied forms of mathematical activity. Second, inasmuch as silence seems to be the *absence* of speech, it seems like a difficult topic of study.

Nevertheless, the fact that holding one's tongue can be both difficult, and necessary, should give some pause in a too quick identification with silence as a lack-of. At least sometimes, silence is a difficult, but important, skill to master. On the other hand, Trninic's (2018) claim, in the epigraph, indicates that, in mathematical activity, silence is closely connected to forms of bodily practice. This contention is strengthened by examining the bodily practices of collaborating mathematicians contained in Petersen (2020b, see figure 2.1). In these mathematicians' collaborative activity, they silently maintained almost the exact same posture and interpersonal configuration for ten seconds (the fingers of the mathematician on the left had moved slightly, as had the right hand of the mathematician on the right), and each mathematician individually maintained that posture for more than those ten seconds. At least in this sort of activity, the silence itself is closely connected to the mathematicians' posture, and it is their posture, as a whole, which connects their silent activity to the discourse preceding and following it (Petersen, 2020b). Similar considerations about the connection between silence and posture can be raised for many of the other ways silence may be present in mathematical activity. For instance, if a student attempts

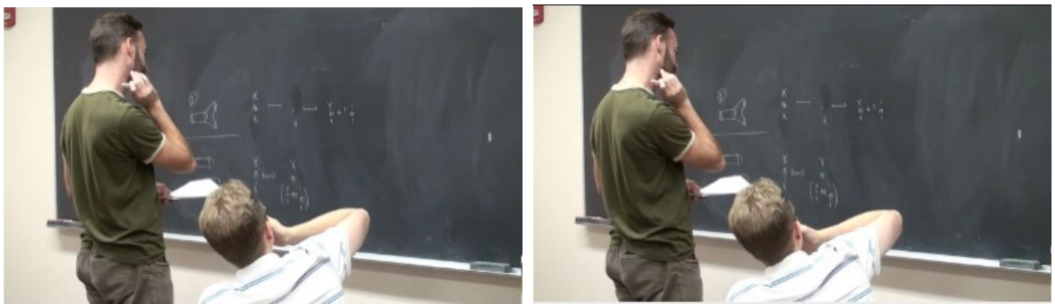


Figure 2.1: Photographs, separated by ten seconds, of two mathematicians collaborating. The mathematician on the left had maintained that same posture for ten seconds prior to the first photograph, whereas the mathematician on the right maintained it for several seconds following the second photograph. Images from Smith (2012).

to hide silently during group work, their silence not only represents an absence from the discourse, but is an aspect of their overall posture of disengagement.

On the other hand, silence is closely connected to discourse and to discursive norms. For instance, the lengthy silence of the mathematicians in figure 2.1 violates the norms of every-day discourse (Stivers et al., 2009). Yet, as Petersen (2020b) shows, during these silences, mathematicians continue to display a form of work on the actions established in the discourse which preceded it. These silences, therefore, are not a gap in the mathematicians' discursive activity, but a particular mode of activity established in and through the discourse. For these reasons, the actions portrayed in figure 2.1 are a part of the novel discursive practices endemic to mathematics (and perhaps related disciplines). Furthermore, as section 2.4.2 will show, students can misinterpret this sort of silence and interrupt it as a violation of norms—or, on the other hand, they may continue to engage in it, without negotiation, even as their peers seek to censure it as a norm violation, thus silencing the censuring student.

This relation of embodied silent practices to discourse shows that an investigation of silent practices in mathematics education needs to be sensitive not only to silence as an individual phenomenon, but also to attend to the connections between silence and discourse.

The next section (section 2.1) will describe the theoretical perspective adopted in my research on silence. Section 2.1.1 argues that silence should be understood from an embodied perspective as a gesture, in its initiation, and a posture, in its continuance. This theorization of silence requires a theorization of communication and of the body. Section 2.1.2 takes a sociocultural perspective on interaction derived from Conversation Analysis and Microethnography, and presents the body as a tool for engaging in various forms of activity. Section 2.2 then discusses a number of ways that mathematics education research may benefit from attention to silence. Following this, section 2.3 develops a methodology for studying silence. This section is divided into three subsections. The first, section 2.3.1 discusses how to record data for research into silence. Then section 2.3.2 discusses early data analysis, including the selection of episodes for further analysis, and transcription conventions that are helpful in selecting passages for deeper analysis. Third, section 2.3.3 concludes the methodology section by showing how an analyst can overlay Conversation Analysis and Microethnography to study silence. One of the main findings of this section is the presentation of transcription conventions that can be used for the detailed analysis of silence in interaction.

Finally, the viability of these methods are shown in section 2.4, in which the methods developed in the prior section are used to analyze a passage in which silence is an important aspect of students' collaboration. In the passage analyzed in this section, two students engaged in active debate about a potential mathematical solution suddenly fell silent for approximately 40 seconds. Following that 40 seconds, one of them began a conversation with a third student, which lasted another minute, during which one of the two group members remained silent. Then, following a lapse in the conversation, the previously silent student talked, and the group returned to the question they were struggling with prior to the silence, and then quickly arrived at a solution. An analysis of the silent students' posture and of the conversation surrounding the silence will show that they treated this silence as an ongoing aspect of their collaborative attempt to find a solution to the problem, and indeed, that mathematical activity conducted in silence was ultimately responsible for their successful solution. On the other hand, the third student read the silence as a lapse in conversation, and attempted to engage his peers in conversation. The two silent students treated this attempt as an interruption of their silence, while the third student seems to have read their persistent silence as something akin to the silent treatment. This section therefore shows the viability of the methods presented in section 2.3 for analyzing silence in conversation, and of distinguishing different readings silence may take in collaboration.

The methods developed in this paper allow a researcher to examine silence in relatively small group work, as an aspect of an individual's behavior, and in its connection to the group's discourse and joint behavior. Because silence is not currently a topic in mathematics education research, prior to detailing a methodology, however, it will be helpful to discuss silence in more detail. This section, therefore, draws from the work of two authors to explore the connections between silence and communication and embodiment, and to connect this theorization to the mathematics education research literature. The following section will expand on ways the study of silence may be helpful for mathematics education research.

## 2.1 Theoretical Perspective

### 2.1.1 Silence

In an excellent literature review of communications research on silence, Acheson (2007) argues that many "Western"[1] authors who have engaged with silence have mistakenly identified it with a lack and a sign of oppression—a judgment seconded by Ephratt (2011). This tendency is visible, for instance, in Freire (1967, p. 54) which pairs silence with passivity and oppression, and opposes it to activity, speech, and agency. Nevertheless, as Acheson argues, treatments of silence that associate it with passivity and oppression and oppose it to speech and agency run the danger of being blind to the multiple valences and meanings that silence has, the ways that it can be active or passive, depending on the context, and the ways it is deeply interconnected with speech. Acheson's (2007) judgement that silence needs to be theorized as a form of active presence are supported by her own empirical work (Acheson, 2008), as well as by Ephratt's analysis of the pragmatics of silence. Further support for this perspective on silence as a form of activity can be drawn from numerous ethnographic studies of communication (e.g., Basso, 1996; Clair, 1998; Esaki, 2016; Lippard, 1988; Nwoye, 1985; Tannen & Saville-Troike, 1985; Vainiomäki, 2004), musicological studies of the functions of silence in music and music therapy (e.g., Harris, 2005; Kim, 2013; Margulis, 2007b, 2007a; Sutton, 2002, 2004; Sutton & McDougal, 2010), studies of architecture (e.g., Bonde & Maines, 2015; Ergin, 2015; Kanngieser, 2011; Meyer, 2015), and philosophical and experiential studies of the phenomenon of silence (e.g.,

[1] Acheson puts scare quotes around "Western" to indicate that though it is hard to find an alternative term, the term is problematic because, in its essentialism, it erases the practices of non-normative members of "Western" societies.

Brownsberger, 2009; Caranfa, 2006; Dauenhauer, 1980; Davies, 2002; Maitland, 2008; Nicholsen, 2002).

For instance, Ergin (2015) examines silence in the Ottoman Topkapi Palace in what was then Constantinople (present day Istanbul). She argues that the palace itself, and the court ceremony followed in the palace, were both designed so that vast silences would communicate the superiority of the Sultan to both his own subjects, and to any visiting diplomats. Inasmuch as these silences marked and communicated a separation between visitors and the Sultan, they were not merely a lack-of, but an audible phenomenon which, through their use in the Sultan's pomp, actively communicated his political position. Furthermore, these silences did not just "happen" as a default absence of prescribed sound. Rather, they had to be actively planned, created and enforced, through architectural decisions, the orchestration of ceremonies, and orders strictly prohibiting talk and noise in certain places and times.

That silence is a phenomenon, actively produced and heard, can also be seen the practices of students and teachers involved in mathematics education. For instance, a student may think silently in deep concentration, a mathematics tutor may wait in silence after asking a question, and a disaffected student may daydream silently during group work. Silence plays a different function in all these situations, but it would not be helpful to theorize any of them as a passive lack of activity: The first student is actively thinking, the tutor is waiting patiently rather than speaking and decreasing the cognitive demand of a question, and the final student is engaged in a form of resistance. Furthermore, though some silences may be individual, embodied activities (for instance, a student who thinks silently while working alone), these silences often are closely connected to discourse: Students sometimes respond to their peers' questions with silent thought (Petersen, 2020c), the tutor is actively *holding* her tongue, and the daydreaming student is actively avoiding speech and participation in a group's work. That silences are an active part of mathematical activity, closely connected to discourse, indicates that research into silence may need to be sensitive both to the silence as a form or aspect of activity (in the first example, the silence is an aspect of mathematical activity, in the third, it is an aspect of a resistance to engaging in mathematical activity), and to its close connection to discourse.

In a series of two papers, Ephratt (2008, 2011) argues that one of the barriers to a strong theorization of silence is the categorization of signs into a "verbal–non-verbal" binary. Since silence does not fit well into either pole, this binary 1) flattens silence into a monolithic unity separate from speech, and 2) reduces silence to passivity. As an alternative to this classificatory scheme, Ephratt proposes a three-fold classificatory scheme. According to her scheme, the four kinds of signs are: 1) Linguistic: Signs that a) neither resemble, nor are naturally connected, to the thing signified and b) are intentionally employed, primarily for their significance. As the name suggests, linguistic signs are word-like, and, words are the paradigmatic example of linguistic signs. 2) Paralinguistic: Signs that a) communicate, and b) are intended to communicate, but c) are not engaged in primarily to communicate. For instance, eye-gaze is extremely important to communication (Rossano, 2013), but since eye-gaze serves the primary purpose of allowing a subject to see, and is only secondarily communicative, eye-gaze is paralinguistic. And finally, 3) Extralinguistic: Signs that a) are naturally connected to the thing they symbolize, but b) are not intended by the one who produces them. For instance, a blush communicates, but we usually don't want to blush; so a blush is extralinguistic.

This classificatory scheme allows a more nuanced treatment of silence. In particular, depending on the specific situation, silences can belong to all three categories. If this is true, then silences should be considered on par with other signs, like words and gestures, and not relegated to a passive lack-of. Examples (the first of which is taken from Ephratt, 2011) of silences that belong, in turn, to each category, is helpful in filling out her argument. Linguistic silence: The silence in a wedding following the exhortation to "speak now or forever hold your peace".—Its primary purpose is to communicate the proposition, stated in unison by each person present at the wedding, "I know no reason why they should not be married." Paralinguistic silence: When a teacher presenting the $\epsilon - \delta$ definition of limit pauses to allow the students to process and catch up, that pause is a paralinguistic silence.—Its primary purpose is not to communicate to the students, but it does, secondarily, communicate that the teacher understands the difficulties involved in understanding such a difficult topic, and is working with the students to facilitate their understanding. Extralinguistic silence: A silence following remark so startlingly rude that it takes the breath away is extralinguistic—it communicates the rudeness of the remark, but the person who responds with silence does not actively choose the silence.

This classificatory scheme is unlikely to translate into a methodology for mathematics education research, since whether particular silences should be classified as linguistic, paralinguistic or extralinguistic is unlikely to be an interesting research question. Rather, Ephratt's (2008, 2011) work demonstrates that silence is a phenomenon in its own right, and that silences are read and responded to as active aspects of discourse. Inasmuch as silence is a phenomenon, and people responding to a particular silence can parse the silence as anything on a spectrum from a word proper (perhaps censuring a claim), to an aspect of some embodied activity, using and parsing silences, in the moment, requires attention to a panoply of actions, and, likely, the existence of learned norms to make interpretation possible. These norms are, then, what Voigt (1989a, 1989b, 1995) calls "norms of interaction", and, inasmuch as they differ in mathematics collaboration and every-day-conversation, the norms and their development, may be interesting research topics.

Acheson's (2008) discussion of silence more easily connects to the mathematics education literature. Drawing off articles that explore the semiology of silence (Clair, 1998; Vainiomäki, 2004), anthropological investigations of Native American uses of silence (Basso, 1996; Carbaugh, 1999), and Merleau-Ponty's phenomenology (1962, 1964, 1968), Acheson argues that silence should be treated from an embodied perspective as a communicational gesture. For instance, if someone is silent in solidarity with a grieving friend, the silence is a form of gesture expressing that solidarity. Moreover, because it is a gesture, the silence is an embodied act undertaken in order to express this solidarity. This theorization of silence as a form of gesture, therefore, links this theorization of silence to mathematics education literature on embodiment and gesture. The treatment of the body will be unpacked further in the section 2.1.2. This section will further explore the relationship between this understanding of gesture, and the mathematics education literature.

Sfard (2008b) defines a gesture as "a body movement fulfilling communicational function" (p. 194). Sfard's definition highlights a difference between the sort of gesture employed in silence, and that employed in mathematics education research.In mathematics education research, gestures are often analyzed as a *motion*, often even a hand motion. For instance, both Soto-Johnson and Troup (2014) and Rasmussen, Stephan, and Allen (2004) restrict their attention to hand movements in which the hand moves from the trunk, and then back. Even Kendon (2004), in his magisterial work on gesture, defines a gesture as a form of action. In their initiation, silences can indeed be understood as a form of action (for an example, see Radford, Bardini, & Sabena, 2007), and so as a form of gesture. But in their long maintenance, as in (Petersen, 2020b, 2020c), they shift from being a gestural motion, to a static aspect of a settled posture. The maintenance of this posture is still volitional, and the continued silences is communicative. Indeed, at times it is an utterance, as when we sit with someone in grief. Like a hand that extends, and then continues statically pointing; in its initiation, silence is a gesture, but in its duration, it is an aspect of posture.

This shift from gestures to postures also implies a different approach to the body. In much of the gesture literature, gestures are studied for their mathematical significance (Radford, Arzarello, Edwards, & Sabena, 2017). Though short silences do indeed signify mathematics (Radford et al., 2007), it is difficult to see how a silent posture could signify mathematics. Rather, Saussy (2014) argues, by shifting from gestures to postures, analytic attention shifts from the construction of meaning (embodied or mental), to the subject's orientation to the world, and the use of the body as a tool for engaging in forms of mathematical activity.

One of the major barriers to continued study of silence is transcriptional. The decision to include particular actions in a transcript and exclude others, and even the decision to order a transcript in a particular way, make theoretical claims (Ochs, 1979). For instance, the usual style of transcription, in which each participant's talk is displayed like in a play script, makes the theoretical assumption that coherent units of talk are not spread across multiple lines and interrupted by lengthy silences.

When studying adult interaction, this judgment is warranted, however, when studying children under the age of two, this assumption is often unwarranted (Ochs, 1979). In a study of silence, it is critical that transcripts legibly display both each participant's silences and any silences of the whole group. Additionally, because it is unclear at the outset whether a silence functions similarly when only one student produces it as when all the participants in interaction produce it, an transcript should differentiate between these two types of silence, rather than reflecting the presupposition that they function similarly by treating them identically. Furthermore, because silence is here theorized as a form of gesture originating in posture, it is important that transcripts be able to display embodied aspects of the actions participants in interaction perform. First, inasmuch as silence is an aspect of posture, it is important that the transcript be able to display other aspects of postural configurations. Second, inasmuch as silences can function as, and be read as, a form of gesture, it is important that transcripts be able to display the moment-to-moment connection between silences and the ongoing flow of conversation.

Previous studies have experimented with notating silences on a musical score (see figure 2.2). Though perhaps better than a play-script style transcript, this style is not sufficient for the study of silence. First, even though rests are notated in music, they can be difficult even for professional musicians to attend to (Turner, 2012). Therefore, these transcripts do not adequately display individual students' silences. Furthermore, if the transcript is read by following a musical line, times when everyone is silent are even more difficult to notice. Moreover, these silences are not differentiated from times when one person is silent. Furthermore, the transcript involves considerable extraneous detail about the rhythm and pitch of talk, and does not display the aspects

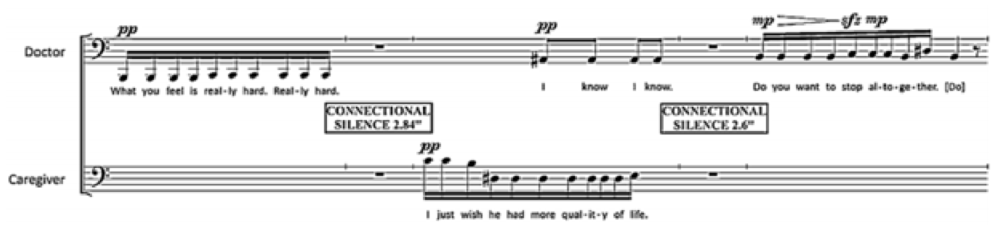



Figure 2.2: A musical-score transcript, from a recent study of silence (Bartels et al., 2016, p. 1589). This figure is intended to highlight the two silences. Nevertheless, it contains numerous extraneous details about the rhythm and pitch of talk. Furthermore, it is unlikely that the doctor was using a basso profundo low B, as the score claims. It seems, therefore, that the pitch information is only intended to display the prosodic contour. Furthermore, the transcript does not display the features of the conversation that lead to the identification of the two silences as "connectional silences".

of interaction that lead to different codings of silences.

There is, therefore, a need for a method of transcription that reflects an adequate theorization of silence, and its relation to other aspects of interaction. One of the major goals, therefore, of this paper is the description of just such an adequate method for the transcription of silences. It is important, first, to theorize the interaction, and the use of the body in various forms of joint and individual activity.

### 2.1.2 Interaction and the Body

My research on silence in mathematical collaboration adopts a situated, socio-cultural perspective on mathematical activity. Furthermore, the understanding of interaction is inspired by the disciplines of Ethnomethodology/Conversation Analysis (EMCA) (Liberman, 2013; Schegloff, 2007), Microethnography (Erickson, 1996, 2004, 2018; McDermott, Gospodinoff, & Aron, 1978), and Ecological Psychology, particularly ecological studies of the coordination of behavior (Chang, Livingstone, Bosnyak, & Trainor, 2017; Fowler, Richardson, Marsh, & Shockley, 2008; Marsh, 2015; Sebanz, Bekkering, & Knoblich, 2006; Schmidt & Richardson, 2008; Shockley, Richardson, &

Dale, 2009; Shockley, Santana, & Fowler, 2003). Greeno (1994) recommended the use of ecological psychology and of conversation analysis in situated analyses of mathematics education, so, in this respect, research on silence is derived from mainstream theorizations of mathematical education.

From a CA perspective, conversation is not understood as an exchange of information or revelation of inner states like beliefs and emotions, or even as being about a particular topic (Schegloff, 1990, 2007). Rather, conversation is understood as consisting of a series of ordered actions, each of which anticipates and delimits the set of normally sanctioned actions other members of the conversation can, at that moment, perform. Furthermore, because there is only a delimited set of actions a second member can perform, the first turn at talk performed an action on the second member of the conversation. For instance, when one person in a conversation invites a second to dinner, the action of inviting delimits the set of normally sanctioned actions the second person can perform to ones like accepting, declining, figuring out the date, etc. This is not to suggest that the invitee is not free to respond as she would. She can, of course, respond with a host of nuanced actions including reluctantly, bitterly, or spitefully accepting, regretfully, scornfully, or insultingly declining, etc. Rather, she must respond, or be subject to censure for violating the norm that she should respond. And indeed, in some circumstances the norm is strong enough that, whatever she does next, that action will be heard as a response to the invitation.

Furthermore, except for the first greetings or summonses, each action in a conversation follows earlier turns of talk which, in turn, have delimited the set of normally sanctioned actions that can follow. Therefore, in a conversation, each turn at talk performs actions on or with other members of the conversation which delimit the set of sanctioned responses, and is in turn, constrained by the actions performed earlier in the conversation (both by the speaker and by any other participants in the conversation).

In outline, these results were established by Garfinkel (1967), through his methodology of "breaching experiments". These experiments are so jarring for unsuspecting participants that Mehan and Wood (1975) claim that, by then, researchers had realized they were unethical. In a breaching experiment, one participant persistently violates a norm they have identified. For instance, in one, when someone expressed a sentiment about movies "like this", their partner persistently attempted to get them to precisely define what "like this" meant, expressing genuine confusion about the scope of the term. When conversational norms are violated in this sort of way, conversationalists will attempt to correct the breach, and, when the experiment persists, often end up getting extremely upset. These results show that, at a given point in a conversation, only particular sorts of actions are in accord with conversational norms, whereas others are subject to censure.

Researchers employing CA, however, have greatly both the precision and depth of the understanding of the structure of conversation. The basic structure of conversation is built on First Pair Part (FPP)–Second Pair Part (SPP) sequence. One person initiates an action like inviting, commenting, noticing, evaluating, etc. and a second party responds in a way constrained by the FPP. For instance, in response to a self-criticism, the second party can agree, disagree, upgrade the assessment, downgrade the assessment, etc. However, again, they cannot do anything imaginable without violating the norm, and opening themselves to censure. This is not to say that conversations are structured as a sequence of unconnected actions and responses. People often need to engage in a third action before they can adequately respond (for instance, they may have misheard something, or may need to settle some details before accepting an invitation). Because these intervening actions can carry on at length, in principle (and to some degree in practice) any number of subsidiary actions can intervene between an FPP and its SPP.

Conversation Analysis, on its own, however, has several important short-comings for the study of silence. First, though there has been some work connecting embodied postures to the ongoing work of conversation (notably Schegloff, 1998), the primary focus is on the actions performed in and through talk. It would, however, be a mistake to assume when a particular student is silent, their actions and postures can best be understood through reference to a conversation. There was, most likely, a conversation that preceded and follows their silence, and it would be a mistake to attempt to read their silence without careful consideration of its relationship to the surrounding conversation. Nevertheless, that their silent actions are best understood as a part of the conversation should be established through careful analysis of a given passage, not assumed beforehand.

Researchers employing microethnography (Erickson, 1996, 2004; McDermott et al., 1978) and ethnomethodology (Liberman, 2013) have shown that just as, in conversation, words are used to perform actions on and with other interactants, in other forms of activity, we employ our bodies to perform various actions. For instance, Liberman showed that, when crossing a busy intersection, expert crossers will employ their body in a way that causes other crossers to hesitate, and so gives them access to the intersection. For instance, pedestrians will talk on a cell phone—while attending peripherally to the intersection—in order to appear inattentive to other's claims on the space, and cause drivers who do not want to run them over to hesitate, and so cede the intersection. Other pedestrians will boldly push baby-strollers into the intersection, knowing that drivers will be particularly attentive to young children, and so will cede the ground. Drivers, on the other hand, will look away from foot-traffic and other drivers, so the other crossers (who do not want to be run over or to be involved in a fender-bender) will hesitate and so cede the territory. All of these actions (talking on a cell phone, boldly pushing a baby stroller into the intersection, looking away) are a form of action performed on other crossers in order to gain access to the intersection. Inasmuch as they are actions performed other people involved in interaction, these actions resemble those performed in conversation. On the other hand, they are not performed with words, but with the body. Because silence is a form of gesture originating in posture, it is important that researchers in silence be able to attend carefully to the actions participants in interaction perform with their bodies, and with their extended silences.

Similarly, researchers employing microethnography show that, in their interaction, people establish and maintain relatively stable postural configurations. These postural configurations inform each actant about the nature of the interaction. Therefore, by adopting particular postures, each member in interaction performs an action which is instrumental in shaping the character of the group's collective activity. For instance, Erickson (2018) examines the postures of a group of elementary students as their teacher pours sand and rocks into a jar of water. Initially the students are seated and lying around the jar, with their gaze fixed on it. Toward the end of the segment, the teacher turns to one particular student and answers a question the student has about the demonstration. As she does so, the gazes of the other students moves from the jar to the teacher-student dyad. Therefore, through her shifted posture, the teacher directed the whole group of students into a different posture. Though the teacher's action, here, was important, each student performs similar work through their posture. For instance, at the beginning of the episode, the teacher noticed that one girl was still engaged with a paper that described the experiment. Before beginning, the teacher quickly asked the student if she was ready, prompting the student to turn her gaze to the demonstration. Therefore, in looking at her paper, the student performed an action which influenced the teacher's reading of the situation, and influenced the teacher's action in the demonstration.

Because of its attention to posture, microethnography is particularly suited to the study of the motionless involved in some of the silences studied previously, (Petersen, 2020b, 2020c), as well as the silences examined in this paper. During these silences, various aspects of posture may, in fact, play important communicative roles, specifying the actions the students are engaged in. For instance, Grafsgaard, Wiggins, Boyer, Wiebe, and Lester (2013) found that mouth dimpling is associated with cognitive effort. If participants in collaboration involving extended silences dimple their mouth, then this gesture is a source of information their peers can observe, and respond to. Nevertheless, in Petersen (2020b), the mathematicians stood beside each other, with their eye-gaze was directed to a board, and cameras were positioned behind the mathematicians. This configuration made it so mouth-dimpling would perhaps have been difficult for peers to observe, and it was usually impossible to analyze, though future studies could position cameras to record it. Furthermore, the mathematicians adopted a fixed posture for an extended period of time. During this time, their posture displayed ongoing engagement with conversationally relevant mathematics, however, precisely for that reason, during these silences, the conversation was temporarily suspended so both mathematicians could engage in non-conversational actions that noticeably served conversational goals. How can their posture during these times be theorized as an aspect of their collaboration?

For this question, studies of synchronization can be helpful. Long-standing research has shown that the posture and limb-movement of people involved in joint action behaves like a linked oscillator, even when the interactants do not explicitly intend to couple their gestures, as do features of their talk like its speed (Cappella & Planalp, 1981; Fowler et al., 2008; Marsh, 2015; Schmidt & Richardson, 2008; Shockley et al., 2009, 2003). It is not entirely clear what function this entrainment occurs, however, the best explanation seems to be that the coordination of these features of interaction serves to facilitate group cohesion, and participation in a common task (Richardson, Dale, & Marsh, 2014). At least in musical performance, this hypothesis has received experimental support (Chang et al., 2017). Researchers found that the postures of professional string quartet players were both aurally and visually linked, and reflected quartet leadership. Though it has not been experimentally confirmed, the hypothesis is that as bodily norms of collaboration are learned, the long-term posture serves subliminally to reinforce engagement with the common task.

Finally, for at least some of these activities, particular forms of bodily activity are employed to give the agent the capacity to engage in specific forms of mathematical activity. For instance, though some features of mathematicians' lengthy silences, during which their gazes are motionlessly fixed on the board facilitate collaboration, it seems unreasonable to assume that the peculiar norms that govern this form of interaction evolved solely to facilitate communication. Rather, because the norms differ markedly from those governing silence in every-day conversation, across cultures (Stivers et al., 2009), it is more parsimonious to assume that the norms governing these forms of interaction evolved to facilitate a peculiar form of individual action, which requires extended silence. (Though it may be impossible, even theoretically, to tease apart which aspects of their activity exist in order to allow each mathematician to engage individually in forms of activity, and which exist in order to allow collaboration.)

The influential French anthropologist Mauss (1934/1973) referred to this sort of use of the body as an deployable means to engage in a particular form of activity a technique of the body, or as Mauss calls it, a *habitus*. Though the construct has been helpfully employed in anthropology (Asad, 2003; Mahmood, 2005), and comparative literature (Saussy, 2016), in mathematics education research, the body is usually investigated for its potential to symbolize mathematics, not for its use in performing mathematical activities. This research, therefore, initiates a new line of inquiry into ways the body is employed in mathematical activity.

## 2.2 Silence in Mathematics Education

The preceding section discussed silence generally, but only provided a few short sketches of silence in mathematics education. This section builds on the previous by describing a number of ways that silence may be an interesting presence in mathematical activity. Because the goal of this section is to stimulate the research, not to present research findings, some of these are based on anecdotal evidence, whereas others are grounded more firmly in the mathematics education research literature.

### 2.2.1 Thinking

Petersen (2020b) argues that the activity of thinking may be an important, physical and communal aspect of mathematical activity. Trninic's (2018) observation that mathematical activity physically resembles Rodin's *Thinker* also suggests that thinking is an embodied component of mathematical activity, though his work makes no claims about its social character. (The claim here is not that cognition is an important aspect of mathematical activity, but that the embodied, interpersonal act of thinking silently is.) Just as the thinking of Rodin's statue is embodied in a particular posture, so the mathematicians in Petersen (2020b) employed distinctive postures. For instance, they motionlessly and silently fixed their eye-gaze on the board for relatively extended periods of time. Furthermore, when enacted during collaboration, this activity depends on behavioral norms and a common orientation toward the blackboard (Petersen, 2020b), and so is not a withdrawal from interaction, but is an aspect of collaboration.

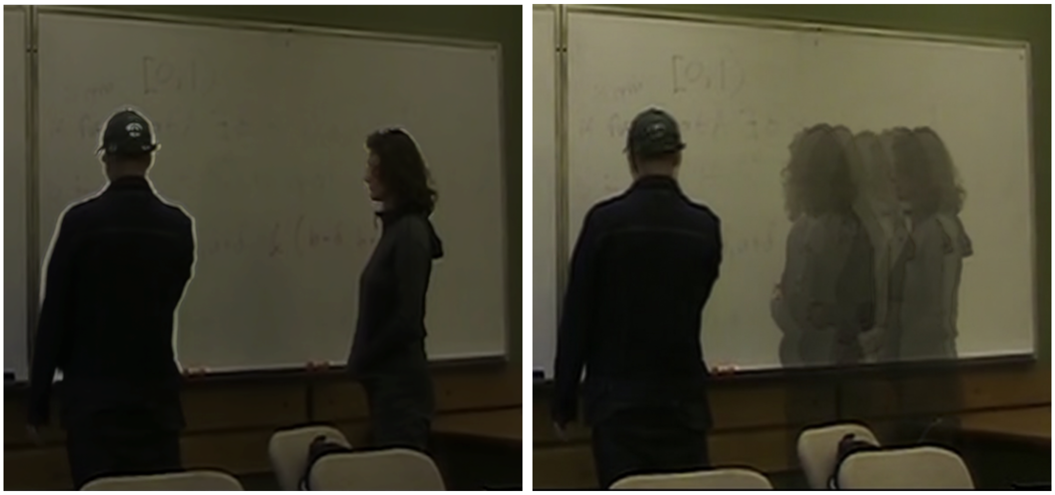
If, as Petersen (2020b) argues, this activity is a part of the work of doing mathematics which requires that the mathematicians employ collaborative norms different from those in every-day conversation, thinking, as a form of embodied, communal activity, is a valid topic for further investigation. For instance, classroom teachers may need to foster thinking activities in their classroom, and to negotiate norms regarding thinking in their classroom. More subtly (and anecdotally), several mathematics professors have recounted that their own thinking activities are an important aspect of teaching mathematics, but, when working directly with students, engaging in thinking activities can pose challenges. If students speech is more easily in accord with classroom norms in an inquiry-oriented classroom than in a lecture-based classroom, then teacher thinking activities in inquiry-oriented classrooms may merit particular investigation.

Similarly, Petersen (2020c) showed that when they are collaborating on proof problems, students may respond to peers' questions by initiating what he called "ruminating" activities, namely, by motionlessly and silently fixing their gaze on the board. As Petersen argued, in two-person collaborations, this sort of response is liable to being read as a failure to respond, and so censured as a norm violation. On the other hand, by initiating it in response to a question, students treat ruminating as an important form of mathematical activity oriented to answering questions. Because of this disconnect between the action of one student, who by censuring devalues an activity, and the other who, by engaging in it valorizes it, these thinking activities pose important challenges for students' on-going collaborative mathematical activity. Furthermore, because these ruminating silences resemble the silences mathematicians employ in their collaborations, in which the gaze is fixed silently and motionlessly on a board (see, for instance, figure 2.1), it is important for students in upper-division under-graduate courses to begin to learn to master utilizing and reading this sort of silent mathematical activity.

There may also be potential here for research into mathematics tutoring. Anecdotes from a coordinator of a tutoring center suggest that not responding too quickly after a student initiates a thinking silence, particularly when the thinking silence is a response to the tutor's question, is important. Nevertheless, this sort of response can be challenging for new tutors. Investigations into the ways tutors respond to thinking, and forms of professional development for mathematics tutors may, therefore, be an important avenue for research.

There is also some question about the methods people use to coordinate their thoughts with peers during face-to-face collaboration. In one of the episodes analyzed in Petersen (2020b), two mathematicians established common knowledge of a problem before one began to think about it at length. Similarly, in the passage analyzed in this paper, two calculus students agreed that there were weaknesses in contrary arguments both had made, and only then began thinking. On the other hand, in one episode in Petersen (2020c), two students were able to coordinate the object of their thinking activities without a common acknowledgement of a problem requiring thought. But it is unclear what methods people utilize to coordinate their thinking activities.

Finally, it may be possible to carry out investigations into the character of thinking activities that are, in different situations, likely to be beneficial (or detrimental) to group work in different ways. The potential for this form of research can be seen in Cavell's (1981) discussion of Cary Grant's acting. (Cary Grant was a silver-screen actor.) According to Cavell's analysis, Grant was particularly skilled at displaying thinking, and doing so in a way that reveals nuances of his character. For instance, in *His Girl Friday*, Grant's character fidgets and shifts his eyes as he thinks, thus displaying his searching journalistic drive for a new angle. This character of Grant's thinking contrasts with the thinking Petersen (2020b) observed in the mathematicians he studied, who he observed in focused concentration that zeroed in on a single point in the perceptual field. There is no indication in Petersen's work that this is the only sort of thinking mathematicians engage in, much less that the sort of thinking Grant's character engages in would be unfitting for mathematical activity. This contrast, however, reveals that the character of thinking is (to some degree) visible, and therefore subject both to research into mathematical activity, and to formation



(a) The first two seconds of Hannah and Seth's thinking activity. White indicates movement.

(b) The subsequent three seconds of Hannah and Seth's thinking activity. Whereas the character of Seth's thinking remained constant, Hannah's changed.

Figure 2.3: The character of Seth (left) and Hannah's (right) thinking activities was similar for the first two seconds (a), but for the subsequent three seconds (b), Seth's thinking remained focused and penetrating, whereas Hannah's became searching.

in the classroom. Furthermore, inasmuch as different kinds of thinking are observably different, different kinds of thinking activities may open up different avenues for response, and so, apart from any speech that follow the different acts of thinking (and that are presumably connected to them), different kinds of thinking activities may be different socially. Finally, these considerations suggest that investigations of thinking activities may be challenging, not because thinking is not social and visibly embodied, but because thinking is in fact a variegated class of activities that need classified and investigated separately.

This potential for observing and so studying different kinds of thinking acts is not only derived from an analysis of fiction (movies), but can be observed in an episode of student collaboration. In this episode two students, with pseudonyms Seth and Hannah, were collaborating on an advanced calculus problem. Just prior to beginning thinking, Seth discovered a problem in their work, and, in response, both students thought for approximately seven seconds. For the first two seconds, the character of both their thinking activities was similar: Both stared motionlessly at the location where something problematic had been written on the board. After two seconds, however, the character of Hannah's thought shifted, and she began searching for a location on the board that offered a clue to the problem, and stepping toward the board to correct the error, once found (see figure 2.3). Furthermore, the contrast between their two types of thinking is not merely apparent from the fact that Hannah moved more than Seth did. After these first five seconds, Seth also began to move. However, in contrast to Hannah, whose motion was relatively fluid and undirected, Seth moved decisively, remained closely focused on new board spaces, and tended to fix his gaze for longer. Therefore, even as they moved, the character of their thinking was visibly different: Seth was engaged in a focused and penetrating form of thought; Hannah, in a more searching and open form of thought. This comparison is not meant as a value judgment on either student's activity. Rather, this short episode is only meant to illustrate the potential for investigations into different kinds of thinking activities—kinds differentiated not by their structure inside a students' head, but by their embodied and social nature.

### 2.2.2 Reading

Petersen (2020c) identified reading—and specifically reading while their peer(s) talked aloud—as a major form of silent mathematical activity. He further argued that because students read while their peer(s) talk—even when they are only collaborating with one peer—reading silences can be interrupted by spoken forms of mathematical activity, and in turn, can themselves interrupt forms of spoken mathematical activity. For instance, in one episode, one student, pseudonymously called Sean, spent the majority of the time he and his partner, called Mary, worked on a proof attempting to read silently from his text book and class notes. As Sean read silently, Mary attempted to talk through a proof at the board. In this episode, Mary's spoken mathematical activity repeatedly distracted Sean from his reading, with the result that he made little or no mathematical contribution to the group's proof production. In another episode that prominently featured reading, two students, Nick and Sam, debated two different readings (and two different solution tacks) of a homework problem. While Sam articulated a critical argument in support of his reading, Nick was reading from his book for evidence in favor of his reading. Because Nick read during Sam's argument, and so did not hear or respond to Sam, the group, as a whole, ignored Sam's argument, and work continued without consideration of Sam's reading.

These two examples show that reading is an important mathematical activity for two reasons. First, by engaging in it, students treat it as a valuable resource for discovering proofs and for engaging in forms of mathematical argumentation. Second, these two episodes show that reading is not a purely individual form of activity, but contributes to a group's communication. Nevertheless, little is known about what factors influence the ways it contributes to group work, or how forms of reading activity can be shaped in the classroom to make a positive contribution a group's work.

### 2.2.3 Writing

As students engage in mathematical activity, they commonly need to write down their arguments or steps in their algebraic process, construct diagrams to support their reasoning, record theorems they need to use, etc. Though students may talk through some steps of the process, writing is often done in silence. Nevertheless, at least in Petersen (2020c), writing was less of a factor in group communication than other forms of mathematical activity, perhaps because it occurs on a shared focus of attention, and so is a more overt activity than reading.

### 2.2.4 Observing

A fourth sort of silent act that may be appropriate to study is observing other students engage in mathematical activity. Three of the four episodes examined in Petersen (2020c) contain periods during which one student silently watched their peer(s) engage in mathematical activity. In the first, a student called (pseudonymously) Sean sat and watched while his peer, called Mary, talked through a geometry problem at the board. In the second, one student, called Allen, spend the majority of the episode watching his peers work. Though he spent much of the time watching, he nevertheless seemed to be engaged in the group's work, and eventually offered one of the best contributions to the group's work. In the third episode contained in Petersen (2020c), a student called Amy had looked up a solution to a homework problem online, but wanted to see how her peers solved the problem, and so spent the whole episode watching them work. Amy only occasionally contributed to her group by, at her peers' request, providing technical support. For instance, her when her peers got confused by their own use of "surjective" and "injective", she helped them remember which means one-to-one and which, onto. But otherwise, she silently observed her peers work throughout the episode.

Petersen (2020c) did not attempt to study this sort of activity, and so his data do not indicate how common this sort of activity is. Nevertheless, his data do show that this activity is a feature of student collaboration. It is unclear, however, whether, and when, this sort of activity is a healthy part of group work, or, indeed, how it contributes to group work and mathematical learning. For instance, Petersen argues that, for Sean, watching Mary was a distraction from his own mathematical activity. On the other hand, for Allen, watching seems to have been a form of legitimate peripheral participation (Lave & Wenger, 1991)—though, homework is itself a form of legitimate peripheral participation (Greiffenhagen, 2008), Alan's activity was a form of peripheral participation in peripheral participation. Finally, looking up a solution significantly decreases the cognitive demand of a proof and so likely interferes with Amy's learning. Nevertheless, actively watching her peers work through a solution seems to ameliorate that problem somewhat—though again, it is unknown how much it helps, or even, whether it does.

These differences between the three instances of this activity suggest that observing peers engage in mathematical activity is a complex activity that students engage in for a variety of reasons. Furthermore, it suggests that, its effect on both student learning and group mathematical activity may be may relate in complex ways to learning and to other forms of mathematical activity. Silent observing may, therefore, merit further research.

### 2.2.5 Negative Silences

On the other hand, students may engage in forms of silence that are antithetical to mathematical learning. For instance, students may disengage, out of frustration, boredom, disaffection, etc. Conversely, students may intentionally or unintentionally silence their peers, so that the second student is no longer authorized (or no longer feels themselves) to speak. Even more sinisterly, a teacher may do likewise—by, for instance, rebuking a question about why a procedure works, or reprimanding students for nonstandard procedures.

There is a long history of studying different forms of student disaffection and alienation in the classroom or in mathematics education more generally (e.g., Gutiérrez, 2018; Martin, 2013; Pais, 2014; Radford, 2012; Williams, 2015). For instance, Roth and Radford (2011) contains discussion of the disengagement of one student, who they call Aurélie. She adopted relatively overt postures, like laying her head down on the desk, or leaning back away from her work and saying that she does not and never will understand (p. 38), which communicated her disengagement and frustration. On the other hand, this work raises the question of whether more subtle forms of silent disengagement (e.g. daydreaming, while remaining physically engaged), are amenable to further study. Bosch and Dmello (in press) used a learning algorithm to train computers to attend to facial cues present in video data to detect mind-wandering in a laboratory reading task, and in the classroom. They report that the sorts of behaviors the computers recognized as mindwandering were hard for human coders (hired through Amazon's MTurk platform) to recognize, and were therefore relatively subtle. Their study therefore indicates that it may be possible for researchers to identify behavioral features of various forms of negative silence. This conclusion, in turn, suggests that the mathematics education research community may benefit from research into various forms of disaffected silence students engage in.

Precisely because these activities seem sometimes to be relatively subtle (and difficult for human coders to detect), it is important that this research be able to attend to subtleties of various silences. This detailed attention to silences would allow researchers to identify features of disaffected silences, and then to examine the effect of disaffected silences on individual learning and group work. For these reasons, research into disaffection in the mathematics classroom may also benefit from the methodology presented in this paper.

## 2.3 Methodology

### 2.3.1 Data Collection

The theoretical perspective argued that silence should be understood as a gesture and an aspect of posture, and that its meaning and use is determined by both the talk preceding and following it, and by the postures individuals adopt during silence. For this reason, it is important that an analysis of silence be based off video-recorded data, not audio alone. Without the video data, it would be difficult to determine the significance of different silences. For instance, (Petersen, 2020c) found that in the collaboration between two students, called Mary and Sean, Sean had not contributed to the group's proof production, and that he was silent for large stretches of the episode. An analysis of the video record revealed that he had read from his book for large portions of the episode, but that he also watched Mary work for a significant portion of the episode. The analysis then argued that he watched Mary work because her spoken mathematical activity repeatedly interrupted his reading activities, and that these interruptions were therefore partially responsible for his lack of contribution. Without the video record, an analyst could note the silences, and, because he once said he was trying to read, could say that Sean sometimes read. But the significance of his reading activity, its distinction from watching activities, the persistence with which he engaged in it, and that Mary's talk interrupted him (often visible in a shift between types of silent activities) are all visible, but inaudible. Had the data only consisted of audio, the analysis could not even have been to analyze what sort of activities Sean engaged in during his silences.

Several different forms of video recording are used in mathematics education research. For instance, for some studies, a camera is zoomed in on a work space, but the bodies of the participants are not visible. In other studies, screen-capture software is used to record computer work and student talk (e.g., Biehler, Frischemeier, & Podworny, 2015). Both these methods are able to record silences, however, the differences from an audio recording do not make material difference in a study of silence. Without a recording of his posture, the forms of activity Sean engaged in in silence, and their relation to the group's activity would be as invisible as they would be inaudible with only an audio recording. Similarly, Petersen (2020b) found that mathematicians use their posture to display ongoing engagement with conversationally relevant mathematics.

For this reason, it is important that video data record gesture and posture of the participants. Ideally, cameras would record the faces of participants so that relatively precise details about gaze could be recorded. (Or even better, participants in studies would be asked to wear eye-tracking glasses.) Other cameras would record both the gesture of the participants, and whatever work-space they utilize. On the other hand, the goal is to record natural behavior. To that end, it is important to minimize the intrusion of the camera on interaction. Depending on the study, these two needs may compete. For instance, some universities have public space set aside for mathematicians' and graduate students' collaboration. It may be possible to record in these locations with only minimal intrusion. On the other hand, eye-tracking glasses or cameras capable of detecting subject's eye movements may need to be obtrusive.

In my research these competing goals were harmonized by having students work on a board, and positioning two cameras, each with a view of the full board, to record all their activity. This set-up allowed all the students' gestures to be recorded, though, it was impossible to read subtleties of eye-gaze with this method. However, other videographic methods have been utilized in research on naturalistic data that would be adequate to a study of silence. For instance, Roth and Radford (2011) and Rasmussen et al. (2004) both used a videographer to record a relatively wide angle on group work, and therefore were able to analyze students' gestures in naturally occurring classroom work. Similar techniques have been used in the Conversation Analysis literature to study tutoring (Schegloff, 1998).

Because the data for a study of silence is unlikely to be drawn from interviews, the researcher is free to collect field notes on silence as the students work. In Petersen (2020c), however, the peculiarities of the study design, and the unique challenges that face studies of silence made the analytic notes mostly unhelpful for further analysis. First, the students who participated were compensated with a half-hour of tutoring immediately following their work, rather than with a monetary compensation. Students reported that this arrangement was helpful, however, it had the unintended result that the researcher had to attend to the mathematical arguments the students made during their work, and so could not devote full attention to the students' silences in the moment. Furthermore, it proved difficult to attend to silence, precisely because attending to silence cuts against the grain of habit formed by listening to speech in every-day conversation. The first difficulty was specific to the design of Petersen (2020c), and so it may be ameliorated in studies that offer students other forms of compensation for their time. The second difficulty, however, is more difficult to overcome, and researchers should expect to have difficulty generating detailed field notes on silence in mathematical collaboration.

One important, unexplored, avenue for further research into silence is the investigation of ways different student backgrounds affect their use of silence, and their apprenticeship into disciplinary uses of silence in mathematics. For instance, Solnit (2012) argues that some men's assertive, talkitive, ignorance (which has come to be called "mansplaining") genders women as silent, deferential, superfluous extras in a world not their own. It goes without saying that this feminine silence differs from the mathematicians' thinking silences. Nevertheless, women gendered into a deferential silence encounter mathematical silences with unique experiences of silence their male colleagues do not face. The fact that the mathematics profession is overwhelmingly male (71% of math department chairs, and 73% of full-time doctoral faculty in the US are male, Golbeck, Barr, & Rose, 2019) adds further complications to gender questions surrounding mathematical silences. Furthermore, the assertive, talkative, ignorance of the men Solnit (2012) discusses pose a threat to the silent thought of their peers, as these men would fill their peers' thoughtful silence with their own ignorance. For both these reasons, it will be important going forward to collect gender information about participants of studies of silence, and to find ways to examine the ways that gender both affects and is affected by mathematical silences.

Similarly, it will be important to find ways to connect students' racial, ethnic, and class backgrounds to their use of silence in the classroom. For instance, Vainiomäki (2004) argues that the phenomenon of silence should be understood as a sign whose significance is given by the specifics of the culture in which it is experienced. For instance, as she argues, in Finland, silence signifies the intimacy of a midnight twilight in summer. Though it is not clear what significance silence will have for students, that silence has different significances for different cultures shows it will be important to explore the meaning silence has for students of different racial and ethnic backgrounds as they encounter mathematical silences. On the other hand, people who, whether because of class, racial, or ethnic reasons (or other reasons) feel unwelcome in the mathematics classroom (Martin, 2013; Gutiérrez, 2013; Mesquita, Pais, & Francois, 2014) may employ silence as a form of resistance ot the project of the classroom, as Erickson (2004) records a student doing.

