2-2-2022

Linear Nearest Neighbor Flocks With All Distinct Agents

Robert G. Lyons
Portland State University

Linear Nearest Neighbor Flocks With All Distinct Agents

by

Robert G. Lyons

A dissertation submitted in partial fulfillment of the
requirements for the degree of

Doctor of Philosophy
in
Mathematical Sciences

Dissertation Committee:
J. J. P. Veerman, Chair
Dacian Daescu
Pui Leung
John Caughman
Wayne Wakeland

Portland State University
2022

Abstract

This dissertation analyzes the global dynamics of 1-dimensional agent arrays with nearest neighbor linear couplings. The equations of motion are second order linear ODE's with constant coefficients. The novel part of this research is that the couplings are different for each agent. We allow the forces to depend on the relative position and relative velocity (damping terms) of the agents, and the coupling magnitudes differ for each agent. Further, we do not assume that the forces are "Newtonian" (i.e., the force due to A on B equals minus the force of B on A) as this assumption does not apply to certain situations, such as traffic modeling. For example, driver A reacting to driver B does not imply the opposite reaction in driver B.

There are no known analytical means to solve these systems, even though they are linear. Relatively little is known about them. To estimate system behavior for large times we find an approximation for eigenvalues that are near the origin. The derivation of the estimate uses (generalized) periodic boundary conditions. We also present some stability conditions. Finally, we compare our estimate to simulated flocks.

Acknowledgments

I'd like to thank Prof. Veerman for all his help and Digimarc for their ongoing support. Professors Daescu, Leung, Caughman and Wakeland all made helpful suggestions. I'm also grateful to Gillian Nance — proof-reader and grammarian extraordinaire.

Finally, thanks to Serge Preston, who convinced me to return to Mathematics research. He was a great Mathematician, a good friend, and a tremendous inspiration.

Table of Contents

List of Figures

1 SEQUENCES OF AGENTS

1.1 Introduction

In this dissertation we study the dynamics of a one-dimensional array of agents that interact with each other. This system could be a model for a physical process or agents equipped with engines. The agents can apply forces that depend on distances or differences in velocity. This means that the equations of motion of our system do not have to be Hamiltonian or even Newtonian (i.e., the force due to A on B equals the minus the force of B on A).

The dynamics of a one-dimensional lattice of coupled agents is a model for many physical systems and so has a long history. If all the agents are identical then connecting nearest neighbors with Hooke's Law results in a simple model of one-dimensional crystals [1]. In this case, if one assumes periodic boundary conditions, then the eigenvectors of the system are the Discrete Fourier Transform basis functions. In the 1950's this nearest neighbor crystal model was extended to include agents of different mass [9].

In the 1950's, simplified "microscopic" traffic models appeared with agents coupled with a force dependent on spatial differences and an added force that is a function of the difference of agent velocities [7] (see [17] for a survey of traffic models). The velocity-dependent force term originated as an empirical law and will play an important part in our discussion. The basic idea is that an automobile agent attempts to follow a leader by attempting to keep his own velocity close to the velocity of his neighbors. Since this pioneering work, the subject of cooperative control has matured considerably [21, 25]. Recent technological advances make automated traffic platoons possible so there has been renewed interest in one-dimensional lattice dynamics. There are several works on both single [18, 19] and double integrator systems

[22, 15, 18]. The results for both single and double integrator systems with nearest neighbor interactions are summarized in [19].

In the one-dimensional traffic platoon, one would like to know whether is it possible, or even reasonable, to have a long platoon consisting of N agents. If we form a caravan of trucks, do we need to break the caravan into separate small chunks or can we form a single caravan of, perhaps, over a thousand trucks? There is a large volume of literature on this topic, but almost all the literature addresses this question by citing an unrealistic case wherein all cars are identical or distributed in some other highly improbable way. Each agent is distinct and may have a unique mass. We can force the forward and backward couplings to have a specific ratio, but it is difficult and certainly impracticable to insist that the force magnitudes are identical for all agents. The entire Section 2 is devoted to this specific problem. Constraints are placed on the ratios of the forward and backward couplings but the weights $g_x^{(\alpha)}, g_v^{(\alpha)}$ are chosen randomly from a distribution.