### 2.3.2 Early Analysis

Once data has been collected, an analyst needs to select episodes for a deeper analysis. In some forms of analysis (e.g., Rasmussen & Stephan, 2008), all the data is transcribed, and analysis proceeds on the transcripts. In an ethnographic micro-analysis, however, the primary data remains the video, and the analyst repeatedly and carefully analyzes the video itself (Erickson & Schultz, 1977; McDermott et al., 1978). Erickson and Schultz (1977) list seven stages of video analysis employed in ethnographic microanalysis. The application of the first two to a study of silence will be described in this section, section 2.3.3 will describe the application of the subsequent four stages to research on silence. The final stage of Erickson and Schultz's analysis is the generalization of the models created in this analysis to other similar cases. This step is not different, in principle, from the prior stages, and so will only be treated in summary. The focus of this and the subsequent section is the methodological principles guiding analysis, however, during both this section and section 2.3.3, these principles will be illustrated through a partial analysis of a short episode of four calculus students collaborating to determine which of three graphs represents the position of a car, which the velocity, and which the acceleration. Following the methodology section, section 2.4 will illustrate the application of these principles through a more full analysis of this same episode.

The first stage in Erickson and Schultz's (1977) methodology is the collection of real-time analytic notes which can serve as an index of different phenomena. In working on Petersen (2020c) this stage proved to be particularly difficult for two reasons. First, I habitually attend to speech, and so found explicitly training my attention on silences particularly challenging. Second, it was extremely difficult to attend to multiple chanels of activity. Nevertheless, as I preceeded to further stages, I discovered that transcripts which highlight silence could prove particularly helpful in uncovering aspects of interaction I had previously missed, and allow me to find and attend to important silent activities. Though the full transcription method outlined in section 2.3.3, however, is too time consuming for early stages of analysis I was able to use more cursory transcripts as a tool to alert myself to silences that they I should attend to as they watched video and so to enable me create more detailed analytic notes. Before collecting analytic notes in stage I of an analysis of silence, therefore, a researcher should transcribe the episode in a manner that highlights silence (for an example, see figure 2.4 below).

The second stage presented in Erickson and Schultz (1977) is the identification of the occasions of theoretical interest, and the diagramming of the time-sequence of interesting phenomena. Though the transcripts outlined in this section are too lengthy for the use in the communication of time-sequences of interesting events in a publication, in preliminary analysis, they can be employed to give a time-line of silence, and, if printed, can be marked to display various silent activities like /thinking/ or /reading/ (where / / enclose preliminary analytic judgments of activities).

The main contribution of this section, therefore, is the presentation a transcription method that can be created relatively quickly, but that can also display silences. These transcripts can then be used both to train the researcher's eye in the first stage of an ethnomethodological investigation, and to display a time line of significant events in the second stage.

Using transcripts as tools for focusing attention on aspects of video-data contrasts with their use in much of mathematics education, but is more consistent with their use in Conversation Analysis (CA) and microethnography. In many methodologies used in mathematics education research, once transcripts have been created, videos are often consulted only to correct or clarify transcripts, but the analysis is conducted on the transcripts themselves. For instance, researchers following the methodology of Rasmussen and Stephan (2008) use the transcripts to identify statements that are used for grounds or warrant, and then again employ the transcript to identify patterns in the sorts of mathematical claims that can be used to support an argument. In Conversation Analysis, on the other hand, the primary data always remains the video (or audio) recordings of a conversation. So for instance, though many stages of analysis are carried out on the transcript, the researcher is encouraged to always return to the recordings to see if there are details missing from the transcript that complicate a particular analysis (Liddicoat, 2011). Transcripts, therefore, from a CA perspective, serve to focus attention on particular aspects of video data, to allow the researcher to see multiple aspects of an episode at the same time, and to communicate data to other researchers.

Similarly, researchers employing microethnography analyze the coordination of posture with extreme precision, and so for an analyst employing microethnography, transcripts are even less central than they are in CA. They are still used, however, they are explicitly described as “illustrative sketches” and “selective renderings with particular heuristic purposes” (Erickson, 2004, p. 29). Indeed, analysts employing microethnography often adopt novel transcription conventions to highlight particular aspects of interaction that video analysis showed important. For instance, in his monograph analyzing interaction, Erickson (2004) adopts three different transcription conventions, depending on the specific aspect of interaction he wished to highlight in a particular part of a study.

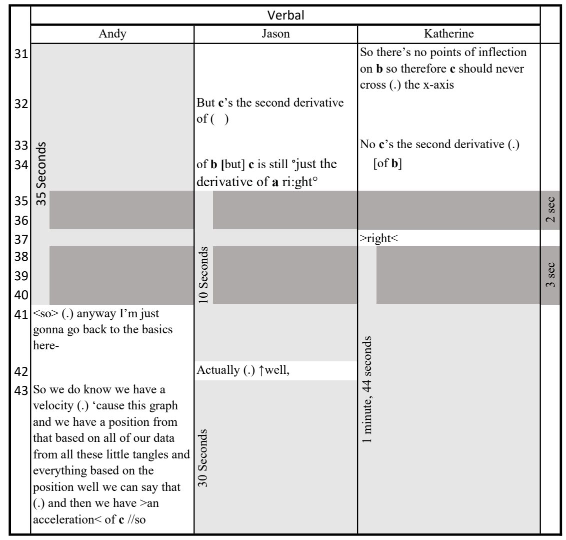
In a study of silence, a transcript therefore, needs to highlight silences, so that an analyst can use them to focus their attention on a video-recording of interaction. The three main principles the transcripts described in this paper are based on (adapted from Ochs, 1979) are: 1) Rather than a play-script transcript in which each turn at talk is given its own line, each speaker is given a column. This change allows lengthy periods of time when one person was silent to be visible. 2) In addition to indicating the duration of silences with measurements of the elapsed time, each second during which all participants in activity are silent is given a full line of transcript. The duration of silences can be indicated in a thin column to the side of the main column. 3) Silences are greyed. Since there are no theoretical reasons to conclude that a silence during which all the students are quiet will function the same as a silence in which only one student is silent, these two kinds of silence are shaded different shades of grey. An example of the employment of these principles in a transcript, see figure 2.4.

In practice, however, these principles require judgements about which silences to shade and highlight. For instance, shading all the lines during which one student is silent would result in a grey transcript, and so the grey portions of the transcript wouldn't carry much information. On the other hand, just as not including silences in a transcript is a judgment that silences are unimportant for the research, not including a particular silence in these transcripts is a judgment that a particular silence is not as interesting. In order to balance these two concerns, the following conventions have been adopted: First, because silences shorter than a second are a normal part of conversation, a row of the transcript is only greyed for silences longer than a full second. Second, a particular student's column is greyed only if either: a) The whole group is silent for over a second (and the student either did not speak immediately prior to the silence, or did not speak immediately following it). b) The student is silent for two or more turns of talk.

The second condition often means that a column is greyed if two or more people speak between a particular student's turn. It is also possible to grey a column if a single student took two distinct turns of talk, with only a short silence between them. For instance, in the following transcript, Alan's column would be greyed, even if the silences were shorter, because he did not talk even though Mary (M) took more than

```
K:   So there's no point of inflection on b so
     therefore c should never cross (.) the
     x-axis
J:   But c's the second derivative of ( )
K:   No c's the second derivative [(.) of b//
J:                                [of b but//
     c is still °just the derivative of a
     ri:ght°
     (2.0)
K:   >right<
     (3.0)
A:   <so> (.) anyway I'm just gonna go back
     to the basics here-
J:   Actually (.) ↑well,
A:   So we do know we have a velocity (.)
     cause this graph and we have a position
     from that based on all of our data from
     all these little tangles and everything
     based on the position well we can say
     that (.) and then we have >an
     acceleration< of c so
```

(a)




(b)

Figure 2.4: Two transcripts of the same sequence of student collaboration. Transcript a) is transcribed according to Conversation Analysis conventions, which follows a play-script format. In b), each participant is given a column, and silences are greyed. Transcript a) minimizes the 5 seconds of silence spread between turns 7 and 9, and gives the impression that the group work followed Andy's redirection, a redirection signified by "so" (Heritage, 1984), in lines 10 and 12. Transcript b) highlights both the five seconds of silence, and Jason and Katherine's 40 seconds of silence. Furthermore, because it highlights Jason and Katherine's lengthy silence, the second transcript raises the prospect that Jason and Katherine's continued silence is indicative of continued joint activity that does not follow Andy. An analyst reading the transcript while watching the video could then check to see how Katherine and Jason acted during the silence, and whether they had turned to listen to Andy or maintained a postural configuration initiated before he spoke. Read vertically, transcript b) also provides a time-line of the group's work, and highlights the shifts in the group's use of silence and speech, as required in the second stage of a microethnographic analysis (Erickson & Schultz, 1977). Specifically, at the beginning, Katherine and Jason were engaged in a back-and-forth conversation, while Andy was silent. For five seconds in the middle, the whole group was silent. Then, for the last 30 seconds, Andy talked while Katherine and Jason continued a conversation. If the transcript continued, then Katherine's silence while Andy and Jason talked would be highlighted. (Even from this transcript, it is clear that Katherine was silent for 1:06 after Jason started talking.) Finally, it would show Katherine's eventual transition from silence to speech.

a single turn of talk.

Transcript 2.1

```
M:   Alan, what would you like to see- what do you think about?
     (1.8 seconds of silence)
M:   Other approaches- There's gotta be stuff that's not like
     graph paper approahes too.
     (1.0 seconds of silence)
M:   Like um you know what we did in our group for the isosceles
     triangle
```

This convention does introduce some researcher judgment into the creation of these transcripts, since one researcher may think a silence came in the middle of a single turn, and another researcher may think it came between turns. This is not a major problem, however. At this stage, the goal of these transcripts is not to analyze data, but to display the data in a way that makes it amenable to analysis. A researcher, therefore, is free to grey a particular student's silence, and later conclude that the silence was not relevant to the paper's research questions. Likewise, a researcher is free to not grey a particular silence, but to attend to the actions a student engages in during it as aspects of back-and-forth conversation. This is especially true because the lengths of silences are visually salient in these transcripts—even when the silence is the result of a lengthy turn—and a researcher can therefore see nuances between silences, with a long "white" silence having a prominence a short "grey" one does not.

The exact duration of the silences during which all the students are silent can be notated in a small column to the right of the columns containing the students speech, and the duration of a single student's silence in a small column on the right hand side of the column representing that student's silence. In earlier versions of the transcripts (including the ones employed in the sample analysis section at the end of this paper), individual students silences were displayed to the right of the column displaying a single student's talk. Because of the way Microsoft Excel treats merged columns, however, it is much easier to edit transcripts if individual students' silences are annotated to the right of the column containing their talk.

The result is a transcript that represents, on the surface level, when students are engaged in four different kinds of activity: 1) Taking a turn at talk. This is visible because there is text in their column when they are talking. 2) Silence, with all their peers. This is visible because a row is greyed for every second the whole group is silent. 3) Silence, even though one of their peers is speaking. This is visible because their column is greyed. 4) Not (currently) speaking, but engaged in a back-and-forth conversation, with no gaps. This is visible, because, in contrast to the rest of the conversation, these rows are simply blank. Additionally, in this format, silences are given more prominence than either talk or silent involvement in a back-and-forth conversation. Finally, because the duration of silences (and of talk) is indicated by distance in the transcript, longer silences are immediately visibly prominent.

At this point in the analysis, the goal is not a precise measurement of the duration of the silences, but a visual outline of the episode. For this reason, it may be advantageous to measure silences using time stamps automatically generated during the transcription process in order to measure silences. At later stages, shorter silences can be counted off (as per standard practice in CA, Liddicoat, 2011), and longer silences measured using an computer program like Audacity or Praat. In my research, I created these transcripts with InqScribe, and then copied the text into Excel. (Though another spreadsheet program like OpenOffice Calc, or Google Sheets would serve as well). Copying the transcripts into a spreadsheet program was not particularly difficult and this technique was practicable. On the other hand, FTW Transcriber allows the creation of transcripts directly in a spreadsheet. Furthermore, in later stages of transcription, the ability to transcribe in Excel while the transcription software open in a second window, and responsive to foot-pedal commands, was invaluable.

Once these transcripts have been created, they can be used in the first stage of a microethnographic analysis of interaction to help focus analytic attention on silences in a video, and to record analytic notes about a video. For instance, if group activity continued, it would be easy to miss that one particular student looked out the window while her peers conducted several lines of argument, or that another student read from their text book for the same amount of time. In a transcript prepared as described in this section, however, those ten seconds of silence would be shaded grey. This means that as a researcher read through the transcripts while watching video, she would see, and anticipate, the coming silence, and so attend specifically to it and the activities students undertake during the silence.

After a researcher has used these transcripts to create analytic notes on the episode, the researcher can use the transcripts in the second stage of a microethnography (Erickson & Schultz, 1977), the creation of a timeline of the interesting activities, and of the transitions between them. Transitions will be particularly important in the next phases of information because, as both Erickson and Schultz (1977) and McDermott et al. (1978) note, at a transition, each member of the party informs all the others (though gesture, word, and posture) that the group is adopting a new postural configuration. This repeated informing ensures that at a transition point semiotic resources abound, and overdetermine the new configuration. Because of this wealth of information, transitions from one configuration to another are particularly analytically fruitful.

Inasmuch as silence is an aspect of posture, part of a particular postural configuration is who speaks during each configuration. Nevertheless, an analyst should be careful. If someone does not speak for a long period, it may be tempting to conclude that they were not authorized to speak, when there are other possible explanations of the silence. This means that someone's new speech after a long silence does not necessarily herald a new postural configuration. Nevertheless, their move from silence to speech (or from speech to silence) is interesting for a study of silence. For instance, in the episode transcribed in figure 2.4, Andy does not speak in lines 31–40. This fact is interesting, however, his speech in line 41 does not necessarily indicate a new configuration—he may have just chosen not to talk, though he could have. Likewise, Katherine and Jason's silence does not necessarily indicate a novel configuration. Nevertheless, both Andy's talk and Katherine and Jason's silence may indicate a novel configuration, and so these transitions merit attention. Furthermore, even if there is no new postural configuration, because silence is unevenly distributed on both sides of the lengthy silence in lines 35–40, an analyst attending to silence should attend to these shifts, and their causes. When these shifts also contain postural shifts microethnography will be a helpful tool in analyzing the transition. When they do not, or only do ambiguously, Conversation Analysis (CA) can be a fruitful tool in analyzing the transition.

### 2.3.3 Analysis

The third phase of a microethnographic analysis listed by Erickson and Schultz (1977) is the precise identification and description of all the relevant transitions, the actions participants engage, and their postures and gestures, and the distribution of talk and listening in the relevant sections. For an analysis of silence, it is important to add a description of the specific silent actions each participant engages in, whether they are listening or engaged in some other silent activity. For this analysis, the transcripts used in figure 2.4 are inadequate, because they do not display the participants' postures and gestures. This section, therefore, presents a more detailed transcription method that does allow an analyst to see and communicate differences in postural configuration, action and gesture types across transitions.

The analytic goal at this stage is to use techniques from Conversation Analysis (CA) (Liddicoat, 2011; Schegloff, 2007) to identify the sorts of conversational actions the subjects are engaged in, and to simultaneously overly that analysis with a microethnographic analysis (Erickson, 1996, 2004; McDermott et al., 1978) of the specific postures and gestures which structure group activity in their silences and speech.

Techniques derived from CA allow the analyst to identify, with precision, the conversational actions each subject engages in, and to understand how these actions relate sequentially. On the other hand, microethnography allows the analyst to carefully identify postural elements of interaction, and to understand what aspects of interaction led to changes between them. Because silence functions, in the moment, as a form of gesture (Acheson, 2008) or even word (Ephratt, 2011), but also, as theorized in this paper, as an aspect of posture, it is important that techniques derived from both CA as well as microethnography be employed in an analysis of silence in collaboration.

The microethnographic analysis of the participants' gestures and postures, during their silence, allows the identification and analysis of the embodied actions (e.g.

thinking or reading etc.) the participants engage in, and the social significance of these postures and acts they embody. For example, in Petersen (2020b), two mathematicians, Joseph and Bill, were silent together for approximately 38 seconds (with two brief episodes of talk in the middle). A CA analysis shows that at the beginning of the silence, Bill's noticing a problem in his and Joseph's work had been conversationally relevant, and that it was precisely the silence that displayed his noticing. Without an analysis of the details of Joseph and Bill's actions in silence, it would be difficult to conclude anything about the precise form of their silent activity. When he noticed, did Bill walk to the board and start writing something on the board? Did he take out a mathematics book and start reading it? Did he start pacing? Did Joseph look at him? Etc. Since it allows a researcher to examine the precise contour of each participant's the gestures and posture, and build models about how participants inform each other about their joint activity, for the task of identifying the specifics of these actions, and their social significance, microethnography is an important analytic tool.

Erickson and Schultz (1977) argue that it can be important to analyze the content of the discourse at this stage of an analysis. Though it has not been a focus of my research into silence, it may therefore be possible to overlay analyses of the mathematical content of the collaboration with these analyses. For instance, in the passage analyzed below, during which three calculus students attempted to identify graphs of position velocity and acceleration, one student, Andy, misread his peers intently-focused thinking silence as an awkward lapse in the conversation. The misreading caused him to attempt to interrupt their silence, even as his peers, Katherine and Jason, persistently continued to engage in it. Because Jason and Katherine remained silent and unresponsive, even as Andy attempted to speak to them, their silence served to "give Andy the silent treatment" and so to alienate him from their mathematical activity. Though the analysis does not engage it deeply, it is notable that Andy attempted to understand the physical motion the graphs represented, whereas, in their interaction, Katherine and Jason attended to mathematical features of the graphs like slope and concavity. That is, Andy's comments were directed to understanding the underlying motion represented by the graphs, whereas in Katherine and Jason's discourse, the motion was merely the occasion for a calculus problem that could have been presented and, and was solved, without any reference to motion (for a discussion of these two ways of approaching this problem, see Petersen, Enoch, & Noll, 2014). Because Katherine and Jason successfully answered the problem without reference to the physics represented by the graphs, and then preceded to another question, Andy's questions were not addressed. Therefore, a researcher analyzing Andy's alienation in this passage may be able to helpfully overlay an analysis of ways his questions about the problem went unanswered with an analysis of ways Katherine and Jason's silence positioned him as an outsider to their mathematical activity.

One of the major barriers to an analysis that overlays microethnographic and CA analyses is transcriptional. CA has a detailed set of transcription conventions that facilitate the transcription of prosody, speech rhythm, short pauses, etc. These transcripts, however, are still presented like lines in a play script, and so do not highlight silences. Researchers employing microethnography place a similar emphasis on the theoretical presuppositions of transcription (Erickson, 2004), but also attempt to tailor their transcripts to the specific phenomenon under study. For instance, when studying rhythmic aspects of conversation, Erickson (2004) notated the rhythms with unpitched musical notation. But there is no method of transcription that can facilitate a careful analysis of silence, together with an analysis of interactants postures and gestures. The procedure outlined in the previous section, can be extended to carefully notate gestures and postures.

Because it is important, at this stage of analysis, to carefully describe participants' gestures (Erickson & Schultz, 1977; McDermott et al., 1978), the transcripts presented in figure 2.4 are modified to include an additional set of columns, containing one column one for each interactant, to the right of the columns transcribing talk (see figure 2.5). This addition allows a researcher to record precisely the actions of each participant during the silences, and so to specify the actions performed in silence, and analyze the social distribution of these actions. For instance, it is their postures that differentiate between Katherine and Jason's activity as Andy talks in lines 41–43 of the transcript in figure 2.5, and Andy's action as he listens to Jason later in the same episode. So that the transcripts continue to draw the eye, lines representing times during which all the interactants are simultaneously silent are greyed across the whole transcript.

The new columns are then used to record each interactant's gestures. Because gestures are responses to other participants actions, and because postures and shifts in posture inform each participant about the nature of the activity they are engaged in (Erickson, 2004; Erickson & Schultz, 1977; McDermott et al., 1978), it is important that the timing of these gestures, relative to speech, be recorded. The timing of the gestures is indicated by superscripts, which are placed in both the speaker's "speech" column and, if they are not speaking, in the "speech" column of the person who produced the gesture. This produces a record of the timing of each gesture, relative both to the speech and to the other activity of the person who produced the gesture.

Furthermore, because each participant's gestures are recorded in their column, when a participant is relatively still, the lack of superscript signifies that lack of motion. CA transcript conventions can be used to record prosody in these transcripts. See figure 2.5 for an example of a passage transcribed according to these conventions. CA transcription conventions are presented in table 2.1.

| Convention | Meaning |
|---|---|
| ( ), e.g. (this) | inaudible talk, analyst's best guess |
| (( )), ((sneezes)) | commentary |
| colon, e.g. ri:ght | extended syllable |
| inward pointing brackets e.g. >fast< | faster speech |
| outward pointing brackets, e.g. <slow> | slower speech |
| brackets, e.g., [overlap] | overlapping speech |
| degree sign, e.g. °quiet° | quiet speech |
| up arrow, e.g. ↑you know | raised pitch |
| numbers in parentheses e.g. (0.2) | silence, number is seconds. |
| dot in parentheses, e.g. (.) | short, unmeasurable silence |
| equals sign, = | immediate or no pause |
| period, e.g. right. | falling intonation |
| semicolon, e.g. right; | rising intonation |
| comma, e.g. right, | continuing intonation |

Table 2.1: Transcript conventions.

These tools are particularly helpful when an analysis requires a detailed accounting of multiple aspects of interaction. On the other hand, for some analyses, they are too detailed, and a more conventional transcript, perhaps following CA conventions, is easier to read. Furthermore, these transcripts do not display eye-gaze direction well. (Though that may be fixable using a combination of shading techniques.) For instance, in the analysis of the collaboration of Fay and Martha in Petersen (2020b), a more conventional transcript, used in conjunction with screen-shots, was more helpful in analyzing Fay's silence than the transcripts described in the previous paragraph were.

| | Verbal | | | | Non-Verbal | | |
|---|---|---|---|---|---|---|---|
| | Andy | Jason | Katherine | | Andy | Jason | Katherine |
| 31 | | | So there's no[z] points of inflection on **b** so therefore[y] **c** should never cross[x] (.) the x-axis[w] | | | | [z]points to graph (occluded) [y]points to tail end of **c** [x]points to x-axis [w]scratches nose |
| 32 | | But **c**'s the second derivative[a] of[z] ( ) | | | | [a]lowers hand from mouth | [z]tilts head back slightly, hand just in front of her mouth |
| 33 | [1] | | No[z] **c**'s the second derivative (.) | | [1]begins to point to graph | | [z]points to graph (occluded) (holds position) |
| 34 | 35 Seconds [1] | of [1]**b** [but] **c** is still °just the derivative of **a** [a]ri:ght°[z] | [of **b**] | | [1]steps toward graph | [a]turns head slightly to right (still at board) then motionless | [z]starts slowly rocking back with rigid body |
| 35 36 | | | | 2 sec | | | |
| 37 | [1] | | [z1]>right< | | [1]raises paper | | [z]pulls arm back (hand still pointing) nods |
| 38 39 40 | [1] | 10 Seconds [a] | [z] | 3 sec | [1]steps back ↓eg to paper | [a]tilts head to right ^squares toward board | [z]lowers hand, tilts head to left. |
| 41 | <so> [a](.) anyway[1] I'm just gonna go back to the basics here[2]- | [a] | | | [1]hands apart [2]steps forward pointing toward graph | [a]^squares hips toward board, crosses arms, tilts head slightly left. | |
| 42 | | Actually (.) ↑well,[z] | 1 minute, 44 seconds [z] | | | | [z]slight pulse of left hand (otherwise motionless) |
| 43 | So we do know we have a [1]velocity [2](.) [3]'cause this graph and we have a position[4] from that based[5] on all of our data from[6] all these[7] little tangles and everything based[8] on the position[a] well[9] we [b]can say that (.) and then[c] we[z] have >an acceleration< of **c**[d10] [so[11] | 30 Seconds [a] [b] [c] [d] | [z] | | [1]points to where they have written "velocity" [2]→eg peers, ↑back to board [3]traces along graph a [4]beats near word "position" then waves at graph [5]→eg to peers [6]↑eg to graph touches graph **a** with pen [7]draws lines down from **a** to **b** [8]traces hand along **b** [9]pulls hand back then gestures forward & turns hand over [10]traces hand along **c** [11]scratches nose | [a]head right [b]weight to right foot ↓eg toward floor [c]↑weight on both feet eg and pelvis toward board [d]turns head very slightly to right | [z]steps back, head straight (eg still board), crosses arms. |

Figure 2.5: The same episode as transcribed in figure 2.4, with gestures added. Superscripts show the coordination of gestures with talk, and, since they are included in each student's column, show the physical activity of each student. This allows an analyst to show that, during their silence, Katherine and Jason remained mostly motionless, and did not turn their eye-gaze toward Andy, or physically respond to him, though he spoke to them at length, and repeatedly looked at them. Though the full analysis of this episode in section 2.4 will look at all students' actions throughout their collaboration, this episode suggests that, for approximately 40 seconds, Katherine and Jason maintained a common postural configuration with their focus oriented toward the mathematics they had just been debating, and so engaged in a similar form of mathematical activity for those forty seconds. Because of their persistence in not moving or speaking in response to Andy, it seems that while they had adopted this configuration, neither motion nor speech was normal. Indeed, Katherine and Jason seem to have censured Andy's attempts to engage them in conversation by not moving or speaking in response to him. On the other hand, it seems that Andy did not respond to that silent activity as a form of activity. Instead he attempted, unsuccessfully, to engage Katherine an Jason in conversation, as if silence in lines 35–40 was problematic.

Once the video data has been transcribed, the researcher should not leave the video data behind to focus on the transcripts. The transcripts facilitate a rigorous description of the videos, and a careful, rigorous, differentiation between different kinds of action, but they cannot replace the video. For instance, in the analysis below, the transcripts allow an analyst to enumerate the gestural and postural differences between the silences, and so to rigorously demonstrate that the silences function differently in the group's activity. But the transcripts are too detailed to be the object of analysis itself.

The fourth stage presented in Erickson and Schultz (1977) is a more superficial analysis of the actions engaged in during the segment of interaction prior to the one described in depth in the third stage. Though Erickson and Schultz do not mention it, an analysis of the actions subsequent to the relevant transitions can also be helpful. In the context of the passage partially transcribed in figure 2.5, this means a more cursory analysis of the sort of action Katherine and Jason engaged in prior to their lengthy silence that begins in line 35 and continues for the next 40 seconds. It also means analyzing Jason's, and then Katherine's behavior once they begin talking, in turn. In the analysis below, Andy's silence in lines 31–40 is less of a focus, however, if it were analyzed in depth, an analyst should attend to all three students' talk before his silence.

CA is a particularly important tool for analyzing the actions that lead to and follow the silence. Because silent actions are performed in the context of a conversation, and in turn, contribute to structure the conversation, the understanding of conversation CA provides is critical for understanding how the silence functions socially. Furthermore, sometimes the aim of a silence is established by the conversation. In these cases, the conversation can be critical for identifying what sort of action the silent participants are engaged in. For instance, in Petersen (2020b), two mathematicians, called Joseph and Bill spent approximately 32 seconds in nearly motionless silence, with only brief interruptions. When considered in itself, the silence looks striking, and an analyst may recognize their activity as "thinking". Though, at a previous stage of analysis, their precise gestures had been described, and the transition analyzed, without understanding the specific conversational actions that preceded their silence, it was impossible to understand precisely why they were thinking at this point in the conversation, or why their thinking took the particular form it did. When the preceding conversation was considered, however, was is clear that Joseph had noticed a problem in their work, and had then spent several turns of talk attempting to help Bill notice the same problem. Furthermore, was is clear that when Bill did notice the problem with their work, this noticing manifested itself as an extended period of thinking. From these results, an analyst can conclude that during their extended thinking episode, Joseph and Bill continued to attempt to resolve a problem with their work they had both noticed earlier just prior to the extended period of thinking. Furthermore, the analyst can conclude that mathematicians communicated the noticing, and made it an aspect of their collaborative activity, not by explicitly commenting on the problem, but by noticeably thinking about it. Finally, the researcher can attend to aspects of their posture (particularly their gaze), and to their use of technology (the blackboard), which facilitated this particular form of thinking.

The fifth step described by Erickson and Schultz (1977) is the construction of a model of the factors that participants attend to when coordinating their interactions, and the testing of the model against the data from the single episode. In this stage, again, the actions participants engage in at transition moments are particularly salient. A detailed description of the methods CA and microethnography use to build models of interaction are beyond the scope of this paper. The reader is referred to published descriptions of the methodologies employed in CA (Liddicoat, 2011; Schegloff, 2007) and microethnography (Erickson, 1996, 2004; Erickson & Schultz, 1977; McDermott et al., 1978; Streeck & Mehus, 2005). A brief description of the principles guiding the methods, however, may be helpful.

Schegloff (1996) lists three characteristics of a successful CA analysis. First, the analysis should describe the actions the participants engage in. Second, it needs to show that the actions, as formulated, are not inventions of the analyst, but the participants themselves orient to the actions. Finally, the analyst needs to identify the aspects of each participant's behavior that allow them to engage in the actions. Following these guidelines, it is important that analyst show that, whatever conclusions are drawn about silence, the participants themselves orient to the phenomena, and that the performance of silence is one the features of the participants action that allow them to engage in the particular interpersonal and mathematics oriented actions they do.

Similar principles guide ethnographic microanalysis, however, there, attention is turned more to the postural configurations which organize interaction. McDermott et al. (1978) lists four means members in interaction utilize to coordinate their interaction, and which therefore an analyst employing microethnography should attend to. 1) Members often simply state what behaviors they will engage in, for instance, announcing that they are now going to read. 2) Members tend to coordinate their postures. 3) Members not only coordinate their postures, but do so in a way that signals, to themselves and to their peers, the activity they are engaged in. 4) Members hold each other accountable.

The sixth and final step of a microanalytic analysis is the application of the model generated in the previous step to other examples of interaction involving the same phenomenon (Erickson & Schultz, 1977). In this stage, a researcher applies the analysis described in step three to other instances of the same phenomenon, but instead of analyzing all the aspects of interaction, they restrict their attention to those that were considered relevant in the model constructed in the fifth step.

## 2.4 Sample Analysis

This section has two goals. First, it will demonstrate that in some situations, attention to silence can reveal facets of group dynamics that are not visible to more conventional analyses. This result will help demonstrate, in practice, the potential utility of methodological techniques that facilitate attention to silence in mathematical activity. Second, this section will show that the transcriptional conventions and methodological principles expounded above can help distinguish different kinds of silence in student collaborations.

The 4 minute 48 second episode analyzed here is taken from an interview of three calculus students, pseudonymously called Katherine, Jason, and Andy. The interview was conducted in a study of the Process Oriented Guided Reinvention (POGIL) curriculum, a quasi-reform calculus curriculum. Interview subjects were asked to figure out which of three functions graphed on a single x-y plane represented the position, velocity, and acceleration of a car. During this episode the students were mostly allowed to work together as a group, without interviewer question, though at one point the interviewer asked the students to move so the camera could pick up their work. For this reason, these students' work approximates naturalistic settings like homework collaboration close enough to illustrate how the methodology can be employed to distinguish different kinds of silence. A full transcript of the episode is included in figure 2.4.2 on page 67. Small portions of the beginning and end of the episode that are not relevant to this analysis are deleted, but the rest of the episode is included. Transcript conventions are listed in table 2.1 on page 57.

Initially, both Andy and Katherine seem to make very little contribution to the group's progress. Only nine of Andy's utterances were more than either mere invitations for other students to speak or cut-off fragments; and two of these are at the very beginning of the episode. At one point Jason discussed the meaning of the graph, and of negative acceleration, with Andy. However, after a four second pause, Andy nominated Katherine to speak, and, after another 5 second pause, she asked what labels they had written on the board earlier. After her question, Jason returned the conversation to the a debate he and Katherine had been engaged in earlier. Katherine what Jason believed the solution was, Jason justified the solution. Following this, Andy proposed they move on to the next question. The group therefore reached the solution after Jason stopped explaining the meaning of the graph to Andy, and without input from Andy.

Though Katherine did make some mathematical contributions to the group's work, she also seems to have only contributed a small amount. In particular, she made an argument about concavity toward the beginning of the episode, but then dropped out of the conversation for a length of time. When she did re-enter the conversation, she asked what labels the students had written on the board, asked Jason what he believed the solution was, and agreed with his assessment. Therefore, she only made three seemingly minor contributions. First, she asked what labels the students had written on the board. Second, she asked Jason what he believed the solution was. Finally, she agreed with his assessment and arguments (several times). She did articulate the solution before any other students (lines 79, 81). But she did so in a question, not a claim, and produced no warrant or backing for the potential claim.

The eventual argument in this section will be that by carefully attending to and differentiating the characters of Katherine's, Jason's, and Andy's silences throughout this episode, a more nuanced perspective will be revealed. This analysis will reveal that though Andy was indeed excluded from the group's interaction, Katherine was not. Rather, in many ways, Katherine's mathematical activity and talk were the driving force behind the group's solution. In particular, after the group's talk had left her and Jason's line of inquiry, she silently continued work on her and Jason's questions for another minute as Jason talked with Andy. Furthermore, Katherine's talk at the end of the episode can be seen as directing the group, and Jason in particular, to a solution.

Furthermore, as Andy began to raise a new line of inquiry, Katherine and Jason continued to silently collaborate on their own line of inquiry for forty seconds. It was only after this lengthy silence that Jason (and not Katherine) began to interact with Andy's questions. Though this silence can be understood as a form of important mathematical activity, it was not without significant downside. Rather, Andy seems to have analyzed it as a lapse in conversation, and consequently sought to interrupt it with a set of questions that lay orthogonal to the questions Katherine and Jason pursued. Additionally, because Katherine and Jason persistently continued their silent work, even as Jason attempted to raise his questions and engage them in conversation, Jason and Katherine's silence silenced Andy. Furthermore, Andy's questions were never answered, so this silencing was not temporary.

This argument will proceed in five stages. First, Jason and Katherine's mutual silence will be described in detail. Second, their actions will be compared with Jason's actions during his silence following his discussion with Andy. Third, Katherine's silence will be compared to Andy's activity during his silence in line 60. Fourth, the activity immediately following Katherine's long silence will be analyzed. These analyses together will establish the claim that, as for the mathematicians' silences in Petersen (2020b), Jason and Katherine's silence was an instance of collaborative activity in a joint project, that Katherine continued that joint project while Jason interacts with Andy. Finally, in light of that conclusion, Andy's speech in line 41 and 43 will be analyzed to argue that, from Andy's perspective, Jason and Katherine are giving him the silent treatment, and therefore their collaborative silence, unintentionally, silences him. Carefully distinguishing between different kinds of silence will be key in this argument.

### 2.4.1 Identification of an Episode: Stage I & II

The first two stages in Erickson and Schultz's (1977) methodology are the creation of analytic notes, and the identification of passages that contain the activity of interest. Because of their importance later, it is important at this stage, to identify, and perhaps chart out, the transitions between kinds of activity. Section 2.3.2 showed that transcripts can be used to aid the creation of analytic notes, and for the identification of transitions between different kinds of activity. Because a miniature version of the transcript displays colors not text (see figure 2.6), a miniature version of a transcript can be particularly helpful in diagramming passages and transition points for deeper analysis in stages three through five.

Figure 2.6 a miniature transcript which serves as a diagram of the group's interaction. In practice, this diagram would be annotated in consultation with the video. At this point, however, it is possible to make the following comments: At the beginning of the episode, the right-most student (Katherine) spoke and was rarely silent for long. Her two peers were silent for stretches, however, conversation seems to flow between the three students. In a full analysis, the video would be consulted to determine whether all three students had adopted a common postural configuration during which each student spoke and listened, in turn. At the first dark grey line, this activity transitioned, and first all the students, then the right two (Jason and Katherine) were silent for an extended period of time. During their silence, the left-most student (Andy) talked, but there was little or no response to him. Eventually the activity of the middle student (Jason) transitioned from silence to speaking. At the same time, the right-most student (Katherine) continued to be silent until the second lengthy cluster of dark grey "mutual silences". The right-most student's silence there is perhaps worth noting and checking against the video. At this point, the activity transitioned again, and two students on the right carried on a conversation till the end of the episode. During that final conversation, the student on the left interjected only a few short comments.

This analysis allows a researcher to identify the following configurations and transition points, which could be noted on the transcript.

- At the beginning all the students were talked. Perhaps the activity in the silences should be checked against the video.

- *Transition*: At the first mutual silence (dark grey) the activity transitions. The two students on the right are silent, while the one on the left talks. The activities in the silence will need analyzed in more depth.
- This configuration continues till the middle student begins talking.
- *Transition*: The middle student transitions from silence to speech, the left-most student continued speaking (though less), the right-most student continued to remain silent.
- This configuration remained stable till the next cluster of mutual silences after which the right most student talked again. The left-most student's silence prior to that mutual silence should be noted, and checked against video.
- *Transition*: Following the last cluster of mutual silences (dark grey), the two right-most students talked while the left-most student remained silent.
- This configuration remained stable till the end of the episode.

Since the focus of this analysis is on Katherine and Jason's silence, the next stage of analysis will carefully describe their activities during it, and the transitions to and from this silence.

### 2.4.2 Jason and Katherine's silence: Stage III

Following a verbal exchange (lines 29–34), Katherine and Jason were silent together for forty seconds, except for two utterances of one and two words respectively

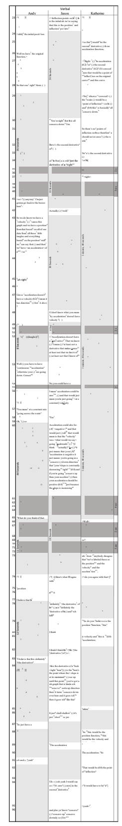


Figure 2.6: A miniature version of the transcript of the episode analyzed in this section. This mini-version of the preliminary transcript can be used to identify important passages. A larger version, does not fit well on a computer screen, and so is more useful for later stages of analysis that identify details of interaction. If an analyst wanted a larger version, this transcript could be printed and annotated by hand.

| | Verbal | | | | Non-Verbal | | | Interviewer Talk |
|---|---|---|---|---|---|---|---|---|
| | Andy | Jason | Katherine | | Andy | Jason | Katherine | |
| 23 | //( )] [1] | //>Inflection points on **b**[a]<], **b** is the initial uh we're saying that this is the position[1] and inflection[y] po//ints[2] | //( )] [y] | | [1]points vaguely at graph [2]beats hand | [a]points to max of b | [z]draws hand back along c to right end [y]inhales, moves hand to left | |
| 24 | //uhh][2] the initial posit//ion | | | | [2]beats hand (same action as above) | | | |
| 25 | | [a] [b] | //so this[z]] would[a] be the second[b] derivative (.) th-an acceleration function; | | | [a]→eg to Kat. [b]↑eg to c (where Kat's pointing) | [z]points to c (holds till b in turn 25) | |
| 26 | Well we have[1] the original function,= | | | | [1]traces **b** stopping at its max | | | |
| 27 | [1] [2] [3] [4] [5] [6] | [a] 30 Seconds | =[z]Right. [1](.) [a]So acceleration (0.2) [2]or[y] (.) the second derivative[3] (0.2)[x] if it crossed [4]zero that would be a point of [w5]inflect↑ion on the original curve?[6] and this curve doesn't= | | [1]beats pointing hand, then rubs nose [2]hand lowered [3]→eg K [4]^> pivots to 45 degrees.[5]eg follows K's hand [6]points to inflection point on **c** | [a]shifts weight slightly | [z]small nod [y]beats with pointing finger [x]points elsewhere on graph (occluded) [w]steps forward and points elsewhere on graph (occluded) | |
| 28 | So that one[1] right[z] there; ( ) | | [z] | | [1]hand to his sholder. | | [z]leans to her left ((away from board, for better view?)) | |
| 29 | [1] [2] [3] | | //No][1] where **c** [z]<crossed> (.) the [2]x-axis (.) would be a >point of inflection[y]< on **b**. (.) and[3] (0.4) this[y] is basically[a] all [x]concave down.[b] | | [1]points back to inflection point on **c**, moves hand to chest, then back. [2]hand to chest [3]steps back drops hands then looks at paper | [a]leans head slightly to left [b]right (dominant) hand to his mouth. | [z]moves hand up and down (occluded) [y]traces hand along **b**. [x]puts pointing hand on top of her head. | |
| 30 | [1] | [z1]You're right[a] that **b** is all concave down.[b] Um | | | [1]↑eg to board, rh across stomach | [a]points at graph (occluded) [b]hand back to mouth (holds) | [z]points to graph (occluded) | |
| 31 | | | So there's no[z] points of inflection on **b** so therefore[y] **c** should never cross[x] (.) the x-axis[w] | | | | [z]points to graph (occluded) [y]points to tail end of **c** [x]points to x-axis [w]scratches nose | |
| 32 | 35 Seconds | But **c**'s the second derivative[a] of[z] ( ) | | | | [a]lowers hand from mouth | [z]tilts head back slightly, hand just in front of her mouth | |
| 33 | ( [1] ) | | No[z] **c**'s the second derivative (.) | | [1]begins to point to graph | | [z]points to graph (occluded) (holds position) | |
| 34 | [1] | of [1]**b** //but] **c** is still °just the derivative of **a** [a]ri:ght°[z] | \\of **b**] | | [1]steps toward graph | [a]turns head slightly to right (still at board) then motionless | [z]starts slowly rocking back with rigid body | |
| 35 | | | 2 Sec | 2 s | | | | |
| 36 | | | | | | | | |
| 37 | [1] | | [z1]>right< | | [1]raises paper | | [z]pulls arm back (hand still pointing) nods | |
| 38 | | 10 Seconds | | 3 sec | | | | |
| 39 | | | | | | | | |
| 40 | [1] | [a] | [z] | | [1]steps back ↓eg to paper | [a]tilts head to right ^squares toward board | [z]lowers hand, tilts head to left. | |
| 41 | <so> [a](.) anyway[1] I'm just gonna go back to the basics here[2]- | [a] | | | [1]hands apart [2]steps forward pointing toward graph | [a]^squares hips toward board, crosses arms, tilts head slightly left. | | |
| 42 | | Actually (.) ↑well,[z] | [z] | | | | [z]slight pulse of left hand (otherwise motionless) | |
| 43 | So we do know we have a [1]velocity [2](.) [3]'cause this graph and we have a position[4] from that based[5] on all of our data from[6] all these[7] little tangles and everything based[8] on the position[a] well[9] we [b]can say that (.) and then[c] we[z] have >an acceleration< of **c**[d10] //so[11] | [a] [b] [c] [d] | [z] 1 minute, 44 seconds | | [1]points to where they have written "velocity" [2]→eg peers, ↑back to board [3]traces along graph a [4]beats near word "position" then waves at graph [5]→eg to peers [6]↑eg to graph touches graph **a** with pen [7]draws lines down from **a** to **b** [8]traces hand along **b** [9]pulls hand back then gestures forward & turns hand over [10]traces hand along **c** [11]scratches nose | [a]head right [b]weight to right foot ↓eg toward floor [c]↑weight on both feet eg and pelvis toward board [d]turns head very slightly to right | [z]steps back, head straight (eg still board), crosses arms. | |
| 44 | [1] [2] | 30 Seconds [a] [b] | [z] | | [1]↓˅pivots at camera (where interviewer is standing) [2]→>pivots part way back | [a]↓˅pivots to camera, steps back [b]↑←^< eg back to board (now 45 degrees) | [z]steps slightly to right (toward board away from J) | //can] you guys make sure[1] that[a] (.) I can see[2b] the [z]gra:ph::, |
| 45 | [1]uh right[2] | | | | [1]shakes sholders [2]↑^pivots to graph | | | |
| 46 | [1] | | | | [1]shifts weight back and forth | | | [1]>so(h)rry thanks.< |

Figure 2.7: Transcript

| | Verbal | | | | Non-Verbal | | | Interviewer Talk |
|---|---|---|---|---|---|---|---|---|
| | Andy | Jason | Katherine | | Andy | Jason | Katherine | |
| 47 | Um so [1]acceleration doesn't[2] have a velocity (0.2)[3] I mean it has direction[4a] (.) but[5] it also ( ) | a | | | [1]weight even, beats hand [2]→ eg peers, beats both hands [3]↑beats right hand [4]beats right hand [5]points near origin then traces hand right | [a]←head A | | |
| 48 | | I I don't know what you mean [1]by acceleration[z] doesn't have velocity [y]( ) | z y | | [1]scratches nose | | [z]steps back and toward board [y]steps forward, immediately rocks back | |
| 49 | ([1a] ) | a | | | [1]shakes head and raises hand | [a]↓eg to paper | | |
| 50 | | 4 Sec | | 2 s | | | | |
| 51 | 1 | a | | | [1]rolls neck | [a]shuffles his papers | | |
| 52 | \\([1] ) ((laughs))[2]] 3 (8 Seconds) | \\[1]Accel]eration doesn't have a [z]deri[a]vative?[2] That we know (.) [b]I mean (.) [y]at least not a derivative that makes sense[3c] at least not that we know of[d] (.) at least not that I know of[ex] | z y x (1 minute, 44 seconds) | | [1]→>eg and pelvis to J [2]crosses arms 3↓eg to paper | [a]←eg A [b]↑eg board [c]←eg A [d]↓eg paper [e]shifts eg to different part of paper, shuffles paper | [z]steps forward, rocks back [y]leans right, leans head to left [x]steps forward | |
| 53 | Well (.) you have to have [1]continuous [2a]acceleration[z] [3]otherwise you (.) [4]are going down. Correct[4b] | a b | z | | [1]↑eg board [2]steps forward [3]points near origin, traces graph **c**. [4]drops hand to graph a (which is negative) → eg J [4]arm across body. | [a]↑eg board [b]twitches head | [z]steps back | |
| 54 | | No you could have a | | | | | | |
| 55 | | | | 1s | | | | |
| 56 | 1 2 3 \\( )] | I mean[1] acceleration could be zero[2az], (.) and that would just mean you're just going[3] \\at a constant] velo[b]city | z | | [1]scratches nose (stops when A looks at him) [2]↑eg board [3]runs hand along x-axis | [a]←eg A then back [b]←eg A | [z]tiny nods, slowly getting larger | |
| 57 | [a]You mean[1] at a constant rate [2]going across the x-axis[z] | | z | | [1]runs hand along x-axis [2]→eg J | [a]eg slowly drifts to board | [z]end of nods | |
| 58 | 1 | [a]Yes[1] | | | [1]scratches chin | [a]←eg A | | |
| 59 | Ok, [1](.) so | | | | [1]↑eg board, drops hand | | | |
| 60 | 1 2 3 4 5 6 7 8 9 10 11 12 (44 Seconds) | Acceleration could also be: (.4)[1] >negative<[2a] and that would just (.) all[3b] that would mean is that the [4]velocity[c] wa:s >what would we say< going [5]backwards[d] (.)[6] °(I think [7e]actually)° [f]no (.) [g]it just means that your (.8)[8] °acceleration is negative it just means you're going in a [h]concave (.) d:own direction[9] that [i]your [j]slope is constantly decreasing[10] right[z] [11](0.8) and if you're going [k]concave up than your accelera-[l] (.) then your acceleration should be positive (0.8)[12] [m]just because the slope is increasing°[n] | z (1 minute, 44 seconds) | | [1]→eg J [2]scratches chin [3]↑eg board, drops hand [4]shifts weight back and forth [5]stops shifting weight, stands slightly more erect than before [6]→eg J [7]↑eg board [8]→eg J [9]↑eg board, rolls neck [10]shifts weight changes hand position slightly [11]leans head right, weight left [12]shakes both arms | [a]steps toward board [b]< pelvis to left ↑eg still board, points toward min of **a** [c]traces hand to right of graph [d]←eg A [e]drops hand [f]↑eg board [g]raises hand (occluded) [h]traises hand in "n" shape. [i]←eg A [j]raises hand traces larger "n" shape. [k]↑eg board, points to minimum of **a** [l]points to inflection point on **c**, traces **c** to right [m]traces along minimum of **a** [n]←eg A, then drops hand. | [z]inhales | |
| 61 | | a | | | | [a]↓eg to paper | | |
| 62 | 1 | | | 4 sec | [1]scratches nose | | | |
| 63 | | | | | | | | |
| 64 | | | z | | | | [z]steps forward. | |
| 65 | [1]What do you think of that. | | | | [1]→eg K (leans to see past J) | | | |
| 66 | | | >Well< | | | | | |
| 67 | 1 2 | a | z | 2 sec | [1]↓eg paper [2]shifts weight back slightly | [a]→ eg K, steps back. | [z]snaps arm with paper in it up, then ↓eg quickly to paper | |
| 68 | | 18 Seconds | | | | | | |
| 69 | | | so[z]- | | | | [z] - here is glottal stop | |
| 70 | 11 Seconds | | | | | | | |
| 71 | 1 | | z | 3 sec | [1]→eg K | | [z]←eg (more slowly) board | |
| 72 | 1 | | z | | [1]↓eg paper | | [z]lips start moving | |

Figure 2.7: Transcript (cont.)

| | Verbal | | | | Non-Verbal | | | Interviewer Talk |
|---|---|---|---|---|---|---|---|---|
| | Andy | Jason | Katherine | | Andy | Jason | Katherine | |
| 73 | [1] | [a] [b] | uh- [z]does [ya]anybody disagree that [x]we've labeled these as the position[bw] and the velocity[w] and the accelera[1]tion[wv], | | [1]steps toward graph | [a]↑eg board [b]leans head left, left hand to mouth. | [z]leans head back, opens eyes wider [y]reaches toward graph [x]steps forward [w]points to each label in turn [v]↓eg J | |
| 74 | //( )] | //[a]( )] Here's what I'll agree with[b] | //>do you agree with that<][z] | | | [a]steps toward graph [b]extends arm | [z]←eg to board (remains through end) | |
| 75 | [1]position | | | | [1]points to the label "position" | | | |
| 76 | | **a**[1az] is | | | [1]withdraws hand | [a]beats hand | [z]folds paper in half (no crease) holds in lh | |
| 77 | I believe that **b**[1] | | | | [1]traces along beginning of **b** | | | |
| 78 | [1] | [z]definitely[a] >the derivative[1] of **b**<[y] (.) **a** is [b]definitely the [c]derivative of **b** (.) and[x] tch hfff[d] | [z] [y] [x] | | [1]lowers hand, >→eg and pelvis to J | [a]points to **b** [b]small beat with hand [c]lowers hand [d]leans head to left | [z]steps back, rests hand on pen tray [y]small firm nods (continues) [x]nods stop | |
| 79 | 12 Seconds [1] [2] | [a] | [1z]So do you [y]believe **c** is the position [a]function; [2]this[x] | | [1]opens mouth [2]eg board scratches nose | [a]^pelvis toward board, straightens head | [z]opens mouth [y]points to **c** [x]steps forward & points to label on **b** | |
| 80 | | I think | | | | | | |
| 81 | [1] [2] | | is velocity and [z]this is [1y](0.8) [2]acceleration; | | [1]stops scratching nose [2]opens mouth, steps forward, points to graph (occluded) | | [z]steps forward & points to label on **a** [y]steps back, scratches neck then, save for scratching fingers, motionless | |
| 82 | | I think I think **b**'s[a]>↑**b**'s ↑the ↑derivative ↑of ↑c<. | | | | [a]points at graph (occluded) then motionless. | | |
| 83 | [1]I believe that **b** is definitely[2] °(the derivative)° | | | | [1]withdraws hand slightly, then returns it 2→ eg right, beats | | | |
| 84 | //( [1] )] [2] [3] [4] [5] | >**b** is the derivative of **c** [a]look //right [1]here[z]] (.) is the [b]here's the point where the< slope is at its maximum[cy] (.) see up until this point[2d] you've got a uh graph that is kinda u:h [3ex]conc:ve[w] sorta up direction then[f] it turns [4]concave down over here and it goes tch[5g] then it goes tch[h] like that[i] | | | [1]steps back →eg J? [2]steps back ↑eg board [3]↓eg down at hands, starts replacing cap on marker in his hand [4]↑eg board, still putting cap on marker [5]finishes putting cap on marker, straightens up. | [a]very small raise of arm (occluded, overall posture remains the same) [b]moves arm (occluded) [c]tucks paper under his arm [d]rh pointing but occluded, lh at origin. [e]traces lh along concave up part of c [f]traces lh along concave down part of c (lh crosses rh, but rh stationary) [g]lh along cd part of c [h]lh along cu part of c (rh still pointing, and occluded) ((tch here is a tongue click)) [i]→eg K. | [z]stops scratching neck, doesn't move hand, pointer finger still extended [y]curls pointer finger (still motionless otherwise) [x]rocks back [w]puts scratching hand on marker tray | |
| 85 | [1] | | [z]mhm:[1] | | [1]→eg K | | [z]firm nod | |
| 86 | [1] [2] | It just[a] shall shallow[1] (.) it's just [b]short[2cz] so jus | | | [1]removes cap from marker [2]steps forward reaches for graph with marker | [a]↑eg board lh to origen (rh still stationary) [b]lh waves along l part of **c** [c]drops lh | [z]opens mouth | |
| 87 | [a]So just have a | | | | | [a]backs slightly away (A was almost touching him) | | |
| 88 | [1] | | [1]So [az]this would be the position function, [by]this would be the velocity and this[x] would be | | [1]Throughout these turns, Andy is writing something illegible on the graph, perhaps labeling intervals of differing concavities. | [a]drops rh, backs further away head leaned left, pelvis square [b]crosses arms | [z]steps forward points to graph **c** (occluded) [y]points to label on **b** [x]points to label on **a** | |
| 89 | | [z]The acceleration | [z] | | At this point, the gestures are no longer important for analysis, and so are dropped. | | [z]withdraws pointing hand, steps back | |
| 90 | | | The acceleration. [z]So | | | | [z]steps forward, points to inflection point on **c** | |
| 91 | a b and c, [z]yeah[a] | | | | | [a]^<opens stance a little | [z]withdraws pointing hand, steps back | |
| 92 | | | [z]That would be (0.8) the point of [y]inflection? | | | | [z]steps forward points to inflection point on **c** [y]withdraws hand to pen tray, steps back | |
| 93 | | | | | | | | |
| 94 | | Uh::: (.) uh yeah I would say so //[z]it's zero[a] (.) zero] in the second [x]derivative[b] | //[z]it would have to be[a] it[y]] | | | [a]steps forward, points with lh to zero of **a** [b]drops hand | [z]nods [y]steps forward, points to graph [x]steps back, nods | |
| 95 | | | >yeah<[z] | | | | [z]scratches sholder | |
| 96 | | and plus ya' know [z]concave[a] (.) °concave up[b] concave downdy so if its°[cw] | | | | [a]points just above c, where it is concave up [b]points to the concave up and concave down parts of c [c]crosses arms, steps back | [z]several small nods [y]straightens up, hand on marker tray [w]nods stop | |

Figure 2.7: Transcript (cont.). Silences at the end are transcribed in less detail because the analytic focus is on the earlier passage (Erickson & Schultz, 1977).

(lines 37, 42). For the last thirty seconds of the silence, they were entirely silent. At the beginning of the silent period, both adopted a similar body posture: Their arms were near their trunk (Katherine had been pointing, but withdraws her hand in line 38); both faced the board, in line 41 Jason tightened his direction toward the board by squaring his hips toward the board; their heads were leaned to the side (line 40 a and z); and both remained nearly motionless for the majority of the time. Because the standing human body is an inverted pendulum, and so is either at an unstable equilibrium (like a pencil balanced on its end, where the potential energy is at a maximum), or in disequilibrium, this last commonality, motionlessness, is itself an activity.

During this silence, Katherine and Jason persistently maintained this configuration in the face of Andy's attempt to shift the configuration, especially its silence, by engaging them in conversation. In transcript line 43, Andy talked, touched the graph (where Katherine and Jason were directing their gaze) six times (1, 3, 4, 6, 7, 10), and twice solicited a response by turning his eye-gaze toward them (2, 5) (Stivers & Rossano, 2010). Yet, even though Andy solicited their attention through his speech, arm movements, and eye-gaze, Katherine and Jason remained motionless, and, except with the possible exception of Jason's speech in line 40, never responded to Andy, even by turning their heads to follow his pointing arm, meeting his eye-gaze, or responding with continuers.

Following Andy's talk in line 43, the researcher asked the students to move so the camera could capture the figure (line 44). In response, Jason and Andy both turned toward the interviewer (1 and a), and then back to the figure (2 and b). When he moved back, Jason again directed his gaze toward the board, but, in order to accommodate the interviewer's request, he was slightly further to the right of his previous position. Furthermore, though still was not looking toward Andy, his pelvis was at 45 degrees, half way toward the board, and half-way toward Andy. That Jason resumed the a similar configuration as before provides further evidence that he remained engaged in the mathematical activity he and Katherine had been engaged in since line 35.

During this short exchange, Katherine twice maintained her posture rather than shift in response to her peers. First, Katherine did not move in response to the interviewer's talk. Though the interviewer had only spoken to Jason and a response was not expected from her or Andy, her lack of response contrasts with Andy's response. The difference between her response and Andy's therefore provides further evidence that, at this time, Katherine's posture displayed engagement in the mathematical activity she and Jason had been engaged in. Second, Jason's movement brought him close enough to Katherine that she could have moved out of his way. Nevertheless, she only moved very slightly (probably less than an inch) to make room for him (line 44, z). That she maintained her posture a second time provides further evidence that just as Jason did, Katherine also remained engaged in the same activity she and Jason had begun in line 35.

Andy then talked again, further soliciting Jason and Katherine's response by directing his eye-gaze toward them (line 47, note 2). Initially both Katherine and Jason maintained the postural configuration adopted in line 35, in which they remain silent and motionless, with their gaze fixed on the board. Once again, because neither responded to Andy, even by shifting their postural configuration or eye-gaze to him, we have further evidence that they continued to engage in, and to display continued engagement in, the mathematical activity begun in line 35. Finally, toward the end of Andy's utterance, Jason turned his head (slightly) towards Andy (line 47, a), and, in line 48, finally began talking to him.

Once he began to speak to Andy, the character of Jason's gestures changed dramatically. After the change he repeatedly moved his eye-gaze between the graph and Andy (line 52, a& c; 56, a& b; 58, a; 60, d, i, n), made large gestures at the graph with his arms (line 60, b, g, h, j,& m), and of course, talked.

Katherine, on the other hand, still did not turn toward Andy, or respond to him, but maintained a relatively stable set of postures, in silence, for approximately another minute, until addressed directly. At that time, (line 63 and following) Katherine redirected the conversation to the problem she and Jason had encountered in lines 31–34. This section will be analyzed later, so the description here is merely a summary. Nevertheless, the fact that she returned the conversation to the question that immediately preceded her and Jason's shift from talking to the postural configuration analyzed in this section, provides another line of evidence that, as she maintained the posture for that minute, and so continued to display engagement with the mathematics the preceding conversation had made relevant, she was, in fact, still engaged in mathematical activity aimed at resolving the issue she and Jason had encountered in lines 31–34.

The character of Katherine and Jason's silent activities can perhaps, be more clearly seen in the fourth stage of an analysis. This analysis will turn to Jason's interaction with Andy following his talk (lines 52–60), the silence (lines 61–72) immediately prior to Katherine's reentry to the conversation, and Katherine and Jason's discussion prior to their silence (lines 29–34) and just following the end of Katherine's silence (lines 73–96).

### 2.4.3 Surrounding Activities: Stage IV

#### Comparison to another mutual silence: An anxious silence

Following Jason's discussion with Andy (line 60), both Andy and Jason fell silent, both fixed their gaze on a representation of the question (this time, the one inscribed on their paper), and both remained relatively motionless for four seconds (line 61–64). In these three aspects, their actions during that silence were similar to Jason's and Katherine's actions during their silence in lines 35–40. However, they differ in two key respects: First, Andy did not remain silent, but turned to Katherine (line 65, 1) and elicited further speech (line 65). Second, once Andy addressed Katherine, Jason did not keep his eye-gaze fixed on the graph, but turned to Katherine (line 67, a), and then followed her with his gaze as she pointed to and stepped toward the board (line 73, a). In calling on Katherine to speak, and in turning toward Katherine and following her as she speaks, both Andy and Jason treat the silent motionlessness as the aberration. They therefore treated and treat speech, and displays of listening, as the norm. Therefore, in contrast to Katherine and Jason's behavior while they had adopted the silent configuration, in this segment, though the whole group was silent, this silence was not treated as part of a configuration consisting, in part, in silence. Rather, the silence was treated as a temporary departure from a configuration in which the participants take turns speaking.

### Comparison to a listening silence

In this second segment, however, no one talked, whereas Andy talked during Katherine and Jason's silence. Can Katherine and Jason's silence be distinguished from the silence of someone who has adopted a postural configuration that displays orientation to discussion, but who is simply not then speaking? Andy's 40 second silence in line 60, while Jason was talking, provides a helpful contrast. Andy remained silent throughout this segment, and did not attempt to speak, or offer any continuers. However, throughout the segment, his eye-gaze moved back and forth between Jason (notes 1, 6, 8) and the graph (notes 3, 7, 9), and he engages in a number of full-body gestures. His numerous gestures display a different configuration and form of activity that Katherine and Jason employed during their lengthy silence. More importantly, his shifting eye-gaze displayed an orientation toward active interaction with Jason about the mathematics represented on the board.

On the other hand, some of the difference between these two segments of silence may be the result of the people involved in it. Namely, in distinction from Jason and Andy's behavior in this segment, during their conversations, Katherine and Jason directed their gazes to the board far more than to each other. Though they are not silent for lengthy periods of time and so silences cannot be compared, during their conversations, Katherine and Jason responded to each other by nodding (line 96), by changing gaze as their partner points to different locations on the graph (lines 21, a; 25, b), by verbally stating agreement (line 85), and even sometimes with their eye-gaze (lines 84, i; 73, v).

That Katherine and Jason's behavior contrasts during while they had adopted the silent configuration demonstrates that during their silence, Katherine and Jason were engaged together in a distinctive form of collaboration.

This comparison with Andy's silence also helps bring the mutuality Jason and Katherine's silence into sharper focus. As Jason talked to Andy in line 60, both Andy and Katherine were silent. In spite of that commonality, they displayed contrasting forms of engagement with the work of solving the mathematical problem. During line 60, both Jason (d, i, n) and Andy (1, 6, 8) directed their gaze toward each other, but neither directed their gaze toward Katherine. Katherine likewise did not turn her gaze toward Jason or Andy. Furthermore, though both Jason and Andy engaged in a number of large gestures (2, 3, 4, 9, 10, 12; a, c, e, g, h, j, k, l, m), save for a short noticeable inhalation (z), Katherine remained motionless throughout. These factors suggest that in spite the silence, Andy and Katherine were not engaged in a similar activity, still less, collaborating during their silence.

Rather, though Andy was silent and Jason talked, during line 60, both displayed continued engagement in a form of activity that involved only the two of them. Three lines of evidence support this conclusion. First, Jason and Andy displayed engagement with each other by turning their eye-gaze toward the other (d, i, n; 1, 6, 8). Second, in contrast to Katherine, both adopted large gestures. Finally, their contrasting lack of eye-gaze toward Katherine displays that only the two of them were engaged the one activity. These three factors show that Jason and Andy were engaged in one mutual activity, in which one participant was silent and one spoke; but they do not treat Katherine as engaged in the same activity.

Furthermore, Katherine's activity contrasts with Andy's. Indeed, if anything, Katherine continued to adopt the silent configuration. The continued stability of her posture provides one line of evidence Katherine continued to display engagement with the problem she and Jason had encountered ins 31–34. This point will be picked up later.

### Katherine and Jason's solution

Following Andy's request that Katherine speak (line 65), she returned to the conversation (line 73) by asking if everyone agreed they had labeled the graphs "position", "velocity" and "acceleration". (These labels were written on the board.) With her talk, Katherine asked about the subject of her and Jason's conversation in lines 29–34, namely, the labels on the board. Furthermore, though no one disagreed that they had written those labels on the board, because she mentioned "disagreement", her talk made a disagreement over those labels conversationally relevant. During her turn, Jason leaned his head to the left (line 73b), as he had while silent (line 43a). He then acknowledged that graph "a is the derivative of b", and, through "tch hfff" articulating his inability to precede. Katherine's question (line 73) and Jason's response (lines 76, 78) therefore, returned the group's work to the aporia he and Katherine encountered prior to the silence. Katherine continued the line of inquiry by, in the form of question, offering a solution to the problem she and Jason had encountered (lines 79, 81). Two aspects of her question are notable. First, Jason's argument (lines 30–34) had been that their previous solution was impossible because graph c is the derivative of a. Katherine's attribution of the solution to Jason then fits with the trajectory of their argument in lines 30–34. Second, this was the first time anyone had suggested the correct solution. Jason then responded to Katherine's proposal by excitedly noting (lines 84) the features of the graph which support Katherine's suggestion—excitedly, since, in line 84, his speech accelerates and raises in pitch.

Though Katherine did offer the initial justification of this proposed solution, she responded to Jason's justification by nodding firmly, by voicing her agreement, and by suggesting, in line 92 and 94, further justifications.

### 2.4.4 A Model for their Actions: Stage V

Jason and Katherine's maintenance of a stable posture over a lengthy time, their returning to that posture after disruptions, and their maintenance of their postures against outside attempts to change them, together indicate that Jason and Katherine positioned themselves and each other in a remarkably stable configuration (McDermott et al., 1978) throughout this time. That they maintained that posture in the face of pressure from both Andy and the interviewer, strengthens the conclusion that this postural configuration was not accidental, but actively adopted as part of their mathematical activity, and that, in this configuration, speaking was a violation of the norm. While in this configuration, they were motionless, or if they engaged in gestures, they were very small. Their hands remained near their trunk. They often leaned their head to the side. And, finally, of course, they remained silent.

On the other hand, the dissimilarity of Andy's and Katherine's silences in line 60 contrasts with the commonality between Katherine and Jason's postures in lines 35–47. Rather, the contention that Andy's and Katherine's contrasting activities during the silence in line 60 displays different forms of mathematical activity, suggests that the unity of Katherine's Jason's postures displays a common form of activity in a joint project during their lengthy silence.

Two further lines of evidence help to establish this hypothesis, and to show what sort of joint project they were engaged in. First, prior to their silence, Katherine and

Jason both acknowledged that the other had given a plausible reading of the graph (Jason, in line 32; Katherine in line 37). The problem with their work was that Graph c was plausibly the second derivative of b, but could not be the derivative of a. The first result provided confirmatory evidence for their provisional solution, recorded in the labels they gave to the graph. The second result provided contrary evidence. Because their posture eye-gaze continued to be directed toward conversationally relevant mathematics, and because mathematicians adopt a similar posture to display continued engagement with conversationally relevant mathematics (Petersen, 2020b), we can conclude that during their silence they continued working on, and displaying continued engagement with, the resolving their different arguments about their tentative solution.

That Jason so quickly, and in response to Katherine's question, returned to the earlier work is evidence that while silent with Katherine, he was working on the project he had prior to the silence, and that he took her to be too. Furthermore, his audibly excitement is evidence that this solution was new for him, and so, in a real sense, Katherine's. Furthermore, that Katherine responded to Jason's direction of the conversation by immediately offering the correct solution to the problem, indicates that she too had been working on the problem she and Jason encountered prior to the silence, and that she took Jason to be as well. Her decisive nods, and her additional justification of that solution provide further evidence that, during the silence, she had continued to work on mathematics that was relevant to her conversation with Jason. That she attributed the solution to Jason provides further evidence that she treated the resolution of this problem as conversationally relevant to her and Jason's continuing mathematical activity. These points therefore further confirm the claim that, during the period of silence, both Jason and Katherine were working on resolving the aporia, and that they saw and heard the other to be working with them.

This consideration of their conversation preceding and following the silence show that during their silence, Jason and Katherine continued to work on a common project, and to display the commonality of the project through their postures. In short, though the shape of their collaboration changes, they continued their collaboration during the silence. Second, though Jason had ceased collaborating with Katherine while he discussed the question with Andy, Katherine continued their project. Finally, Katherine and Jason resumed their collaboration when she finally broke her silence. At that point, Katherine stated the correct solution (though she attributed it to Jason), and Jason pointed out the relevant features which supported that conclusion, features which Katherine did not consent to as if learning them from Jason, but seemed almost to judge correct. Therefore, not only she continue to collaborate with Jason during the silence, in contrast to what an analysis that left aside silence seemed to suggest, Katherine was in a large measure, responsible for the group's eventual solution.

What of Andy's activity? If Jason and Katherine positioned themselves and each other in the silent configuration, what was Andy doing at the same time? Andy's response to the lengthy silences in lines 36–40, was similar to his response to the lengthy silence in lines 61–64. Following lines 36–40, he initiated a new line of conversation himself; following lines 61–64, he nominated a new person, Katherine, to speak. That is, in both, he nominated a new speaker who had not been immediately involved in the previous conversation. Since he responds to the silence in lines 61–64 as an aberration from a speaking configuration, it is reasonable to conclude his speech in line 41 reflects a similar orientation to the preceding silence as an aberration from a speaking configuration. His subsequent action bears that hypothesis out: During his lengthy talk that followed, he repeatedly solicited attention from the other two students, both by approaching and pointing at the graph, and by directing his eye-gaze toward them. Therefore, though Jason and Katherine positioned themselves and each other in the silent configuration, Andy acted as if the whole group was still in a speaking configuration.

If this reading is correct, then, from Andy's perspective, Jason and Katherine's actions during their silence were an extended departure from the norm: As he did as he listened to Jason, during speaking configurations, listening participants respond to the eye-gaze and gestures of speaking participants, if not in word, then with their eye-gaze. Their failure to do so is tantamount to giving Andy the silent treatment. Though he spoke, they did not respond. Thus, precisely through a miscommunication regarding the use of silence, and its relation to speech, what was, from one perspective a collaborative pursuit of a solution was, from the other perspective, an instance of silencing.

### 2.4.5 Discussion

This focus on silence adds several important facets to an understanding of the group's mathematical activity during this passage. First, without attention to the Katherine and Jason's mutual silence, and Katherine's lengthy continuation of that silence, it would have seemed that Jason was largely responsible for the group's progress, and Andy and Katherine both played minor roles. However, once attention is turned to the silences, we can see that Jason and Katherine did not stop acting when they fell silent, nor did the group, as a whole, consider the questions Andy raised following the initial silence. Rather, though Andy kept talking, Jason and Katherine continued their collaborative attempt to resolve the aporia raised in their discussion preceding their silence. After Jason was drawn into the discussion with Andy, Katherine continued the work she and Jason had been engaged in, and when nominated to speak, she suggested the correct solution, strongly agreed with Jason's assessment, and showed another feature that supports the correct solution. From this perspective, the group's solution is not Jason's solution, mostly independent of the contribution of his peers, but Katherine and Jason's collaborative solution.

Second, and more sinisterly, we can understand a little more about how Andy was silenced. Throughout the interview he was cut off, and not allowed to speak. Even when he "interacted" with Jason about his questions, the interaction was mostly a monologue. When he attempted to voice his questions, however, because of a miscommunication regarding the function of his group-mates' silence, and the sort of activity the group was engaged in, he was given the silent treatment. In this, Katherine and Jason's silence silenced Andy.

## 2.5 Conclusions

This paper drew off microethnography (Erickson, 1996, 2004; Erickson & Schultz, 1977; McDermott et al., 1978) and Conversation Analysis (Liddicoat, 2011; Sacks et al., 1974; Schegloff, 2007) to present a methodology that can be used for the investigation of silence, and different forms of silent activity, in mathematics collaborations. The presentation of this methodology followed and adapted the seven stages Erickson and Schultz (1977) list in a microethnographic investigation. The viability of this methodology was then demonstrated through the analysis of a passage of three students calculus collaborations.

The most salient feature of this paper is a transcription method that can be used to highlight silencess in interaction. This is a needed contribution because existing transcription methods obscure silences ratheer than highlighting them. Furthermore, the solution presented in this paper is a non-trivial extension of transcription conventions not derived from Ochs (1979), and which are not commonly used in mathematics education research literature. This transcription method is then able to aid research at all stages in analysis. In the beginning stages of analysis, the transcript can be employed as a tool to help a researcher overcome the difficulty in attending to silences, and as a physical outline of the flow of silences in an episode. The transcripts can also be used to record and organize analytic notes on the episode. This technique allows the researcher to see the temporal relations between different analytic notes, and how they relate to various silences. Finally, as the researcher identifies passages for more in-depth analysis, the exact timing of gestures and postural shifts can be transcribed in considerable detail. The detail of these transcripts then allows a researcher to carefully distinguish between types of silences, to draw conclusions about the different silences participants employed intheir mathematical activity, and to analyze the responses of all the group-members to these different silences. Furthermore, they allow the rigorous communication of these silences in a format that is suitable for publication in print.

The results in the final section show the power of the methodology presented in this paper in distinguishing between different kinds of silence. This episode contained Katherine and Jason's lengthy "thinking" silence, as well as several of Andy's silences. The character and function of Katherine and Jason's silence was rigorously distinguished from the character of Andy's silence. Therefore, by following the methodology established in this paper, an analyst can rigorously distinguish between active, collaborative silences, like Katherine and Jason's, and listening silences, like Andy's. Additionally, Andy responded differently to Katherine and Jason's silence than either of his peers did. The ability to methodologically distinguish their responses illustrates the power of the methodology presented in this paper to analyze students' different responses to various silent activities the students engage in. These tools therefore, can be helpfully employed in furthering mathematics education research. Furthermore, the analysis illustrated how the stages of a microethnography (Erickson & Schultz, 1977) can be applied to an analysis of silent activity.

Though it is not the topic of this paper, quantitative analyses of silence may be possible using time series analysis (Lütkepohl, 2005). These methods have proven fruitful in analyzing similar forms of activity quantitatively. For instance, after using motion-tracking technology to obtain a time-series of their limb movements, Chang et al. (2017) and Chang, Kragness, Livingstone, Bosnyak, and Trainor (2019) used Granger causality to examine the ways members of professional string quartets coordinate their behaviors and music, whereas other researchers have examined entropy in forms of bodily collaboration (Glowinski et al., 2013). It may be possible to use these methods to examine the physical aspects silent thinking in mathematics quantitatively. For instance, researchers perhaps use motion tracking technology to create a time-series of limb-movments, and could then use forms of time series analysis to determine how often students engage in various forms of silent activity, and how their peers respond to them.

# 3

# Silence in the Collaboration of Students Enrolled in Introductory-Level Proof Classes

> [C]ommunicative behavior consists of both sounds and silences, and. . . adequate description and interpretation. . . of communication requires that we understand the structure, meaning, and functions of silence as well as of sound. (Saville-Troike, 1985, p. 4)

Niss (2007) concluded his analysis of the state of mathematics education research by claiming that the mathematics education community may benefit by focusing more on silent contemplation and less on speech. His focus is on the practice of mathematics education researchers, not students and classrooms; but his comment is suggestive for the mathematics classroom: Could our classrooms benefit from attention to careful, silent contemplation? This refrain was recently taken up by Lim (2017) who argues that, though it is important for students to have a voice in the classroom, Lim more attention could be paid to introverted students who value silence and careful thought, and the challenges they face in thinking carefully, in silence.

Petersen (2015) and Petersen (2018a), both analyze the same passage and contain a very brief case-study of naturally occurring student interaction that at least partially verifies Lim's concerns: In the episode, two students who had been debating proposed solutions to a calculus problem suddenly stopped talking, and stared together at a problem they had written on the board. It seems that, for these two students, the silence was an aspect of their agonistic mathematical collaboration. A third student, however, seems to have read the silence as a lapse in the conversation that needed filled, and persistently worked to engage his peers in conversation not immediately connected to the preceding debate. He eventually succeeded in engaging one of them, though the other remained silent for over a minute before returning the group to the line of inquiry they had been pursuing before two of them fell silent. In this short episode, we see students working to debate and verbally engage each other; but we also see the same students attempting, with partial success, to temporarily set aside their discussion and work in silence. Furthermore, we see how interactively challenging this setting aside debate for silence can be: As Petersen (2018a) argues, there is some evidence that, for the two students who attempted to work silently, the third student's attempt to engage them was an interruption, whereas, their attempt to work silently silenced him. Petersen (2015) and Petersen (2018a) suggest that a sort of silent contemplation may indeed be an important aspect of student mathematical activity, but that it is communicationally challenging in a discursive setting, both for the students who would be silent, and for the students who would talk.

These hypotheses are supported by three further lines of evidence. First, Petersen (2017a) argues that, in contrast to every-day conversation, in which silences are minimized (Stivers et al., 2009, see also, Erickson, 2004; Liddicoat, 2011), in face-to-face collaboration between mathematicians, extensive silences are normal. That mathematicians seem to employ extensive silences suggests that activities involving silence may be an important for giving mathematicians the ability to engage in mathematical activity, and so, that these activities are well-adapted for doing mathematics at all levels. Second, Petersen (2018a) gives a breif anecdote: During a tutoring session, he realized that a solution he had worked out in advance was incorrect and to fix it, he needed to pause and think in silence. The students, however, like the third student mentioned above, read that silence not as an aspect of mathematical work, but as a lapse in conversation and repeatedly attempted to engage him in conversation. In that situation, Petersen found it impossible to engage in the required mathematical activity; and in order to help the students, he had to intentionally create space for silent thought. Finally, in the process of recruiting groups of students to participate in the study reported in this paper, students in introductory proof classes were asked whether they preferred to work alone or in a group. Out of 72 students surveyed, a plurality claimed they preferred or strongly preferred to work alone (43.1%, as opposed to 38.9% who preferred or strongly preferred to work in a group). Of those that preferred or strongly preferred to work alone, 35.5% claimed that a reason for their preference was the ability to concentrate in quiet. One student's comment is particularly articulate: "Proof problems require me to focus uninterrupted for fairly long lengths of time and I find that working in groups is not conducive to my success." These preferences, and the student's comment, are further evidence for Lim's (2017) claim that silent contemplative aspects of mathematical activity, and of proof related activities in particular, may be important to include in the classroom, and important avenues of mathematics education research.

None of these claims should be taken to suggest that students should not be collaborating in or out of the classroom, or that students should be reduced to silence. They do, however, suggest two important hypotheses, both of which fit with existing directions in research in mathematics education. First, given this last student's claim, and Petersen's (2017a) finding that mathematicians utilize silence extensively, even in their collaborations; attending to the ways students use silence in their mathematical activities related to proof may lead to a fruitful proof intervention, (see Stylianides, Stylianides, & Weber, 2017 for the importance of proof interventions). This sort of intervention may have a secondary impact of helping students, like those cited above who prefer solitary concentration to group activity, to better participate in courses that require extensive student discussion. Second, the difference between the norms of conversation and mathematical collaboration, as well as the difficulty both the students analyzed in Petersen (2015) and Petersen (2017a), as well as Petersen himself (2018a), had engaging in needed silences while working in a group, suggest that apprentice mathematicians need to learn the peculiar discourse conventions that govern face-to-face communication in the mathematics profession (for research on discourse conventions of mathematics, see, Lew & Mejía-Ramos, 2019).

This leads to the research question for this study: How do students, when collaborating on their homework, employ silence, in ways that are counter to the norms of every-day discourse, and how do those practices affect the group dynamics and their ability to make progress in discovering and writing their proofs?

## 3.1 Literature Review

Lingard (2012) argues that silence is an unexplored theme running through and connecting much medical education research—e.g. an oncologist who gives a terminal cancer diagnosis may need to use silences to build empathy and solidarity with their patients and their families, but avoid using silence to alienate the grieving. Though it isn't a theme running through mathematics education research, silence has nevertheless been an unexplored background presence in a number of classic studies. One paper with the notable presence of silence is Steffe (2003), a study of the development of the fractional schemes of two fifth-graders, Jason and Laura, as the teacher/researcher implemented 15 research "protocols" over the course of a year. Though it receives minimal explicit comment, silence shows up throughout the paper. For instance, in the first protocol, Jason and Laura respond to their teacher's instruction by "sitt[ing] quietly for approximately twelve seconds" (p. 243) before beginning to use manipulatives. Though the transcript is a little ambiguous, two turns later, they seem to again sit quietly together for another 15 seconds, without interacting with manipulatives. Later in the same protocol, they respond to further instruction with an additional ten seconds of silence. Finally, during the second-to-last protocol (which Laura was absent for) Jason is silent and seemingly motionless for 25 seconds, and, on his next turn, is silent and seemingly motionless for 20, 50(!), and finally 10 seconds. Silence is also present in several other important articles: The teacher/researcher in Olive and Steffe (2001) waits for a detailed student response for 25 and later 10 seconds. Tzur (1999) (analyzing the same class as Steffe, 2003) records an instance during which a student waited approximately six seconds before answering a question, in a later episode two children are silent together for about 15 seconds before the teacher/researcher restarts the discussion with the children, and a still later episode contains student silences of 15, 5 and 6 seconds in close succession. Though Steffe (2002) does not mention the duration, he records that two students, Jason and Patrick, "sat in silent concentration" (p. 271). And, from a very different perspective, Radford (2010) argues that a 2.5 seconds of silent, motionless eye-contact between a student and a teacher was a critical aesthetic experience that helped open the student to "a new way of seeing" mathematics (p. 6).

One tradition in the education literature that has given sustained attention to silence is teacher wait-time (e.g., Tobin, 1986, 1987; Tobin & Capie, 1983; for a recent mathematics education example, see Paoletti et al., 2018). Researchers working in this tradition timed pauses following both teacher questions and student answers. They found large gains to student contributions to the classroom when there were relatively long pauses following both teacher and student speech (Tobin, 1986), and recommended that, when teachers want to include complex cognitive questions in their classroom, they employ wait-times between three and five seconds (Tobin, 1987). Researchers further theorized that the mechanism behind the gains was that students' mental circuitry needed a relatively long time to process the information in the conversation, and so when a teacher spoke too quickly, students were not able to contribute as much as they could have (Tobin & Capie, 1983). More granular research on classroom interaction challenged this theorized information processing mechanism and provided a more precise picture of how teacher wait time influences classroom discourse: Ingram and Elliott (2014, 2016) found the conversational norm that silences are minimized, and are often heard as a negative evaluation of the preceding turn in talk, were operative in the classroom, and so students heard the teacher's silence as a negative evaluation of the their short, not as high-quality-as-would-be-preferred, answer, and so, as a prompt for further elaboration. Ingram and Elliott hypothesize that therefore lengthy wait-time could be useful to establish classroom norms of participation and answer quality; but that after those norms have been established, long wait-times following student answers would likely still be heard as a negative evaluation of the (now high-quality) answer, and so may even be counter-productive.

That a more granular analysis was able to challenge the theorized mechanism behind the gains in wait-time, and give a more nuanced treatment of wait-time in the classroom shows that if silence is to be a topic of research, more granular methodologies for representing silence than those employed in the wait-time literature are required. Nevertheless, wait-time research remains the only literature we are aware of to give sustained attention to silence in the mathematics classroom.

Silence itself, however, seems difficult to study in more detail, for two reasons: First, silence seems to be the lack of speech or of sound and not a phenomenon in its own right—and if it is not a phenomenon in its own right, how can we attend to it? Second, silence does not regularly signify anything mathematical, and if it does, as in Radford et al. (2007), it only does so accidentally. It seems therefore, silence should be addressed when it happens to come up, but should not be a topic of research in its own right. This section will begin to address these two issues, the theoretical perspective, section 3.2, will address questions of why silence could be a topic of research in more detail, and methodological questions will be addressed in section 3.3.2.

### 3.1.1 Silence

Just as the black of letters on a white page and the darkness of night are not a mere lack of visual stimulus; silence is not a mere absence or a lack of auditory stimulus (Chrétien, 2004), but a phenomenon, actively heard with our ears, that both frames sounds and words, and is in turn framed by sounds and words (Acheson, 2008; Ephratt, 2011). So, for instance, Handel often underscores dramatic moments in his music with lengthy silences (Harris, 2005), which are heard as dramatic parts of the music because of the notes that surrounds them (see also, Kim, 2013). On the other hand, we only recognize the unity of the performance of a piece of music because the sound is bookended by silences (Dauenhauer, 1980). These two examples show further, that color and meaning of a silence is dependent on the posture and gestures employed during it and the particular sounds that bracket it (see also, Acheson, 2008; Margulis, 2007a, 2007b). Finally, silence is not a default state—a mere lack of vocal production—but is actively produced. Thus, for instance, silence can be a design feature of buildings (Bonde & Maines, 2015; Ergin, 2015; Kanngieser, 2011; Meyer, 2015); and we are all familiar with how difficult it can be to hold our tongue.

That silence is a multivalent phenomenon, at times difficult to produce, implies that when a student suddenly breaks off her, or his, speech with a peer, and silently stares at the board (see section 3.4.4), she, or he, is not simply ceasing to speak, but adopting a particular activity aimed (presumably) at achieving a mathematical goal, and a particular social practice, whose significance his, or her, peers need to read. Similarly, when a student stops talking and reads from the textbook but remains physically present (see section 3.4.1), he, or she, is adopting a particular stance toward his, or her, peers, a stance that they need to read and respond to, and that presents its own peculiar interactive challenges.

In every-day Anglo-English conversation silences longer than a second are minimized and marked (Erickson, 2004; Liddicoat, 2011). That they are minimized means that in every-day conversation, maintaining a silence for longer than a second violates the conversational norms, and so if the conversation lapses for over a second, one of the participants in the conversation will quickly start speaking. That they are marked implies that they often bear semantic information: For instance, following an invitation, a negative response is often delayed, and the delay itself heard as a negative response (Liddicoat, 2011). Similarly, in a classroom, a common form of interaction is that, following a student's answer to a teacher's question, the teacher evaluates a student answer (Sinclair & Coulthard, 1975). In this context, a delayed teacher evaluation of a student response is often heard as a negative evaluation of the student's answer, causing the student to further elaborate the answer (Ingram & Elliott, 2014, 2016).

In order to decide between two competing theoretical explanations for the Anglo-American English norm that responses follow less than a second of silence, Stivers et al. (2009) carried out a large-scale cross-cultural replication of the result that silences are minimized. The two hypotheses Stivers et al. sought to judge between were, first, that this norm is based on particular cultural or linguistic features of Anglo-American English; and second, that it is based on the structure of the human language system itself. These two hypotheses make different predictions regarding the cultural variation in delay following a question, and a cross-cultural analysis of delay following a question can help decide between these hypotheses. Stivers et al. collected a corpus of recordings of conversation from a variety of cultural-types and language-types, and measured the delay between "yes-no" questions and answers. Yes-no questions were selected to control for differences between question types in the different corpora. They found that though there is some cultural variation in response time to "yes-no" questions, that variation was on the order of tenths of seconds, and, in every language, a response could be expected within a second of the question. These results provide strong evidence against the hypothesis that the norm that wait-time following a yes-no no question is shorter than a second is a cultural phenomenon, and in favor of the hypothesis that the norm is based on innate biological features of the human language system, which is to say, that the norm is a cultural universal. They furthermore verified, using a corpus of Dutch conversations, that there is no difference in response time following questions and following non-questions. This last result suggests that delay-time following a yes-no question is a valid model of delay-time more generally, and so that their results can be generalized to response time in conversation. To summarize their results: Though there are some specific social interactions in which longer wait times are expected (see e.g., Basso, 1996; Lippard, 1988), and though there is some small variation in response time between cultures, the norm that, in every-day conversation, silences longer than a second are minimized seems to be based on biological features of the human language system, and so to be a cultural universal.

In contrast, in mathematical collaborations, lengthy silences (in excess of 15 seconds), even following questions, are in accord with interactional norms, and may even be common (Petersen, 2020b). Employing a novel perspective as thought as an embodied and social activity, Petersen examined the interactional norms that allow mathematicians to engage in lengthy periods of silent thought during mathematical conversations. He found that lengthy silences (he recorded silences in excess of 20 seconds) were normal at all times in mathematicians' conversation, including following yes/no questions. For instance, Petersen (2018b) presented an example from two mathematicians' collaboration in which one mathematician's question is followed by, in turn: 15 seconds of silence, the other mathematician asking a question (without response), 11.6 seconds of silence, the first mathematician saying "hm!" and finally another 7.5 seconds of silence. Since this episode contains pauses of 15 seconds and 11.6 seconds following questions, and no attempt to address the violation of a norm, this episode gives evidence that the norm that responses follow less than a second after a question was not operative in this collaboration. Furthermore, in that episode, the mathematician who asked the first question seems to have been actively seeking to have his peer notice a that their mathematical work had run into a problem, and a silent response to his question. These results show that, in contrast to everyday conversation, lengthy silences are normal in mathematicians' collaboration, and furthermore, that mathematicians treat these silences as an important aspect of collaboration.

That these silences occured during conversation, and following questions, raises the question of how the continuity of a conversation is maintained across these silences. In ordinary conversation, people use features of their talk to organize the conversation into a coherent whole, and to establish a common topic. It is therefore unclear how a conversation can maintain a common focus across lengthy silences. Petersen (2020b) found that mathematicians maintain a coherent conversation through their posture. Specifically, during these thinking silences, mathematicians remained (nearly motionless) and fixed their gaze on the board, and so on a record of their conversation.

There is some evidence that mathematicians and seek to model this sort of silent activity for students: Canagarajah (2018) found an undergraduate mathematics teacher, Tan, who, during his lectures, modeled extensive silent attentiveness to mathematics he had written on the board. Canagarajah does not describe this in more detail, but an example can be found in Looney, Jia, and Kimura's (2017) analysis of the same data: After erasing the board while looking at his notes, Tan lifted his gaze to the remaining board work and remained silent and seemingly motionless for five seconds, he then raised his left hand to his chin, and remained silent for an additional second before beginning a new episode in his lecture.

Petersen (2020b) analyzed this these silences as part of the embodied, social, activity of thinking, without reference to what was occuring in the mathemaicians' heads. Inspite of this precedent, in this study the language of "thinking" is avoided, for two reasons. First, "thinking", understood as an internal process has been extensively studied in the mathematics education literature. Even Sfard (2008a) defines thinking as an interior process—the novelty of her perspective lies in treating it as an internalized form of conversation. But Petersen (2020b) required bracketing any processes internal to the mathematicians' and attending only to external, social aspects of thought. The analytic focus of (Petersen, 2020b) was on interactional norms, not the specific activities mathematicians engaged in. This focus allowed thinking to be a background presence in an analysis that was clearly focused on interaction. The research question in this paper, however, students' silent activities. For this reason, identifying student one kind of activity with thinking would put the analytic focus on student thinking—but not on "student thinking" as the phrase is usually used in the mathematics education research literature. For this reason, it was decided that the any theoretical pay-off of "thinking" would be outweighed by the confusing use of the term. Second, the research questions in this study require that student actions be specified as precisely as possible. But I know of no empirical work that shows that, from an embodied and interactional perspective, thinking is a single species of activity rather than a genus of related activities. This means that "thinking" is not a precise enough term for a study seeking to specify particular activities. Indeed, using "thinking" in this study would risk creating two kinds of theoretical confusion down the road. 1) Disparate activities could studied under a single term, and so conflated. 2)

Interesting kinds of thinking activities could be excluded from further analysis as "not thinking". On the other hand, becaue the focus in Petersen (2020b) was on norms, not activities, there was no need to precisely specify mathematicians' activities, and a generic term was sufficiently precise.

Nevertheless, Petersen's (2020b) and Canagarajah's (2018) findings suggest two important results for undergraduate mathematics education: First, as they are apprenticed into the mathematics profession, students learn to employ novel forms of thoughtful silences, in their collaborations, and to respond to these silences; and some mathematicians seem to intentionally work to initiate students into these distinctive activities. These thoughtful silences likely involve relatively little motion and a gaze fixed on a written mathematics. Second, that mathematicians use extensive silences in their collaborations, and, at least sometimes, model them in their lectures, suggests that employing particular sorts of silence during collaboration can be an effective means of solving a mathematical problem, and so one that students could do well to employ. Petersen (2015) contains a brief analysis of two calculus students' use of silence, but we are aware of no article-length studies of silence in students' mathematical activity.

### 3.1.2 Proof

Activities related to proof form an important cornerstone of mathematicians" work, and students' apprenticeship into proving and/or justifying activities is an important goal of the mathematics classroom, throughout the curriculum (Dawkins & Weber, 2017; Staples, Bartlo, & Thanheiser, 2012; Stylianides et al., 2017). Nevertheless, activities surrounding proof and justification remain major challenges to many students, even in dedicated proof classes (Selden & Selden, 2007), and both Stylianides and Stylianides (2017) and Stylianides et al. (2017) call for the development of interventions to improve student proof practices. Inasmuch as it is normal for mathematicians to use silence extensively in their activity, even in their joint activity (Petersen, 2017a, 2018b), it seems reasonable to hypothesize that the sorts of silences mathematicians' employ in their joint activity are effective in bolstering their capacity to realize the affordances for proof present in written or oral representations of mathematics (for discussions of affordances, see, Baggs & Chemero, in press; Chemero, 2009; Gibson, 1979/2015, for seminal mathematics education papers that address affordances, see, Greeno, 1994; Greeno & MSMAP, 1998, for more recent papers, see, Abrahamson & Sánchez-García, 2016; Fukawa-Connelly & Newton, 2014; Gresalfi, 2013; Gresalfi, Barnes, & Cross, 2012), and that therefore, an intervention that seeks to determine how these practices can inform student proof and justification activities may be helpful. Such an intervention would have two important goals: First, helping students to practice a sort of activity that is hypothesized to be helpful in discovering proofs, and second, helping them to respond appropriately when their peers engage in that activity.

Though such an intervention would not touch on the mathematical aspects of proving activity, there is some precedent for an intervention that addresses important bodily components of proof activities that do not themselves symbolize mathematics. Specifically, Savic (2015) found that mathematicians overcome impasses in their proof productions by stepping away from mathematics, and raised the prospect of interventions aimed at teaching students to employ similar practices. Like the interventions Savic discusses, an intervention focused on aspects of mathematical collaboration conducted in silence would aim to help students adopt practices that, though not explicitly mathematical, are helpful in providing the material means for mathematical agents to engage in overtly mathematical practices.

In most courses students are not apprentice mathematicians (Staples et al., 2012). In introductory proof courses, however, one of the goals of instruction is the apprenticeship of students into the disciplinary specific practices of the mathematics community—whether the student's trajectory leads toward more central participation in the mathematics community through original contribution to mathematical scholarship (Dawkins & Weber, 2017; Greiffenhagen, 2008; Johnson, Caughman, Fredericks, & Gibson, 2013), or through teaching secondary or community-college mathematics (Even, 2011; Wasserman, Weber, Fukawa-Connelly, & McGuffey, in press). For these students, initiation into the sorts of silences mathematicians employ in their proving collaborations is hypothesized to be an important aspect of their apprenticeship into the mathematical community. Specifically, apprenticeship into the linguistic conventions of a community is an important aspect of an apprenticeship (Lew & Mejía-Ramos, 2019), and if, as Petersen (2017a, 2018b) suggest, mathematicians utilize peculiar linguistic conventions for silence in their collaboration, then an important aspect of students' apprenticeship into the mathematics community is learning to utilize and respond to silence, as mathematicians do.

This research is in its infancy, and an proof intervention based on silence may not prove effective. Nevertheless, because mathematicians employ extensive silences in their collaborations (Petersen, 2017a, 2018b), it is reasonable to hypothesize that these practices are helpful in allowing mathematicians to do mathematics. Therefore, it is further reasonable to hypothesize that an silence-based proof intervention could be helpful for students. Similarly, though we have reason to think that in their collaboration, students' use of silence will follow different linguistic conventions than mathematicians does, we have little knowledge of their apprenticeship into the uses of silence in mathematicians collaborations. This paper therefore advances existing scholarship in two directions: First, it makes the first steps toward laying a groundwork for a novel intervention into student proof-related activities. Second, it begins to lay the groundwork for an investigation into students' apprenticeship into the peculiar conventions of the use of silence in the mathematics community.