More specifically, in this work, we shall analyze a one-dimensional lattice of agents with linear nearest neighbor couplings determined by the distance and the velocity difference between neighbors. We will assume equations of motion where the force on an agent is linear in position and velocity differences (double integrator system). We shall not assume that the forces are "Newtonian" (i.e., the force due to A on B equals the minus the force of B on A).

In previous work [5, 6] it was shown that if the agents are identical and the system has periodic boundary conditions, then the equations of motion are solvable. This system has equations of motion given by the ODE,

$$\frac{d}{dt}\begin{pmatrix} z \\ \dot{z} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -g_x \mathbf{L}_x & -g_v \mathbf{L}_v \end{pmatrix} \begin{pmatrix} z \\ \dot{z} \end{pmatrix} , \tag{1}$$

where z is the position vector, $\mathbf{L}_x, \mathbf{L}_v$ are row-sum zero circulant tri-diagonal matrices and g_x, g_v are scales for the two matrices. We add negative signs so that the stability conditions have positive g_x and g_v. Since the matrices are circulant, $g_x\mathbf{L}_x$ and $g_v\mathbf{L}_v$ commute. This is instrumental in finding solutions and deriving the conditions of stability. If the forces are extended to include next-nearest neighbor terms, then assuming periodic boundary conditions, the equations of motion are, also, given by equation (1). As in the nearest neighbor case, $\mathbf{L}_x$ and $\mathbf{L}_v$ are row-sum zero circulant matrices, but this time, they have 5 non-zero diagonals. The solutions are more complicated, as are the conditions of stability, [16]. For both systems, the matrices $\mathbf{L}_x$ and $\mathbf{L}_v$ commute. In both these cases, the characteristic polynomial has a double root at 0, which corresponds to the stable configuration where all agents are moving at a constant velocity. The asymptotic behavior of the system may be found by expanding the discrete zero locus around this point. On stable systems, roots near the origin dominate the long-term behavior of the system, as other roots have larger negative real components, and so decay faster. Expanding the characteristic equation near the origin yields an approximation to the signal velocity and a dispersion term.

However, in this work, we do not assume that the agents are identical. Instead, we introduce a repeating sequence of p distinct agents duplicating this string and use an extension of periodic boundary conditions, first described in [3]. In particular, let $A_0, \cdots, A_{p-1}$ be p agent types organized in a one-dimensional lattice,

$$A_{p-1} \leftrightarrow A_{p-2} \leftrightarrow \cdots \leftrightarrow A_1 \leftrightarrow A_0.$$

We then repeat this $p-$sequence q times to get a total of $N = pq$ agents. In the general form for this system, the matrices $\mathbf{L}_x$ and $\mathbf{L}_v$, **do not commute**. The case for $p = 2$ and $p = 3$ is analyzed in [3] for both nearest neighbor and next nearest neighbor interactions. Since $\mathbf{L}_x$ and $\mathbf{L}_v$, do not commute the system is considerably more difficult to analyze, but some conditions necessary for stability are derived.

In this dissertation, we present a variety of tools to analyze this general system. The goal is first to understand the periodic case and then to use these results to shed light on the general system of N agents traveling on the real line. To pursue the dynamics of a general system we start with the system in equation (1), where $\mathbf{L}_x$ and $\mathbf{L}_v$ are the circulant matrices with -1 on the diagonal. The matrix $\mathbf{L}_x$ is assumed to have $1/2$ on the sub and super diagonals. The matrix $\mathbf{L}_v$ is assumed to have fixed values $\rho_{x,+}$ on the super-diagonal and $\rho_{x,-}$ on the sub-diagonal. This is the system in [6] except that we extend this by scaling each row by a distinct value, which is the same as taking distinct weights $g_x^{(\alpha)}$ and $g_v^{(\alpha)}$. In this case equation (1) becomes