## 3.2 Theoretical Perspective

This study adopts a sociocultural perspective toward learning mathematics (Lave & Wenger, 1991; Rasmussen, Zandieh, King, & Teppo, 2005; Sfard, 1998), and drawing from Mauss (1934/1973), employs a novel embodied perspective on mathematical activity. Following a sociocultural perspective, mathematics is not understood as a body of knowledge, but as a form of labor with distinct goals, activities, tools and behaviors asociated with it (Roth & Radford, 2011). Indeed, from this perspective, because different have used mathematics, have put it to different ends (Nasr, 1968), and have employed different technologies and practices in their mathematical activity (Greiffenhagen, 2008), it may not be accurate to refer to mathematics as a distinct form of activity, but to various ethnomathematics, including the one studied in this paper, formal academic mathematics (Pinxten & François, 2011).

Because mathematics is not a body of knowledge, but a set of practices, from this perspective, learning is not treated as the acquisition of concepts, but as the apprenticeship into communities, and so as learning to participate in a variety of culturally given practices and activities (Sfard, 1998). The process of apprenticeship and movement toward more "full participation" in a community is characterized through what Lave and Wenger (1991) term "legitimate peripheral participation", namely, the performance of activities that are connected to their community's practices, and which serve of as the means for moving toward more full participation. For example, upper-division and graduate mathematics students are apprentice mathematicians, and the mathematical activities students engage in as they complete of proofs on their homework are a form of legitimate peripheral participation in the mathematics community. On the other hand, mathematicians' proof productions are a form of full participation (Greiffenhagen, 2008).

Following Rasmussen et al.'s (2005), seminal paper, the term "mathematical activity" has been employed to refer to the sorts of activities they enumerate, (e.g. symbolizing, etc.). Rasmussen and colleagues, however, adopted the term "activity", which earlier seminal papers written from a sociocultural perspective had used to describe the practical consequences of treating learning as participation (e.g., Lave & Wenger, 1991; Sfard, 1998, 2001), to signify a shared theoretical perspective with these earlier authors, and do not explicitly connect "mathematical activity" to the sort of activities (e.g. "'justifying" and "symbolizing") they consider in the paper. "Mathematical activity" can, therefore, be more broadly employed to describe the actions of people legitimately participating in the mathematics community, whether peripherally or fully (as it has in e.g., Cobb, 1995; Gainsburg, 2007; Greeno, 1997; Meira, 1998). For instance, Savic (2015) found that in order to overcome a proving impasse, many mathematicians will sometimes intentionally step away from their work, and, for instance, go to lunch, knowing that this break from the problem may help them find a solution when they return. Because stepping away from their work is an aspect of their proving activity, these mathematicians could be described as engaging in a proving mathematical activity even when they step away from mathematics. Likewise, in this paper, when people are legitimately participating in mathematics, they will be described as engaged in mathematical activity.

This sociocultural perspective is augmented by a embodied construct drawn from anthropology (Mauss, 1934/1973, 1926/2007, see also Crossley, 1995, 2007), "techniques of the body". Though the construct is new to the mathematics education literature, researchers from disciplines like sports science (Sánchez García & Spencer, 2013), orality (Saussy, 2016), and ritual studies (Mahmood, 2001) have helpfully employed Mauss' construct in examining the way the body is utilized in the various activities they studied.

A technique of the body is defined as a traditioned way of utilizing the body that is physically and psychologically important for engaging in particular activities or accomplishing particular ends. For instance, the specific set of actions that constitute a particular swimming technique like the Australian crawl or the breaststroke constitute a technique of the body, as do the specific breathing techniques, postures, and motions tai chi practitioners employ (Brown & Jennings, 2013), and the particular postures and practices enjoined by the Shari'a like times and postures for prayer, or wearing a veil (Mahmood, 2005). These latter two examples raise intriguing questions for research into the ways mathematical activity is embodied: Embodied research in the mathematics education literature focuses on meaningful gestures and the interplay between material artefacts and the development of mathematical meaning (Radford et al., 2017). But for at least some practitioners of tai chi and some Muslims, these techniques are not employed as meaningful symbols, but as the means for cultivating inner acts like self-control, and inner peace (for some practitioners of tai chi), or proper orientation toward God and neighbor (for some Muslims). These examples raise the prospect of a shift in perspective from gestures agents employs to symbolize mathematics, to postures which an agent inhabits, and reflect—or better, are—her orientation toward mathematics and her peers (Saussy, 2014, 17:00–17:30; see also Schegloff, 1998): Postures, that is, that an agent employs not for their meaning, but as the physical means for engaging in mathematical activity. This finding shows that Since silence can be theorized as an aspect of posture (Petersen, 2018b), this perspective is well-suited for a study of silence.

This is a novel approach to embodiment in mathematics education, however, there are significant results from Machine Learning that illustrate ways posture is an aspect of students' orientation to mathematics. For instance, Grafsgaard et al. (2013) found that when students interact with an automated tutor, mouth dimpling positively predicted active learning and mental effort, whereas Grafsgaard et al. (2014) found that sitting with face close to the computer screen is predictive of both engagement and frustration, but sitting far from the screen may be an aspect of satisfaction with the achievement of a goal. These results are not theorized, so it is unclear exactly why students dimpled their mouth when exerting mental effort, or sat close to the screen when actively engaged. Rather, these examples show that particular postures are an aspects of different mathematical activities, and that these postures are legible to social-scientific methodologies as aspects of those activities.

Though Mauss (1934/1973) does not discuss ways techniques of the body, and in particular, postural techniques, are employed in social situations, the concept can be usefully expanded for an analysis of the various ways people interact (Crossley, 1995; Schegloff, 1998). Social behavior is governed by various normal or conventional ways of acting and holding the body, which Voigt (1995) called "patterns of interaction". That behavior is governed by these norms implies that social agents generally uphold them, and furthermore, are expected to uphold them. This regularity of behavior therefore gives agents opportunities for acting on each other's actions. For instance, in his analysis of ways pedestrians, drivers, skateboarders and cyclists cross a busy intersection at the University of Oregon, Liberman (2013) found that people aiming to cross at busy times will skillfully adopt postures in order to cause other crossers to yield. For instance, while attending peripherally, skilled crossers will sometimes look away from oncoming traffic so that other crossers, all of whom follow the norm that collisions should be avoided and crossers should yield to people who cannot see them, will make space for them to cross.

McDermott et al. (1978) show that norms which facilitate joint action not only govern bodily, but the postural configurations adopted in those actions. Specifically, participants in joint activity adopt postural configurations which serve to inform each particpant about the nature of the interaction, in the moment, and which are therefore normatively enforced. Furthermore, these postures are used to position each member of the interaction in a particular configuration. For this reason, the postural configurations are not only normatively enforced, but are a means of enforcing norms. So, for instance, McDermott et al. show that the behavior and postures of a seemingly chaotic group of elementary students is in fact coordinated and rule-governed. In an episode they analyzed, a teacher calls on a particular student seated at a table with several of her peers the teacher, to read. Following this action, several of the students turn away from the table, and the teacher uses both her words and her posture to call their displayed attention back to the table. As her peers moved their gaze around, the first student sat ready to read, with her book open, thus displaying and positioning the other students so their inattention was inattention *to her impending reading*. Finally, as the teacher organized the students, the reading student fixed her eyes on the teacher until the teacher gave her a subtle sign to begin reading.

Though numerous aspects of posture were at play in this example, one of the most significant was the eye-gaze of the teacher and the students, since, through their gaze, they displayed the direction of their attention. Research on eye-gaze can help fill out another important communicational feature of eye-gaze: itss function in soliciting a response (Gobel, Kim, & Richardson, 2015). For example, in American English conversation, when a speaker's gaze is directed toward her conversation partner, a response usually comes in about 200 ms, as opposed to 400ms when a speaker's gaze is directed elsewhere (Stivers et al., 2009). More generally, during some activities like eating, silence may be normatively permitted. During these activities, when one party wishes to initiate speech, simply talking may not be enough. In that situation, eye-gaze is one resource speakers utilize to solicit speech (Stivers & Rossano, 2010). Indeed, eye-gaze alone can solicit speech, even during silence (Rossano, 2013).

There is little precedent for a study from an embodied perspective that attends to aspects of mathematical activity that are not overtly connected to mathematical meaning. For instance, all the studies Radford et al. (2017) survey relate immediately to mathematical meaning.There is precedent, however, for research from a sociocultural perspective that attends to aspects of mathematical activity that, like silence, are not overtly mathematical. First, in their attempt to understand why Ari and

Gur's collaboration were not successful, Sfard and Kieran (2001; see also, Kieran, 2001; Sfard, 2001) turned their attention to non-mathematical aspects of students' joint activity that make communication ineffective or effective. To this end, they developed a methodology, *preoccupational analysis*, which abstracted away from the mathematical content of student discourse, and instead, represents who an utterance is directed toward, and whether a utterance is oriented to mathematical objects. Ryve (2004, 2006) extended this methodology to include information about types of mathematical discourse and used it to analyze linear algebra students' discourse. More recently, in their analysis of a group of students working on a probability worksheet, Liljedahl and Andrà (2015) extended this methodology by incorporating information about eye-gaze (see also, Andrà & Liljedahl, 2014). Though preoccupational analysis does not seem helpful in studying silence, these papers demonstrate the importance of attending to aspects of joint mathematical activity that, like silence, influence student ability to engage in mathematics, but that are not themselves overtly mathematical.

Lerman's (2001) sociocultural research provides a second precedent for attention to aspects of mathematical collaboration that are not overtly mathematical. He argues that that working from a sociocultural perspective "requires examining the resources. . . that the teacher, texts, peers and others supply as well as the ideas that emerge in joint activity" (p. 102). Though his research attends to aspects of classroom discourse like gender, race, and power, which are important for very different reasons than silence is, this quote provides further justification for studying aspects of mathematical activity, like silence or Savic's (2015) stepping away from mathematics, that are not overtly mathematical, but that students and mathematicians employ to give themselves the resources to undertake overtly mathematical aspects of mathematical activity.

Finally, Lew and Mejía-Ramos (2019) investigated linguistic conventions that mathematicians use in writing and evaluating student proofs. They found that mathematicians expect proofs to be written in an academic register, to follow rules of English grammar like proper capitalization, to use and introduce variables and other objects properly, and that the context of a proof affects its formality. On the other hand, many students were unaware of these conventions. Because proofs that do not conform to these conventions are not seen by mathematicians as good proofs, knowledge of these conventions is an important aspect of learning to be a mathematician. Nevertheless, these conventions are not directly related to mathematical meaning: Indeed, students could in theory produce attempted proofs that contain formally valid justifications of a theorem, and that evidence strong conceptual understanding, but that violate all these conventions.

There has been little research into the techniques of the body employed in the mathematics profession, one example can be found in Savic's (2015) investigation of how mathematicians overcome a proving impassé (for another example, see Ochs, Gonzales, & Jacoby, 1996). Savic found that, in order to overcome a proving impasse, mathematicians will intentionally step away from that mathematics, with the knowledge that when they return, they will be able to see possibilities they cannot at the time. Because this is a bodily technique mathematicians engage in as part of their mathematical practice, so that they have the capacity to engage in proving activities, it is a technique of the body, in Mauss' sense. Similarly, the actions of the mathematicians Petersen (2017a, 2017b, 2018a, 2018b) analyzed seem to be part of a technique of the body endemic to the mathematics profession, in two different ways.

First, the silences are aspects of particular postures which the mathematicians seem to employ as the bodily means for engaging in kinds of mathematical activity. Second, the mathematicians exhibit a technique of the body in their embodied responses to their peers silences.

## 3.3 Methodology

### 3.3.1 Data Collection

When collaborating on homework, student work is free from the background classroom noise that is present in small group classroom discussions, and there are only a small number (2–4) of students involved. These two aspects of their homework collaboration make their collaborations ideal for studying the use of silence in student mathematical collaborations.

Groups of students that regularly collaborated on their homework were recruited from junior level, proof-based, mathematics courses at a large research university in the Western United States, to have their collaborations video-taped. As compensation for each hour of their time, students were offered a half-hour of tutoring, immediately following their video-taped collaborations. Several students commented that the found the compensation helpful for their classwork. Three groups participated in the study: 1) “The Geometry Group: A group of 2 to 4 students (group size varied depending on the week) from a ”Modern College Geometry” class, whose course description lists topics in Euclidean and non-Euclidean geometry. This group participated for an hour on each of four separate weeks. Two students, Sean and Mary, participated every week, but were sometimes joined by Alan and Isaac.[1] 2) “The Algebra Group”: A group of four students drawn from two ”Intro to Group Theory” sections with the same instructor and homework assignments, Sam, Nick, Amy and Olivia who participated for an hour on each of two separate weeks. The students organized this group, so it is likely they were all from the same section, however, no data was collected on which section the participants were from. 3) “The Analysis Group”: A group of two students, from the first term of a two term ”Introduction to Mathematical Analysis” sequence, Seth and Hannah, who participated once, for an hour. Seven sessions were recorded, totaling approximately seven hours. The analysis and algebra classes were lecture-based, the geometry course was a reform course.

Because these students were all at least Junior level college students, they had been working homework, and likely collaborating on it, for approximately 15 years. This extensive experience means that, though the students are, in some sense, novice mathematicians, learning the discipline-specific behaviors mathematicians employ in their work, they are simultaneously experts at homework collaboration.

Since the goal of this study is to investigate student collaboration, as it exists, not as it could be given particular teaching moves, during the recording sessions, the researcher attempted to keep his interaction with the students to a minimum. Students were asked to work on a white-board, and generally did so without interruption. Occasionally the students asked the researcher technical questions about the video-taping (e.g. “If we move to that board, will the camera still capture our work”), or new students came into the room and had to sign paperwork; but during the work sessions, the researcher did not comment on the students’ mathematics, but allowed

[1] All names are pseudonyms.

the students work as they chose.

Two video cameras and an audio-recorder were used to record student activity. Cameras were positioned behind students, and at the far left and right of the room. This allowed the cameras to capture student work from two different angles, so it is less likely that student activity was obscured or too distant to be recorded. On the rare occasions that someone worked privately with pencil and paper, or the students shifted their work to a side-board, camera angles were adjusted to capture student work. During data analysis, the right camera functioned as a primary camera, and the left camera as a secondary camera: If audio on the right camera was unintelligible, or if student gestures were obscured, the researcher consulted the left camera; however, the majority of the analysis was conducted with the video recorded by the right camera.

### 3.3.2 Transcription

One of the major barriers facing research into silence is transcriptional: The usual transcripts, in which talk is recorded as in a play-script either drops silence entirely, or only briefly mentions it—even when lengthy—and so renders the silences invisible and unanalyzable. For instance, Steffe (2003)'s description of Jason's silences of 20, 50, and 10 seconds (discussed in section 3.1.1) take up about two lines in the transcript; and Steffe's description of the student posture during the silences is longer than is typical: If they are not omitted entirely, silences are often denoted, as in Tzur (1999), simply by stating their duration.

This is not a criticism of Steffe (2003) and Tzur (1999). Ochs (1979) showed that transcriptional choices make theoretical presuppositions. For instance, she argued a play-script style transcript assumes that utterances are relatively delimited and utterances of participants alternate, whereas individual utterances of small ($< 2$ years old) children often contain lengthy gaps, and are spread over multiple turns of adult talk, with the result that a play-script transcript hid the coherence of small children's vocal productions, whereas a columnar transcript allows small children's utterances to be seen and analyzed. On the other hand, because English reads left-to-right, in a columnar transcript, if an adult's utterances were placed to the left of a child's, the transcript indicated that the adult initiated interactions and the child only responded, and concealed times when the child initiated a particular form of interaction; therefore, it was better to put a child's utterances to the left of an adult's. (For examples of columnar transcripts, see figure 3.2.) Because Steffe and Tzur were interested in student construction of mathematics, lengthy transcriptions of these silences would have been inconsistent with their theoretical perspective, and would not have added to their analysis.

Nevertheless, the silences in Steffe (2003); Tzur (1999) could be potentially interesting to research. Because Steffe; Tzur singled out these episodes, we can conclude that they were important for the students' construction of fractional schemes. Yet in these critical moments, the teacher/researcher, Dr. Ron Tzur, allowed students to respond to his questions with silence for 5, 6 (twice), 10, 12, 15 (twice), 20, 25, and 50 seconds. Furthermore, during these silences, the students do not seem to have been engaged in overtly mathematical activities like working with manipulatives. We can therefore ask a series of questions about Dr. Tzur's responses to the students silences: What aspects of his students' posture and activity informed his expert judgment that these particular silences, which all violate both usual conversational norms (Stivers et al., 2009), and usual classroom norms (Ingram & Elliott, 2014, 2016), should not be interrupted? How would a less experienced teacher/researcher have responded to these silences, and how could they learn to notice the aspects of student behavior Dr. Tzur was responding to? Is a skillful reading of silences like these an important skill for a teacher, math tutor, or researcher conducting teaching experiments (Steffe & Thompson, 2000)? These questions are all potentially interesting, however, a play-script style transcript is inadequate for addressing them or other similar questions about mathematics education.

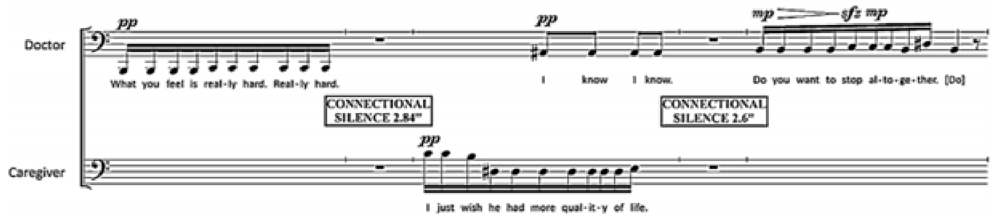
Though play-script transcripts are sufficient for most mathematical educational research, in studies of silence, a transcript should highlight individual and collective silences. Furthermore, in order to understand and differentiate silences, it is important that transcripts be able to notate the gestures that co-occur with that silence, and the relative timing of the gestures. Eye-gaze and posture are particularly important to include.

Previous studies of silence have experimented with transcription conventions, for instance, placing text on a musical score (e.g. Bartels et al., 2016, see figure 3.1b). This procedure, though better than a play-script in making silences visible has three major draw-backs: First, these transcripts highlight rhythmic aspects of speech that are not relevant to silence, per se. Second, it would be difficult to notate gestures and postural changes on these transcripts. Finally, though the silences are at least visible, they are hardly highlighted: Indeed, professional musicians sometimes have difficulty giving notated rests their full value (Margulis, 2007a).

In order to better highlight silences and allow other gestures to be included in the transcript, the following transcription conventions were adopted for all episodes

R: So you started with [calling it a] ninth, and we said, "All right, we don't want the ninth, but we want a name for that one [9/8]." Linda gave it one [name], "One eighth and one." Can you think of a fraction name for that one [9/8] without using "one"?
L: [After about 15 seconds] Nine eights.
R: "Nine eights?" What do you [Jordan] say? [Five second pause] Why would you [Linda] call it "nine eights"?

(a) A segment of a transcript from Tzur (1999, p. 406), containing silences of 15 and 5 seconds. This transcript is well suited for a constructivist analysis, but silences are not given enough province to allow research to focus on them.




(b) A musical-score transcript, from a recent study of silence (Bartels et al., 2016, p. 1589), highlighting two silences. The silences are not much more prominent than in Tzur (1999), and lots of extraneous information is included.

Figure 3.1: Two transcripts containing silence. The first is from a study of student construction of fractional schemes. The second is from a study of silence in patient/doctor interaction.

included in this analysis, adapted, in part, from (Ochs, 1979): Each participant is given their own column, arranged left-to-right to reflect the predominate physical arrangement of the participants, and distinct utterances are shown in rows. To make the duration of silences palpable, each full second of silence is given a row in the transcript, and its duration is noted. Line numbers are marked, for ease of reference. For an example, see figure 3.2a. Though silences are given space, they still are not very prominent, and without further support, the eye is still drawn to the text. In order to notate the silences themselves, and so a means of highlighting the silences is required. This presents a definitional challenge: How prominent must a silence be to be included in the transcript? In order to balance the tension between transcribing too much and too little, two conventions were adopted. First, since mutual silences longer than a second are unusual (Stivers et al., 2009), and so they are highlighted

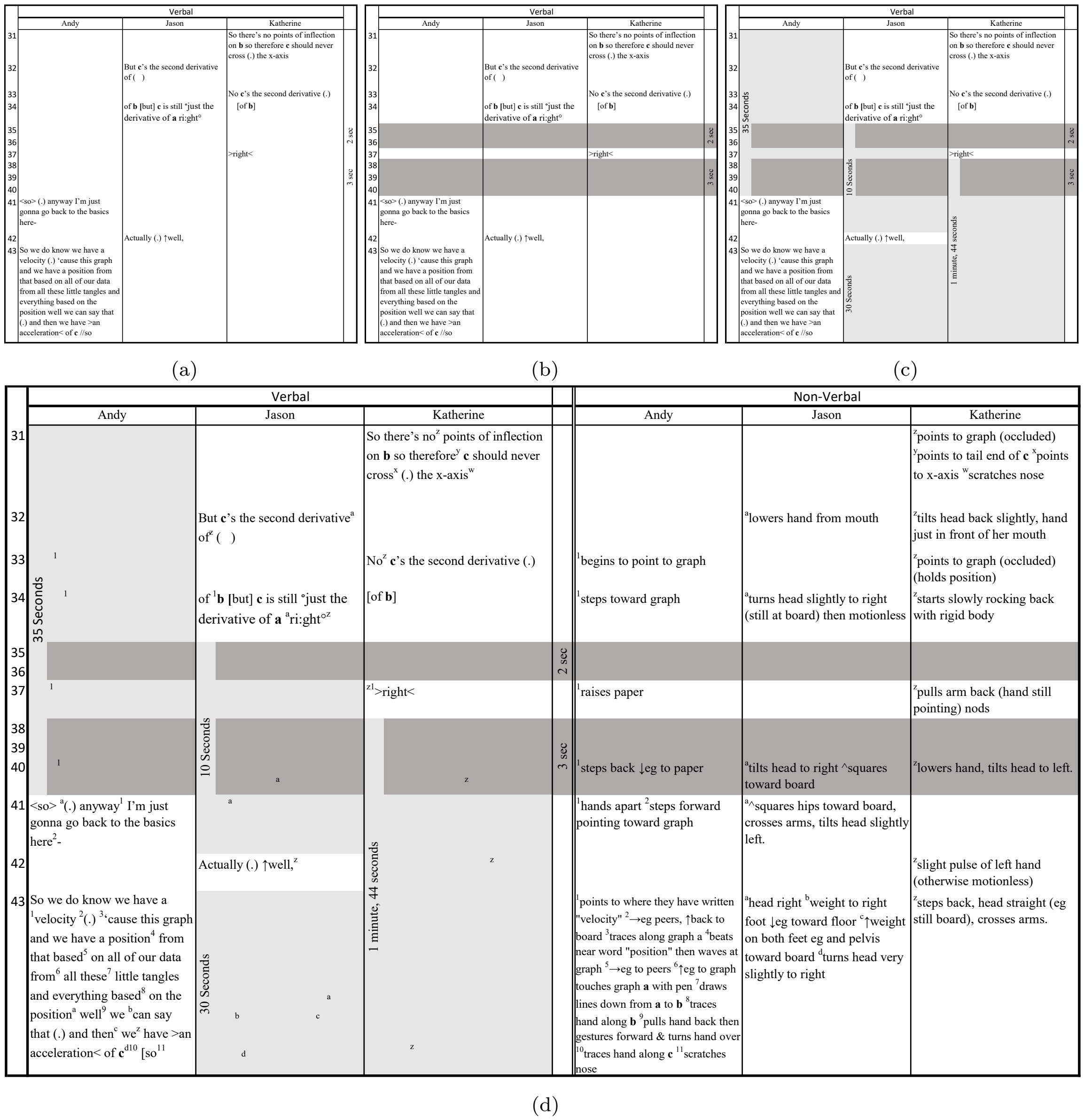



Figure 3.2: a-d: The same episode with, in a-c, progressively more visual salience given to silence. a) Each second of silence is given a line in the transcript. b) "Mutual" silences are highlighted. c) Each participant's silences are highlighted. At this point, silence is prominent in the transcript, but the question naturally arises: What happened during these silences? The final solution, following Ochs (1979), is to include columns for each participant's non-verbal gestures, and (if necessary) to use superscripts to indicate their timing. Here, this convention allows an analyst to see that during their silences, Jason and Katherine adopt similar postures, and even though he speaks, makes eye-contact with them, and walks forward to the board, do not shift their eye-gaze to Andy.

gray in the transcript, see figure 3.2b. This convention, however, ignores times when one participant has dropped out of the conversation. To include individual students silences, times when there are more than two utterances between instances of a particular participant's speech, or a mutual silence, are highlighted light gray, and duration is indicated on the left, see figure 3.2c. At this point, silences are included in the transcript, but the question naturally arises: How can different kinds of silences be distinguished? What were students doing in these silences? To allow the transcript to address that question, a second set of columns is added, where behaviors can be notated. When timing is important, footnotes can be used to coordinate verbal and non-verbal aspects of interaction. For an example, see figure 3.2d.

Prosodic elements, and silences shorter than a second, are notated according to the conventions of Conversation Analysis (CA). A complete list is contained in table 3.1.

| Convention | Meaning |
|---|---|
| ( ), e.g. (this) | inaudible talk, analyst's best guess |
| (( )), ((sneezes)) | commentary |
| colon, e.g. ri:ght | extended syllable |
| inward pointing brackets e.g. >fast< | faster speech |
| outward pointing brackets, e.g. <slow> | slower speech |
| brackets, e.g., [overlap] | overlapping speech |
| degree sign, e.g. °quiet° | quiet speech |
| up arrow, e.g. ↑you know | raised pitch |
| numbers in parentheses e.g. (0.2) | silence, number is seconds. |
| dot in parentheses, e.g. (.) | short, unmeasurable silence |
| equals sign, = | immediate or no pause |
| period, e.g. right. | falling intonation |
| semicolon, e.g. right; | rising intonation |
| comma, e.g. right, | continuing intonation |

Table 3.1: Transcript conventions. Conventions specific to a given transcript are listed before the transcript.

These transcripts are extremely helpful for initial analysis of silences, and for extensive fine-detail work; however, they are cumbersome for analysis of longer passages, for analyses that want to highlight other features of interaction and downplay silence (e.g. eye-gaze), and for less detailed analyses. When this type of transcript was not suitable for the sort of analysis preliminary rounds of data analysis suggested was necessary, CA transcripts, with highlighting and underlining to indicate particular postural elements of student work, were adopted.

### 3.3.3 Data Analysis

Following data collection, all interviews were watched, and an annotated timeline of the videos was created. Conversation between three or more participants is considerably more complicated than conversation between two participants (Schegloff, 1995). For this reason, conversations between two students are more suitable for early investigations of the presence of silence in student collaborations, and so the non-exclusive focus of these notes was on episodes involving two students. Furthermore, though there are times when students stopped working on homework (for instance, to discuss lunch plans), because the students came to these sessions to collaborate on homework, the default assumption in early rounds of analysis was that, at any given time, the students were engaged in mathematical activity. As the analysis progressed, when there was a potential for doubt as to whether a student was engaged in mathematical activity, the claim that a student remained engaged in mathematical activity was analyzed in more depth, and these arguments are contained in the results section.

Across cultures and language types, the conversational norm is that silences longer than a second are minimized (Stivers et al., 2009). Furthermore, because norms are enforced when they are violated, it is reasonable to hypothesize that if there is a silence longer than a second, students will enforce that norm, and it will be difficult for students to practice the sorts of silence recorded in (Petersen, 2017a). This means that if all the students are silent for longer than a second, they are violating the norms of every-day conversation. Furthermore, because norms are enforced when they are violated, it is reasonable to hypothesize that if there is a silence longer than a second, students will enforce that norm, and it will be difficult for students to practice the sorts of silence recorded in (Petersen, 2017a). For both these reasons, when there are silences longer than a second, students' mathematical activity is important for addressing the research questions for this study: "How do students, when collaborating on their homework, employ silence, in ways that are counter to the norms of every-day discourse, and how do those practices affect the group dynamics and their ability to make progress in discovering and writing their proofs."

Several small earlier analyses of silence in mathematics collaboration were used to help further refine this criterion by identifying what aspects of the students' techniques of the body which seemed likely to merit further attention in the early rounds of analysis. First, (Petersen, 2017a, 2017b, 2018b) showed that in some of their silences mathematicians remain physically tense and motionless, with their eye-gaze fixed on a symbolic representation of their mathematics. That mathematicians engage this sort of silence is evidence that novice mathematicians may sometimes engage in a similar, though less developed, form of activity.

Furthermore, body techniques are both individual and peer directed: They give an agent the ability to engage in particular actions; and, through habituated social norms, they facilitate the coordination of interpersonal activities. It seems reasonable to hypothesize that because the mathematicians' use of silence violates every-day social norms and because norms for responses to these silences can only develop as these silences themselves are practiced, when students engage in this sort of silent mathematical activity, their peers do not always recognize the silence as important mathematical activity, and instead hear the silence as it would be in every-day conversation—a whole that needs filled.

These hypotheses are further supported by the episode analyzed in Petersen (2015) and Petersen (2018a) (as discussed in the Introduction, and transcribed in Section 3.2). In the episode, two community college calculus students suddenly cut off a debate and fell silent. During this silence, they jointly adopted a posture similar to the mathematicians, namely, during the silence they remained physically tense, and oriented their gaze toward mathematics they had written on the board. Furthermore, a third student did not treat this silence as an important mathematical activity, but "drew [his peer] into" (Petersen, 2015, p. 874) an unrelated discussion, while the other two students treated his speech as an interruption (Petersen, 2018a).

These considerations of prior research on students' and mathematicians' uses of silence during face-to-face interaction, indicated that it may be profitable to look for times when, at an individual level, students: 1) Are silent for longer than a second, 2) square their hips to the board, fix their gaze on one location on the board, stop moving, and/or cease responding to their group-mates. Furthermore, at a group-level, to focus on times when: A) Two or more students are undertake the same sort of silent activity at the same time, or, B) peers seemed to interrupt the silence.

Though the hypothesis was that students would engage in this motionless, eye-gaze fixed on the board silence, both individually and, less frequently, in concert; but that they would also have difficulty responding well when their peers engaged in this sort of activity; few episodes were identified that fell into these categories. Indeed, on closer analysis, one of the activities that seemed to contain an interruption of silent activity did not, but instead contained two students undertaking similar, though not quite identical, silent activities together.

Following this initial round of analysis, large sections of the analysis group's one homework session were flagged as containing silence that merited further exploration, along with two segments, each approximately 15 seconds, from the geometry group's work on a single problem. That is, at this point, the majority of the geometry group's four recorded episodes—particularly all the times involving two students—and none of the algebra group's two recorded episodes were flagged as containing silence that merited further analysis. In order to analyze a balanced selection of groups, the criteria for selecting episodes were relaxed. The remaining videos were re-watched, and the analytic notes were inspected for further episodes that contained silence, even if the above criteria were not all met. For instance, at one point one of the geometry students, Sean, wrote a completed proof on the board, and asked his group-mate, Mary, to check it, and fix any bugs. She then went to the board to read through the proof, while he sat at a table watching her and answering her questions. When Mary attempted to understand the specific claims made in the proof and to figure out how to debug an aspect she finds problematic, she was silent for relatively lengthy stretches. However, since Sean was seated watching, while she was alone at the board (thus signaling a different roles in the group's mathematical activity), they did not adopt similar postures, nor did Sean interrupt Mary's silences; and so this episode was not initially identified as meriting analysis. At this point, however, it was included.

This process facilitated the selection of student work on five problems, each episode between 10 and 20 minutes long, for transcription and more detailed analysis. The episodes selected contained the following problems: 1) The analysis group attempted to prove that $[0, 1)$ is neither open nor closed. 2) The Geometry group attempted to explain the SAS congruence theorem to a hypothetical high school student who wants to understand why it works. 3) The geometry group debugged a completed proof that Sean brought to the session. 4) The geometry group attempted to prove that the perimeter of any polygon is greater than twice the length of the polygon's longest side. 5) The algebra group attempted to prove that if $G$ and $H$ are groups, $f$ is a homomorphism from $G$ to $H$, and $K \leq H$, then $f^{-1}(K)$ is a subgroup of $G$. That is, that the inverse image of a subgroup is a subgroup.

These five episodes were transcribed, copied into excel, and mutual and individual silences measured and notated on the transcript. Because earlier rounds of analysis indicated that the precise timing of the interaction and gestures in the analysis group's work were important, the non-verbal gestures for that episode were transcribed.

This transcription process both began, and facilitated the completion of another round of more detailed analysis of the silences of each participant. Both during, and following this transcription process, additional rounds of analytic notes focusing on the work of each group, and the ways the members used and/or responded to silence, and the sorts of silence present in each group's work, were collected on the selected episodes. The transcription of silences was especially helpful in collecting further notes, as the transcripts highlighted activities that had been more difficult to see previously. For instance, the geometry group's work in the fourth episode was selected because the earlier notes had noted that there was potentially some silence as the group wrote the problem on the board. The transcript, however, highlighted the fact that Sean spends a large fraction of the episode silent; and, when the videos were watched with that fact in mind, and the precise timing of his silences notated, it became apparent that, at a number of times, Sean's reading activity was interrupted by Mary's activity. At this stage, it was decided that conclusions that could be drawn from the third episode were (in which Mary debugs a proof Sean brought in to the meeting) was a subset of the conclusions that could be drawn from the fourth (in which they attempt to prove a theorem by induction), and so the third episode was dropped from further analysis. This decision was made because in both episodes, Mary engaged in the same sort of behavior, but Sean's activity was less interesting in the dropped episode. This resulted in four episodes for a more close analysis.

These episodes were analyzed using the techniques of Conversation Analysis (CA) (Liddicoat, 2011; Schegloff, 2007) and closely related fields like Ethnomethodology (Garfinkel, 1967; Liberman, 2013; Mehan & Wood, 1975), and ethnographic micro-analysis (Erickson, 2004, 1996; Garcez, 2008; McDermott et al., 1978). At least since Greeno (1994) recommended CA as an important tool for studying the social aspects of situated learning, these techniques have been not infrequently recommended and employed in mathematics education literature. For instance, Roth (2006) and Roth and Hsu (2009) recommended CA for analysts looking to understand social and power relations in action, Abrahamson, Flood, Miele, and Siu (2019) argued that CA could be helpful in analyzing the work of visually impaired students. And, turning to its use, Roth and Radford (2011) employed it extensively in their cultural-historical analysis of student learning. Likewise, ethnographic microanalysis has often been employed in mathematics education literature that take an embodied perspective on learning mathematics. For instance, Nemirovsky, Rasmussen, Sweeky, and Wawro (2012) employ it in their investigation of how students understanding of complex analysis is an embodied, navigational, know-how (for other studies, see, Nemirovsky & Smith, 2013; Soto-Johnson & Troup, 2014).

The goal of these analyses is to display specimens of social interaction, and the methods members in the social interaction employ to achieve their tasks and coordinate their behaviors. The guiding assumption is that social situations are not guided by some pre-programmed agenda, but are the result of moment-by-moment adjustments to the observable behavior of each participant in interaction, made possible through agreed-upon and always present norms for interaction. Precisely because the methods members in a social engagement use in to coordinate their activities are observable to all members of the interaction, they are also amenable to observation and discovery by an analyst. The analytic goal, however, is not to develop analytical constructs that are academically interesting, but to identify and enumerate the methods the members in a social situation employ to act, and to coordinate their activity.

Schegloff (1996, p. 173–173) gives three elements that any such analysis of the actions people engage in interaction must contain:

1. First, the account needs to describe the actions participants are engaged in, and, when possible, test any claims against seemingly deviant cases.
2. Second, the analysis needs to show that the participants in interaction themselves orient to the actions the analyst describes; and, correspondingly, that the analyst is not merely employing their own analytical constructs.
3. Finally, the analyst must give an account of what aspects of a participant's behavior allow that participant to accomplish the action in question—whether

their peers recognize the action or not.

These methodologies are fitting for a study of the way silence and other postural techniques of the body are employed in doing mathematics because, as Schegloff (1998) notes, the particular postural configurations that people use are so intimately connected to their structuring of the conversation, that CA could be said to be analyzing a particular aspect of participants' techniques of the body.

Finally, because disproportionate amount of time was spent analyzing their collaboration, a number of techniques specific to the analysis of Sean and Mary's geometry episode were employed. These techniques will be described in a subsection for that episode. Note that a focus on CA is very closely tied to CA methodologies.

### Sean and Mary

Previous rounds of analysis had flagged several of both Sean's and Mary's behaviors surrounding silence as both locally important for their interaction, and persistent throughout the episode. A CA approach to their interaction could help elucidate the local importance of silence in their interaction, however, because Conversation Analysis is a tool for attending to sequences of talk, particularly through the turn-pair, CA was less helpful in showing that these features of their interaction persisted throughout the episode, and shaped their interactional dynamics, on a large scale. In order to assess the persistency of these features of their interaction, four additional methods were adopted for analyzing their work.

First, though the length of all their silences—both collaborative and individual—had been calculated during the transcription, the duration of their silences, and their activities during these silences, had never been tabulated. In order to assess whether the their behavior differed in the length of silences, all their silences longer than 25 seconds were tabulated, and their activity during that silence described; and all their silences between 15 and 25 seconds were counted.

Second, earlier analysis had suggested that several features of their behavior were key. First, it seemed Sean had attempted to read several times, but that his eye-gaze repeatedly shifted from his reading to the board in response to Mary's address. Second, it seemed that Mary several times adopted a distinctive postural stance toward the mathematics symbolized on the board. In order to obtain a big-picture view of these activities, and tabulate the detail of the interactions, time stamps for: a) Sean's eye-gaze direction, b) times Sean's eye-gaze shifted up in response to Mary's address, c) times Mary silently stared at the board, and d) the beginning and end of Sean's and Mary's speech and silence were tabulated. Following this tabulation, the math applet Geogebra was used to create a timeline of approximately 11 minutes of Sean and Mary's work, during which they engaged in the majority of their mathematical activity.

Third, previous rounds of analysis had indicated that Sean made few mathematical contributions to his and Mary's proof production. Though this question is similar to the sort addressed by Rasmussen and Stephan (2008), here, what is at issue initially is whether and how much Sean contributed, not the specific mathematical character of his contributions, a methodology based off Toulmin's argumentation scheme, is not helpful. Instead, to determine how much Sean contributed, all nine conversational sequences that contained Sean's non-backchanneling contributions to the discussion were coded for whether they contributed mathematically to his and Mary's proof production. The possible codes were "Likely", "Unlikely", and "Possibly". If Sean's utterances merely provided material support that could have been provided by someone with a copy of their textbook, but without any knowledge of the mathematics, (e.g.: double-checking a diagram matched one in the book, reading requested theorems aloud), it was not coded as contributing mathematically to Sean and Mary's proof production. Otherwise, if there was any indication that an utterance contributed mathematically to the proof production, it was coded at least "Possibly"; and if a clear case could not be made that it did not contribute, it was coded "Likely".

It was at this point that, an approach based off Conversation Analysis was employed to zoom-in on a relatively short fragment from Sean and Mary's interaction that the earlier analysis had identified as critical in determining Sean's orientation to reading at the beginning of the episode.

## 3.4 Results

The research questions for this study are: How do students, when collaborating on their homework, employ silence, in ways that are counter to the norms of every-day discourse, and how do those practices affect the group dynamics and their ability to make progress in discovering and writing their proofs? These questions are answered through detailed analysis of four episodes of student collaboration.

The first episode, and the one which receives the most sustained attention, is an attempted proof by induction by two geometry students, Sean and Mary. It will be used to develop three main findings: 1) In their group collaborations, students may attempt to produce proofs by concurrently a) talking things through (together) at the board, and b) silently reading through their book or notes. In this group, these approaches were concurrent because the two students engaged in both types of activity at the same time, did not object to the other student's activity, and even explicitly sanctioned the other sort of activity—while continuing to engage in their own. The concurrence of these two types of activities, furthermore, violates ordinary conversational norms, which states that people should attend to each other, and make eye-contact with speakers. 2) When students adopt two tacks like this, the mathematical activity of a student who attempts to talk through a problem can interrupt the mathematical activity of a student who attempts to read, limiting the second student's eventual contribution to the proof production. Indeed, in this example, the group's work was almost entirely the result of the speaking student's work. 3) Contrary to the norms of conversation, even when students are primarily oriented to the public space of the board, they sometimes employ distinctive silences, which resemble mathematicians silences, in their work. This third finding suggests the possibility that, at least at these times, a peer's orientation to silent mathematical activity, whether reading from a private text or other silences, may be beneficial to a student who attempts to engage in this sort of silent activity.

For lack of a better term, this second sort of silent activity will be called a ruminating silence. This term is adopted in preference to "thinking silence" because this research does not attend to what is going on in the students' head during these silences, but to the social function of these silences, and the social challenges with engaging in these silences. The discussion section will examine in more depth how this activity relates to the action of thinking. Though ruminating silences only make a relatively minor appearance in this first section, they will increasingly be the focus of attention in subsequent sections.

The second section will analyze a segment of student collaboration that contrasts with the first. Whereas the first section shows that spoken mathematical activity, in its attempted commonality, can be a threat to a particular sort of silent mathematical activity; the second section shows that silent reading activity can pose a threat to spoken mathematical activity. It will further raise a few further complicating points regarding ruminating silences. Specifically, sometimes when students engage in a ruminating silences, they seem less aware of the actions their peers engage in, and the significance of these actions, than they would otherwise be. And furthermore, this lack of awareness can damage the coherence of group work.

A third section will briefly explore the way students respond when their peer adopts a ruminating silence that runs counter to the norms of every-day conversation. Specifically, though students adopt these silences themselves, they can interrupt the silence as a violation of conversational norms, and so fail to allow their peers to practice them.

A final section will attend to ruminating silences in the work of something of a model group. In this group's work, though the students exhibited very strong mathematical and group practices like making sure both understood any relevant definitions before attempting a proof, the results suggest ruminating silences exerted something of a centrifugal force on the coherence of their joint activity that the students had to work to overcome, even when both students oriented to a ruminating silence as an important form of mathematical activity.

### 3.4.1 Mary and Sean: Reading and Ruminating

In this episode, two students, Mary and Sean, sought to use induction on the number of sides of the polygon to prove that the perimeter of a polygon is always longer than twice the length of its longest side. For time reasons, their investigation was limited to the base case, namely, triangles. During the episode, they—or rather Mary—successfully proved the base case, but did not have time to continue to the induction step.

The focus of this analysis is how the two students interacted during their work. The analysis will follow the following steps:

First, in order to orient the reader to the episode, and introduce questions the rest of the analysis will seek to address, three preliminary points will be established in the section that begins on page 131. The first two points orient the reader to the silence in the episode, the third raises questions about the two students' mathematical interaction. First, the students were silent simultaneously for a large portion of their work. Second, throughout their work Sean's silences were far longer than Mary's were. Finally, Sean made few, if any, mathematical contributions to the group's final proof. Rather, their proof production was almost entirely the result of Mary's work.

This final point raises a question which the following two sections will attempt to answer: What features of their collaboration led to Sean's lack of mathematical contribution to their final proof production. This argument will follow two steps. First, the section that begins on page 135, will establish the claim that, for large sections of the students' work, Mary and Sean concurrently adopted different tacks for solving the problem. Specifically, for extended periods of time, Sean sat a little away from the board, where he worked silently on his tablet computer, seemingly reading from an electronic copy of the course textbook or from a list of theorems and definitions he had typed up. At the same time, Mary worked at the board, and attempted to talk through the problem. Furthermore, neither student objected to this situation, and Mary recognized Sean's activity as a legitimate form of activity to pursue as she worked. At the same time, Mary's talk can be seen to interrupt Sean's activity on a number of occasions, though, it is unclear at this stage whether this was an occasional interruption that only affected Sean's work at a few discrete points, or whether her talk disrupted his work substantially.

This result that the students adopted concurrent strategies is a significant finding in its own right, however, the section beginning on page 141 adds a second step to the analysis of Sean's lack of contribution to the group's proof production. In order to see how persistent Sean's tack was, the argument in this section zooms in on a segment relatively early in Sean and Mary's work to show that Sean persistently tried to approach the problem in silence, but that Mary's talk made silent mathematical activity difficult for him. These two points together establish the claim that Mary's talk inadvertently created an inhospitable environment for Sean's silent mathematical activity, and therefore, provide an explanation for his lack of mathematical contribution to the group's proof production. Namely, Sean's consistent behavioral strategy was to begin a proof attempt in silence, but that strategy was consistently interrupted by Mary's talk. This is a second major finding of the paper.

Though Mary's silences were far shorter than Sean's, Mary was also silent for much of the episode. A final subsection (beginning on page 152) analyzes one aspect of Mary's silences: While she was working at the public space of the board, Mary repeatedly stopped moving and stared at the board for over a second. These silences, called here "ruminating silences" resemble mathematicians' silences (Petersen, 2017b, 2018b) and are, in Mary's work, clustered just prior to her successful proof production (see figure 3.3 on page 137). This final subsection analyzes the character of these silences, while the interactive significance of this sort of silence is mostly postponed to subsequent sections. That said, because these silences are oriented to the public space of the board and occur during the middle of Mary's talk, but violate the norms of every-day conversation (Stivers et al., 2009), it is conjectured that Sean's silent reading activity at this time provided a hospitable environment for Mary's mathematical activity, in which she could both talk as to a partner, but not be interrupted when she needed to be silent.

### Introduction

In contrast to other episodes involving Mary and Sean, a large fraction—43.6%—of this episode was spent in collaborative silences over a second long. Furthermore, though both students were silent for a large portion of the episode, Mary's silences usually came in relatively short bursts, whereas Sean's silences were far more lengthy. This can be seen in table 3.2, which describes each students' silences that are over 25 seconds long, and enumerates their silences that were between 15 and 25 seconds.

Table 3.2a shows all of Sean's silences longer than 25 seconds and the total number of Sean's silences between 15 and 25 seconds. Summing the length of the silences longer than 25 seconds, Sean spent 5:44, or 27.3% of the total episode in silences longer than 25 seconds. Furthermore, Sean spent at least 7:59 or 38.1% of the total episode in a silence longer than 15 seconds. Sean Table 3.2b lists Mary's one silence longer than 25 seconds, and enumerates her one silence between 15 and 25 seconds long. Mary spent 28.1 seconds or 2.2% of the total episode in silences longer than 25 seconds, and no more than 53.1 seconds or 4.8% of their total work time in silences longer than 15 seconds. These estimate are calculated by counting Sean's 9 silences

| Time | Duration | Activity |
|---|---|---|
| −143 | 27.2 | Writing |
| −114 | 92.3 | Writing |
| 45 | 27.3 | Reading, Watching Mary |
| 88 | 28.7 | Reading, Watching Mary |
| 173 | 29.5 | Reading |
| 438 | 60.1 | Reading |
| 604 | 39.1 | Reading, Watching Mary |
| 718 | 39.8 | Watching Mary |
| Number between 15 s and 25 s: 9 | | |

(a) Sean's silences

| Time | Duration | Activity |
|---|---|---|
| 453 | 28.1 | Writing, Ruminating |
| Number between 15 s and 25 s: 1 | | |

(b) Mary's silences

Table 3.2: Sean and Mary's silences longer than 25 seconds (top rows), and between 15 and 25 seconds (bottom row). Group work began approximately 2:20 prior to Sean's first silence, and ended approximately 1:08 after his last silence. For silences longer than 25 seconds, the time of the beginning of the silence, its duration, and a short description of the silent student's activity during the silence are all tabulated. The time is the number of seconds from the beginning of figure 3.3. Sean's first two silences, which were only separated by "mhm", were before the start of figure 3.3 and so have negative values for the starting time. Note: Mary's 28.1 s silence was in the middle of Sean's 60.1 s silence. Silences between 15 and 25 seconds long are counted, but are not described.

between 15 and 25 seconds long as 15 seconds, and Mary's one as 25 seconds. The estimate, therefore, underestimates Sean's time, and overestimates Mary's time. With one exception, these silences span most of the group's work time. That exception is the 3:55 between Sean's 29.5s silence and his 60.1s silence, which will later be seen to contain lengthy silences from Sean and shorter silences from Mary. Furthermore, by both measures, Sean spent almost an order of magnitude more time in lengthy silences than Mary did. It therefore follows that throughout the group's work Sean's silences were far longer, and Mary's far shorter.

Fifteen and twenty-five seconds are relatively arbitrary cut-offs, and are mostly chosen so a reader can get a sense of the two students' activities, without making the table too long (which would result from decreasing the 25 second cut-off) or making either category too wide (which would result from decreasing the 15 second cut-off to 10s). Since the times the students spent in silences at each of these two levels differs by almost an order of magnitude, however, changing the cut-offs does not make a material difference to the description of the student's activity.

The students began their work seated at a table, but quickly stood up and moved to a board, where Mary directed Sean to write the triangle inequality (since, as Mary said, "that's going to be how we prove our base case") and Mary wrote the definitions of perimeter and polygon. Prior to beginning to write these definitions the students mostly shifted around and were not physically engaged with any mathematics. Furthermore before they wrote on the board, only two mathematical contributions were offered: Frist, Mary said they should use induction, and Sean quickly concurred; second, Sean asked whether they needed to prove the theorem for any polygon, or only for simple polygons, and Mary quickly said that, in this case—and especially when working on the base case, when all polygons are simple—it didn't matter. The first contribution set the trajectory for their proof, but was raised and resolved extremely quickly, while the second did not inform their proof any further. Furthermore, though Mary talked about the definitions as she wrote them, there was little mathematical work until the students finished writing these definitions and theorems on the board.

About 10 minutes and 45 seconds after Sean finished writing the triangle inequality on the board, Mary articulated the body of a successful proof of the base case, and the rest of the episode was spent tying up loose ends. In particular, Mary was a little confused about the precise reading of the triangle inequality, and the students needed to confirm that the triangle inequality was able to bear the load her proof required it to.[2] In this section, the students looked in their book to see the diagram the book used to illustrate the triangle inequality, but Mary only adopted one, short, ruminating silence. Furthermore, Sean read little, and with little interplay with Mary's work. For these two reasons, this section is less interesting for addressing this study's research questions.

Because their work was most interesting and productive between the 10:45 after Sean finished writing on the board and before Mary articulated a successful proof of the base case, the main body of the analysis focuses on this portion of the students' work. The most striking feature of this period is that though Sean seems to have been engaged in mathematical activity for much of the episode, the proof production was almost entirely Mary's work.

In order to determine whether Sean contributed mathematically to Mary's and his proof production, conversational sequences that contained Sean's non-back-channeling utterances were coded for whether Sean made a mathematical contribution to the group's progress. Eight of the nine sequences were coded "Unlikely", and the ninth was coded "Possible". These codes were independently checked by a second researcher who agreed with eight of the nine codes. In the one sequence coded "Possible", Sean perhaps helped generate a justification for the false and purely nominal claim that the longest side of a scalene triangle could be called the hypotenuse, though he may

[2] The triangle inequality is: "In any triangle $ABC$, $|AB - BC| < |AC| < |AB| + |BC|$". Mary was unsure whether AC is a specific side (the shortest side of the triangle, the longest side of the triangle, or the side of intermediate length), whereas her proof required that AC be the longest side of the triangle. After the end of the section included in this analysis, the students confirmed that in the book's diagram illustrating the triangle inequality, AC is the longest side and so the inequality can bear the weight her proof requires. They never realize that AC is any side.

have merely been helping Mary recall her earlier verbal justification as she wrote it down. Even if Sean did contribute mathematically to this justification, the claim was not used in their eventual, successful, proof.

Both the fact eight of nine sequences were coded "Unlikely" one "Possible" and none "Likely", and the fact that Sean's one potential mathematical contribution was a justification of a false and purely nominal statement, indicates that though we assume that Sean was engaged in mathematical activity throughout this problem, he did not contribute to his and Mary's proof production, and the final proof can be viewed as entirely Mary's work.

It may be tempting to assume that Sean did not contribute mathematically to their proof production because he was silent for so much more time. This assumption, however, should be resisted. First, his silences themselves need explanation, so that claim only shifts the question. More importantly, however, a student may be silent because there are engaged in careful thought, whereas a key idea may only require a few short utterances to communicate. Furthermore, table 3.2 shows Sean was engaged in mathematical activity throughout his lengthy silences. For these reasons, we should not assume his silence accounts for his lack of contribution, but seek to ascertain why his silent mathematical activity was less productive than Mary's spoken activity.

**Different Concurrent Tacks**

A large portion of the whole episode was spent in mutual silences—silences which, because both students are silent during them, are therefore the work of both students. Nevertheless, the two students simultaneously adopted different behavioral strategies throughout their work: Mary often stood at the board and attempted to talk her way through the problem while Sean simultaneously attempted to solve the problem by silently reading through the theorems and definitions he had on his tablet, though at other times he watched Mary work. Rather than negotiating a common strategy, Sean and Mary pursued these two strategies concurrently, and treated this arrangement as, in some sense, legitimate.

As table 3.2 has already indicated, Sean repeatedly read silently from his tablet while Mary talked about her work. For instance, he read silently for 29.5 seconds, beginning at the 173 second mark, even though Mary continued to speak, at least intermittently, during this time.

Figure 3.3 provides a graphical display of students' activity between the time Sean finished writing on the board, and the time Mary finished articulating a proof of the base case. Mary's activity is shown by the top two channels, and Sean's by the bottom two. The top channel shows Mary's ruminating silences, which will be examined below. At this point, however, it is important to note that these occurred repeatedly throughout the episode, and two large clusters of them come in the two minutes prior to her eventual articulation of a proof, which happened around the 630 mark in the figure (the figure ends when Sean agreed with her formulation). The next two lines show the times each students was silent (orange) or spoke (green). Gaps signify back-and forth conversation with no pauses over a second long. Finally, the bottom line shows Sean's eye-gaze: Blue indicates his eyes were on the common space of the board, whereas red indicates his eyes were directed to his private space, namely, to his tablet. Gaps indicate times when his eye-gaze was elsewhere, for instance, as he walked to a chair and sat down.

This figure further confirms the claim that Sean's silences were consistently longer


Figure 3.3: A timeline, in seconds, of Sean and Mary's work. In order for a segment to be coded as silence, it must have contained at least one second of mutual silence, so gaps represent back-and-forth conversation. Triangles represent times Sean shifted his eye-gaze in response to Mary's address, or responded verbally without raising his gaze. Black triangles represent times Sean's action was a response to clear features of Mary's address (e.g. she asked "right", or shifted her gaze to Sean (see, Stivers et al., 2009), dark grey triangles represent times it seemed likely Sean was responding to Mary, light grey triangles represent times Mary was speaking, but there is nothing about Mary's speech that solicited a response. Sean's speech during the lengthy times his eye-gaze is down is: (1) "( )" (2) "yeah" (3) "a°( ) hhh°" (4) "hh no not really h((eg down))eh" (5) "I'm ( ) through the book right now actually" (6) "°s- s- ( a )°" (7) ((whistles)) (8) "hm" (9) "mhm" (10) "s- s-" (11) "mhm". The small gap in the bottom line between 220s and 230s represents a time Sean walked to a table and sat down. He remained seated till the very end of this figure, when, in response to Mary's proof, he got up and came to her work on the board.

At the beginning of the episode, Mary was still copying definitions onto the board, and while she worked at the board, she occasionally consulted her book or a page of notes. By the end of the first minute, however she stopped consulting them, and, except for a brief pause to drink coffee (around 540 s), Mary's eye-gaze was on the board or Sean the rest of the episode.

than Mary's. For instance, during the first two minutes displayed in the figure, Sean repeatedly engaged in silences over 10 seconds long, whereas Mary's first silence over ten seconds long didn't occur until approximately 150 seconds into the students' work, and the second one, until just after 370 seconds into their work. Furthermore, this figure sheds light on the 3:55 gap in Sean's activity shown in table 3.2. That time follows Sean's 29.5 second silence graphed at 173 seconds (the start of line 2) and ends with his 60.1 second silence at 438 seconds (in the middle of line 3). During that time, Sean repeatedly engaged in silences over 10 seconds long—many containing Mary's talk—whereas Mary only engaged in 2 silences over ten seconds long during this time, and both started after the 360 second mark.

The key information figure 3.3 shows is the students' different behavioral strategies to solving the problem. As the bottom channel shows, for extended periods of time, Sean turned his eye-gaze away from the board to the private space of his tablet. The first extended period began at approximately the 82 second mark, and continued, with a short interruption in his eye-gaze, for just over 20 seconds. He then kept his eye-gaze on the board, with two short exceptions, for approximately the next 1:10, during which time he talked with Mary. (This passage is examined in detail in the next section.) His eye-gaze then returned to his tablet for almost 30 seconds before shifting back up to the board. With a few relatively short exceptions his eye-gaze remained on the board for the next 3 and a half minutes. (During the three times his eye-gaze was down between 243 and 275, he was looking up a theorem for Mary. This resulted in his talk during and after the time his eye-gaze was down. But the same pattern of eye-gaze on tablet coinciding with speech does not occur elsewhere.) Just prior to the 420 second mark, however, Sean's eye-gaze again shifted down to his tablet where, with two short interruptions, it remained for approximately the next 1:07. Then his eye-gaze again shifted up to Mary, where it remained for almost 30 seconds before shifting back to tablet for approximately the next 1:45.

Though Sean turned his eye-gaze to his tablet throughout the episode—and especially toward the end—Mary worked almost the whole time at the common space of the board. Indeed, she did so with enough consistently that including it in figure 3.3 would not be informative. As she worked at the board, Mary sometimes "ruminated", sometimes wrote at the board (for instance, table 3.2 shows she wrote for most of her long silence beginning around the 453 second mark), and as figure 3.3 shows, she talked through the problem. For instance, during the minute between 540 seconds and 600 seconds, though Sean's eye-gaze was on his tablet almost the entire time, Mary talked through the problem and ruminated silently. Sean did speak briefly several times during this minute and at least twice his talk to be in response to Mary's (9 and 11 from figure 3.3), however, Sean did not even look up from his work as he spoke.

This analysis of their behavior indicates that rather than negotiating a single coherent group tack, Sean and Mary adopted these two behavioral strategies simultaneously. Furthermore, there is evidence that they both treated the arrangement as in accord with interactional norms. First, moving his eye-gaze and talking in response to Mary's work, rather than asking her not to talk to him, Sean treated Mary's simultaneous pursuit of her tack as legitimate. For instance, Figure 3.3 shows that around at around 202 seconds, and again at 496 seconds, Sean had been reading for a lengthy period of time, when, in response to Mary's address, he looked up, and responded to her. Furthermore, at (2), (9) and (11) in Figure 3.3, he responded briefly to Mary, even while he continued reading.

Likewise, Mary treated Sean's continued reading as legitimate.

Transcript 3.1

```
      (3.0)
M:    ((Eye-gaze to Sean, both nod))
      (2.5)
M:    Ideas from here;
S:    hh no not really ((eye-gaze down, Mary's to the board)) hehh
      (1.2) I'm ( ) through the book ((Mary's eye-gaze to Sean)
      right now actually
M:    ((Mary's eye-gaze to board)) Oh ok ((Mary takes a drink of
      coffee, and then turns again to the board))
```

This episode begins around at approximately 525 in Figure 3.3—note Sean's talk here is (4), (5), and (6). Sean's claim that he was reading through the book is a little confusing since he had been watching Mary work for just over 30 seconds. Mary's response to his claim, however, is clear: Both by her "oh, ok", and by returning to work at the board, Mary, here, treated Sean's private reading as a legitimate activity for him to engage in, even simultaneously with her attempt talk through the problem at the common space of the board.

This shows that, at least for part of the episode, rather than negotiating a strategy they both pursued, Mary and Sean concurrently adopted different behavioral tacks, and treated that arrangement as legitimate. Mary worked on the board, whereas Sean worked silently by reading from an electronic copy of the textbook on his tablet. This is not to say that this arrangement had no interactional effects. For instance, though there are other interpretations, it is possible to interpret times when Sean's eye-gaze was on the board as times Mary's spoken work distracted him from engaging in his own work. The data presented so far does not support this hypothesis, and it is not offered as a conclusion about Sean and Mary's work, but as a relatively simple example of the sort of interaction that could occur even when two students concurrently adopt different behavioral tacks in their work on a problem.

### Sean's Silences

The fact that Sean and Mary attempted to work, on one hand, in silence and by talking through the problem does not itself explain why Sean did not contribute mathematically to Mary's proof production. There is no theoretical reason to suggest that he could not have discovered a key idea while reading. That Sean and Mary concurrently adopted these two tacks to the problem suggests one line of inquiry for determining why Sean did not contribute to the group's proof production: Interactional effects hindered Sean's private work on his tablet.

This hypothesis is plausible since, as figure 3.3 shows, Mary's address interrupted Sean's reading work multiple times: In response to Mary, his eye-gaze clearly shifted from reading six times. And he twice responded "mhm", and once "yeah" without looking up. On the other hand, precisely because of Sean's lengthy silences, and because she never had to pursue a tack he suggested, Mary was able to work uninterrupted during this whole episode. That is to say, Sean's work on solving the problem, conducted almost exclusively by reading in silence, was repeatedly interrupted, whereas Mary's was uninterrupted. These facts suggest a reason Sean's activity resulted in so much less mathematical production than Mary's did: Mary's speech repeatedly distracted and interrupted Sean's work, whereas Sean's silence did not bother Mary, and perhaps even provided a fruitful environment for her to work in.

However, since Sean never objected to Mary's address, and since there is a long stretch of time during which he did not read (341s–419s in figure 3.3), an alternative suggestion is plausible: He did not attempt to read until relatively late in the episode, and so he spent considerably less time working on the problem than Mary did—and therefore her speech had only a small effect on his work. To investigate this hypothesis, more detailed attention to their work from approximately the 81s until approximately the 201s is required. This passage contains two long segments (63s–82s and 143s–173s) during which Sean's eye-gaze was directed toward the board, the longest segment of sustained speech from Sean (130s–175s) as well as four clear interruptions. If, during that time, Sean did not persistently attempt to read, then the hypothesis that Sean only began working from his book relatively late, is plausible. On the other hand, if Sean can be shown to have persistently attempted to read from his book, only to be repeatedly interrupted, then there is evidence for the hypothesis that he did not contribute substantially because he was repeatedly interrupted, and therefore the collaborative environment hindered his ability to sustain engagement in mathematical activity for long periods of time is established.

In the following transcript, punctuation is used following CA usage, with the additional signs: 1) A thin, tight, blue $\overline{\text{overline}}$ indicates that Sean's eye-gaze was on the board. 2) A thick red $\underline{\text{underline}}$ indicates that Sean's eye-gaze was on his tablet, where he had all their theorems and definitions, and seemed to have a copy of their course textbook. 3) Superscript $^{s>}$, $^{s<}$, and $^{m>}$ indicate that Sean (s), or Mary (m) shifted their gaze right (signified by “>”) to the triangle inequality, or left (<) to where Mary worked at the beginning of the transcript where these symbols are necessary. 4) Arrows indicate lines during which Sean's eye-gaze shifts up. So for instance, in line 5, his eye-gaze was on the board ($\overline{\text{blue overline}}$), but, at the beginning of the line, he shifted his gaze to the right ($^{s>}$). The segment analyzed begins around 62s (figure 3.3), and finishes just after before 180s. During this episode, Mary drew a triangle to illustrate the triangle inequality, and then had Sean verify that her drawing was accurate.

Transcript 3.2

1 M: So our base case
*S's gaze goes to where Mary is working.*
2 (1.2)
3 M: °if we're going to do a proof by induction°
4 (5.2)
*Mary writes ''proof by induction'' on the board, finishing and stepping back as Sean answers ''yeah'' in line 6.*
5 M: $^{s>}$right;
6 S: $^{s<}$yeah$^{m>}$
7 (0.2)$^{s>}$(0.6)
8 M: So twice the length of its longest side
9 (4.4)
*Mary repeatedly shifts her weight during this pause. Her gaze, however, remained on the statement of the triangle inequality. Sean was already motionless with his gaze fixed in the same location as Mary's, a posture he maintained until his gaze dropped.*
10 M: Triangle ABC
11 (0.6)(0.4)
*During the above pause, M walks to the board and starts drawing just to the immediate bottom right of the triangle inequality. S looks down as M walks in front of him.*
12 M: so let's go ahead
13 (2.2)
*M draws a triangle with a horizontal base, and all three sides nearly equal in length.*
14 → M: Does that look like triangle ABC;
15 S: °sure°
16 (2.7)
17 M: Uh (0.2) let's make sure (0.4) the length of one side AC
*M fixes her gaze on the statement of the triangle inequality, while holding her right hand just under the triangle.*
18 (2.7)
19 M: I guess we should make as- (0.2) triangle ABC look like
*M erases the triangle she had drawn, and draws another triangle with a much longer horizontal side.*
20 (5.8)
21 → M: Does that look (0.9) I don't know (0.8) so this is bounded
22 above so this would be
23 (3.0)
24 M: °oh shoot°
25 (1.1)

M: Like so; (0.2)
S: Sure
(3.5)
M: eh
(3.0)
M: I wanna draw it so it lines up with the triangle with the
inequality
(1.0)
M: right; (0.4)
S: Yeah (0.5) yep (.) I'm just gonna
(0.8)
M: Is that
S: °AB plus well like visually° huhu (0.8) yeah (0.6) AB equals
BC (.) eeh close enough
(1.4)
⇒ M: Eh you can redraw it but
S: No it's good
⇒ M: That's good
S: Yeah
(0.8)
⇒ M: So is it the smallest side that hangs out in between
(0.8)
S: Sh(ould be) (0.9) so that's your AC right there
M: Yeah
(0.8)
S: °So AB (0.3) AC (0.8) can't that should be greater (.) Yep
(0.6) And then (0.4) absolute value o:f the: (0.8)
difference of those two:°
M: So it would (0.2) we think about vector pro[ject\\ion u::hh
S: [mhm
right
(1.0)
S: Mhm
M: Ok
(0.7)
S: Yeah
M: Ok
(1.4)
S: Alright
(1.1)
M: That looks right (0.5)
S: Yeah (0.7) Yeah
(3.4)

At the beginning of the episode (~62 seconds in figure 3.3) Sean's eye-gaze was on his tablet, however, in response to Mary's change-of-state token "so" and her articulation that she was beginning to work on the base-case, Sean shifted his eye-gaze to Mary, and the portion of the board where she was writing. His eye-gaze remained fixed on her work until line 5 ($\sim$ 10 seconds later), where he briefly looked right at the triangle inequality—which he, at Mary's instruction, had written "because" as Mary had said earlier, "that's going to be how we prove our base case." Shortly after Sean shifted his eye-gaze back to Mary's writing, Mary stepped back from the board (line 7), and shifted her eye-gaze right to the triangle inequality, where the focus of her work remained throughout this transcript. In response to Mary's shifted eye-gaze (line 5), Sean shifted his eye-gaze back to the triangle inequality, where it remained until line 11, or about 5 seconds. During that time, the two students spend 4 and a half seconds silently looking at the triangle inequality: Sean, standing motionless, with his hips squared to the board; Mary, shifting her weight repeatedly.

Up to this point, there was a high amount of coordination of their activity: Mary's vocalization of the fact that she was beginning a new line of work made that line, and its newness, common knowledge, and so could, at least potentially, serve to coordinate the group's activity on the new tack. Sean responded to Mary's articulation of the new tack by looking up from his tablet, and fixing his gaze on her writing, thereby allowing her vocalization to coordinate their focus. That his gaze is focused on her work provides some evidence that he had joined her work. It is possible, however, that internally he continued another tack. Two lines of evidence support the assertion that he had more thoroughly coordinated his activity with Mary's: First, he glanced right at the triangle inequality (line 5)—which he had written so they could use it as the foundation for their proof by induction—just as Mary was ready to shift her attention to it; thereby evincing an anticipation of her work. Second, in response to her shifted gaze, he shifted his gaze more permanently (line 7). These three lines of evidence: His initial looking up at Mary, his anticipation of her movement, and his following her activity, are evidence that, through the first nine lines, Sean had joined Mary in a common, coordinated activity.

On the other hand, if we accept the hypothesis that they had joined in a common activity, there is a clear division of labor: Mary, on the one hand, wrote on the board, and talked quietly about what she was writing. Sean, on the other hand, watched her write, and, presumably was expected to comment on it and double-check that Mary's writing was accurate. Though Sean was authorized to approach the board (he does so eventually, in line 36), until he does so, or Mary steps away from it (as she does here, in line 6), this division of labor remains.

Following the silence, however, something changed: Mary stepped to the board, and continued speaking, sometimes addressing Sean directly, whereas Sean, at least for a while, turned his eye-gaze away from Mary's work and back to his tablet. Both students' actions after this transition, and their changed orientation (or lack of orientation) to their joint activity are analyzed, in turn. After Sean's eye-gaze shifted back to his tablet, how did Mary orient to their joint activity? How did Sean?—assuming, and this assumption will be problematized, that he continued to orient to the work as joint activity.

Four lines of evidence indicate that Mary continued to orient to their activity as joint, though with the same division of labor: she writes and draws, while he evaluates her drawing, and helps make sure her drawing is correct.

First, her use of a plural pronoun in line 19 "we should make [sure]" (and perhaps also in the contraction "let's" in line 17) ascribes responsibility for the accuracy of her drawing to both students. This evidence may be counterbalanced by her use of the singular "I" in line 31. On a closer examination, however, the variation in personal pronoun fits very well with the hypothesis that she is oriented to joint activity, with a division of labor. Specifically, whereas making sure the diagram is accurate is, on this hypothesis, their joint responsibility, writing on the board is hers alone. That is to say, she uses the plural "we" to refer to what is, on this hypothesis, joint in their work, and the singular "I" to refer to what is hers part of the joint work. Because her pronoun usage fits well with the hypothesized division of labor in their joint activity, rather than counterbalancing a conclusion drawn from the plural pronoun use, the variation is further evidence for the hypothesis that she continues to orient to a joint activity, with a division of labor.

Second, Mary continues to talk full voice—that is, she does not sub-vocalize—as if someone is listening. Since there is no one else on hand to listen (she never turned to the interviewer), that she speaks as to a listener indicates that she continues to orient Sean as engaged with her in a common activity. Furthermore, she does not just talk aloud, she specifically addresses Sean, and solicits his input. Though it is possible that she addresses him knowing she is interrupting his separate activity, she later ($\sim$533s in fig 3.3) accepts that he is trying to read as reason for him not to have any input, and so it is unlikely that she is doing otherwise here. Therefore, the fact that she continues to speak full voice provides a second line of evidence for the hypothesis that she continued to orient to joint activity.

Third, she did not just address Sean, but addressed him from the earliest stages of her work. For instance, in line 14 of the transcript, she had drawn a triangle, but she had not labeled the vertices, or made any verbal claims about the triangle.

Because her work was in its very early stages, she does not seem to have been seeking a judgment that she was correct, but what might be called a continuer, a short utterance like ”yeah” that would demonstrate Sean’s attention and amount to a claim her productions at the black board were common knowledge. If this reading is correct, it is in turn evidence that she treated their work as still coordinated. The contrary hypothesis is that she saw they were not coordinated, but was attempting to reestablish the coordination. But this is less likely, since she doesn’t even look at him—whereas responses are much more likely, when the speaker looks at the addressee (Stivers & Rossano, 2010)—and further, because she is satisfied with his somewhat perfunctory response.

Finally, in lines 39-44 she was relatively persistent in her question: She repeated her question twice, in spite of his reassurance that her drawing is fine, and then, zeroed her question in on a specific part of her drawing (line 44). This persistence indicates that, from Mary’s perspective, Sean’s more cursory response violated a conversational norm that he contribute to their joint activity by checking her diagram, alert to possible problems. And this, in turn, shows that she was still treating their activity as a joint activity.

These four points together provide strong evidence that Mary continued to orient to her drawing as a collaborative project, with both students oriented to the same task, though with the following division of labor: For the most part, Mary works at the board, drawing, while Sean watches her draw, and makes sure her work is accurate. For much of the transcript (lines 14b, 21b-39a, 46b-67), Sean was physically oriented to Mary’s work, and he occasionally approached the board and commented on Mary’s work (lines 36, 49). Therefore, at least part of the time, Mary’s orientation and Sean’s seem to be aligned. On the other hand, Sean repeatedly stops attending to Mary's work, and reads from his tablet. How can Sean's activity during this transcript be accounted for?

The argument in this section will be that though Sean at times stopped reading and jointed Mary in her orientation to a joint activity focused on her work on the board, this orientation was a slightly disprefered second option; and his first option was for both students to independently pursue a proof: her, by working at the board, him, by reading.

Several lines of evidence support this reading. First, Sean looked down as Mary steps to the board, perhaps even in response to her motion. If Mary's stepping to the board signified that she was returning to the sort of labor she had pursued before turning to the triangle inequality—that is, writing at the board, and potentially talking about her writing—then Sean's shift of his gaze from Mary's work indicates that, whatever his attitude to Mary's tack, he was not interested in continuing his half of the work in the division of labor.

Second, if, as was argued above, Mary's full-voice speech was an attempt to make her activity common knowledge, then Sean's continued averted gaze during her speech, even as her speech described not just mathematics, but her activity (e.g. in line 17 and 19 Mary's utterances describe her drawing, and reasons for changing it), and his corresponding inattention to Mary's speech was itself a small resistance to joining Mary in the work of drawing the triangle, and therefore further evidences a preference for working from his book.

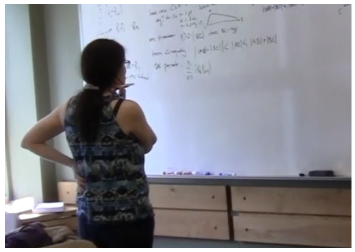
Third, though Sean did respond to Mary's question "does that look. . . " in line 14, first by looking up, then by saying "sure", several factors indicate that he treated this response as something of an interruption. First, his low volume and use of sure (which seems to signify less commitment to the answer than "yes." would have) indicate a relatively weak response to this question. Furthermore, by criticizing and redrawing the triangle in the subsequent lines (e.g., line 17), Mary treated his response, as not sufficiently strong to verify the drawing was good. This response indicates that she also read his quiet "sure" as a weak response. Finally, the speed that he returned his gaze to his tablet—averted even during his answer—indicates an attempt to quickly return to his work. The quiet weak response, and the speed he returned to his tablet together indicate that, at this time, Sean treated his engagement with Mary as an interruption of his work on his tablet.

Fourth, after he evaluated Mary's drawing as "close enough" in line 39, he was relatively persistent in working on the tablet. That is, Mary had to repeat the question three times before he responded without either turning his gaze immediately back down, or not even looking up at all. Just as Mary's persistence in asking the question evidences that she was treating the drawing as a joint activity they were both responsible for, Sean's persistence in not giving further evaluation is an act of resistance to joining her in the joint activity of drawing the triangle, and so expresses a preference for working in the private space of his tablet, and not with her.

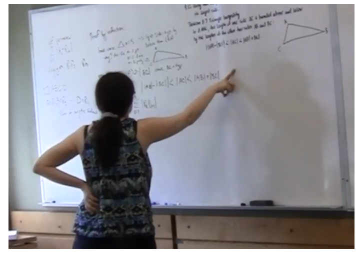
One objection that could be raised to this analysis is that in line 21 Sean looked up in response to Mary's question, and resumed his part in the division of labor, until he eventually evaluates her drawing as "good enough" (lines 36-7). That he looks up, however, only suggests that he was willing to watch Mary, and that her work, and the potential for it to be their joint work, is something of a distraction from his work reading—a point that reinforces the thesis of this section, that Mary's vocal work distracted him from reading. Mary's question in line 21, however, was not completed until line 32. Until it was not clear what Sean was answering, and so he would not have able to provide any sort of an answer. Since we see elsewhere (lines 14, 42, 44, 46) that, even as Sean works from his tablet, he responds to Mary when she addresses him, it is not surprising that Sean continued to look up until he could answer, nor that, since he has been looking for longer, he was able to give a more thorough answer to Mary's question. That said, even when he gives an answer, Mary's persistence in asking for further evaluation is evidence that he has not been as attentive as she would like, and so, even here, we see a slight preference for reading.

These four points together show that, even at this early stage, Sean attempted to work from his tablet, but was interrupted by Mary's address—an address that, given Mary's orientation to her and Sean's joint work is entirely appropriate. They show further that that, from the beginning of the episode, Sean had attempted to work from his tablet, but pursuing that work was challenging precisely because of the way Mary pursued her work, or rather, from her perspective, their work. He later was able to pursue his tack for a relatively lengthy amount of time, but, because of his late start, and because, even then, Mary's talk sometimes interrupted him, it is not surprising that Sean did not contribute to his and Mary's proof construction.

Sean's failure to contribute, then, was, at least in part, the result of the two students' different behavioral tacks: Mary worked aloud at the board, on, what seemed to her, to be a joint activity; whereas Sean worked in silence. This difference of orientation that left him vulnerable to interruption from her work, while protecting her from interruption from his.



(a) 562s – 566s, line 4 in the transcript on page 153.



(b) 594s – 600s, line 22 in the transcript on page 153.

Figure 3.4: Mary remains silent and motionless. Times are from figure 3.3.

### Mary's Silences

This analysis suffices for Sean's silences, and shows the ways that his silence and Mary's speech made achieving uninterrupted work difficult for Sean. But what about Mary? The group was silent, together, for 43% of the episode. What did Mary do while she was silent? Setting aside silences that involved large body movements, like writing at the board, were her silences an important part of her mathematical activity?

On numerous occasions, clustered just before she found a proof, she fixed her gaze on the board, and remained silent and motionless, often with her hips squared to the board, for more than a second (the times she engaged in this sort of activity are shown in the top line of figure 3.3). She engaged in this activity for a total of approximately 94 seconds, all but 2 during this 10:45 episode. Furthermore, at least five other students (Sean, Alan, Sam, Seth, Hannah) engaged in similar activities, the activity is similar to the mathematicians' silences (Petersen, 2017b, 2018b), and, at least sometimes, this activity immediately preceded a key insight. Because Sean was silently reading most of the times Mary adopted this posture, there was little interactional "interference" with Mary's action, and so attending to this posture in this episode will be helpful in sketching its contours; while ways peers responded to this sort of activity will be analyzed in sections 3.4.2, 3.4.3, and 3.4.4.

Several features of Mary's behavior here, all visible in the following transcript, are noteworthy. First, unlike Sean's silences, during which he physically withdrew from the common space of the board, Mary remained oriented to the common space of the board during her silences. Second, because Mary fixed her eye-gaze on the board, her silences display (or at least claim) an ongoing engagement with mathematics—though, perhaps in a way that students are not attuned to notice. Third, though Mary did sometimes use self-directed speech (lines 1 and 3 in the transcript below), Mary's silences were in the midst of full-voice speech (e.g. lines 24–26 below; her posture there is shown in fig 3.4b). Third, even in the midst of her silences, Mary sometimes turned to and addresses Sean (line 17 below).

In this transcript, magenta highlighting indicates that Mary was engaged in a ruminating silence, whereas, no highlighting indicates that Mary is not. In order to give visual prominence to silent time, highlighted silences are expanded so each highlighted character represents 0.1 seconds.

Transcript 3.3

| | | |
|---|---|---|
| 1 | M: | °AB minus AC° |
| 2 | | (1.8) |
| 3 | M: | °(seven)° |
| | | **Fig 3.4a** ▼ |
| 4 | | (4.4) |
| 5 | M: | Well I could always do a triangle inequality |
| 6 | | (1.3) |
| 7 | M: | subtract things; |
| 8 | | (1.3) |
| 9 | M: | So I can always get BC alone by subtracting stuff around abc |
| 10 | | abc abc abc |
| 11 | S: | hm |
| 12 | | (1.9) |
| 13 | M: | I cou:ld |
| 14 | | (2.0) |
| 15 | M: | a:dd (0.3) AC to everything |

```
         (1.6)
M:   right;
     (0.5)
S:   mhm
         (2.5)
S:   s-s-
     (0.7)
M:   m
       Fig 3.4b
       ▼
                         (7.9)
M:   cause that would give me the perimeter over here (0.3)
```

These features of Mary's behavior indicates that, unlike Sean who had, to some degree, withdrawn from interaction to his private space, Mary's ruminating silences were a part of her attempt to interactively find a proof—in this respect, they are more akin to speech than to Sean's silences. That is to say, Sean's physical posture, including the direction of his eye-gaze to text only he could read and his physical distance from the common space of the board, indicated that though Mary was present, he was attempting to work privately. Mary's silence on the other hand, was an aspect of her attempted public interaction with Sean. That her silences were an aspect of attempt to find a proof collaboratively has several consequences:

First, if Mary's speech posed a challenge for Sean's attempt to work on the problem, even though he had attempted to partially withdraw from interaction, there is reason to suspect that ruminating silences present an interactional challenge. This claim is strengthened because Mary's silences violate the every-day English norm that silences are minimized, and silences longer than a second are marked (Erickson, 2004; Liddicoat, 2011; Stivers et al., 2009). Though we do not see it in this episode—Sean was largely silently reading when Mary engaged in these activities, see 3.3—there are good reasons to suspect that in other episodes, when students engage in the sort of activity Mary does, the silences are threatened by the speech of other group members.

If that hypothesis is true, Sean's lengthy silences provided a helpful environment for this aspect of Mary's mathematical activity.

Second, Mary's silences were a feature of her Sean-oriented discourse, and so Sean's silences were, in many respects, a suitable response to these silences. That is, his silences allowed to continue whatever work is accomplished during the these silences, without interruption. If it is true that Sean's silences provided a suitable response to Mary's silences, then two results follow: First, since silence in response to silence violates the norms of everyday conversation, a silent response to a ruminating silence, is a learned activity. Second, Sean's silent activity was not just a neutral backdrop that allowed Mary to work uninterrupted, but, at least at a few times, was an ideal environment for her labor.

### Conclusion

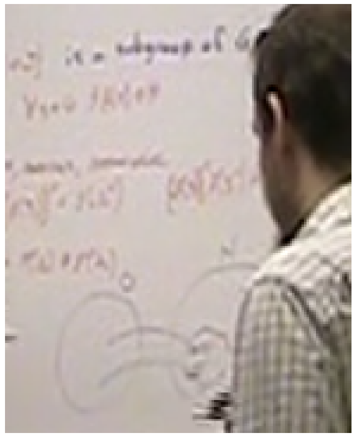
In this episode, Mary produced a proof, largely without any mathematical contribution from Sean. This striking difference in their contributions to the proof can be traced, at least in part, to their different approaches to silent mathematical activity: Sean approached the proof primarily by reading silently from the book and his notes, whereas Mary attempted to talk her way through a proof at the common space of the board. This difference meant that Sean's activity was repeatedly subjected to interruptions from Mary's, whether through her speech, or more commonly, through her direct address to Sean, who she took to be immediately collaborating with her, and so had considerably less time to pursue a line of inquiry. On the other hand, because he mostly spoke when Mary engaged him, Sean never interrupted Mary's work, and she was able to pursue a coherent tack throughout the episode: This includes times when, during her common work, she stopped and stared at the board, an activity that raises important interactional questions, questions which will be raised and addressed in subsequent sections.

### 3.4.2 Sam and Nick (and Amy): More on Reading

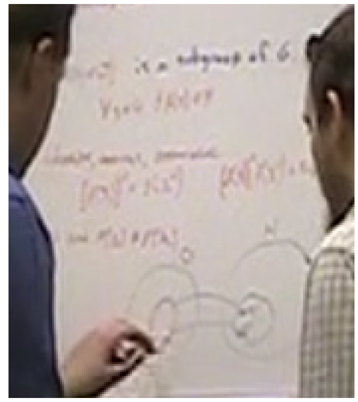
In this section, three students in an introductory Group Theory course, Sam Nick and Amy, attempted to prove the following theorem: Let $G$ and $H$ be groups, and $J$ a subgroup of $H$. Show $f^{-1}(J) = \{x \in G | f(x) \in J\}$ is a subgroup of $G$.

Before the session, Amy had looked up a proof online, but wanted to see if the proof she found was like the one her peers would come up with. She occasionally made a few small comments—e.g. reminding Nick and Sam of the definitions of "injective" and "surjective" when they were confused about which was which—but for the majority of the episode, she watched her peers work. For that reason, this section is treated as two-person collaboration between Sam and Nick.

Throughout the beginning of their work session, Nick and Sam took different mathematical tacks to the problem. Nick consistently attempted to focus the group's attention on $f$, in an attempt to show that it is invertible (see lines 14-21 in the transcript on page 3.4, which follow an earlier statement to the same effect, and anticipate at least two later statements), and was inattentive to the set $f^{-1}(J)$. The problem asks them to show $f^{-1}(J) = \ldots$, which, when terminated there, seems to ask them to prove something about $f^{-1}$, so this is an intelligent reading of the problem. Sam, on the other hand, repeatedly attempted to focus the group's attention on the set $f^{-1}(J)$, and then, to show that the group axioms hold for elements of this set (see lines 10 & 27–30 of the transcript on page 3.4). In addition to the times shown in



(a) Nick's diagram of the problem is at the bottom. The left oval symbolizes $G$, the right, $H$. The small oval inside $H$ is $J$, and two elements are shown mapping from $G$ to $J$.



(b) Sam (left) adds the circle inside $G$ symbolizing $f^{-1}(J)$.

Figure 3.5: Sam and Nick's coconstructed diagram of the problem.

this transcript, he attempted to focus the group's attention on $f^{-1}(J)$ several times in later portions of their work session, including one time where he explicitly added arrows to the diagram showing that $f$ may not be one-to-one.

The student's different interpretations are reflected in a diagram they co-constructed at the beginning of the episode analyzed here, see figure 3.5. Nick's reading is reflected in figure 3.5a, which he drew without Sam's assistance. In this initial version of the diagram, $f$ is a one-to-one mapping from $G$ to $J$, but $f^{-1}(J)$ is not included in the diagram. Sam's reading, on the other hand, is reflected in his addition of an oval symbolizing $f^{-1}(J)$ to what had been Nick's diagram (see figure 3.5b).

In Nick's reading and Sam's ruminating silences, these different readings of the problem came to a head.

The following transcript contains their discussion as they drew and discussed diagram shown in figure 3.5. During this discussion, Sam engaged in a lengthy ruminating silence and Nick walked to a table where his book was laying, picked it up, and read. Critically, Nick's reading activity caused him not to hear Sam's explanation of the problem and the difficulty he had in finding a proof (lines 27–35). On the other hand, Sam's ruminating silence, though oriented to the discussion nevertheless, is not straightforwardly a turn in the conversation. In the following transcript, black underline indicates that Nick's eye-gaze was on his book, a green underline indicates that though he was not looking at his book, he was working on reading (e.g. walking to pick up the book, walking back to the board where he would stand as he read, book in hand, with his eye-gaze not following Sam's gestures), and magenta highlighting indicates Sam was engaged in a ruminating silence (again, ruminating silences are expanded to 1 character per tenth of a second). Though this episode was the most clear instance of the sorts of interaction examined below, the sorts of patterns examined in this section occurred several other times during Sam and Nick's collaboration.

Transcript 3.4

N: I- I keep just seeing it (.) i- in terms of visuals you have
(1.0) G and (0.7) H
(2.3) ((Drawing))
N: mmm (0.3) J (0.6) here
(2.1) ((Still Drawing))
N: and uh we map from G to H and
S: mhm
N: uh (0.3) you can also map you know some of it is going to
land in J ((draws)) (0.6)
S: Yeah we're trying to pick the specific subgroup that does so
(0.5)
N: ok
S: Yeah (0.3)
N: So: (0.3) ok (0.3) yeah (0.2) and then we also want to s:ay
that the (.) uh inverse (.) of the function that is here
will (0.3) put them back in the same place in e- in G
(0.9)
S: n: not necessarily; ((eye-gaze to N)) cause that would be an
isomorphism ((N turns back to pick up his book)) wouldn't
it;
N: Well it sounds to me like that's what they're asking for
(2.7)
N: it says (.) if:

```
     (0.5)
N:   [let me put my sunglasses down] ((turns back, removes his
     sunglasses from head))
S:   [>s o<=   a  l  l      a  l  l] (0.2) I think this this
     image covers it right we need to prove that this ((Nick
     turns and walks to board)) is truly a subgroup of G (0.7)
     but then to do that as Amy said we need to show that it’s
     closed under identity and inverses and associative but then
     how do we refer to its elements arbitrarily (0.7) and I
     think that- (.) I think that’s what I’m trying to get onto
     here I just know that I’m not using the right notation for
     it .hhh (.) I don’t think this is right hhh
```

Roth (2003) argues that graphs function to focus the attention of interactants on specific features of otherwise abstract objects, and, in this, to highlight points of disagreement. If Roth is correct, Sam and Nick’s co-constructed diagram (figure 3.5) could serve to bring their disagreement over the solution strategy into focus, and make the strategy an explicit object of negotiation. Indeed, Sam repeatedly appealed to the diagram to support his reading of the problem (line 10, when he draws the oval symbolizing $f^{-1}(J)$; and lines 27-29, pointing to the representation of the set), and Nick appealed to the diagram in line 21, and then turned to the book to substantiate his reading. It seems, therefore, that the diagram at least began to focus their disagreement. The students, however, never successfully discussed their different readings of the problem, nor did they agree on a common tack to pursue. Following this episode, Nick continued to read in his book, and occasionally to claim that it seemed to him that they need to show the function was injective. Sam, on the other hand, continued to attempt to find a way to refer to arbitrary elements. Indeed, it is unclear if they ever resolved the question of what tack to pursue. What accounts for this failure to use the graph to discuss their different readings of the problem?

Part of the issue here was that Nick was reading rather than listening when Sam articulated his reading of the problem, and the tack he believed they should follow (lines 27-32). This problem can be analyzed through the lens of each student's activity. On the one hand, Nick did not listen when Sam talked. On the other hand, Sam talked to Nick while Nick was visibly reading and not listening. Because Sam's talk in line 27 was immediately preceded by his ruminating silence (lines 21–24), analyzing this passage from the perspective of Nick's reading and Sam's rumination can throw light on both reading and ruminating silences.

In line 21, Nick reiterated his claim, from lines 14–16, that they were supposed to show $f$ was invertible, and began to walk to his book to find evidence for this claim. Because Nick was visibly working to substantiate his claim, a response was, perhaps, not required at this point. Nevertheless, in response Sam fixed his gaze on the board for over three seconds—that is, he responded by ruminating. Several factors support this reading of Sam's rumination as a response to Nick's claim. First, the mathematicians in Petersen (2020b) used their eye-gaze to display ongoing engagement with conversationally relevant mathematics, even during their silence. Similarly, the direction of Sam's eye-gaze displays an engagement with conversationally relevant mathematics. Furthermore, its initiation displays a *new* engagement with conversationally relevant mathematics, and Sam's rumination is initiated when a response could be expected. Third, Sam's activity here is not an isolated incident but part of a broader pattern of rumination in response to a peer. Specifically, several other students ruminated at a time when a response could be expected, at least twice following direct questions. Finally, Sam's talk in lines 27–35 fits this model of Sam's ruminating activity, since it commences as soon as Sam stopped ruminating, and is hearable as a response to Nick's claim in line 21. The fact that it is uttered at a time Nick likely cannot hear it only makes this fit between model and Sam's talk closer, since it indicates that, following the rumination, Sam didn't even take time to see if his talk would be heard.

If the hypothesis that Sam's rumination in lines 21–24 was a response to line 21, is accepted, this passage provides further confirmatory evidence that rumination is ordered to immediate interaction. (As opposed to Sean's reading, which was part of a temporary withdrawal from interaction.) On the other hand, this rumination provides more nuance to the way ruminating silences relate to the surrounding talk. Specifically, it was somewhat askance to the discourse, since 1) we have no evidence that Nick "heard" it, and 2) it resolved in speech at a problematic time for achieving communicational ends.

Turning to Nick's reading activity: The group did not immediately debate adopting Sam's strategy (and in fact, never adopted it), in part because Nick was reading when Sam attempted to articulate his reading of the problem. This interference of reading activity suggests that reading can pose a temporary threat to verbalized forms of mathematical activity. In particular, a reading student may not hear talk that is critical in a peer's argument. Furthermore, this disconnect did not only affect their reading of the problem. In lines 30-32 Sam was seeking aid in addressing a particular issue that he has already faced, that he returned to several times later in the episode, and that the group never addressed. For that reason, Nick's reading—though not treated by the students as contrary to collaborative norms—posed a threat to Sam's ability to advocate for the group's adoption of his preferred strategy—and so, could in some sense, be said to have silenced Sam.

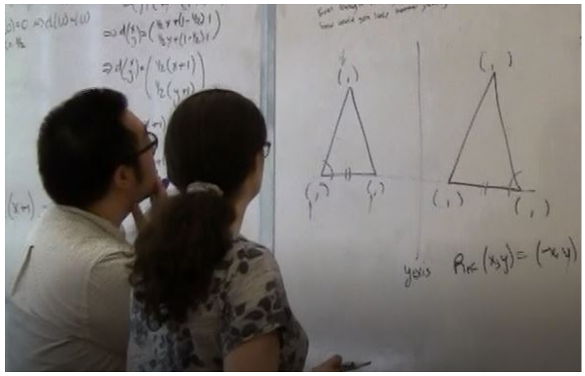


Figure 3.6: Alan (left) ruminating in line 8 of transcript 3.5.

#### Conclusion

Inasmuch as Sam ruminated in response to Nick, this section has given further confirmatory evidence that rumination is not a withdrawal from face-to-face interaction, but is instead a communicative aspect of the face-to-face interaction, whose goals are caught up in the conversation itself. (In a better world, Sam's rumination may have even served as a demonstration that his eventual verbal response was not hasty, and so that he had taken Nick's claim in line 21 seriously.) Furthermore, it shows that though rumination is a part of conversation, it can stand somewhat askance to discourse, since the speech it aims at may not be well timed in relation to peers' ongoing activity. Finally, it shows that just as Mary's spoken mathematical activity posed a threat to Sean's reading activity, so Nick's reading activity posed a threat to Sam's spoken activity.

### 3.4.3 Alan, Mary, and Sean: More on Ruminating

In this section, three students, Alan, Mary, and Sean, (and eventually a fourth student, Isaac) worked on the following problem: "You're a high-school teacher, and a student comes to you asking why the Side-Angle-Side (SAS) congruence theorem holds. How would you answer?” For the majority of the time before Isaac arrived, Mary and Sean worked together on the problem, and Alan watched their collaboration—though he eventually offers the only solution that explained the SAS theorem rather than its converse. Because he largely watched Sean and Mary work, it may be tempting to conclude Alan was passive during the group’s work on this problem, however, he seems to have been actively watching Sean and Mary work. For instance, at one point he moved to get a better view of the board, and at others he laughed at appropriate times. For these reasons, it is probably better to interpret Alan’s work as a form of legitimate periphery participation. (Though this sort of activity could be studied in more depth, it is outside the scope of this study.)

Because the majority of the dialogue in this episode is between Sean and Mary, it is instructive to compare the silence in this episode to the silence in the induction episode analyzed above. In contrast to that episode, in which 43.6% of the episode was spent in mutual silences, in this episode, only 13.7% of the time before Isaac came in was spent in mutual silences. Furthermore, in that episode, 18% of the mutual silence was spent in silences under 1.5 seconds long, and 29% of the silence was in silences over 5 seconds long; on the other hand, in this episode 35% of the mutual silence was spent in mutual silences under 1.5 seconds long, and only 20% of the time was spent in silences over 5 seconds long (a single 8 second silence).

Though the students talked for much of the episode, with little silence between them, there were two significant ruminating silences. The first, which is analyzed here, shows some of the communicational difficulties that can arise around ruminating silences. In particular, one student can enforce the norm that silences should be minimized, by talking to a ruminating peer. In the second silence, which is not analyzed but is worth mentioning, Alan ruminated for several seconds while Sean, Mary and Isaac talked before offering the best answer the students came up with, and, as noted above, the only one that correctly explained the SAS theorem rather than its converse.

As above, magenta highlighting indicates times when Alan has his gaze silently and motionlessly fixed on the board.

Transcript 3.5

M: Well I know but this is like something I'd consider with my
kid at home too
S: Yeah yeah
(2.5)((Mary's eye-gaze to Alan))(0.4)
S: [Ok so
M: [Alan; what would you like to see ((Mary eye-gaze to board))
what do you think about
(0.8)(1.0)
M: Other approaches there's gotta be stuff that's no:t like
graph paper ((Mary eye-gaze Alan)) approaches too
(1.0)
M: Like um ((Eye-gaze to board)) (.) ↑you know what we u:h did
in our group for the ((eye-gaze Alan)) um isosceles triangle
(0.8)
S: ((clears throat))
(1.2)
M: [proof
A: [((smiles & shakes head))
(0.7)
S: mhm
M: ((steps to the board to write)) cause we could also do
something like that we could take a triangle
(1.0)
M: right let it be an isosceles triangle;
A: mhm
(1.5)
M: And if we just
(1.0)
M: right; ((eye-gaze to Sean))

Just prior to this section, Sean and Mary had been discussing how they could use a mirror to demonstrate the (converse of) the SAS congruence theorem, and whether it would be wise to trust high school students with mirrors. In line 3 of the transcript, their conversation lapsed. Both Sean and Mary responded to the lapse at the same time, and both seem to have initiated a new line of inquiry. Though Sean's did not develop further, "ok" and "so" are both used in conversation to indicate the introduction of a new conversational tack (Bolden, 2006; Heritage, 1984). Mary, on the other hand, initiated a new tack by looking at and addressing Alan, who had been silently watching Sean and Mary work.

Alan responded to Mary not verbally, but by silently fixing his gaze on the board and placing his hand on his mouth. This hand gesture seems to display "thinking", broadly defined, as seen by its use in the thinking emoticon. Furthermore, Petersen (2020b) found that in their extended silences, mathematicians fix their eye-gaze on the board in order to display an ongoing engagement with the conversationally relevant mathematics. Finally, this posture conforms to the pattern seen in both Mary's ruminations and especially Sam's. In these students' activity, rumination seems to be an aspect of conversation, in which posture and eye-gaze are used to display ongoing engagement with conversationally relevant mathematics. Indeed, just as Alan's rumination seems to be a response to Mary's address, Sam's rumination was a response to Nick's talk. These lines of evidence support the hypothesis that Sam's ruminating activity was a response to Mary's address.

If this initiation of a new ruminating activity, with its concomitant display of a new orientation to conversationally relevant mathematics, constituted a response to Mary's inquiry, then Alan's activity was in accord with the strong conversational norm that a response should follow a yes/no question in under a second, especially when, as here, a speaker's gaze is on the recipient (Stivers et al., 2009).

On the other hand, Alan's response violates the conversational norm that silences longer than a second should be minimized. Additionally if an analyst—including one of the participants in the collaboration—only attended to the auditory channel of the interaction, Alan's activity would seem to violate the norm that responses should follow questions in under a second. These two facts show that it may be easy for a peer to misread Alan's response as a violation of conversational norms, and so, to enforce the norm that a *verbal* response should come less than a second after a question. This seems to be how Mary analyzed and responded to Alan's silent response. Specifically, she twice responded to Alan's silence as contrary to the conversational norm by reiterating her question, with variations, and soliciting a response with her eye-gaze (for eye-gaze soliciting response, see Stivers et al., 2009). First, after the 1.8 second silence in line 8, Mary offered a possible direction a response could take ("not graph paper"), and nominated Alan to speak by her eye-gaze (lines 9–10). Second, following another second of silence (in line 11) she used the second person "you" to addresses Alan, suggested a new tack he could pursue, and again used her eye-gaze to nominate him to speak (lines 12–13).

This disjunction between Alan's conformity to the norm that a response should come after one second (through his silent ruminating) and Mary's enforcing the same norm, as if Alan had violated it, is critical. The students' actions show that though Alan's ruminating activity was a response to Mary's query, and though his posture demonstrated (or at least claimed) that he had adopted new form of mathematical activity, and was newly engaged with conversationally relevant mathematics, his activity was liable to interpretation as a failure to engage properly in conversationally relevant mathematics. This double valence of his action, in turn, provides an existence proof that, at least at times, when students are collaborating on mathematics, it can be interactionally difficult for students to engage in ruminating activities.

Following these two attempts to repair the lack of a response, Mary stepped back to the board, and initiated a new conversational tack herself, this time with Sean, not Alan (see her eye-gaze in line 29). Though Alan did shake his head (very slightly) in line 18, perhaps indicating that he did not have an idea—though only after Mary had repeatedly addressed him—the data suggest the hypothesis that Alan was not able to give a response precisely because Mary's attempt to enforce the norm that (verbal) responses should follow questions within a second interrupted his ruminating activity, and she then, rather than waiting for Alan's verbal response, initiated her own tack. Though it cannot be fully substantiated, if this hypothesis is true, then Mary's reading of Alan's ruminating as a norm violation provides a partial explanation for Alan's continued peripheral participation in his group's work. On the other hand, that result seems less generalizable than the result from the previous paragraph; namely, that, because they ruminate in silence, with relatively low-key actions, it can be interactionally challenging for students to engage in ruminating activities.

### Conclusion

By showing that Alan's ruminating silence was a response to Mary's address, this section again confirms the results established in section 3.4.1 that students engage in ruminating silences are an aspect of face-to-face collaboration, not, like reading can be, a temporary withdrawal from conversation. Furthermore, both this section and section 3.4.2 showed that the initiation of a ruminating silence displays a new engagement with conversationally relevant mathematics, and so can constitute a response to an address. This second example strengthens this existence proof.

Additionally, this section showed that parsing these responses can be difficult for students. Rather than analyzing a ruminating silence as a form of response, peers may read it as a norm violation, and seek to enforce the norms that silences be minimized and responses follow addresses by interrupting the ruminating silences. Because students treat ruminating silences as an important aspect of mathematical activity by engaging in ruminating silences, and as an important means of finding and articulating an adequate response by engaging in them in response to a peers' address, this result suggests that conversational norms may make it difficult for students to engage in a key aspect of mathematical activity. Additionally, Petersen (2020b) shows that, in their collaborations, extended silences which resemble ruminating silences are in accord with interactional norms, and that mathematicians use their gaze to display continued with conversationally relevant mathematics. This implies that mathematicians' collaborations are supported by specific norms and conventions for engaging in silent ruminating activities. Lew and Mejía-Ramos (2019) found that students may need to learn to follow the specific linguistic conventions that govern mathematical proof writing. Similarly, Mary's difficulty reading Alan's ruminating silence as a response indicates that students may need to learn specific linguistic conventions governing rumination in mathematical collaborations.

### 3.4.4 Seth and Hannah: Ruminating and Commonality

In this section, Seth and Hannah worked on a homework problem requiring them to show that the half-open unit interval $[0, 1)$ is neither open nor closed. They began their work by writing this problem on the top left of the board, and so setting an overall trajectory for their work (see Greiffenhagen, 2014). They then wrote the definition of open (in their words, "for all $a \in A, \exists \delta > 0\ B(a, \delta) \subset A$") and of closed on the board. They did not, however, at this time, attempt to continue the proof directly, but paused at length to make sure they were able to accurately utilize the definitions. In the passage analyzed in this section, they examined the definition of open by proving that the open interval $(a, b)$ is open. Afterwards, they showed that a closed interval is closed. Only after they had successfully shown that open intervals are open and closed intervals closed, did they work on the homework problem itself.

Perhaps because of the notational ambiguity between $a$ in the definition of an open set, and in their open interval, Hannah and Seth collaboratively unpacked the definition of open as meaning that "$(a - \delta, a + \delta)$"—which Hannah first said aloud, and Seth reiterated, and wrote on the board—"is", as Seth said, "a subset, even if $a$ is the lower bound." Though this statement is false, since $a$ is not an element of $(a, b)$, Seth realized their mistake, replaced "$(a, b)$" with "$(x, y)$", and correctly used the definition to prove the interval $(x, y)$ is open. Following this successful proof, Seth correctly applied the definition of open to other sets (for instance, he correctly applied the definition while proving the complement of the closed unit interval is open). It therefore seems that in this episode he solidified a correct understanding of open. Furthermore, both students returned to the definition of open to make sure Hannah understood, and only proceeded to the main proof, which they completed collaboratively, after Hannah confirmed that at that point she had come to understand—as she said, "That makes sense to me now. I didn't get that before."

Just prior to this segment, Hannah had verified with Seth that, on their reasoning, the open ball centered at $b$ would also be a subset of $(a, b)$, and wrote "$(b - \delta, b + \delta)$

| | Verbal | | | Non-Verbal | |
|---|---|---|---|---|---|
| | Seth | Hannah | | | |
| 20 | [a]If a is the (0.4) lower bound [b]and [1]b is the upper bound[c] (0.9) [2d]since [3]a and [4]b are not [5e]in the set, | 9.2 seconds a b c d e | | [1]moves hand to "b" in "(a,b)" [2]leans l, [3]traces finger along "(". [4]traces finger along ")". [5]eg H | [a]writes ", etc." [b]finishes writing, head l [c]stands straight, eg to hands, replaces cap on pen [d]eg r, rhw/pen toward tray [e]eg l (where S points) |
| 21 | [1] | [1a]mm | | [1]eg back to "(a,b)", rh begins to move l. | [a]hands in sweater pockets, shakes elbows once |
| 22 | [1]then, | | | [1]beats on "a". | |
| 23 | 14.4 seconds | 7.6 seconds | 7.0 seconds | | |
| 24 | | | | | |
| 25 | | a b | | | [a]reaches for pen [b]hand to trunk, steps forward [c]looks r, drifts forward. [d]eg to S's hand [e]eg r [f]glances l, eg up&r [g]eg l |
| 26 | | c | | | |
| 27 | 1 | d e | | [1]eg r [2]leans r, slides finger r [3]stops slidding [4]eg up, rh moves toward chin | |
| 28 | 2 | f | | | |
| 29 | 3 4 | g | | | |
| 30 | 1 2 3 4 | Oh[1] (0.5) await a minute. (° [b] °) (0.[c]6) [d]°well this is- should [2]be the [3e]problem), [4f]this is the definition. (0.[g]6) [h]right°; | | [1]eg up, rh at chin, which he pats [2]eg H's hand [3]steps back [4]eg down (not following H's hand) to where it was before moving (l. 30, 1). | [a]rh follows eg, to "[" in "[0,1)" [b]eg right taps "[" [c]eg back l, rh to sweater pocket [d]reaches toward ")" [e]taps pen twice on ")". [f]Rh gestures down past everything lower on the board [g]rh to sweater pocket [h]removes cap from pen reaches toward board |
| 31 | wait[1] I'm doing something wrong[2] now (0.2) because that doesn't quite fit | a | | [1]head twitches r, then he steps r& slightly back [2]steps l &slightly back | [a]writes on board (obscured) |
| 32 | | 11.5 seconds a | 3.4 sec | | [a]finishes writing, steps back [b]finishes stepping, eg r [c]eg l, then down |
| 33 | | b | | | |
| 34 | | c | | | |
| 35 | cause a minus delta [a](0.8)[1] (is not in there) | a | | [1]turns right | [a]eg up |
| 36 | [1] | | 2.6 sec | [1]walks r | [a]eg up &r [b]eg l &down |
| 37 | [1] | a | | [1]still walking | |
| 38 | [1] | b | | [1]still walking | |

etc." on the board. As she completed that, Seth continued, in lines 20–22 unpacking the definition, with an incomplete utterance. Incomplete statements are often used to solicit someone else to complete the utterance (Lerner, 1991). For instance, rather than asking what the integral of $x^2dx$ is, a teacher may state "The integral of $x^2dx$ is..." expecting a student to finish the utterance. Here, however, that does not seem to be the purpose of the incomplete utterance for three reasons. First, Hannah does not talk at all, let alone complete it, for seven seconds, and Seth does not treat Hannah's missing completion as a norm violation. Second, as the analysis will show, Seth did not treat Hannah's eventual utterance as a legitimate completion, but, in line 31, refocused the group's attention on the same point it had been when he began to be silent. Finally, Hannah's eventual utterance in line 30 begins "oh" which functions in conversation to indicate that the following talk does not flow immediately from the preceding work, but initiates a new line of inquiry (Bolden, 2006; Heritage, 1984).

How then did Seth and Hannah orient to the incomplete utterance? What sort of actions to they pursue following it? And how do the differences in their orientations contribute to group function? Attending to these questions will help sharpen the understanding of ruminating activities.

Their actions that followed the incomplete utterance contrast markedly with those that preceded it. Prior to the incomplete utterance, both engaged in relatively large motions (line 20, notes, 1, 2, 3, 4, a, b, c, d, e; line, notes 21, 1, a; line 22, note, 1), and solicited interaction (line 20, note, 5) or responded verbally (Hannah's talk in line 21). Following the incomplete utterance, both Seth and Hannah motionlessly fixed their eye-gaze on the text "$(a, b)$ (line 21, notes e and, 1) which they had written on the board. (And Seth continued to point at $a$.) That is, the students ruminated and displayed continued silent engagement with the open interval $(a, b)$, in concert.

Furthermore, for Seth, this motionless silence was not an isolated instance: he returned to the same posture two turns later (lines 32-34). Additionally, in contrast to the interactive aspects of his prior action, when Hannah spoke in line 30, his eye-gaze only followed her briefly (line 30 notes 2 to 4), beginning well after the start of her turn, and ending well before the end of her turn. Additionally, though he addressed her in line 31, in contrast to his address in line 20 (note 5), in line 31 he did not solicit a response with his eye-gaze. Third, his directive "wait" (line 31), indicates that he treated the new direction Hannah had initiated with "oh" in line 30 as erroneous. Finally, this directive constituted a refusal to respond to her question the line prior. It was only then that he redirected the group's work to the interval $(a, b)$, and took up again the same posture as before. These four points together show that not only did Seth engage in ruminating activities following his incomplete utterance, he treated rumination as an important activity for him to engage in as he addressed the problem with their work, namely, "$a$ minus $\delta$ is not in there" (line 35). This result further underscores the importance of rumination activities in student work. Furthermore, the fact that he had to explicitly work to return to a ruminating activity provides further evidence for the conclusion of the last section, namely, that there can be interactional challenges to engaging in ruminating activities (though, in this segment, Seth was easily able to resolve the challenges).

Hannah's behavior, on the other hand, contrasts with Seth's. After the first two seconds (that is, two seconds before Seth moved at all), Hannah picked up a pen (note a), thus orienting toward the production of new semiotic content, stepped to the board (note b), and shifted her gaze rapidly through different things the students had written on the board (notes c, d, e, f, g): First her eye-gaze shifted to her statement "$(b - \delta, b + \delta)$ etc.", and then then to a sequence of other locations that are difficult to identify, none of which was held long. Her eye-gaze finally rested on the original statement of the problem. As she moved to the board and shifted her gaze, though she continued to display an engagement with conversationally relevant mathematics—and her activity could probably be described as thinking about a solution—her thinking was a restless searching thought, but not rumination.

Shortly after her eye-gaze rested on the original statement of the problem, Hannah uttered the change of state marker “oh”, which makes public a new realization and line of conversation (Bolden, 2006; Heritage, 1984); and the directive “wait a minute”, which indicates that someone had been moving, but needed to stop, wait (as does Seth’s “wait” in line 31), and attend to whatever new tack the “oh” signified. But who had been in motion?—Seth had been motionless!

Her speech following “Oh (0.5) wait a minute” is unintelligible, and then, since “well” is another change of state token, perhaps interrupted. Her gestures, however, are more informative of her action initiated at the beginning of line 30. After her directive, she pointed specifically to the left closed bracket in the original statement of the problem (line 30, notes a and b). Notably, Seth’s incomplete utterance had just indicated a problem with the open left endpoint of an interval, whereas Hannah pointed to the closed left endpoint of the interval in the problem statement. Furthermore, if the interval Seth should be working with were closed, then, contrary to Seth’s claim in line 20, $a$ would be in the set. (In her talk following the 0.6 second pause she seems to have been asking for confirmation that she had pointed to something that wasn’t immediately relevant, since they weren’t trying to solve the problem itself, but to understand how to use the definitions they would need to solve the problem.) This gesture, therefore, may indicate that at the start of line 30, Hannah was seeking to address the problem Seth had run into during his incomplete utterance, and was beginning to claim, plausibly, that Seth was working with the wrong interval: They should be working with an interval that was closed on the left. If this interpretation is correct, it was precisely Seth’s ongoing, motionless, rumination that Hannah treated as a form of motion that needed to (temporarily) stop so she could articulate a possible solution to the difficulty.

On this interpretation, then, she oriented to the silence as a time of mutual silent mathematical activity, during which both students attempted to address a problem with the left end-point of the interval $(a, b)$, and to Seth's rumination as a form of mathematical activity. Therefore, in contrast to Mary's response to Alan's silence (analyzed above), there was not a conflict in the norms the students followed. Rather, both students oriented to the silence as a time for mathematical activity, and to Seth's lengthy rumination as an important form of mathematical activity, addressing a common problem.

Nevertheless, Hannah incorrectly identified the problem with the left end-point. The problem Seth had run into, as he says in line 35, was that $a-\delta$ is not contained in $(a, b)$, for any delta, and so the open interval seems not to be open. This problem was a fundamental problem with the procedures Seth employed to interpret the definition of open, and therefore cannot be addressed by merely finding typos in their work (as Hannah attempted to). That is, her attention to the left-end point of the interval in the original problem is not just unhelpful because, as she noted, they're working on the definition, not the problem. Even had she correctly identified a problem with their work, she would not have found a solution to the problem Seth had encountered. On the other hand, it is difficult to see how she could have known more about the problem than she shows—something was wrong with the left end-point of the set $(a, b)$: Seth had not articulated anything further.

Prior to the silence, Seth and Hannah had been able to collaboratively unpack the definition of open by applying it to the point $a$. Furthermore, as Hannah was writing, she turned to the same section of the board as Seth (e.g. line 20 note e). And when not actively collaborating, the students were able to work in parallel toward the same goal: For instance, Hannah had earlier correctly interpreted the "for all" in the definition of open, suggested the statement they had produced should also hold at the other endpoint of the interval, and at any other point in the interval, and written those facts on the board. Though Hannah was able to see where the problem lay, since Seth did not articulate the problem, Hannah could not know its details or helpfully work on resolving the problem. Therefore, Seth's ruminating silence posed a challenge to the commonality of the students' mathematical activity. Furthermore, since articulating the problem is incompatible with ruminating about the problem. This result, therefore, implies that ruminating activities exert a sort of centrifugal force on group cohesion, pushing group members onto different tacks.

#### Conclusion

The previous three sections analyzed ways that one student's silent mathematical can interfere with another student's spoken mathematical activity, and the ways one student's spoken mathematical activity can interfere with another student's silent mathematical activity. Inasmuch as it showed that students treat reading and ruminating as normally sanctioned forms of mathematical activity, both by explicitly sanctioning them and by engaging in them even in response to their peers, these sections showed that students treat ruminating and reading as important aspects of their mathematical work. Because Seth not only ruminated, but stopped Mary's talk, in order that he could continue to ruminate, this episode provides further evidence that students treat rumination as an important form of mathematical activity. Furthermore, in this episode, Seth's rumination was directly connected to his noticing a problem with their work, his solving the problem, and ultimately, his learning the meaning of the definition of open.

Furthermore, the students' collaboration in this section contrasts with the students' collaboration in the previous episodes. In this episode, Hannah correctly reads Seth's rumination as a form of mathematical activity, and directs her attention to the precise location of the problem. Furthermore, not only does Hannah attend to the precise location of the problem, she joins Seth in rumination. Nevertheless, even with Hannah's skillful reading of Seth's ruminating activity, and the students' skillful coordination of their ruminations, the ruminating itself exert something of a centrifugal force on the students' collaboration. They were able to overcome the centrifugal force, so the point is not that the rumination harmed their communication. Rather, the point is that they had to engage in activities to overcome the centrifugal force of their ruminations, and that, without those actions to overcome that centrifugal force, the coordinated ruminations would have directed the students onto different tacks.

## 3.5 Conclusions

Previous studies of silence in mathematics collaboration, together with survey data collected in the course of this study, suggest two reasons investigations of silence in mathematics education may be important. First, there is evidence that mathematical activity requires lengthy periods of focused silent concentration (Petersen, 2020b). Moreover, Lim (2017) argues that these forms of silent mathematical activity are a neglected topic in mathematics education research and practice. Furthermore, survey data collected with this study indicates that a significant minority of students prefer to avoid group work becacuse the pressures to talk and listen in a group prevent them from engaging in the lengthy forms of silent concentration necessary for successfully learning new material. These results indicate an intervention targeting both social and individual aspects of sielnce in proof may be fruitful (for the importance of proof-related interventions, see Stylianides et al., 2017).

Second, in their collaborations, mathematicians follow linguistic conventions for the use of silence that differ from those found in every-day conversation (Petersen, 2020b). Both previous research (Petersen, 2015, 2018a) and annecdotal acounts of teachers' difficulty thinking while tutoring suggest that students do not consistently follow these conventions. Lew and Mejía-Ramos (2019) argued that since since students often do not know to follow the linguistic conventions governing the style of proofs, these conventions are be a barrier to student apprenticeship into the mathematics profession. Similarly, if the linguistic conventions governing the use of silence in face-to-face mathematical collaboration differ from those students follow in their group work, then these conventions may likewise pose a barrier to student apprenticeship into the mathematics profession.

Both the possibility of a fruitful intervention into proving practices, and the difference between the linguistic conventions followed in mathematicians' and students' collaborations, indicate the importance of an investigation of silence in student mathematical collaborations. This study seeks to initiate that inquiry by answering two research questions: First, how do students, when collaborating on their homework, employ silence, in ways that are counter to the norms of every-day discourse. Second, how do those practices affect the group dynamics and students' ability to make progress in discovering and writing their proofs?

In answer to these questions, this study has five main findings. First, it showed that as they collaborate on their homework, students engage in (at least) two forms of silent mathematical activity, reading and, what was called "ruminating". Reading silently is a sufficiently well-known phenomenon that it does not require further description. Rumination, however, in this paper, refers to a time when a student silently fixes their gaze to the board, and stops moving. Because rumination has not been previously identified, the identification of rumination as an important form of mathematical activity is a second key finding of this paper.

Colloquially, we might say that ruminating students are "thinking" (as Petersen, 2020b, does). "Thinking", however, has strong connotations of inner processes, whereas, in this paper, the focus is on the students' embodied activity, and the social implications of rumination. Therefore, in order to avoid these conotations, the term "thinking" was not employed in this study. Furthermore, when describing the kinds of activities students engage in, the activities should be specified as precisely as possible. Since there is no research from an embodied, social, perspective into different kinds of thinking, and there therefore may be interactionally different species of thinking, it would be a theoretical error to assume that the term "thinking" best specifies a particular activity.

Third, this study found that when at least one student engages in silent mathematical activity, the students may, unproblematically, pursue their work in parallel, without negotiating a common tack. These parallel tacks are unproblematic inasmuch as the students do not treat it as a norm violation, and sometimes even explicitly confirm that a reading peer is pursuing legitimate activity, and so is excused from conversation. Furthermore, even as their peers engage in silent reading activities, students may continue to talk, as if to their peer were listening (perhaps putting further conversation on offer).

Fourth, even though reading and talk are both treated as normal forms of mathematical activity, the interplay them can have negative effects on both the reading activities, and the group's disourse. First, talk can prove distracting for reading students, and limit a reading student's ability to contribute to a group's proof production. Second, reading can prevent student from hearing their peer's arguments and even responses, so that critical claims and arguments are articulated, but do not become a part of the group's common work.

Fifth, students not only ruminate, several of their practices treat it as an important form of mathematical activity. Specifically, they ruminate at critical times in their work, like just before finding a key question, they ruminate in response to each other's address, and they at least occasionally seek to quiet their peers talk so they can ruminate. Additionally, students' ruminating activities resemble mathematicians' behavior when they are silently engaged in thining about mathematics in their collaborations (Petersen, 2020b). That said, rumination presents its own challenges for mathematical collaboration. First, when students' ruminate, their activity is similar to activities that violate conversational norms that silence be minimized. This inaudibility of rumination is particularly problematic when students respond to their peers questions by initiating a ruminating activity. The initiation of a ruminating displays a new engagement with conversationally relevant mathematics, and so could be hearable as a response. Nevertheless, because rumination can be difficult to hear as rumination, and so as a response, students sometimes treat the initiation of rumination as a norm violation, and seek to enforce the normally expected talk by speaking to, and so interrupting, the ruminating student. Furthermore, even if a peer hears a rumination, and use its display of attention to a particular place on the board to coordinate their thinking with their peers, rumination exerts a centrifugal force on student group work.

Several important questions need further exploration. First, though this research identified two silent activities, and noted ways that they interact with mathematical speech, no evidence is presented about the prevalence of the activities. How often do students, when collaborating, engage these activities, and how often do the sorts of interactions between them and mathematics speech occur? Second, this research was conducted in only one setting: students in small groups—usually just groups of 2—collaborating on homework problems, at the board. Do similar activities occur in the classroom, or when students collaborate on paper at a table? Third, how is gender implicated in these activities?—one mathematician at a conference noted, without further explanation, that gender is an important complicating factor re: silence and mathematics. Fourth, some students in the introductory survey noted that they preferred to work alone because of the quiet, yet silence was present here, and is far more present in mathematicians' collaborations: How can we foster positive collaboration between students who prefer to work in quiet, and students who prefer to work in groups—to the benefit of both. Finally, though mathematicians utilize silence far more extensively than these students did, nothing is known about how students come to adopt the practices of mathematics.

Finally, it may be worth comparing these results with Petersen (2020c), which found that like the mathematicians here, students in junior-level proof classes engage in a form of mathematical activity involving the motionless fixation their gaze on the board. Unlike this study, however, Petersen (2020c) refrained from characterizing this activity as “thinking”, and instead described the students activity as “rumination” rather than “thinking”. This terminological difference is due to the different research questions in the two papers: (Petersen, 2020c) examined student activities, and so a precise specification of the activities was important. Because empirical research has not determined whether, from a social perspective, thinking is a unitary phenomenon, it was better to use the specific term “rumination” in that paper rather than the generic “thinking”. In this paper, however, since the focus is on norms, and mathematicians’ activities are only analyzed inasmuch as they can shed light on norms, a generic term is better. Nevertheless, it can be very tentatively suggested that, for mathematicians, mathematical thinking is rumination, and perhaps other closely related forms of thinking.

# 4

## Norms for Silent Thinking Activities in Mathematicians' Collaborations

> What makes my *Thinker* think is that he thinks not only with his brain, with his knitted brow, his distended nostrils and compressed lips, but with every muscle of his arms, back and legs, with his clenched fist and gripping toes.
>
> –Auguste Rodin

Thinking, theorized as a private, interior act, has been a major focus of mathematics education research (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Sfard, 2008a; Steffe & Thompson, 2000). For instance, Steffe and Thompson (2000) spell out a very influential methodology for investigating student thinking. The goal of their methodology was to use public statements to gain access to what they saw as an interior state and act, namely, thinking. So for instance, on their reading, students' mathematics are merely "indicated by what they say and do" (p. 268). The researcher's job then, is to "look behind what students say and do" (p. 269) through models of students' interior ways of thinking. Even when Sfard (2008a) takes up thinking from a sociocultural perspective, it is defined in terms of a private and invisible activity, namely, "an *individualized* version of the (interpersonal) communicating" (p. 81, emphasis mine). The revolutionary aspect of her work lies in redefining thinking so that collective activity precedes thinking, rather than being the result of thinking. But thinking itself is still theorized as an internal act.

At the same time, there has been some pushback against an exclusive focus on the interior act of thinking. For instance, researchers writing from a sociocultural perspective have argued that mathematics is a particular activity practiced by members of a community of practice that students are apprenticed into (Lave & Wenger, 1991; Sfard, 1998). Others have examined particular activities mathematics students engage in, e.g., justifying, defining, conjecturing (Rasmussen et al., 2004). Similarly researchers writing from an embodied perspective have investigated ways that mathematical activity is constituted by various activities (Radford et al., 2017). Researchers from these perspectives often contrast their approach with research that focuses on thinking. For instance, in order to focus on mathematics as a cultural production, Rasmussen et al. (2005) deliberately shifted the focus of research from thinking to mathematical activity. Likewise Nemirovsky, Kelton, and Rhodehamel (2013) attempted to provide a thoroughgoing reformulation of mathematical activity as a form of embodied practice, akin to playing a piano or organ, rather than as thinking.

It is not entirely clear, however, that thinking itself cannot be theorized as an embodied, public, communal, act; and analyzed using techniques of ethnomethodology and Conversation Analysis (CA). These tools would allow a researcher to attend to public, embodied aspects of thinking, without reference to its character "inside the head". In order for an act of thinking to be analyzed from this perspective, people would have to recognize and orient to a public acts of thinking, during the interaction itself; and an analyst would attend to the ways thinking, as a public act, shapes and is shaped by other actions people engage in during the interaction. To take an example somewhat removed from face-to-face interaction and which therefore risks introducing some confusion, researchers utilizing an ethnomethodological perspective on thinking could study how the thinking emoji, 🤔, is used in text-message conversations. Similarly, in face-to-face collaboration: If there are forms of embodied behavior that co-participants in a given activity recognize as "thinking" and use to structure parts of their interaction, then, precisely for that reason, those acts of thinking are amenable to an ethnomethodological analysis of the way public acts of thinking structure forms of face-to-face interaction. From this perspective, therefore, an analyst could not say that people always think while listening or speaking. Rather, people think while listening or speaking if there are aspects of their listening or speaking that other members of the conversation recognize as thinking, and which therefore structure the conversation itself. There are therefore two hypotheses that need established before thinking can be analyzed from an ethnomethodological/CA perspective. First, in order for people to recognize an act of thinking, the thinking needs to be recognizably embodied. Second, people need to orient to the thinking in their collaborations.

The embodied nature of thinking is perhaps easier to see than its public, communal nature, and so makes a better entry point for inquiry. Chess, for instance, is a physically demanding activity. A grandmaster may burn approximately to 6,000 calories on game day—by sitting and playing chess. World Champion Magnus Carlsen even employs sports scientists to help him adopt an optimal diet for the physical rigors of chess, and to make fine adjustments to his posture that allow him to minimize unnecessary energy loss from for instance, holding his head up. He also refuses to review strategies on game day because doing so uses valuable strength required for the match (Kumar, 2019).

It may be true that it is not the thinking, per se, of competitive chess that is physically demanding, but the stress inherent in the focused competitiveness of the thinking. Mathematicians, however, are unlikely to think in the same way grandmasters do, and so mathematical thinking will not be as physically demanding as competition chess. While this is an important observation, it does not show that thinking is not an embodied activity, but rather suggests that different forms of thinking are embodied differently. The competitive, stressful, thinking of chess (and presumably of other similar games) is physically different from the thinking involved in, say, reminiscing. Mathematicians' thinking is also, presumably, different from both reminiscing and playing chess—what ways it differs is an empirical matter. If there are in fact physical differences between kinds of thinking, then it is likely that different acts of thinking are available for analysis, by social scientists, but more significantly, by people engaged in face-to-face collaboration. That is to say, if different kinds of thinking are physically different, then thinking may not only be externally embodied, but also may be a socially relevant aspect of face-to-face encounters.

That thinking sometimes has external components can be seen in some of the postural commonalities between people engaged in different sorts of acts of thinking, postural commonalities that can be read as "thinking". For instance, Rodin's statue is scratching his chin, a posture world champion Carlsen often takes in games (for instance, GRENKE Chess, 2019, 4:58:00), as do the mathematicians in this study (see figure 4.7), and a mathematics lecturer in Looney et al. (2017) as he thinks about the next step in his lecture (see also Canagarajah, 2018, who analyses this same lecturer's thinking activity as modeling mathematical activity for students). Indeed, this sort of hand-gesture is iconic enough that Looney et al. (2017) describe it as a "thinking gesture"—that is, as a gesture that shows thinking—without further citation.

In a provocative footnote Schegloff (2007) helps further open up the possibility of theorizing thinking as an interactional achievement. Schegloff's point of departure is the dual nature of noticing as both a psychological and interactional achievement. On the one hand, noticing can be understood as a private, internal psychological act. To take an example from the mathematics education literature: Dietiker, Males, Amador, and Earnest (2018), rightly, treat pedagogical implementation as the culmination of acts of noticing, and so as an aspect of noticing. Yet two of their three aspects of noticing (curricular attending and interpreting) are conducted in private, and the last (curricular responding) is based off these earlier two. On the other hand, noticing is an interactional act. For instance, conversations are structured, in part, by what Schegloff (2007) calls a *conversational preference* for noticing over telling, so conversationalists will work to make their peers notice something—and realize that noticing is relevant—rather than tell them about it. Furthermore (interactionally) noticing the wrong thing can violate cultural norms for what sort of thing should be noticed, by whom, and so be treated as picky or an insult. Conversely noticing the right thing can express solidarity. Indeed, even an object of mutual knowledge can be noticed in order to be picky, insulting, comforting, solicitous, etc. Schegloff claimed these considerations had wide-reaching implications, and offered them as a way of "stimulating reflection on, and... inquiry into... other aspects of the organization of conduct whose understanding may be deepened by seeing their connection to practices of talk-in-interaction" (Schegloff, 2007, p. 88). Inasmuch as mathematical activity involves the activity of thinking, thinking is an aspect of the organization of conduct, at least for individuals; and therefore, following Schegloff's suggestion, our understanding of the practice of mathematical thinking may be deepened by attending to thinking, not as a private interior act, but as an embodied and publically observable activity connected to talk-in-interaction, and therefore as an aspect of interaction.

Inasmuch as thinking is temporally extended, posturally embodied, act, when people work together, thinking is, like noticing, socially achieved. There may, on the one hand, be times when thinking is projected and called for (and too fast a response is hearable as hasty or insultingly thoughtless). At other times thinking may be insulting. For instance, thinking may indicate that something that should be readily agreed on is not obvious (e.g. that an action was praiseworthy), and we are all familiar with trying to talk to someone persistently lost in thought. Likewise, there may be particular actions people take to making thinking relevant now, and ways of responding to thinking that valorize or censure different kinds (careful, hasty) of thought. For instance Steffe (2003) and Tzur (1999), analyzing the same data, contain transcripts of lengthy silences in response to Dr. Tzur's questions, and analyze the silences as time the students were concentrating or thinking. These silences violate the ordinary norms of discourse (Sacks et al., 1974; Stivers et al., 2009), and so these silences were not a mere absence of talk. Rather, Dr. Tzur actively chose to let these particular silent thinking activities continue, and so valorized that type of thinking. Furthermore, if these students had attempted to think in a similar way when collaborating with their peers or with other teachers, precisely because the silences violate conversational norms, doing so may have been extremely difficult. The social achievement of thinking has drawn some attention in the literature. For instance, Goodwin and Goodwin (1986) found that, when people are searching for a word they cannot remember, one practice they use to solicit help, is adopting particular postures which they called “thinking face”.

At the same time, thinking is often conducted in silence. This is seen, again, in Rodin’s statue, who is clearly silent. But it is also shown in the common instructional practice of giving students an extended period of silent “private think time” before they begin discussing a new problem. But lengthy silences violate norms of conversation, and so, during face-to-face dyadic collaborations (what Goffman [1966] calls *“focused activity”*), extended periods of silent thought present an interactional challenge. That is to say, the act of thinking silently is part of a common collaborative project, and whatever actions people engage as they think are common achievements. If people think silently during focused dyadic interaction, it is only because other participants did not enforce a norm that silences are minimized, or otherwise seek to engage the thinker in further conversation. And so, in turn, it is only because each participant chooses to respond to this “thinking” as in accord with the norms governing their work that anyone is able to think. During face-to-face conversations, people are only able to “take time to think” (that is, since they are “taking time” to think silently), because one displays their activity for the their peer to read, and the other reads the fact that the other is thinking, and responds accordingly.

This study extends these results about thinking by inquiring into mathematicians’ thinking as a form embodied, temporally extended, silent, mathematical activity, by attending to one aspect of thought: Silence. Previous research (Carlson & Bloom, 2005) has shown that, in their individual problem-solving sessions, mathematicians go through cycles of silent work. Furthermore, (Petersen, 2017a) argued that this silent activity persists even in mathematicians' collaborations. During at least some of these the silences Petersen presented, they seem to motionlessly stare at the board (Petersen, 2017b, 2018b), and so, presumably, their activity during these times can be broadly, and pre-theoretically, referred to as thinking. These facts make silence a natural entry-point into an investigation into the relation between thinking and face-to-face conversations in mathematical collaboration. The research question this paper addresses is: "What are the norms governing silence in mathematical collaboration?" A second, subsidiary, question, asks what hypotheses for the functions, both interpersonal and communicational of the particular silent practices mathematicians employ can be generated.

## 4.1 Theoretical Perspective

This study takes an embodied interactional perspective on the mathematical activity of thinking (Sfard, 1998). The previous section argued thinking can be theorized as an embodied, interactional activity, closely related to talk-in-interaction. The lens used to gain a perspective on the activity of thinking, however, is silence. This choice of a lens may, however, seem to raise more questions than it answers: Is not silence merely the lack of speech or of sound—and perhaps even a form of oppressive passivity, as in (Freire, 1967, p. 54)—and not a phenomenon in its own right? This section therefore will need to theorize silence as a phenomenon in its own right, and connect it to embodied and interactional work. The argument will therefore precede in three steps. First, it discusses silence as a phenomenon in its own right. Second, it connects silence to the body and to interaction. Finally, it turns to the theory behind the methodology employed in this paper.

Silence is not a mere absence or a lack (Acheson, 2008; Ephratt, 2011), but a phenomenon, actively heard with our ears, that both frames sounds and words, and is in turn framed by sounds and words (Acheson, 2008; Chrétien, 2004). So, for instance, as Dauenhauer (1980) notes, a performance of music is only heard as a unity because of the silences that bracket it. On the other hand, Handel often underscores dramatic moments in his music with lengthy silences (Harris, 2005), which are only heard as dramatic parts of the music because they are surrounded by sound (see also Kim, 2013). Nor is silence one-dimensional: The sorts of sounds that bracket a particular silence, and the posture and gestures employed during a silence, give a particular color and meaning to silences (Acheson, 2008; Margulis, 2007a, 2007b). Finally, silence is not a default state, but is actively produced. Thus, for instance, silence can be a design feature of buildings (Bonde & Maines, 2015; Ergin, 2015; Kanngieser, 2011; Meyer, 2015); and we are all familiar with how difficult holding our tongue can be. These considerations do not show that the silence of silent thinking may be an important aspect of mathematical thinking, but they do show that the silences of silent thought are not a sort of background default, but are a presence, and so are potentially amenable to analysis, and potentially influence the shape of collaboration.

Furthermore, because silences are actively produced and heard, they can bear particular meaning (Acheson, 2008; Ephratt, 2011). Though they are not words, they can therefore perhaps be understood as a particular sort of gesture; a gesture which we can perform in concert with others, or alone while others are speaking (Acheson, 2008). For instance, Quaker worship is structured by lengthy collaborative silences (Lippard, 1988), and the bond of a nursing mother with her infant can be strengthened through mutual eye-contact and silence (Maitland, 2008); more negatively, the children Philips (1983) studied on the Warm Springs Indian Reservation communicated that they were actively listening, not through eye-contact or back-channeling (e.g. "mhm"), but through silence, leading to communicational failures with their Anglo teachers.

Sfard (2008b) defines a gesture as a form of communicational movement. This definition fits well with actions like pointing at the board, but it is hard to describe the sort of lengthy silences mathematicians engage in as a sort of movement. It may, therefore, be helpful to shift perspective slightly, from gestures to postures. Mathematicians' silences are not a form of movement that mathematicians' employ to convey meaning, but a posture which they inhabit (this distinction between gesture and posture is from Saussy, 2014). The silence of the mathematicians' silent thinking activities are one aspect of their postures, along with things like the direction they are facing, how much they sway, where their eye-gaze is directed, etc.

It may be helpful to distinguish between two different types of silence. First, one person can remain silent while the other talks. In everyday two-person conversation, when someone needs to talk at length, they usually give short responses called "continuers" (Schegloff, 1981) that serve to demonstrate knowledge that, at that point in the conversation, longer turns are inappropriate, and to claim an understanding of the preceding talk. This means that, in two-person conversation, neither participant is usually silent for long stretches of time, and that if one person remains silent for long periods of time, that silence may be marked. On the other hand, just as someone may not object when they recognize that they have been interrupted (Sacks, 1992) (since doing so may open one to the criticism of being picky), someone in a conversation may not orient to a continuer that they notice is missing. Furthermore, even if the other person explicitly objected, the objection itself does not end the silence. For this reason, if someone remains silent for long segments of a conversation, their silence may violate conversational norms, but each person is able to be silent on their own.

### 4.1.1 Conversation Analysis

This study draws off the methodology of Conversation Analysis (CA) to study thinking in its connection to talk-in-interaction. CA was developed by Sacks and his colleagues (Sacks, 1992) to study how social organization is done through talk, and what aspects of conversation members of conversation attend to in order to analyze the conversation and determine their conversational course of action. One of the fundamental theoretical insights that lead to CA was the realization that conversation is not only about particular objects, but that each participant's talk can be understood as part of a *sequence* of actions that do things (Sacks, 1992; Schegloff, 2007). Schegloff (p. 2) illustrates this principle with the following example (numbers in parentheses indicate silence, in seconds, and punctuation indicates *prosodic* elements of speech):

```
Mom  = hhh Whooh! It is so hot tuhnight. *Would somebody like
          some more ice tea. ((* = voice fades throughout TCU))
          (0.8)
Wes: Uh(b)- (0.4) I('ll) take some more ice.
```

Here, Mom's statement should not be understood as *about* iced tea, so that Wes' response changes the topic. Rather, Mom's statement should be understood as doing (or making) an offer (that addresses the heat), and Wes' response as a valid acceptance of an alternative solution to the heat problem.

Each of these inter-personal acts are both conditioned by the previous turns in talk, and project a limited set of normal types of action through which other participants in the conversation have the opportunity and normative obligation to exercise their agency. Furthermore, the projective force of an utterance is often strong enough, that: 1) A missing response is conspicuously absent, and even gives a particular response. For instance, following some requests, a lack of an answer is hearable as a decline, whereas, following a question like "am I bothering you", silence functions as "yes". 2) Propositionally unrelated responses can be heard as responses. For instance, to use an example from Schegloff (2007), "Isn't it supposed to rain?" can serve as a response to a question like "Shouldn't we leave for the ball game?".

These considerations mean that, in the above example, Wes could have accepted or declined the offer, he could (as he did) accept an alternative, etc. But he couldn't do just anything without his response being censured as a violation of a norm. For instance, as Garfinkel (1967) demonstrated, if he adopted a theoretical interest in precisely delimiting the meaning of the terms "hot" and "tonight", mom would probably have treated his response as extremely rude, and gotten upset at him. More mild responses—both declining the offer—include saying nothing, and claiming that a first glass had cooled him down sufficiently.

Furthermore, speakers can take actions during their turn at talk that perform conversationally important actions like nominating a subsequent speaker, or soliciting a response. For instance, Stivers and Rossano (2010) found that, even when people are not engaged in active conversation, one of the resources a participant in the conversation can utilize to make response more relevant is to turn their eye-gaze toward the person they want to respond. Rossano (2013) shows that this is not only a resource speakers can utilize to pick a conversation back up, it is also a resource utilized within a conversation to solicit a response at this moment of time, an effect that (Stivers et al., 2009) found holds across cultural and language types.

This is not to say that participants in conversation cannot respond in ways that redirect the questions, delay an answer, or even violate the project established by other participants. Rather, if they do, the fact that their response isn't one of the immediately projected kinds actions needs to be set up with certain purposively employed discursive moves. For instance, if a participant is going to introduce new material that is not related to the projected actions, they will preface their utterance with particles, called *change of state tokens* whose function is communicating that the following utterance is not directly projected by the previous turns. For instance, in conversation, *turn-initial* "so" indicates the introduction of new material that has been latent in the conversation. Turn-initial "oh" is used to displays a change of state in the speaker's situation (often, their acquisition of new knowledge), and so the introduction of new material (Bolden, 2006; Heritage, 1984). And "well", indicates that the following talk may be obliquely relevant to the projected action, but will privilege the speaker's perspective over the projects of other members of the conversation (Heritage, 2015).

Furthermore, talk is not only sequenced, it is sequenced in a particular way: Most conversational actions are initiated by one participant, and a second participant follows the first's talk by responding. The talk that initiates an action is called a *first pair part (FFP)*, the talk that responds, a *second pair part (SPP)* (Liddicoat, 2011; Schegloff, 2007). So for instance, in the above example, mom's talk initiates the action of offering and so is an FPP, whereas Wes's talk responds to Mom's initiative, and so is a SPP. Additional actions can come be inserted at each step in this exchange.

When actions precede an FPP (generally by preparing for it, as for instance, if, in the example, mom had said “Excuse me” to get everyone’s attention before her offer), they are called *pre-expansions*. When an action comes between FPP and SPP it is called an *insert expansion* (in this example, if Wes had said “what was that?”, his talk would have initiated an insert expansion). Finally, when they follow a SPP, they are called a *post-expansion* (in this example, mom could have responded with the post-expansion “no problem”).

Additionally, silence has different meanings at different points in a conversation. To make that point, another distinction, from Goffman (1966) is necessary: That between *unfocused activity*, in which people are together, but they do not share the same cognitive and perceptual focus, and *focused activity*, in which they do (Couper-Kuhlen, 2010; Schegloff, 2010a). So for instance, people who work together at a common office are engaged in unfocused activity, as are many people during meals. On the other hand, people engaged in ongoing conversation are engaged in focused activity. While silence is contrary to the norms of conversation, and seemingly to forms of focused activity involving talk, it is not contrary to the norms of unfocused activity: People sharing a meal can eat, people sharing an office can work. Critically, focused activity can transition into unfocused activity, and it generally does so following SPP’s or post-expansions. This means that, following an SPP or post-expansion, silence is not always problematic (if it indicates that a conversation has transitioned into unfocused activity). On the other hand, following other turns in a conversation, like an FPP, silence is problematic, in every-day conversation.

These facts have important implications for an analysis of mathematicians’ silences. To make these considerations and their implications concrete consider the following episode of two mathematicians Matt and Bart, collaborating. This episode is the entirety of the "Bart Pushed Back" episode from Smith (2012). In the transcript, punctuation is not prosodic, but Smith's interpretation of the utterances. Arrows highlight lengthy silences.

```
   Brt: And these things
        (3.0)
   Brt: are the same, right?
        (1.0)
   Brt: Is that correct?
        (3.0)
   Mtt: ( ) (0.5) ( )
 →      (8.0)
   Brt: True?
 →      (27.0)
   Mtt: Well, let me just show you-
   Brt: Oh of course it is, it's zero! Uh, I mean that's the kernel
        (.) so these things are zero. (.) u_q vanishes on this
        intersection.
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Though the transcript is short, this episode shows a violation of the cross-cultural norm that silences should be less than a second long, particularly following FPP's like polar (yes/no) questions (Stivers et al., 2009). Nevertheless, an initial analysis of this passage suggests that Bart's repeated reiteration of his question in lines 5 and 9 was an attempt to repair a missing response, and so this norm is not entirely absent here. That is, his polar question in lines 1 and 3 seems to have projected an immediate response from Matt. Furthermore, since Bart waited about 14 seconds before the second reiteration of the question, and, with no further attempt to repair the missing response, for the subsequent 27 seconds; whatever actions Matt (and Bart) engaged in during those 41 ($= 14 + 27$) seconds were also projected by the preceding activity (whether by Bart's question, or by actions Smith does not record), and so, in this context, are normal. Indeed, the participants treated this activity, undertaken in silence, as important enough that engagement in it could override the normativity of a projected answer to Bart's question. Nevertheless, though Bart's actions during the 27 seconds of silence were normal, his use of "well" indicates that, without this change of state token, his eventual speech was not, otherwise, sanctioned, but instead initiated a departure from the actions Bart's question or the preceding talk had projected, in favor of a mathematical project he saw as, in some sense, relevant to their work. Finally, Bart's oh-preface of his final utterance (l. 12–14) indicates that the utterance is the result of something new, likely some new realization on knowledge that bears on his work.

An analysis of the norms for the usage of silence in mathematicians' collaboration can be judged by how fully it allow a description of this passage to be filled out. What did Matt and Bart do during these silences? What sorts of silent actions (in their specificity) were projected by Bart's and/or the preceding speech? What, precisely, was it about Matt's silent actions that made them normal, in this situation? Which utterances project those actions (for instance, is Matt's silence here projected by Bart's questions in lines 1 and 3, or by some earlier talk)? Does the silence function as what CA analysts would call an insert expansion, that is roughly, that whatever else is going on during it, this particular sort of silence can be semantically interpreted as "I will respond, but first. . . "? Or is the silence itself projected, so that these particular silent actions function somewhat like that in the exchange N: "Please pass the salt." M: (passes the salt)? Why did both Matt and Bart's sanctioned silent activity precede utterances that the participants treated as not sanctioned? An analysis of the norms can be judged by the quality of answer it facilitates for this sort of question.

Furthermore, if we temporarily grant the assumption (and this assumption will be argued for in more depth below) that one of the overriding norms in mathematical collaboration is that all the participants work toward mathematical progress, the answers to the research question provides a subset of the actions that, when they are collaborating, mathematicians treat as normal, sanctioned, mathematical activity. That is to say, addressing the research questions in this paper is one way of addressing empirically what actions constitute mathematical activity.

## 4.2 Literature Review

In her description of her pedagogical strategies, Lampert (1990) identified thinking, along with other activities like explaining, answering and revising, as of central importance to the culture of a mathematics classroom focused on student debate and rigorous, student-led justification. On her analysis, just as an important goal of the mathematics classroom is training students to recognize and employ the sorts of explanations and answers to be treated as mathematically appropriate, so the meaning of mathematical thinking is an important objective of the mathematics classroom, negotiated between teacher and student.

In practice, Lampert's (1990) attention to thinking consists in describing the public classroom work, through which the students as a classroom worked-through the issues a question or problem raised, as a form of public thinking. And this is certainly a salutary practice. She did not, however, thematize the act of thinking silently, and indeed objected to classroom silence. This decision seems to be in-line with Lampert's attempt to make reasoning a collective activity of the whole classroom, since thinking is often theorized in terms of interior processes. For instance, Steffe and Thompson (2000) laid out a methodology for using students' talk and actions to formulate and test hypotheses about the private cause of their actions, namely, their thoughts. Likewise, from a Vygotskian perspective (Vygotsky, 1978), as in Sfard (2008a), external, interpersonal activities like communication are internalized into psychological processes like thought (Ernest, 2008).

There is reason to think, however, that the act of thinking silently is also subject to classroom negotiation, both in its character and its relation to other mathematical activities (as will be argued). Several avenues of research support this claim, both outside and inside the mathematics education research literature.

The first analysis that suggests that the character of thought may be subject, if not to formation and classroom negotiation, to public observation comes from film criticism. Cavell (1981) argued that a salient feature of Cary Grant's acting in his classic silver-screen films is his "photogenic tendency to thoughtfulness, some inner concentration of intellectual energy" (p. 164). In recognizing Grant's portrayal of an "inner concentration of intellectual energy", Cavell aligns with a theorization of thinking as an internal process, as in, for example, Sfard (2008a), as discussed above. On the other hand, Grant was an actor, and therefore Cavell's observation draws attention not primarily to the content of Grant's characters' thinking, but to its *photogenic* quality. The observation of Grant's photogenic thoughtfulness is suggestive, and, in noting the embodied, postural, character of thinking Cavell makes an observation similar to both Rodin's description of his *Thinker* which serves as an epigraph to this paper, and to Goodwin and Goodwin's (1986) argument that when people need to think about a word choice, they signal this with a facial expression they call "thinking face". This leads to an assumption that is fundamental to this study: Thinking is an embodied activity, undertaken with particular postures, and can therefore be observed.

Cavell's (1981) continued analysis of Grant's work, however, enables a deeper understanding of the social character of thinking. According to Cavell, Grant's thoughtfulness has a different character in his films. For instance, in *Bringing up Baby* and *Monkey Business* Grant shines in his portrayal of a thoughtful, but absent-minded, professor. In *His Girl Friday*, on the other hand, the kind of thinking Walter (Grant's character) engages in, is legible in his "drumming or fidgeting fingers and his shifting eyes" (p. 164). These characteristics distinguish Walter as thinking "incessant[ly] and compulsive[ly]" looking for a new angle to take so he can catch the world unawares, without its mask, and so capture the next big news scoop. This analysis suggests that not only is the fact of thinking embodied, and therefore observable, but that first, the precise character of silent thinking can be seen and observed, and in that, its connection to other activities is also legible. If this reading of the visibility of Cary Grant's characters' thinking is true, then silent thinking's character and relation to other mathematical activities is at least potentially amenable to formation in the classroom.

This hypothesis that the character of silent thinking can be seen and observed by teacher and peer, and even shaped by teacher activity, has some support in the mathematics education literature. First, Lim (2017) argues that silent, private, engagement with mathematics is an important mathematical activity, and, indeed, of mathematical discourse. Further, Lim argues introverted students, in particular, may gain particular benefit from engaging in silent mathematical activities. On Lim's reading, not only are silent aspects of mathematical activity important, they can be fostered or hindered by the shape of the classroom: For instance, a classroom that (rightly) centers discussion may also fail to similarly center important activities that involve silent, private, thinking about challenging questions that arise in the course of mathematics discourse. On the other hand, a teacher can center silent engagement with specific mathematical questions, and so mathematically benefit the whole classroom, but especially, according to Lim, introverted students who may find too much talking alienating.

One of the most interesting sets of papers that can be read to show the importance of the act of thinking in the mathematics classroom are Steffe (2003) and Tzur (1999). Both papers analyzing elementary students work understanding fractions during a year-long teaching experiment. Though the public act of thinking is not the subject of these papers, both the researchers, and, seemingly, the classroom teacher, Dr. Ron Tzur, interpreted particular student acts as important, silent thinking activities. For instance, in Steffe (2003), after Dr. Tzur posed a question to a student, the student sat "silently for approximately twenty seconds" erased something "and again [sat] silently *in deep concentration* for approximately 50 seconds" (p. 276, emphasis mine). Similarly, earlier in the same paper, the same student responded to Dr. Tzur's question of which fraction out of a collection of fractions should be ordered after six eighteenths by "put[ting] his head down and think[ing]" (p. 247), an action that Dr. Tzur explicitly valorized by telling him to "think of the next one" in response. On the other hand, in (Tzur, 1999, p. 406) two students think together for approximately 15 seconds before Dr. Tzur acted as if this act of thinking would best end, by talking to the students.

The focus of both Steffe (2003) and Tzur (1999) is student construction of fraction schemes, and so neither paper examines the thinking activities described in the previous paragraph. It is clear, however, that in their work of constructing these fraction schemes, students had to engage in multiple thinking activities. Furthermore, these thinking silences are much longer than the one second that is the approximate maximum length of normal silence in every-day conversation (Erickson, 2004; Liddicoat, 2011; Stivers et al., 2009). This suggests that Dr. Tzur's decision to remain silent with the students reflects his exercise of expert judgment. Finally, Dr. Tzur responded differently to these three different thinking activities: In the first case, he allowed a student to think silently for 70 seconds, in the second, though the student thought for less time, he valorized the student's action, but in the third case, he judged that it would be better to suggest a new tack than to allow the students to continue thinking. Since he is an expert teacher, it is reasonable to suggest that Dr. Tzur was responding to differences between these situations. If this hypothesis is accepted, then these papers sketch an example of an expert mathematician responding to different acts of silent thinking. Furthermore, in that response, he did some work forming the character of silent thinking, and its relation to other mathematical activities.

Two other sources point to the importance of thinking activities, and of teacher shaping those activities. First is the practice of wait-time (e.g., Tobin, 1986, 1987; Tobin & Capie, 1983). These studies found that when teachers give students at least three seconds following questions, the quality of student responses increases. For instance, students give more complete answers and ask more probing questions. These gains were theorized to be the result of student need for processing time to respond well or formulate questions, and so as time to think through answers. Indeed, some authors argued that these silences should be not be referred as "wait-time" but as "think-time" (Stahl, 1994), terminology that was used in practitioner papers at least as early as Gambrell (1980). The description of these silences, however, is monochrome, generally consisting merely in a measure of its duration with no further information about the silence. Furthermore, if the silences in Dr. Tzur's classroom (Steffe, 2003; Tzur, 1999) are an indication, they are sometimes too short for the sort of thinking activity required in the mathematics classroom (at least one of the silences in Dr. Tzur's classroom was almost 20 times longer than the recommended wait-time). Nevertheless, this literature does indicate that there may be ways for teachers to shape the character of student thinking activities. (Though, see Ingram & Elliott, 2014, 2016 for a different interpretation of the phenomenon.)

Second, though there does not seem to be any literature explicitly examining the practice, it is common for reform mathematics teachers to give their students several minutes of "private think time" following teacher questions, and perhaps again following brief discussions with peers (Schoolcraft, 2015). These practices are important, but, in contrast to the silences in Dr. Tzur's classroom (Steffe, 2003; Tzur, 1999), which were part of the students' public, face-to-face engagement with teacher and peer, these practices are treated as "private". Students, however, do not only need to think while they work by themselves, but also as their teacher or peers raise important questions. And these thinking activities will not be private. Nevertheless, these practices again show that classroom time needs to be given to thinking activities, and so, that at least to a small degree, student thinking activities are shaped by classroom practices.

What, though, does mathematical thinking *look* like? And how can it function in interaction? Lampert (1990) argues that the mathematics classroom should take its cues from the practice of mathematicians. Though there has been some push-back against using this standard in every classroom (Staples et al., 2012), mathematicians practices should more directly inform the work of upper-division undergraduate students. Indeed, just as there are specific linguistic conventions of mathematical writing (Lew & Mejía-Ramos, 2019), there may be linguistic conventions for oral mathematical collaboration that students need to learn to participate as members of the mathematics profession. Furthermore, even for other classes, though mathematicians' practices will not so directly inform students' practices, a knowledge of mathematicians' thinking practices may be a helpful place for researchers to start investigating thinking activities.

## 4.3 Methodology

### 4.3.1 Data Collection

The data for this study are taken from the transcripts included in Smith's (2012) doctoral dissertation, a phenomenological study of the "ways in which mathematicians structure their experiences of struggle while working in pairs in person on a current problem" (p. 46). In order to answer this research question, Smith used two cameras, one on each side of a room, to record approximately ten hours of collaboration between pairs of mathematicians. He selected nine episodes, totaling 21 minutes and 43 seconds that, after repeatedly watching his videos, seemed to be strong examples of mathematicians struggling; and analyzed them in depth. (His dissertation is an analysis of these nine episodes.) The transcripts he included to support his analysis consist 1,029 rows of data. Each row consists of: 1) screen-shots from both cameras, annotated with arrows to indicate gestures; 2) time stamps; and 3) the dialogue, or if no one spoke, "[pause]" (see figure 4.1 for an example). Thus, the transcripts

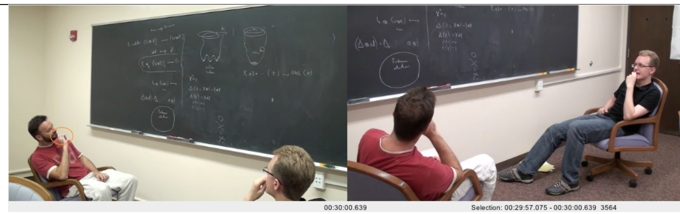
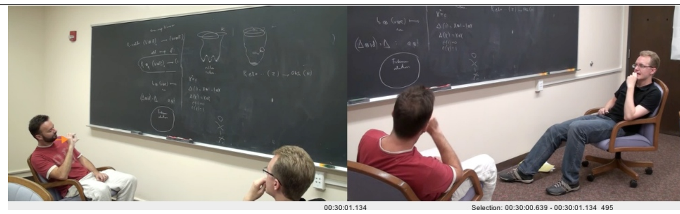
| Row | Gesture & timestamp | Transcript |
|---|---|---|
| 9 |  | interesting thing; I mean the, the [pause] the, um, |
| 10 |  | [pause] |

Figure 4.1: Page 261 of Smith (2012). One of sixty-nine pages of Smith's transcription of his "Odd Scalars" episode, containing rows nine and ten out of one hundred thirty-seven total rows. Small orange circles and arrows highlight Joseph's hand. Bill remained motionless.

come close to low-frame rate (0.79 frames per second), captioned, silent films of the episodes Smith studied. The extreme detail of these transcripts makes them ideal for a careful reanalysis. The text of his dissertation also contains descriptions of each episode which sometimes describe actions that were important preludes or postludes the episodes his analysis focused on.

Smith decided Ph.D. candidates were sufficiently advanced to be a close proxy for mathematicians, and that the meetings between Ph.D. candidates and their advisors were a) naturally occurring, b) likely to contain struggle, and c) amenable to recording because of their regular occurrence. He and his advisor, Ricardo Nemirovsky, contacted mathematicians at a large southwestern US research university and asked if they had any ABD students who were willing to participate in their research (Nemirovsky & Smith, 2011, 2013).

Two mathematician/PhD candidate pairs were willing to participate in the study (all names are Smith's pseudonyms): 1) Joseph, who works in topology, with his student Bill. They were working to “flesh out” (p. 50) Bill’s dissertation. 2) Fay, who studies graph theory, particularly random graphs, and her student Martha, as Martha’s dissertation “nears completion” (p. 50). Additionally, Matt, an algebraic geometer, responded that his colleague Bart would be visiting from Europe in order to collaborate with Matt. Their collaboration is also included in Smith’s study.

Smith’s dissertation includes transcripts of five episodes of Joseph and Bill’s collaboration, totaling of 12 minutes 5 seconds; three episodes of Matt and Bart’s collaboration, totaling 6 minutes 57 seconds; and one episode of Fay and Martha’s collaboration, totaling 2 minutes 38 seconds.

A word search shows that Smith’s dissertation contains only one mention of the word “silence”: “After another few seconds of essentially motionless silence Joseph suddenly brings his hands forward” (p. 76). And only one mention of “silent”, stating that the published mathematics education literature is “generally silent” (p. 4) about the activity of doing mathematics. In the paragraph preceding the mention of “silence”, Smith notes that a mathematician “froze”. Freezing seems to include silence, so it is possible that silence is indirectly present in his work. Nevertheless, it is not directly thermalized, and Smith did not use the presence or absence of silence as a criterion in selecting episodes to analyze.

On the other hand, Petersen (2017a) showed that significant portions of these collaborations are silent (in some of Smith’s episodes, both mathematicians were simultaneously silent for over half the episode, and over 20% of the episode was spent in silences longer than 15 seconds). That Smith chose his data without reference to silence, and that, nevertheless, silence was a prominent, unanalyzed, presence in Smith’s data, makes them amenable to a re-analysis focusing on silence, and his extensive transcripts make such a re-analysis feasible. Furthermore, that he selected these passages without reference to silence gives some indication that these silences are a general phenomenon.

## 4.4 Transcription

Though Smith's transcripts are extremely detailed, 1) precisely because of their detail, it is difficult to use them to get a sense of the overall sweep of an episode, and 2) silences can only be seen through the relatively laborious process of recording their initial and final time stamps. For instance, both utterances and lengthy silences are usually spread over multiple pages of transcript (and even short silences can only be seen to be short by referencing at least two pages of the transcript). Flipping between pages is too time-consuming to make these transcripts the primary transcripts used in this study. Therefore, it was necessary to re-transcribe Smith's transcripts in a format that 1) allowed the entire episode to be read, and 2) highlighted silences. A typical play-script style transcript, for instance, would do a much better job of allowing an analyst to see the general shape of the episode, and the interaction in it. These transcripts, however, would not allow the analyst to see the silences—and some of Smith's (2012) episodes are almost entirely silence—and so a new transcription method is necessary.

To that end, the following conventions, adapted from Ochs (1979), were adopted in the transcripts used to analyze the data reported in this paper. First, rather than showing who spoke by giving their name before their utterance, the transcript was put into a table, and each of the two participants was given their own column. When the participants kept a relatively uniform positional configuration (with the same one

| | Verbal | | | Non-Verbal | |
|---|---|---|---|---|---|
| | Joseph | Bill | | Joseph | Bill |
| 1 | [1a] [2]But think- [2b]Yeah I think [4c]what I'm most [5]interested in is setting [6d]$B_1$ to be zero ((laugh)) (.) See | | | [1]seated eg board hands in lap [2]raises rh [3]rh vertical [4]rotates rh [5]pulls rh back a little, leans head to right [6]eg moves [7]leans head back | [a]seated eg board rh in lap lh on chin [b]rotates lh slightly [c]curls up one finger [d]eg moves |
| 2 | | [a]Yeah | | | [a]straightens head eg left not at J. |
| 3 | It's this [1]interesting thing I mean the, the (.) the um | | | [1]rh "o" shape | |
| 4 | [1] | | 4.7 seconds | [1]moves rh forward slightly | |
| 5 | [1] | [a] | | [1]straightens head, rh to mouth. | [a]eg place, tilts head left |
| 6 | | ~16 seconds | | | |
| 7 | | | | | |
| 8 | | | | | |
| 9 | [1]You know so [2]a a map between [3]supervector spaces is just a [4]functorial system [5]of things like [6]this true [7]for any B [8]there there ex[9]ists something [10]like that. | | | [1]leans head back rh forward ~6in. [2]eg moves [3]raises lh and points pronated, fingers spread, toward place. [4]eg B [5]eg board [6]rotates lh 90deg [7]pronates lh, eg B [8]lh into c shape, eg board [9]lowers lh slightly, eg B [10]lh eye, bends right wrist. | |
| 10 | | Yeah | | | |
| 11 | It's the [1a]the compatibility between different *B* s [2b]which really gives us [3c]the restriction. But certainly, in particular we could [4]just set *B* to be trivial. ((laughs)) And then an odd map would just be ze[5]ro. | | | [1]leans head back slightly, hands open 45 degrees sholder height and width. [2]flexes rwrist, lh to leg. [3]eg moves, raises rh (elbow still resting on chair). [4]points along eg with pronated lh, fingers spread. [5]lh to nose (lose fist). | [a]rotates chair, eg moves some not at J (would be if J were forward). [b]rotates chair back (eg doesn't move). |
| 12 | [1] | Right | | [1]lh to leg | |
| 13 | | | ~3-5 s | | |
| 14 | | | | | |
| 15 | [1] | | | [1]raises rwrist (straight w/ arm) | |
| 16 | So I think [1]I mean I [2]I don't quite understand how this can be possible but (.) assuming that the | 1 minute 39.5 seconds | | [1]lowers rwrist. [2]rh to leg (elbow straightens) | |
| 17 | | | ~25 seconds | | |
| 18 | | | | | |
| 19 | | | | | |
| 20 | | | | | |
| 21 | | [a] | | | [a]eg moves |
| 22 | | | | | |

Figure 4.2: Twenty-two, out of seventy-three rows of my re-transcription of Smith's (2012) "Odd Scalars" episode. Figure 4.1 comprises most of rows 3 and 4 of this transcript.

on the left and the same on the right throughout), the speech of the participant who stood to the left was put in the left column. The duration of mutual silences (when both participants were silent together) was notated in the following ways: First, a blank line was entered for each second of silence. Second, those lines of the transcript were greyed-in (to give them visual prominence). Third, the duration of the silence was notated in a small column to the right of the columns of speech. Since it may be possible for one participant to be interestingly silent even while the other talks, when one person was silent across two of their partner's utterances, or across a second of mutual silence, the lines in which they were silent were greyed-in in the transcript (in a lighter grey than for mutual silences), and the duration of the silence indicated in a thin column to the left of that participant's speech. Finally, because it is natural to ask what each mathematician was doing during the silences, their movements (gestures and whole-body movements) were recorded in a set of columns to the right of the columns containing the mathematicians' speech, and the timing of the gestures was indicated by superscripts. (See figure 4.2 for an example.)

As these transcripts were being prepared, the type of interaction in passages of mathematical work were tagged and passages that contained mostly conversation-like interaction (either back-and-forth conversation, or one person talking at length and the other showing engagement with their talk) were identified as less likely to merit careful attention. These passages, therefore, were transcribed into columns (so the sweep of the episode could still be seen), but silences and other gestures were not transcribed. So for instance, it is hard to use Smith's transcripts to find passages containing lengthy back-and-forth conversations, but as the spoken interaction was transcribed, these passages were identified as unlikely meriting further analysis, and the transcription process was halted at that point.

The standard in Conversation Analysis is to use punctuation to indicate the *prosodic contour* of talk rather than the analyst's judgment of the syntactic structure of talk. For instance, in the CA literature, a period indicates falling intonation. Smith (2012), however, did not use CA conventions in his transcript, and so, in this paper, punctuation reflects Smith's judgment of the structure of the preceding talk. A period, for instance, indicates that Smith heard the preceding talk as a statement. Because the focus of this paper is the silence, this is not a fatal flaw in Smith's corpus. Since the talk surrounding silences is analyzed in this paper, however, conclusions that may be overturned on subtleties of prosody should be given less weight than they otherwise would be.

Finally, Liddicoat (2011) recommends using more detailed descriptions of interaction in transcripts used for data analysis than for publication. When analyzing data, extra information is important, since it may turn out to be a critical point. Aspects of interaction, however, may turn out to be irrelevant to a particular analysis, and so add details that only distract a reader from the point an author is making. Therefore, as transcripts were prepared for publication, when the transcripts described above contained extraneous details, simplified versions of the transcripts were prepared for the publication. The transcripts described above, however, were used in the analysis of all the examples presented in this paper.

### 4.4.1 Data Analysis

Because Smith (2012), an independent researcher, selected these episodes as examples of intense mathematical struggle, the assumption throughout this paper is that these episodes are examples of high-level, deeply-engaged, mathematical activity. The fundamental methodological question this section addresses is how to read the mathematicians' silences during their mathematical activity, not whether they were engaged in mathematical activity.

The first step of analysis, therefore, was identifying sequences with significant presences of silence, so research could focus on these episodes. Initially, a four-fold tagging system was developed to help identify passages for further attention:

1. Mutual silence: Both mathematicians silent, thinking about mathematics. The second criterion rules out passages in which one mathematician silently erased the board or fluidly wrote mathematics on the board.

2. One speaking one listening: One mathematician talks at length, the other displays engagement with the speaking mathematician. There are two ways mathematicians can display engagement: (a) Regularly directing their eye-gaze to the speaking mathematician or at something they reference. (b) Responding with what conversation analysts call *"continuers"* (Schegloff, 1981) short words like "uh" and "um" which display engagement and claim understanding.

3. One speaking one thinking: One mathematician talks at length, the other does not shift their eye-gaze in response to the first, does not respond with continuers, but displays engagement with mathematics (chiefly by their eye-gaze).

4. Back-and-forth conversation. Both mathematicians speak in turn.

Because the goal, at this step, was to identify relatively long passages that merited further attention, and to allow research to focus on different kinds of interaction, short deviations from a particular tag were ignored. For instance, in the passage shown in

4.5 on 223, rather than single out line 36 as "one speaking, one thinking", lines 21-56 were tagged as "mutual silence". As this work proceeded, two more categories were developed: "Board work" and "neither speaking". For instance, at one point both mathematicians looked through their bags for their notes. Neither spoke as they did so, so it is infelicitous to tag this passage as "back-and-forth conversation", however, they were searching in their bags for notes, not actively thinking about mathematics (though they may have been thinking as they searched, that thinking was not socially apparent). As passages were given different tags, brief descriptions of the mathematicians' behavior during each clip were recorded. Finally, because the focus of this study is on silent thinking, when identified as one-speaking-one listening, back-and-forth conversation, were transcribed to a table, but brief silences were not measured exactly, and gestures were not all identified.

Silence following a first pair part (FPP) is different from silence following a second pair part (SPP) or post-expansion, since SPP's and post-expansions could initiate a transition to a sort of unfocused activity, whereas, in focused interaction (as in these episodes), lengthy silences following FPP's violate conversational norms (Stivers et al., 2009). Furthermore, lengthy, unproblematic, silences following FPP's seem to violate the turn-structure which is foundational to the discipline of Conversation Analysis (Couper-Kuhlen, 2010; Schegloff, 2010a), and so require especially careful analytic attention. Therefore, all the passages tagged as "mutual silence" were identified as being immediately preceded by an FPP, an SPP, or a post-expansion. In tagging these, it became clear that several of the mutual silences were immediately preceded by incomplete FPP's (which may be a feature of turn design, or the turn may have cut off part-way through), and these were identified. A short description of the turn preceding the mutual silence was also tabulated.

The mutual silences preceded by FPP's were further coded for whether the missing SPP was repaired, and whether it was ever eventually supplied. When the FPP was incomplete, the mutual silence it passage was further coded for whether the missing conclusion of the FPP was repaired, and whether it was ever completed.

These can be made concrete through reference to the transcript in Section 4.1.1 on p. 196, and reproduced here:

Brt: And these things
(3.0)
Brt: are the same, right?
(1.0)
Brt: Is that correct?
(3.0)
Mtt: ( ) (0.5) ( )
→ (8.0)
Brt: True?
→ (27.0)
Mtt: Well, let me just show you-
Brt: Oh of course it is, it's zero! Uh, I mean that's the kernel
(.) so these things are zero. (.) $u_q$ vanishes on this
intersection.

Since neither mathematician spoke during the lengthy silence, and they were engaged in mathematical activity (e.g., not searching in their bag of notes, or writing on the board) throughout, the silence was tagged as "Mutual Silence". The silence was further identified as preceded by a FPP, that was repaired, and with no eventual SPP (this is the only example of a repair following a FPP that initiated a mutual silence). On the other hand, a 25 second silence in the "Odd Scalars" episode is preceded by an incomplete FPP ("So I think I mean I- I don't quite understand how this can be possible but (.) assuming that the"). Here, the fact that the FPP was incomplete was never repaired, and the FPP was never finished. (The next talk, 25 seconds later, was from the same mathematician, who said "I- You know I- I sort of want to argue that these just give the zero map that's ( )".) During the 25 second silence, neither mathematician directed their eye-gaze toward the other, or changed their posture, and so there was also no non-verbal attempt to repair the incomplete FPP.)

This procedure allowed the development of a differentiated corpus of episodes for analysis. At this point, extensive analytic notes were collected, with a focus on 1) passages tagged "Mutual Silence" that were preceded by FPP's, 2) passages tagged as "One speaking one listening", and 3) passages tagged as either "Mutual Silence" or "One speaking one listening" with uncommon features (like pacing) of the silent mathematician's behavior. First, drawing off Smith's (2012) transcripts and the transcripts prepared for this study, the mathematicians' actions were described in more depth, particularly with reference to the way each action was related to actions of the other mathematician. Furthermore, a preliminary hypothesis was developed regarding the norms for collaboration. This hypothesis was subsequently refined, but its general outline was preserved throughout the remaining analysis. The guiding principle of this analysis was determining what action the mathematicians perform at given times, and why they perform that particular action then.

These notes also identified passages that were helpful in establishing a thesis about the norms governing silent thought in mathematicians' collaboration. For instance, a passage from Smith's (2012) "tubes" episode was selected as amenable for further analysis because it 1) contains a very lengthy silence 2) allows for extensive analysis of the mathematicians' actions preceding their silence, 3) the silence is preceded by a question, 4) the silence is briefly interrupted by a seemingly unrelated question, and 5) Smith's (2012) verbal description of the passage allows some analytic attention to the way the silence concluded.

Finally, a comprehensive analysis of the chosen passages, utilizing CA methodologies, was conducted. Unlike a quantitative study, in which the methodology shows the validity of the analysis, whose results are then displayed in the results section, in a more ethnographic study like this one, though reporting a methodology is critical, the results section should contain enough information to allow a reader to, at least preliminarily, judge the competency of the analysis (McDermott et al., 1978). So for instance, Schegloff (2010a) objected to another CA paper, Stivers and Rossano (2010) on the basis that the evidence presented in their transcripts does not support their classification of episodes. For instance, in one passage, in Stivers and Rossano (2010) Kim commented on her addition of raisins to a salad, and Mark responded by saying "love raisins". Stivers and Rossano glossed Mark's response as "confirm[ing] liking the addition of raisins" (p. 24). Schegloff argued that in fact, though Mark claimed to like raisins, he specifically avoided saying that he liked the *addition* of raisins. Schegloff notes that though this point does not undermine Stivers and Rossano's argument, it does call into question "the adequacy of" Stivers and Rossano's "analysis of the extract as an interactional exchange with its own integrity" (p. 43). The fact that Stivers and Rossano (2010) contained a detailed enough transcript to support Schegloff's objection shows that it is important for analysts practicing CA to include enough detail in their episodes that the integrity of each individual episode is visible on the transcript itself. Furthermore, that Schegloff thought the point worth raising, even though it was not material to Stivers and Rossano's argument, shows that Conversation Analysts display their methodological rigor through the analysis of passages which are transcribed in sufficient detail for a colleague to see the problems with readings that are methodologically suspect.

## 4.5 Results

Petersen (2017a) showed that, while collaborating, mathematicians' discourse has numerous, lengthy, collaboratively-produced silences in it. His results, however, need to be considered somewhat preliminary, since he did not determine where in conversational sequences these silences lie. Since post-expansions and second pair parts (SPP's) can serve to close sequences of talk, there is not necessarily any relevant speech following them, if mathematicians' collaborative silences usually follow post-expansions and SPP's, then mathematicians work could be said, in some sense, to alternate between relatively well-defined times of more something like individual work, and times of conversation. On the other hand, if these silences often follow first pair parts (FPP's), then, because the FPP projects a response, then mathematicians' conversation, itself, follows different norms than those that govern every-day-conversation.

In the corpus examined here, lengthy silences follow both FPP's and SPP's, whereas only one, approximately 2 second silence, follows a post-expansion. Furthermore, though 10, 13, 16, and 17 second silences follow SPP's, FPP's are followed by a 42 second silence (interrupted by three brief reiterations of the FPP that began it), and, separated from each other by only one turn at talk, 15 and 20 second silences (with one syllable in the middle of the 20 second silence). Additionally, of the 15 mutual silences that followed FPP's, and in which, therefore, an SPP would have been missing every-day conversation, a mathematician only repaired the lack of an SPP once. Furthermore, on only one occasion was there eventually a SPP at all (though sometimes the SPP may have come after the end of the transcript). For instance, in the "Tubes" episode (analyzed in the next section), Joseph asked a question which was followed by 15.1 seconds of silence. Bill then suggested a problem in their work, concluding his utterance with the tag-question “right”. This was in turn followed by 11.6 seconds of silence, Josephs utterance ”Hm!”, 7.5 seconds of silence, and the end of Smith’s transcript. Both Joseph’s question, and Bill’s tag-question were FPP’s, however, however neither was followed, immediately or even eventually, by an SPP. Moreover, neither mathematician repaired the missing SPP. These results indicate that whatever conversation exists in mathematicians’ collaborations, it is governed by a different set of norms than every-day conversation is. For that reason, not only the mathematicians’ actions directly connected to mutual silences, but their activity when one is speaking must be analyzed.

Furthermore, because 1) extensive silences following FPP’s more strongly violate the norms of every-day-conversation, 2) the generic norms minimizing silence after SPP’s and post-expansions are also present following FPP’s, and 3) depending on the circumstances, lengthy silences following SPP’s may not even be problematic (Szymanski, 1999), it makes the most sense to begin unravelling the norms governing silence in mathematicians’ collaborations by examining an example of a lengthy silence following a FPP.

### 4.5.1 Tubes

In this example, Joseph and his doctoral student, Bill were working on a topic from topology or knot theory. This episode began with Bill explaining a recent email from a colleague that addressed his and Joseph’s prior work. Joseph then set a focus for his and Bill’s subsequent work by suggesting they try to explicitly work through an example (Smith, 2012, p. 84). (Conversation analysts distinguish several different

(a) (b)

Figure 4.3: a) Joseph's geometric diagram of two different situations that should have different algebraic interpretations. In both diagrams, "time" is to the right. At any given time, what is important is not the three-dimensional figures, but the vertical cross-section. Top: Two loops, one disappears, then the other splits. Bottom: Two loops, one splits, then the other disappears. These two situations should differ algebraically by a sign, however, their eventual algebraic representations are identical. (Smith, 2012). b) Joseph's algebraic representation of the two situations. The top expression is complete, but before writing the last step in the bottom expression, he realized it would be the same as the final term in the top expression, even though the two terms should differ by a sign. (Smith, 2012).

ways people can set a topic for conversation [Schegloff, 2007, p. 169].) Smith does not include a transcript of Joseph's initiation, so it is not clear what sort of discursive action he employed to do so. Fortunately, the means he used to initiate this sequence is not relevant to this paper's research questions. Seated, he then drew a diagram of two different situations (figure 4.3), stood, and began writing these two situations algebraically, using their new notation. While Joseph wrote, their interaction resembled a conversation in which one person takes an extended turn: Bill's eye-gaze was fixed on Joseph's work, and he responded to Joseph's talk with continuers. Where transcript 4.1 begins, Joseph had just finished the second term in the bottom expression, stepped back and looked up at the last term of the top expression. Throughout the transcript his eye-gaze moves to both expressions and both diagrams.

Transcript 4.1

```
1    Jos: Ha. I notice I don't understand.
2    Bil: ((Chuckles))
3    Jos: Uh.
```

```
 4        (1.1)
 5   Bil: Yes.
 6   Jos: Uh
 7        (1.5)
 8   Jos: And then-
 9        (0.3)
10   Bil: Cap ‘em out.
11        (1.1)
12   Jos: So why is this ((Gaze to bottom expression)) minus at all?
```

In line 1, Joseph laughed, and announced that he noticed he did not understand. Since he 1) was ready to write the last term of the bottom expression, 2) looked at the last term of the top expression (which should not be the same, but would have been), and 3) spent the next several seconds looking at different things he had written, it seems that, here, Joseph noticed that the two expressions would be the same, and so something was wrong. Rather than announcing his lack of understanding in line 1, Joseph could have initiated a search for the source of the trouble with a FPP that told Bill the problem explicitly. His actions, however, conform to the conversational preference for noticing over telling (Schegloff, 2007). According to this preference organization, when there is some feature of the environment that one participant wants to tell the other about (e.g. a new haircut), people will generally work to make it so the other person will notice it, and indeed, before that, to realize that noticing is conversationally relevant (e.g., by asking “do you notice anything different about me?”). Bill responded to Joseph’s claimed noticing by laughing (line 2), which, since it echoed Joseph’s laughter in the previous line, claimed to follow the problem. Bill’s turn in line 5, however, is more ambiguous: Yes what? Joseph responded by continuing to look, and then, in line 8, with an incomplete utterance that may be designed to elicit a completion from Bill (Koshik, 2002; Lerner, 1991), though without the prosody, it is difficult to tell for sure. Either way, Bill responded to the incomplete utterance by, in line 10, finishing Joseph's utterance. Therefore, Joseph's incomplete utterance "and then–" again made Bill's noticing relevant. If Joseph were still just working to write the bottom expression, Bill's response in line 10 would give the correct completion to Joseph's previous utterance. Joseph, however, had already noticed (line 1) that something was wrong with what he would write, and was looking to figure out how to address the problem. In that context, Bill's utterance in line 10 is a failed noticing. Joseph oriented to this failure by, in lines 12–13, asking a question that made noticing the problem even more relevant, but which, even still, does not quite constitute a telling.

In every-day conversation, this sort of question that simultaneously makes noticing relevant often receives a double response, (e.g. "Why yes! You got a haircut. It looks lovely.") At the least, there is a strong preference for a question that makes a noticing relevant to be followed by a noticing FPP; and these questions normatively receive a response.

Bill's response, however, did not consist in a verbal noticing, but in 15 seconds of silence, which served to display his noticing.

Transcript 4.2

```
12    Jos: So why is this minus at all?
13 →       (15.1)
14    Bil: Yeah so I think where we're losing something right in these
15         is the tensors. Right?
```

In line 13 of transcript 4.2, Bill suggested a place to look for a solution to the problem he had already noticed. But that act cannot also be the act of publicly noticing, since it's 15 seconds late. Ten and a half seconds after Joseph's question Bill began moving toward the board, but that too is 10 seconds too late. Just after Joseph's question, a tiny act is visible on Smith's screen captures: Bill closed his slightly-open mouth and frowned (figure 4.4). But this motion was tiny and hardly visible in the screen capture (figure 4.4b–c), while Bill's head seems to have been slightly behind Joseph's, and Joseph was looking at the board. It is perhaps possible that Joseph could see and Bill out of the corner of his eye, and respond to the frown. But because his eye-gaze was directed away, the more likely explanation is that the frown did not, here, communicate Bill's realization to Joseph.

But then how did Bill publicly notice? Bill's response in line 10 of 4.1 completed Joseph's utterance in line 8, but in that, displayed a failure to notice. Similarly, Bill could have responded to Joseph's question in line 12 by explaining why "it was minus at all", or by displaying a noticing of the problem. Following line 12, Joseph ceased to attempt to make noticing relevant, and so acted as if Bill's response in line 13 manifested noticing.

Bill's response, however, was silence. Therefore Joseph treated Bill's silent activity as a display of noticing. Furthermore, when, as in line 12, turns initiate two actions (e.g., asking a question and making noticing relevant), the SPP usually responds to both actions (Schegloff, 2007). Therefore, Bill's silence in line 13 not only displayed a noticing, but also answered Joseph's question (with the proposition "I don't know"). Therefore, that the sort of lengthy, motionless, silences that follow Joseph's question are a normal, expected, kind of mathematical activity. Or, perhaps stated differently, lengthy silences devoted to thinking intently about conversationally relevant mathematics are a normal, expected, intended, aspect of mathematical collaboration; and these silences do not only serve the private activity of each mathematician. Rather, their initiation can be communicative, the thinking silences are themselves a public aspect of the group's common activity. Indeed, it would not be a stretch to say that, in

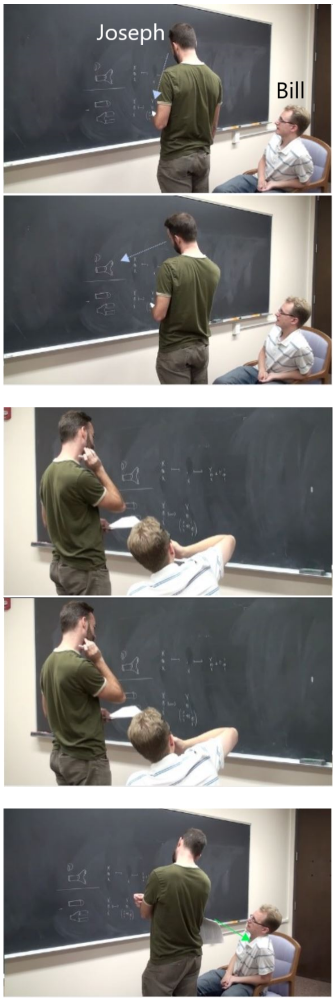
(a) (b) (c) (d)

Figure 4.4: a) Approximately 1.5 sec before line 12 . b) End of line 12. c) 1.5 seconds after line 12. d) 3 seconds after line 12.

his attempt to make Bill's noticing relevant, Joseph was holding Bill accountable for a silence during which, as we shall see, he publically thought about conversationally relevant mathematics.

Exactly what this silence consists of is a central concern of this paper, and so it is explored in more detail. One remark, however, can now be made before further analysis: Inasmuch as Bill's silence was a fitting response to Joseph's attempt to make noticing relevant, and the two mathematicians subsequently engaged in extremely lengthy silences, this exchange is part of the work Joseph and Bill do to construct face-to-face mathematical activity as involving more than just talk.

Prior to Bill's noticing the problem Joseph's eye-gaze had repeatedly moved around the board, and, at first, this activity continued during the silence. After his question, Joseph moved his eye-gaze around the board 5 times (figure 4.5b, lines 21–26), then lifted his right hand to his mouth (figure 4.5b line 26) and after 4 motionless seconds, moved his eye-gaze to the top geometric diagram a final time (figure 4.5a, second from the top). His gaze then remained fixed to the top left of the board for the next 20 seconds (figure 4.5a, second from the top through second from the bottom, b lines 30–47). Aside very small movements of his lips and jaw, Bill initially remained motionless (figure 4.5a top two screenshots, b lines 20–35). About the same time Joseph last moved his eye-gaze, Bill opened his mouth, leaned forward toward




| Line | Verbal: Joseph | Verbal: Bill | Verbal duration | Non-Verbal duration | Non-Verbal (1) | Non-Verbal (a) |
|---|---|---|---|---|---|---|
| 20 | [1]So why is this [2]minus at all? | | | | [1]lowers chin slightly [2]eg moves | |
| 21 | | | 30.7 second silence (lines 21–47); 19.5 seconds (lines 21–35) | 15.1 seconds (lines 21–35) | | |
| 22 | 1 | | | | [1]leans head right, eg to place | |
| 23 | 1 2 | | | | [1]eg moves [2]straightens head | |
| 24 | 1 | | | | [1]eg moves [2]straightens head | |
| 25 | | | | | | |
| 26 | 1 | | | | [1]raises hand to chin, scratches | |
| 27 | | | | | | |
| 28 | | | | | | |
| 29 | | | | | | |
| 30 | 1 | a b | | | [1]eg moves | [a]opens mouth [b]opens sholders |
| 31 | | a | | | | [a]slides lh down leg, raises rh |
| 32 | | a | | | | [a]leans head left |
| 33 | | | | | | |
| 34 | 1 | | | | [1]moves fingers slightly | |
| 35 | | | | | | |
| 36 | | [a]Yeah so I think where we're losing [b]something right in these [c]is [d]the tensors. [e]Right? | | | | [a]extends rh index finger (pointing toward board) [b]reaches out to place [c]moves finger side to side [d]rh to his mouth [e]slight movement of rh |
| 37 | | a | 20.1 seconds (lines 37–56) | 11.6 seconds (lines 37–47) | | [a]drops rh from nose to chin |
| 38 | | | | | | |
| 39 | | | | | | |
| 40 | | | | | | |
| 41 | | | | | | |
| 42 | | | | | | |
| 43 | | | | | | |
| 44 | | | | | | |
| 45 | | | | | | |
| 46 | | a | | | | [a]drops rh to leg |
| 47 | | | | | | |
| 48 | [1] Hm! | | | | [1]turns over lh, crosses arms. | |
| 49 | 1 | | | 7.5 seconds (lines 49–56) | [1]leans back | |
| 50 | 1 | | | | [1]leans head right, moves | |
| 51 | | | | | | |
| 52 | | | | | | |
| 53 | | | | | | |
| 54 | | a | | | | [a]straightens head slightly |
| 55 | | a | | | | [a]leans head left, opens sholders. |
| 56 | | | | | | |
| 57 | | ((Bill suggests a change in notation, and they change topics.)) | | | | |

(a) (b)

Figure 4.5: (a) Top to bottom: Beginning of line 21. Ten seconds later at the beginning of line 30. (The blue and green arrows are Smith's (2012) reading of Joseph's eye-gaze and postural shifts, respectively.) Beginning of line 37. Ten seconds later at the beginning of line 46. End of line 56 (Smith's last screen-shot). (The green arrow is Smith's annotation of Bill's movement.) The camera angle is different, but in the second screenshot, Joseph had just adopted the posture he kept through the subsequent two screenshots. During the 10 seconds between lines 37 and 46 neither moves. (b) A transcript of part of the Tubes episode. Line 57 is described in the body of Smith's dissertation, but is not included in his transcript.

the board, and began to slowly extend his hand toward the place the bottom algebraic expression, where he had been looking whole time (figure 4.5b line 30). Four more seconds later, he reached all the way to the board, suggested they focus their attention on the tensors, and used a tag-question to ask Joseph if that suggestion was correct (figure 4.5b line 36). As noted, Joseph's gaze was then on the top geometric diagram (figure 4.5a second from the top)—about as far away from Bill's pointing as it could be—and he did not move in response to Bill's talk (figure 4.5b line 36). Bill then withdrew his hand to his mouth, and both mathematicians stared at the board motionlessly for approximately 10 seconds (figure 4.5a second and third from the top, b lines 37–45) before Bill dropped his right hand to his leg (figure 4.5b lines 46). Joseph then shifted his posture and eye-gaze (figure 4.5a bottom, b lines 49 & 50), and said "Hm!". Four seconds later Bill leaned back (figure 4.5b lines 54 & 55), and then after an additional two seconds suggested they had reached a dead-end and should move on (figure 4.5b line 57). Joseph agreed, and they moved away from this tack.

Schegloff (1998) argues that participants in face-to-face interaction display their differential commitment to multiple activities through what he called, their body *torque*: Hips are squared to their primary focus, their head may turn away toward a secondary more temporary focus of activity, their eyes may be turned further still toward a tertiary even more temporary focus of activity. Attending to Joseph and Bill's body torque can helpfully illuminate their precise work during the extended silences.

Throughout this episode, Joseph and Bill displayed a primary focus on the board—that is, on conversationally relevant mathematics (figure 4.5a). Bill may have been able to turn his head toward Joseph without too much body torque, but when he looked at the board, his head was aligned with his chest. Joseph did turn his head to look around the board. Through that motion, however, he maintained minimal body torque. Even as his head moved, his shoulders remained squarely oriented to the board throughout the episode, and his head did not turn much from his trunk (figure 4.5a). Looking at Bill would have involved either a change of posture, or relatively high body torque. That is, throughout the episode, both mathematicians displayed a primary focus on the conversationally relevant mathematics Joseph had just written on the board, and a, at best, secondary orientation to conversation with the other mathematician. On the other hand, though Joseph moved his head, and so displayed a focus on different aspects of conversationally relevant mathematics, though his eventual motionless gaze indicated extended attention to the top geometric diagram, his trunk never displayed a preferential focus for one particular aspect of the conversationally relevant mathematics.

Four important points follow: First, their posture served to coordinate their activity, indeed, we could say, to coordinate their thinking: Whereas before, their coordination could be both heard and seen, now it could only be seen. But it could be seen. Second, though more evidence is needed, because their activity remained focused on conversationally relevant mathematics, and Bill had to explicitly suggest a different tack—and Joseph to agree to it—before they could leave the work on the problem they noticed, it seems likely that Joseph's question, and the problem he and Bill noticed, remained conversationally relevant throughout the following silence. This means a sudden change of topic would violate the norms of their collaboration. Third, the fact that their work continues a mutual project begun earlier in their conversation indicates that this silent attention demonstrably is aimed at producing more conversation that continues the project, in its mutuality. Finally, if these last two hypotheses are true, not only did their alignment of head and trunk publicly display the fact that the mathematicians were privately thinking intently and at length about conversationally relevant mathematics. Rather, these observable actions were irreducible aspects of the joint activity of thinking mathematically while collaborating.

An analysis of Bill's suggested focus ("the tensors here") and tag-question ("right?"), can further this analysis of the norms governing mathematicians collaborations. Specifically, it will show that, at times in mathematics collaboration, responses are not normally required, and that, instead, a continued focus on conversationally relevant mathematics is normal, even after a question.

The key point is that Joseph's lack of response in lines 37–47 of the transcript in Figure 4.5 was treated as unproblematic. Earlier, some statements seemed to still normatively require a response. For instance, the mathematicians continued working on the same topic they established before Smith's (2012) transcript began, and Bill offered continuers as Joseph talked. At this point, however, it does not seem that a response was normatively required: In lines 37–47, Joseph responded—if it can be called responding—by maintaining the exact same posture, with his eye-gaze directed toward a different portion of the board than the one Bill suggested they focus on. And in the same lines, Bill responded—again, if it can be called responding—to Joseph's "missing" response by staring intently at the section of the board he suggested they focus on, and not turning toward Joseph at all. That is, Joseph did not respond to Bill, at all, and Bill treated that lack of response as wholly unproblematic, and entirely in accord with conversational norms. The term in the CA literature for times when responses are not normatively required, and their absence is not accountable is a state of *incipient talk* (Schegloff & Sacks, 1973). It is not entirely uncommon when people are engaged in activities like eating dinner, riding together in a car, or working together in the same office, activities in which people are merely copresent, and not engaged in in activity with a single common "focus of cognitive and visual attention" (Goffman, 1966, p. 89). In these situations, however, the conversation is usually said to have lapsed as the copresent people go about their own business. Here, however, the conversation had not lapsed, but both mathematicians were engaged in conversationally relevant activity—again, conversationally relevant in that the activity was the continuation of activity initiated by previous conversation, but also, in that the activity aimed at speech that would achieve the goals the mathematicians had conversationally established earlier. That mathematicians, at times, are in a state of incipient talk in an unlapsed conversation has important consequences.

First, however, it will be important to dwell for just a moment more on the claim that, at this time in Joseph and Bill's mathematical collaboration, speech no longer normatively required a response, but only put particular actions on offer. Though responses may have been one normatively sanctioned action, they were not the only sanctioned type of action: Continuing demonstrably to think about conversationally relevant mathematics is, at least at times "like this", also a normatively sanctioned action. (Scare quotes are used for "like this" because what sort of situation is "like this" is an empirical one, and needs investigated.) Furthermore, the normative lack of a response does work to construct mathematical collaboration as the sort of thing that includes something very much like conversation, but also, in the midst of that conversation, includes something conducted in silence.

The claim that a conversation can be put on hold, and interrupted by a form of silence, in which responses are not required, is remarkable, and raises a number of questions. Two questions are important here: First, what situations is "thinking about conversationally relevant mathematics" a sanctioned activity? Can, for instance, a mathematician unilaterally begin thinking about mathematics, while their partner talks (so that, in a sense, whenever mathematicians converse, they are in fact, in a state of incipient talk)? Second, what does it even mean to suggest that parts of a conversation are carried out in a state of incipient talk—the terms seem to be contradictory.

One last point is worth touching before proceeding. If motionless focus on conversationally relevant mathematics can serve to display ongoing mathematical activity, conducted in silence, then relaxing and turning a little from the board could, theoretically, serve as a display of the cessation of thought about conversationally relevant mathematics, and so could serve as an invitation to new lines of spoken activity. It is noteworthy, though not conclusive, that prior to Bill's suggestion of a new tack, both mathematicians very slightly disengage from conversationally relevant mathematics. In particular, Joseph put his notes out of sight and leaned back slightly; and Bill returned to a posture he had before reaching forward to engage with the mathematics. On the other hand, it is possible that the length of the silence itself had suggested that this was not a fruitful line of inquiry. Further episodes will help cast some (not completely decisive) light on whether disengagement from the board can serve as a signal that mathematical activity has ceased.

To restate the conclusions of this section: Whereas, lengthy pauses during sequences of talk are contrary to the norms of every-day conversation, in this section we found that a lengthy pause, together with a postural configuration that shows continued thinking about conversationally relevant mathematics, is not only in accord with the norms of mathematical collaboration, but sometimes mathematicians can aim at such silences, and treat them as legitimate responses. Second, there is some evidence that mathematicians can signal that the silent thinking is fruitless by slightly disengaging from conversationally relevant mathematics. Furthermore, while mathematicians display that they are thinking about conversationally relevant mathematics, even though the conversation seems not to lapse and only particular actions remain relevant, the mathematicians are in a state of incipient talk, where talk and questions can, unproblematically, be left entirely without response.

### 4.5.2 Graph Confining

The previous section argued that when mathematicians display thought about conversationally relevant mathematics, talk only puts a response on offer, a response is not normatively required. But it did not consider the transition from a silence that displays attention to a pper, to a silence that does not display attention to a peer. Furthermore, since a display of attention is normal in every-day conversation, and sometimes present in mathematical collaboration, silences that do not display attention to a peer may display inattention to the peer's ongoing talk, and thus to display thought in preference to listening. It will argue that, in their collaborations, when a conversation makes a particular mathematical point conversationally relevant, a mathematician may stop displaying listening to their peer's ongoing talk, and instead display continued thought about conversationally relevant mathematics.

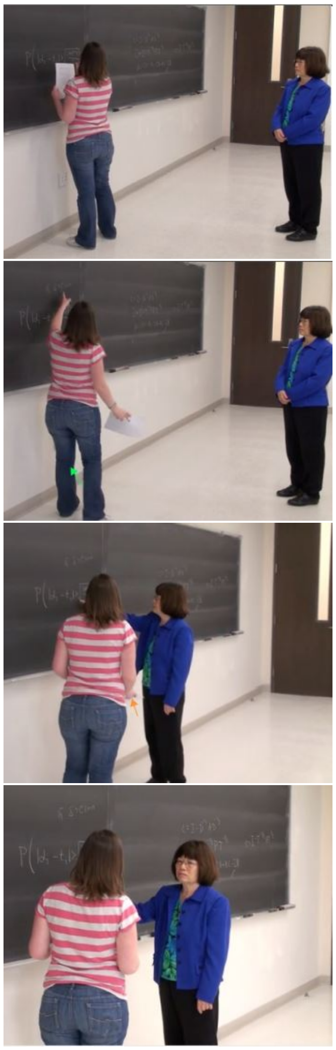
This claim raises questions about the nature of mathematical collaboration. Specifically, if talk does not normatively require a response, but only puts it on offer, can mathematical collaboration be described as a conversation at all? This section utilizes the construct of a *preference structure* from CA to show how it is possible that mathematical collaboration can be based on talk-interaction, and yet, talk may only put a response on offer. In particular, this section will argue that a display of thinking (and not listening) is a dispreferred response, whereas a response that displays continued attention to the peer's talk is preferred. Nevertheless, both are in accord with interactional norms.

In this episode, Martha (wearing a stripped sweater), a PhD candidate whose dissertation "near[ed] completion" (Smith, 2012, p. 50) had come to her advisor, Fay (wearing a blue jacket), to ask about a difficulty she had with an idea she'd gotten from a published paper. During the episode, Martha held the paper in her hand. Most of the details of the mathematical context are not important to the analysis in this paper. The one important point, as Smith (2012) explains, is that Martha's work suggested that, if $\delta$ is large enough to satisfy the top inequality, the bottom expression is small (written as $< \epsilon$); whereas, in the bottom expression, if $\delta$ is *small*, the whole expression is also small. (Martha explains this in lines 16–20 of figure 4.6c.) That is, the top inequality suggests that $\delta$ should be large, the bottom says it needs to be small.

Smith's (2012) transcript begins where the one included in this paper does, and, unlike the previous episode, he does not verbally describe their earlier talk. For those reasons, it is even more difficult than in the previous episode to determine precisely what had preceded. Two lines of evidence, however, indicate that they had established a topic for their inquiry, and that topic remained mathematically relevant throughout

$$\text{if } \delta > C \ln n$$
$$P\left(|d_i - t_i| > \sqrt{\frac{3\ln(3n/\epsilon)}{\delta}}\right) < \epsilon$$

(a)



(b)

1 M: This (($\delta$ in the bottom expression)) is of
2 minimum degree
3 (0.5)
4 M: This ((points to $\delta$ in bottom expression)) is
5 just a thing we use in ((steps back and
6 looks just above what she has written))
7 comparison. Right?
8 (0.8) ((During the silence, M steps to the
9 board and writes the top equation where she
10 had been looking))
11 M: And who says that if $\delta$ is bigger than some
12 constant $\log n$ then if you make your constant
13 large enough you can make this small as you
14 want. Right?
15 (0.3)
16 M: But I don't think that works! If you make
17 your constant larger, (.) then-
18 F: ((Begins to step to board)) I, I-
19 M: You don't care- This ((points to $\delta$ in bottom
20 expression)) is a smaller value!
21 F: Even- This ((The base of the square-root
22 sign)) is not the right form.
23 (1.0)
24 M: No?
25 F: No.
26 (1.5)
27 F: For me actually
28 (0.8)

(c)

Figure 4.6: a) At the beginning of the transcript Martha (left) had just finished writing everything to the left of the inequality in the bottom equation. During the silence in lines 8, she adds the top equation above it. At "small" in line 13, she adds "$< \epsilon$" to the bottom equation. b) Top to bottom: The beginning of the transcript. Line 16, just after "think". Beginning of line 21. Fay maintains this posture till "then" in line 21, 20 seconds after the start of line 1. Beginning of line 23. c) A transcript of the episode. Tight blue underline indicates that Martha's eye-gaze was on Fay. Lower red underline indicates Fay's eye-gaze was on Martha. All other times their eye-gazes were on the board.

this section. First, Smith, a member of the mathematics community, claims that this episode was focused on a particular topic. Though this is not as strong of evidence as a description of the establishment of the topic, since Smith is a member of the community he studied, and studied these episodes extensively, his judgment carries some authority. Second, the mathematics Martha wrote on the board remained a visual and conversational focus of all of both mathematicians' activity throughout this episode. That their activity was coordinated indicates that something had been done to coordinate the focus of their activity. This coordination can be accomplished conversationally through a topic proffering sequence (Schegloff, 2007), and so their coordination makes a preceding topic proffering sequence plausible. However, the precise method they used to coordinate interaction is not relevant for the analysis in this paper. Their coordination provides a second line of evidence that the mathematics written on the board was conversationally relevant, whereas other mathematics, or non-mathematical topics, were not.

At the beginning of the transcript, Martha had drawn the left part of the bottom expression in figure 4.6a, and was in the process of explaining a problem she had with it. As she talked, she wrote the top statement, and then added "$< \epsilon$" to the bottom equation (figure 4.6a). Fay's gaze was, at the beginning, fixed on the expression Martha had just completed writing (4.6b, top). Unlike both Bill in the previous episode and every-day conversation, however, as Martha talked, Fay did not respond with what Schegloff (1981) calls "continuers", words like "yeah", "uh huh" and the like, minimal turns that serve to displayed an understanding that, at this point, a longer turn would be inappropriate, and to claim to understand or agree with the preceding talk. Furthermore, Fay remained almost completely motionless throughout

Martha's work (in total, for over 20 seconds) (figure 4.6b top two screenshots). Fay's motionlessness had the effect that her gaze did not follow Martha's work to the top equation, nor to the right as she wrote "$< \epsilon$" (see figure 4.64.6b).

That Fay did not respond with continuers or follow Martha's work, but remained silent and motionless throughout Martha's talk is noteworthy in itself. Continuers display attention to the specific shape, and ongoing work of producing, a conversation. Because of the missing continuers, therefore, Fay's action was at best ambiguous: While her gaze may have continued to display (or at least claimed) continuing engagement with conversationally relevant mathematics, in the absence of these continuers, nothing displayed continuing engagement with the precise shape of the ongoing conversation. Three further features of Fay's activity make continued engagement with the precise shape of the conversation even less likely. First, Martha twice solicited her responses (lines 7 and 14 of the transcript in figure 4.6c), and yet Fay did not respond, or even move. Second, Martha wrote, and then pointed at top statement (second from top in figure4.6b), but Fay's gaze remained fixed on the bottom expression. Finally, on multiple occasions, Martha turned and used her eye-contact to solicit a response from Fay (Rossano, 2013; Stivers et al., 2009; Stivers & Rossano, 2010), who did not respond, or move. Altogether these points indicate that, at least for part of this episode, Fay displayed (or perhaps claimed) work on conversationally relevant mathematics, and further, displayed that she was not attending to the specific shape of the ongoing conversation. This conclusion can perhaps be stated as: Sometime during their conversation, Fay stopped displaying ongoing commitment to the developing shape of the conversation, and instead displayed continuing thought about something that had been conversationally relevant.

Their subsequent interaction bears this interpretation out. While Martha was speaking in line 17 (and not at a *Transition Relevant Place* Liddicoat, 2011) Fay began to talk, line 18, and stepped toward the board. Martha responded with surprise (line 19, and Smith's punctuation of her utterance in lines 19–20 with an exclamation mark), objected to Fay's action (line 19), and finished her explanation of the problem. Fay, who had by then finished walking to the board and pointed at the $\delta$ in the bottom expression (Figure 4.6b, third from top), did not so much respond as, with "even" (line 21), claim that there was a more fundamental problem with Martha's formula than the one Martha had noticed. Fay then looked at Martha while Martha looked at the equation Fay was then pointing at (Figure 4.6b, bottom), with the same posture (including tipped head) Joseph used to display thought about conversationally relevant mathematics. After that, Martha asked about Fay's claim (line 24), and Fay responded by, in one word, reiterating her claim (line 25), and then again looking at Martha for a second and a half while Martha looked at the formula Fay had singled out (line 26). Fay then glanced back at the board while she started an abortive turn (line 27), and then looked back at Martha for almost another second of silence while Martha still looked thoughtfully at the equation (line 28).

Martha's surprise requires a more detailed treatment, and will be examined in more detail below. The other details, however, more directly confirm the hypothesis that, sometime during the discussion (either toward the beginning of Smith's (2012) transcript, or before it began) Fay stopped displaying an ongoing commitment to the development of the conversation, and instead displayed continuing thought about an earlier point in the conversation that, in some sense, remained conversationally relevant. First, the timing of Fay's initial speech (line 18 of 4.6c), just after Martha's "then", indicates that Fay started her turn when Martha's was not projectable as complete (Lerner, 1991), displayed an inattention to the preceding turn of talk—and Martha's objection and surprise (lines 19–20 of 4.6c) treated it as such. Likewise, that Martha dropped her protest, and displayed continued thought about the formula Fay identified as problematic, indicated that she treated Martha's response—though not its timing—as legitimate responses to the forgoing conversation. Martha's response made her previously acceptance of Fay's speech, and so implicitly, also the preceding silent thought and inattention to the continuing conversation, more explicit.

These lines of evidence indicate, first, that during Martha's description of the problem with her formula, Fay had displayed continued thought about mathematics earlier conversation had made relevant. Second, in disengaging with the conversation, Fay constructed collaborative mathematical activity as the sort of thing that sometimes involves thinking about relevant mathematics in preference to displaying a listening to an ongoing conversation. Finally, if collaborative mathematical activity is the sort of thing that involves thinking about relevant mathematics in preference to listening, this fact suggests, though it does not conclusively prove, that in mathematical collaboration, thinking about relevant mathematics in preference to continuing conversation is normally sanctioned. This extends the conclusions of the previous section: The previous section showed that when mathematicians display continued thought about relevant mathematics, talk only puts conversation on offer, but, unlike conversation, a response is not normally required. This section suggests that, even in ongoing conversation at boards, mathematical talk (perhaps even all mathematical talk after the beginning of a conversation!) only puts responses on offer, and that that ceasing to engage with the conversation, in preference for displayed thought about relevant mathematics, is a normally sanctioned activity.

This conclusion, however, is a remarkable claim, and needs both more evidence to support it, and raises theoretical questions: What does it mean to say that when mathematicians collaborate, their talk both establishes certain sorts of action as conversationally relevant, yet simultaneously, can only put responses on offer? That theoretical apparatus will be developed in the examination of Martha's objection (line 19–20 of 4.6c) to Fay's talk. However, because the conclusion of the preceding paragraph is central to this paper, before proceeding, it is helpful to spend some time dwelling on the claim, and explicitly unpacking it. The claim is that when mathematicians collaborate at the board, two courses of action are normally sanctioned: First, mathematicians can continue attending to their peers and aligning their actions with the actions their peer is currently engaging in. Stated more generally, there are, then, two *kinds* of relevant action mathematics can engage in: First, they can align with the activity their peer is currently seeking to accomplish, by engaging directly with it. Second, they can align with an activity they or their peer were previously seeking to accomplish, by continuing to think about it (thus displaying a relative inattention to their peer's ongoing talk). Verbal responses (answering, continuers, etc.) generally (though, as Fay's eventual speech here, not always) align with their peer's continued talk. Lengthy silences, on the other hand, either display the second kind of activity, namely, continued alignment with actions initiated in previous talk, or are ambiguous.

Martha's surprised objection (lines 19–20 of 4.6c) to Fay's movement toward the board and initiation of a turn (line 21 of 4.6c), however, seems to undercut that analysis. She does eventually accept Fay's speech—and it is worth noting that Fay is

Martha's advisor so it is possible a power differential influences that acceptance. But in these lines, her objection that Fay's action embodies "not car[ing]" seems to be a censure of Fay's timing (and perhaps also her earlier "not hearing". Furthermore, and her completing of her turn seems to indicate that what she had to say was conversationally important, and so that Fay could be sanctioned for not listening. Martha's response is therefore evidence that Fay's action in the preceding line (line 18 of 4.6c) is contrary to the norms of mathematical collaboration.

The Conversation Analysis literature, however, contains another sort of action that would receive that sort of censure. This is found in the phenomenon of the preference organization. In ordinary conversation, it is often the case that there are two *kinds* of canonical responses to a FPP, one that is usually delivered quickly, and another that is usually delayed, or if not delayed, is subject to censure (Liddicoat, 2011). For instance, if someone invites a friend to dinner, the friend can accept or reject the invitation. An acceptance usually follows quickly after the invitation, but, on the other hand, a rejection is usually delayed—indeed a delay, either "um" or silence can be heard as a rejection—and a quick rejection is very strong, and can be heard as rude. For this reason, accepting an invitation is called a preferred response, and rejecting it is called the dispreferred response. It needs to be stressed that preference is a feature of the structure of conversations—of, as it were, the conversational landscape in which people act—not of the psychological preferences of the participants. You may receive an invitation you would very much like to decline. But if you decline quickly, your action will be seen as rude—indeed, you could decline quickly in order to insult the person who invited you. Because, even in this situation, a prompt declension is likely to be heard as rude, the preference for acceptance still structures this situation, and is, as it were, the landscape in which all parties act.

Setting aside, for a moment, the question of what a preference organization would mean in this context, the hypothesis that, when mathematicians collaborate at the board, there interaction is structured by a preference organization, can help explain Martha's objection (lines 19–20 of 4.6c), and to explain how responses can be, and simultaneously, not be, conversationally relevant. Stated plainly, the hypothesis is that ongoing engagement with the talk of their peer is the preferred response; thinking about mathematics a previous line of conversation made relevant is the dispreferred response. Under this hypothesis, Martha's objection is not to the action Fay begins with her turn at talk, nor to her preceding "thinking"—if it were, why would she abandon the objection so quickly—but to its precise timing and manner of delivery. Martha's objection shows that she analyzed Fay's timing—just as Martha was coming to the conclusion of her talk—as harsh; but her subsequent acceptance of Martha's talk reveals that Fay treated Martha's talk as conversationally relevant.

This hypothesis also shows how talk can both make particular actions relevant, and simultaneously, only put talk on offer. At the beginning of a conversation, there is no conversationally relevant mathematics, and so the only possible kinds responses are those entailed by ordinary conversation. As the conversation progresses, however, mathematics becomes conversationally relevant, and so thinking about that mathematics (and not engaging with further talk) is normally sanctioned, though dispreferred. On the other hand, inasmuch as a mathematician adopts the preferred action of conversational engagement, the normal rules of conversation remain relevant.

The last question that needs addressed is what it means to say that a preference organization is operative here. Can a display of thought be said to be delayed relative to conversational engagement? If the display of thought is in part accomplished through silence, then, the answer is perhaps. However, that response is not necessary. Preference organizations are not only defined in terms of the structure of responses, but also in terms of whether a SPP furthers or confounds an action begun in the FPP (Schegloff, 2007, 2010b). When FPP's make two kinds of response relevant it is often the case that one that furthers the actions the FPP began, and one that interferes with interferes with the action. The response that continues the project is called the preferred response, whereas the response that interferes with the project is called the dispreferred response (Schegloff, 2007). For instance, if someone invites their friend over for dinner, the invitation works toward a shared meal. Accepting the invitation continues that project, whereas rejecting the invitation interferes with that project. For that reason, accepting the invitation is the preferred response, and rejecting the invitation the dispreferred response. (Schegloff's and Liddicoat's treatments of preference organizations are not in conflict. Schegloff also gives Liddicoat's 2011, definition, and discusses the conversational interplay resulting from the two different kinds of preference organizations.)

Then, because displayed silent thought does not align with the actions embodied by the previous turn of talk, it is a dispreferred response. On the other hand, a verbal that embodies continued alignment with the previous turn of talk is, by definition, a preferred response. This description of the data also helps to make sense of the theoretical questions. The participants in mathematical collaboration at the board are engaged in focused activity, and throughout their work, at least some aspects of the conversation remain conversationally relevant, as was shown above with the cursory examination of Matt and Bart's collaboration. On the other hand, one of the conversationally relevant actions is to think silently about something earlier conversation made relevant, and, to "tune out" further speech. When, however, people engage in conversation, something like ordinary rules for conversation still apply: First pair parts receive answering second pair parts, and can, provisionally, be expected to.

### 4.5.3 Pacing, Averting Gaze

At this point one questions remains. It is unclear whether staring motionlessly at the board is the only activity through which mathematicians display silent thought about mathematics, or whether it is one of a class of actions. This section will show that sort of activity is not unique, but is one of a class of activities. In particular, mathematicians sometimes display engagement with mathematics by averting their gaze from the board (Figure 4.7b–e), and by pacing 4.7c–d). Smith's (2012) corpus is relatively small, and so it is unclear whether there are other activities that mathematicians use to display engagement with mathematics. Furthermore, even averting the gaze and pacing only occur rarely in the corpus, and their precise use cannot be examined in detail. It is unclear, for instance, whether these practices merely display silent thought about conversationally relevant mathematics, or if they function somewhat analogously to change of state tokens and indicate that mathematicians are pursuing potentially interesting mathematics that is not directly conversationally relevant.

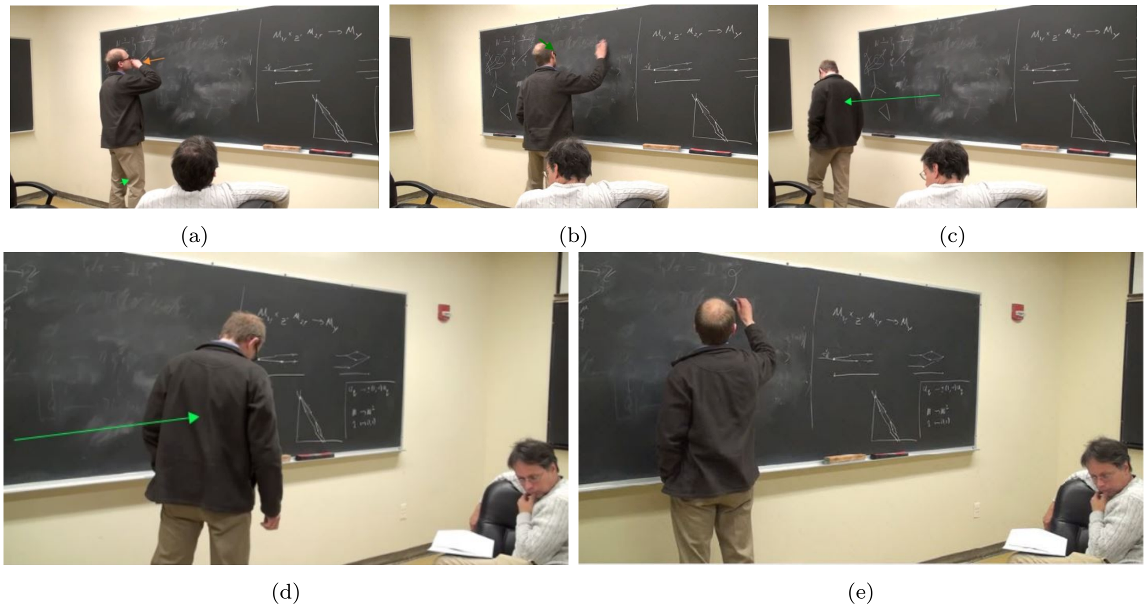
This episode features the work of two algebraic geometers, Matt and Bart. Both mathematicians had coauthored articles together, and, at the time the video was recorded, Bart was visiting Matt for a week from Germany. In this episode, they were working on to determine what happens to curves on algebraic surfaces, called varieties, when multiple surfaces are attached together simultaneously.

In the following transcript Bart (standing) had just finished erasing the board, and he and Matt (seated) were discussing a new tack. Smith (2012) argues that Bart's erasing did not merely clear space on the board, but specifically signaled the imitation of a new tack. Smith's argument, therefore, implies that though the board was blank, by directing their gaze to the board, the mathematicians displayed engagement with conversationally relevant mathematics. Nevertheless, though both mathematicians did direct their gaze to the board (Figure 4.7a), Bart also paced (Tanscript 4.3 line 8, Figure 4.7c–4.7d), with his gaze away from the board, and Matt directed his gaze away from the board (Transcript 4.3 lines 3–11, Figure 4.7b–4.7d).

Transcript 4.3

Mat: Yeah, I mean, so I- I guess the first question is, can we at
least get the curve at the local ( )? (.) Yeah, so
Bar: Yeah, exactly so I'm, I'm, uh, slightly lost. (.) Um
(3.9) ((Bart stands away from the board, 4.7a, then steps
forward and erases a small mark.))
Bar: Okay
Mat: ((Looks down, see figure 4.7b-e))
(6.9) ((B paces during this silence))
Bar: So let's, ((starts writing)) let's say we have two curves
that we want to
(2.25) ((Matt looks up part-way through.))
Bar: ((Finishes writing, steps back))
Bar: Right?
Mat: Mhm
Bar: Um.

This work occurs with the relevant portion of the board blank, and so initially, it does not seem like their gaze (lines 4–5 in transcript 4.3) display continued engagement with conversationally relevant—there is no conversationally relevant mathematics on the board(!) Smith (2012), however, argues that the deliberateness of Bart's erasing shows this work was not a simple preparatory action to some generic to-be-determined mathematical activity, but prepared the board for a new sort of mathematics, and



(a) (b) (c) (d) (e)

Figure 4.7: Matt (seated) and Bart. a) "um" in line 3. b) Beginning of line 8. c) Two seconds into line 8. d) 5.4 seconds into line 8. e) Beginning of line 11. Arrows are Smith's (2012) annotation of movement. Notice how little Matt moved: His fingers were a little different in (b) than they were later, but by (c) his posture was almost identical to (d) and (e).

so signified and furthered a distinct shift in strategy (p. 108–109). If Smith's reading is correct, then the conversationally relevant mathematics was, precisely, a new tack to the problem they had been working on. Furthermore, the fact that the board had been prepared to display work in pursuit of a new mathematical tack suggests that by directing their gaze toward the prepared board, the mathematicians displayed their continued engagement with conversationally relevant mathematics, namely, to the new tack Bart had prepared through his erasing.

Following Matt's first pursuit of this new tack with a topic proffer (l. 1, accepted in l. 3 with "Yeah, exactly"), the emptiness of the board, displayed not only a generic new tack, but a specific, though new, tack: "Get[ting] the curve at the local ( )." During the silence in lines 4–5, Matt's and Bart's gazes toward a board prepared to record this new tack (and Bart's physical interaction with the board) displayed continued engagement with conversationally relevant mathematics. Furthermore, both Matt and Bart seem to have indicated their need for silent engagement with the mathematics: Matt with his incomplete utterance “Yeah, so-” (Lerner, 1991); and Bart with his claim to be confused, and his “um” (which can be used to initiate silences even in every-day conversation Schegloff 2010b).

With those considerations, the first silence further illustrates the conclusions of Section 4.5.1. After Bart’s “okay” (line 6 in Transcript 4.3, after erasing a spot from the board), the mathematicians engage in two new types of activity, neither of which fits well with the conclusions of the preceding section. Matt put his hand on his mouth and, rather than using his gaze to display continued engagement with mathematics, averted his gaze from the board. Shortly after Matt averted his gaze, Bart began, as Smith describes it, “pacing slowly as though in deep thought” (p. 110), with his head down (and so, likewise, not using his gaze to display continued engagement with mathematics). This configuration was maintained for about seven seconds (see figure 4.7). Following the silence, Bart turned to the board and began talking about something he wrote on the board. After Bart had written for a while, Matt looked up at the board, and the mathematicians proceeded to work together on what Bart had drawn.

In this passage, the mathematicians engage in thinking activities with two kinds of physical embodiment, pacing, and averting the gaze, neither of which overtly displays continued thought about conversationally relevant mathematics. Indeed, in both the mathematicians’ gaze seems to be specifically *averted* from conversationally relevant mathematics. It is important to ask, then, how these specific postures function conversationally. Unfortunately, neither activity occurs frequently in the corpus, and so definitive explanations of these activities is difficult: This is the only time in Smith's (2012) corpus a mathematician unambiguously paced silently. (Bart is the only mathematician who paced.) Two other times he may have started briefly to pace silently, however, Smith's (2012) transcripts have too few images to make a clear judgment. And Bart paced while he talked once.

Similarly, there are only two other examples of a mathematician averting their gaze: First, toward the end of the "odd scalar" episode, Bill, who had spent the overwhelming majority of the episode staring silently at the board, both while Joseph spoke and while he was similarly silent, but had just turned his eye-gaze to a section of the board near Joseph and to an area Joseph was gesturing at and taking about, averted his gaze while Joseph talked. Second, in the "Adjusting a Triangle" episode, approximately 2 seconds into a short 3 second "mutual silence", Matt averted his gaze for the remainder of the silence. Prior to this silence, their activity had been coded as "one speaking one thinking", since Matt had not responded to Bart's talk. After averting his gaze, Matt stepped back from the board, looked at it for a few moments, and then sat down (the latter actions while the other mathematician, Bart, talked).

Nevertheless, conclusions can be drawn, the first relatively definitive, the second far more conjectural. Though the mathematicians' posture does not display continued thought *about conversationally relevant mathematics*, it does display continued thought. Thus, for instance, as noted above, Smith (2012) read Matt as engaged in "deep thought" during his pacing. Likewise, Bart rested his hand by his mouth, the same rough hand posture Rodin's The Thinker adopts, and one that chess players like world champion Carlson adopt while thinking about their move. Furthermore, like all the mathematicians whose activity was analyzed earlier, Matt remains almost motionless (see, for instance, the fixity of Joseph's and Bill's posture in figure 4.5).

It is tempting to say, then, that in mathematical collaboration, after a topic is proffered, if a mathematician displays thinking, it is assumed by other members of the conversation that they are thinking about conversationally relevant mathematics. This hypothesis, however, is insufficient, since these activities are so rare in Smith's (2012) corpus. In contrast with those activities, in the majority cases in the corpus, the mathematicians not only display thinking, but thinking about conversationally relevant mathematics: This activity contrasts with that, and that is not a sort of intensification of this activity. It seems then, that this activity displays a sort of engagement with mathematics that contrasts with the more common sort of engagement: Perhaps, for instance, it functions something like a change of state token (like "so"), and displays a shift in engagement to not immediately relevant mathematics. That conclusion theory may be too hasty, however, it seems reasonable to hypothesize that this sort of activity displays engagement with mathematics in a way that is the result of a problem (mathematical or conversational) in the mathematics displayed on the board.

These activities also show that the sort of engagement seen in the previous two sections is not the only sort of normal silent mathematical activity, but is the most common in a class of activities that display engagement with conversationally relevant mathematics.

## 4.6 Conclusions

In an attempt to initiate inquiry into the interpersonal character of mathematical thinking, this paper investigated the norms governing extended silent thinking in

**Transition from talk to mutual silence:**

- Normal conversational trajectory continued in silence.
- Body posture normally displays ongoing engagement with conversationally relevant math.
- Silence itself is communicative.
- Mutual thinking silence can be goal of conversational moves.

**Activity during thinking silences:**

- Brief talk only puts conversation on offer, it does not normally require an answer.
- Most common posture in this corpus:
  - Motionless fixation of gaze on record of past conversation.
- Other postures:
  - Averting the gaze.
  - Pacing.
- Specific significance of these postures is unclear.

**Silent thinking during peer's talk:**

- Even as someone talks, a mathematician can cease displaying listening, & begin displaying silent thought.
- Silent thought normally continues conversational trajectory.
- But one party's talk can continue even as peer displays silent thinking.
- Display of continued engagement with conversation (e.g., continuers, responses, etc.) preferred.
- Thinking silently dispreferred.
- But both in accord with norms.

Table 4.1: Summary of Results

face-to-face mathematical collaboration. Because, tautologically, the mathematicians did not speak as they thougt silently, this investigation into the norms governing silent interaction is also, necessarily, an investigation into the embodied practices mathematicians employ in their thinking, and which commuicate the character of their thinking. The long-term goal of this research is to open lines of investigation into the social achievement of thinking in the classroom, and into the establishment of classroom norms and practices that make extended thinking possible during social exchanges.

Previous research suggested that in their collaborations, mathematicians engage in lengthy silences, during which the mathematicians seem to be thinking. Furthermore, because of their length, these silences seem to violate the norms governing every-day conversation, and so point to a different set of norms governing face-to-face mathematical collaboration (Petersen, 2017a). Nevertheless, this research only attended to the duration of silences during active mathematical collaborations, and so was not able to rigorously study the norms or the activities of mathematicians in those silences. Furthermore, the way that silences violate the norms of conversation depends on the position in the conversation in which they occur. At some places in a conversation, notably following a Second Pair Part (SPP) or a post-expansion, a silence may be an unproblematic signal that the conversation has temporarily lapsed as co-present people go about non-conversational activities (Schegloff, 2010a; Couper-Kuhlen, 2010). At other points in a conversation, for instance, following First Pair Part (FPP) like a question, a silence violates conversational norms. The previous research into mathematicians' silences therefore leads to two competing interpretations. First, it was possible that mathematicians' collaborations are normally divided into periods of conversation, during which silence is problematic, and periods of non-conversational, silent, mathematical activity. On the other, it was possible that that the conversations themselves contain lengthy silences.

This paper's findings are summarized in table 4.1. First, this paper showed interactional norms permit mathematicians to engage in thinking silences at all points in a conversation, including following FPP's. Indeed, though the data set analyzed in this paper is too small to admit rigorous statistical analysis, some of the longest silences in the corpus follow FPP's, where silence would normally be disallowed. Furthermore, not only are silences in accord with the norms *following* any type of turn, participants do not even violate norms when they display an engagement in silent thinking activities while their partner is actively talking to the thinking mathematician.

This result should not be understood to mean that a conversation can normally be suspended for thinking at any point in the conversation. Rather, the thinking seems to be normally connected to the talk that preceeds and follows it. For instance, a mathematician may engage in conversational moves designed to lead their peer to display thinking activities. Indeed, a mathematician may communicate noticing by displaying silent thinking activities.

The results, however, are stronger than that. In a conversation, participants' talk establishes a trajectory for the conversation, and so restrains the sort of talk that can normally occur later in the conversation. In mathematicians' collaboations, these normal trajectories of conversation are not interrupted by silent thinking. Rather, during silent thought, postures which display ongoing engagement with conversationally relevant mathematics are employed to maintain these trajectories that earlier conversation had established as normal. This is the key, and central finding of this paper.

Limitations in the data analyzed in this study prevented the identification of all the postures mathematicians utilize to display ongoing engagement with the trajectory of a converation. Nevertheless, the most common such posture in this corpus involving the fixation of the gaze on a record the conversation, and holding the rest of the body in a tense motionlessness. Furthermore, the blackboard plays a key role in this norm, since one of the main ways mathematicians' display continued engagement with conversationally relevant mathematics is by fixing their gaze on a blackboard which contains a record of the mathematical content of the conversation, and remaining almost completely motionless. Nevertheless, there are other sorts of activities that display continued engagement with conversationally relevant mathematics, including chin scratching, averting the gaze from the board, and pacing. These activities, however, occured too infrequently in the data to be analyzed in depth. Still, in the majority of cases in this corpus, mathematicians' displayed ongoing engagement with conversationally relevant mathematics by motionlessly fixing their gaze on a blackboard that contained a record of their conversation.

These results show that in mathematicians' collaborations, thinking silently is not a form of private activity existing in a sort of hiatus of the conversation, but is closely connected to the public process of conversation, and to convesrational norms. For this reason, this research opens up the possibility of social investigations of mathematical thining, the postures people use to embody that thinking, and the development of norms for utilizing those postures in conversation.

Finally, the results raise a paradox: The suggestion that mathematicians can stop listening and start thinking even while their partner speaks, suggests that responses are not normally expected during mathematicians' collaborations. At the same time, this result would make the idea of mathematical conversation nonsensical (since conversations depend on members responding to the actions their peers engage in). Furthermore, the fact that thinking activities can be a response, and continue projects established in the conversation shows that conversational norms are not suspended.

This paper resolves this paradox by suggesting that, in mathematical collaborations, there is a strong preference for listening to a peer, however, continuing to think about an aspect of mathematics the conversation established as relevant, is also in accord with mathematical norms.

Since mathematicians are not chatting, but engaged in a form of collaborative goal-directed activity, that the norms governing matheamtical collaboration differ from those of every-day conversation is not entirely surprising. Nevertheless, unlike many other forms of goal-oriented activity (like playing soccer, building a boat, carrying a board, cooking dinner, performing music in an ensemble) in face-to-face converation, mathematicians acheive their goals through talk. From this perspective, mathematical collaboration is akin to conversation. For instance Sfard and Kieran (2001) began their research under the assumption that "many school subjects, and mathematics among them, are best learned in an interactive way, through conversation with others" (p. 42). They ultimately concluded that mathematical conversations did not automatically result in learning; however, they at the end of their paper, they continued to identify mathematical interaction with conversation. This paper challenges that identification. There may be forms of goal-directed activity whose use of silence stand somewhere between mathematicians' collaborations and converation, and these may prove fruitful grounds for introducing students to the distinctive uses of silence in mathematical collaboration. But, though mathematicians' accomplish their goals through talk, and though their talk follows many of the norms that govern every-day conversation (as this paper has shown), their activity also differs from conversation through its tolerance, and even requirement for, lengthy periods of silent thought, at times when the interaction has not lapsed.

This study suggests several avenues for additional research. First, if these results are going to inform teaching, it is important that researchers attend to thinking activities in student collaborations. Several avenues of investigation present themselves. First, a researcher could investigate the development of norms governing silent thinking activities in student collaborations. Second, a researcher could investigate ways that students and teachers make time to think silently, when they are actively engaged in conversation with their peers. Third, survey data reported in Petersen (2020c) suggests that some students prefer to work alone, precisely because mathematical activity requires silent thought. A researcher could therefore investigate ways the necessity to think makes group-work difficult for students, leads to disaffection with forms of instruction that require group work, and instructional moves teachers can employ to help better include those who acutely feel a need to think mathematically, in their classroom.

On the other hand, this paper only begins an investigation into the embodied, public character of mathematicians' thinking activities. A researcher could therefore investigate the different postures mathematicians employ in their thinking activities, and the different uses, both individually and interpersonally, of these postures. Furthermore, these thinking activities could be distinguished from other forms of thinking. For instance, the mathematicians in this study seemed to employ an intensely focused form of thinking, whereas, in Cavell's (1981) analysis, Carey Grant's character in *Monkey Business*, Walter, employs a searching, but unfocused sort of thinking in his journalism. These differences do not show a difference between mathematics and journalism, but they do suggest that a more thoroughgoing investigation of the specific nature of mathematical thought could be profitable.

# 5

# Conclusions

This dissertation presented a methodology for using a ethnographic micronalysis (Erickson, 1996; Erickson & Schultz, 1977; McDermott et al., 1978) in the study of silence in collaboration, and two careful case-studies of silence during mathematical activity. The largest contribution of the methodology paper is the explication of a novel transcription method that can be used, at all stages in data analysis, to focus analytic attention on activities performed while one or more person is silent.

The second paper examined silent activities in collaboration of students in junior-level mathematics courses. This paper identified two activities that students silently engage in, reading and ruminating. Reading is a well-known activity, rumination, however, refers to a particular kind of thinking (understood from an embodied and sociocultural perspective, not from a psychological perspective) in which someone fixes their gaze on the board and remains motionlessly silent for several seconds. Furthermore, this paper identifies ways these activities affecct group dynamics. For instance, student will sometimes, unproblematically, talk and read simultaneously. When that happens, the spoken mathematical activity can be a distraction for the reading student, leading to his diminished contribution. Though reading is a well-known activity, this paper is the first to identify rumination. The identification of rumination is, therefore, the most important result of this paper.

The final paper examined the social acheivement of thinking in mathematicians' collaboration. Though thinking is often examined from a psychological perspective, as an activity inside the head. However accurate this description is, it is only half the story: When people think, they can be seen to think. That fact means that thinking is an embodied activity—that is it is not like the heartbeat or the internal processing of sounds into words, which involve purely internal aspects of the body, but is akin to activities like climbing stairs, rolling dice, or gesturing, which are externally embodied, and visible. Furthermore, thinking is (often) temporally extended. For instance, as this paper shows, mathematicias are silent for extended periods of thought. These two facts, thinking's embodiement and its temporal duration, show that thinking is not a purely internal act, available to psychological investigation but invisible to an ethnomethodological account of action. Rather, thinking can be approached as an interactional acheivement, and studied as a form of interaction. This paper begins that investigation by attending to the norms governing silence (or silent thinking) in mahtematical collaborations. Though this paper has several important results regarding the norms of silent thinking in mathematicians' collaborations, its most significant contribution is opening thinking to investigation from an interactional perspective.

At this point, there are several avenues for further research. First, as noted in Petersen (2020a), it will be important to address questions of the relationship between gender, race, ethnicity and class and silence in mathematics collaboration. Silence is often gendered (Solnit, 2012), and, like other signs its significance is socially constructed. This result suggests that students from different backgrounds may have very different prior experiences with silence, and so different experiences of mathematical silences.

Second, researchers could more thoroughly survey forms of mathematical thinking: The thinking examined in Petersen (2020b) closely resembeled rumination. But are there other forms of mathematical thinking? Petersen (2020b) argued that mathematicians use several different forms of activity to display thought about conversationally relevant mathematics. Researchers can more explore more fully the social significance of these different activities. Why, for instance, in those instances, did the mathematicians pace or avert their gaze, rather than fixing their gaze on the blackboard?

Third, mathematicians have anecdotally told me that they have noticed that their forms of interation learned during mathematical collaboration inform their social interation in the rest of their lives—for instance, they use more careful silent thought than their peers or colleagues from other disciplines tend to. A researcher could therefore investigate the way distinctive forms of mathematical interaction both shape mathematicians' subjectivities more broadly, and reasons people with different dispositions and experiences of silence gravitate toward and away from mathematics.

Fourth, mathematics teachers need to engage in mathematical activity during their coursework (Johnson, 2013). If mathematical activity requires silent thought, as Petersen (2020b) suggests, then mathematics teachers need to think silently in their coursework—an observation I have observed in my own tutoring, and that several mathematicians have noted in their own teaching. What methods do mathematicians use to gain time to think when teaching? And, furthermore, do different teaching styles provide different affordances for thinking? For instance, in a lecture classroom, the teacher has the floor, unless she nomiates a student to speak, whereas in a reform classroom, a student may have a more natural right to the floow. This difference may mean that it is easier for a teacher in a lecture course to think silently in response to a student question than it is for a teacher in a reform classroom. If so, then one skill reform teachers may need to learn in their classroom is how to gain time for thinking.

Similarly, teachers need to learn to recognize and valorize silent forms of mathematical activity in their students. Specifically, Tzur (1999) and Steffe (2003) record students' extremely long thinking silences as they worked through the conceptual meaning of fractions, but do not analyze those silences further. This raises questions like the following: How do silent thinking activities show up when small children work on mathematics? How does the social requirement that people answer relatively quickly make silent thinking difficult for elementary students? How can skilled teachers recognize and valorize these activities, while recognizing and responding appropriately to silences that are not part of thinking activities?

There may also be avenues for quantitative analysis of silent mathematical activity. There is a growing literature on behavioral synchrony. For instance, researchers have attended to the ways members of string quartets entrain each other's movements, using both visual and audible cues (Chang et al., 2017). When mathematicians collaborate, do they similarly entrain each other's movements? If so, does that entrainment show that thinking is even more communal than is often thought? And, again if so, how do students learn to appropriately respond to their peers' movements—especially when their peers are more experienced mathematics practitioners?

Third, annecdotally, the bodily practices mathematicians employ while listening to mathematics seminars resemble the bodily pratices they employ while collaborating in silence. I plan therefore to study the bodily and note-taking practices of mathematicians as they attend mathematis seminars. This research is intended to link up with research into students' learning in lecture classes and note-taking practices (e.g., Lew & Mejía-Ramos, 2019). In particular, since mathematicians regularly attend seminars in which one mathematician lectures about their research, attending to mathematical lectures is a practiced, disciplined, form of mathematical activity that students apprentice into. This means that, though students often have extensive practice listening to lectures, they may need to learn the techniques for listening to mathematical lectures. Furthermore, though there are obvious differences between a mathematics seminar and the classroom—for instance, mathematics professors do not need to prepare for a test over seminar topics—attending to the ways mathematicians listen to lectures may aid research into the techniques students can employ to learn from mathematical lectures.

Finally, I hope to link up my work in statistics education and ethnomathematics to my dissertation work.

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## Appendix A. Technical Terms

*Change of state token:* A short term like "oh" or "so" that indicates a change in the conversation, for instance, in the epistemic status of one of the participants.

*First pair part:* A turn that initiates an action, for instance, inviting, commenting, evaluating, or noticing.

*FPP:* Abbreviation for First pair part.

*Focused interaction:* The sort of interaction in which people in spatial proximity display a shared focus. Examples include: Dancing, conversation, etc.

*Incipient talk:* A time during which a conversation has been initiated, and so can be resumed without a greeting, but the conversation has lapsed.

*Insert expansion:* A turn or sequence that goes between an FPP and an SPP. A repair sequence, during which one speaker corrects a problem in the speaking or hearing of the FPP, is one example of an insert expansion. But participants could, perform a number of actions as an insert expansion. For instance, before accepting a dinner invitation, they could clarify the menu (to make sure dietary restrictions will be honored).

*Positioning:* McDermott et al.'s (1978) for a postural configuration. They used the term "positioning" to emphasize 1) that postural configurations are dynamic, and 2) that they serve the purpose of communicating the nature of the task a group is engaged in, and giving each participant a position in that task.

*Post expansion:* A turn or sequence that follows an SPP. For instance, following a query and response, "thanks!" is a post-expansion. Post expansions can be used to signal a return to a state of incipient talk.

*Postural configuration:* A steady arrangement of the postures of participants in joint activity. This term clearly describes the physical meaning of the phenomenon it describes. Nevertheless, it seems, wrongly, to indicate that this configuration is static; when in fact, it is the result of on-going negotiation, and maintenance, and actively contributes to participants cooperation in an activity.

*Pre-expansion:* A turn or sequence that goes before an FPP. For instance, "can I ask you a question?" initiates a pre-expansion that indicates the following question will require multiple turns to set up.

*Preference structure:* A feature of the conversation, and not of the internal preferences of the members of a conversation. Often there are two kinds of SPP's, one that furthers the action initiated by the FPP (e.g., accepting an invitation, or agreeing with an assessment) and one that opposes it (e.g., declining an invitation, or disagreeing with an assessment). The response that furthers the action initiated by an FPP is said to be preferred, the one that opposes it is said to be dispreferred. Furthermore, preferred responses are usually given quickly following the FPP, whereas dispreferred responses are usually delayed (and the delay is hearable as a dispreferred response).

*Prosodic contour:* The contour of the pitch and rhythm of talk.

*Second pair part:* A turn that responds to the action initiated in a FPP.

*SPP:* Abbreviation of second pair part.

*Sequence:* Conversation follows a particular sequential order, with some turns of talk initiating actions (FPP's), others responding to them (SPP's), and still more prefacing the initiation of an action (pre-expansions), commenting on the FPP after it is uttered or prefacing a response (insert expansions), or commenting following a response (post-expansion). Furthermore, these sequences are themselves sequenced in particular ways with certain kinds of actions being accomplished at beginnings of conversations, others toward the middle, and others toward the end.

*Torque:* A description of the angle between trunk orientation, shoulder orientation, and head orientation. A posture is said to have high torque when the head is turned away from the shoulders, and the shoulders from the trunk. The direction of the trunk indicates the primary focus of activity, whereas shoulders and head indicate secondary, temporary focuses of attention.

*Transition relevance place:* A time in a conversation when a second speaker may normally begin talking.

*TRP:* Abbreviation for "transition relevance place".

*Turn-initial:* Occurring at the beginning of a turn of talk. Sometimes particles have different meanings at the beginning of a turn and during a turn. For instance, turn-initial "so" is a change of state token (and functions somewhat like "oh!"), whereas, in a turn, it is not.

*Unfocused interaction:* Interaction without a common focus of activity. Information about how to act is gained through quick glances, not through sustained postural configurations. Examples include crossing the street, standing in an elevator with someone, and eating a meal.