$$\frac{d}{dt}\begin{pmatrix} z \\ \dot{z} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{G}_x\mathbf{L}_x & -\mathbf{G}_v\mathbf{L}_v \end{pmatrix}\begin{pmatrix} z \\ \dot{z} \end{pmatrix}, \tag{2}$$

where $\mathbf{G}_x$ and $\mathbf{G}_v$ are diagonal matrices with positive real values. Again, we note that $\mathbf{G}_x\mathbf{L}_x$ and $\mathbf{G}_v\mathbf{L}_v$ do not commute, so this system is more difficult to analyze. In Section 2.5 we analyze the dynamics by expanding the characteristic polynomial root locus around the double root at 0. As in the problems above, the asymptotic behavior of the system is given by roots near the double root at 0. Expansion around 0 results in expressions for the signal velocity and a dispersion term, given in Theorem (2.5.1).

The expansion, used in Section 2.5, requires an extension of the periodic boundary condition first found in [5] and [3]. We take the p distinct agents and repeat them q times. This guarantees that the discrete locus is well approximated by a continuous curve as q gets large, and this allows us to use a Taylor expansion. In the simulations in section 2.8 we will set $q = 1$ and show that the results apply well to the general case of $N = p$ distinct agents.

The stability of this general system is complicated. If the system is stable, then all the eigenvalues of the linear operator must have eigenvalues in the left half complex plane (e.g. the real part cannot be positive). This is called "Hurwitz Stability" and is captured in the following definition.

Definition 1.1.1. *Given a linear dynamical system that is governed by a matrix* $\mathbf{M}$*. The system is called Hurwitz stable if all the eigenvalues of* $\mathbf{M}$ *lie in the open left half complex plane except for a possible double eigenvalue at the complex origin.*

In Section 2.7 we discuss and derive some necessary conditions for stability. If one adds additional constraints one can derive more general stability conditions.

1.1.1 Graph Laplacians

The Laplacians in the one-dimensional problem are circulant in the periodic case and almost circulant for other boundary conditions. This simplified structure is a result of the simplified configuration where agents are on either $\mathbb{R}^1$ or S^1 and agents interact with their k-nearest neighbors. A more general system assigns agents to a node of a graph and interactions are determined by the graph edges. For details on Graph Theory, see [8, 13]. The Laplacian on a graph is a model for disease [23], agent dynamics [24, 27], diffusion and Markov Chains [14, 8]. In these applications, the equations of motion have some variant of the following linear system,

$$\frac{dx_k}{dt} = -\sum_j L_{kj} x_j, \tag{3}$$

where L_{kj} is the graph Laplacian. The graph Laplacian is a row-sum zero matrix. It's a generalization of the discrete Laplacian. For a function $f : \mathbb{R} \to \mathbb{R}$ the discrete derivative is

$$f'(x) = \frac{1}{\Delta} \left(f(x + \Delta/2) - f(x - \Delta/2) \right).$$

The second derivative is

$$f''(x) = \frac{1}{\Delta} \left(f'(x + \Delta/2) - f'(x - \Delta/2) \right) = \frac{1}{\Delta^2} \left(f(x + \Delta) - 2f(x) + f(x - \Delta) \right)$$

In the discrete case we set $\Delta = 1$ and get the row-sum zero pattern,

$$f''_i = [1, -2, 1] \left[f_{i+1}, f_i, f_{i-1} \right]^T.$$

In [28] the structure of the Laplacian is related to the structure of the underlying directed graph. In a directed graph, an edge is an ordered pair of nodes (i, j), which we denote $i \to j$. To every edge (i, j) we assign a positive weight w_{ji}. To construct the Laplacian, we first construct the combinatorial adjacency matrix Q_{ji} where the value Q_{ji} is the edge weight for the edge $j \to i$. We divide each non-zero row of Q_{ij} by the sum of the weights in that row and get the normalized adjacency matrix, S. The Laplacian is $L = E - S$ where E is the identity on non-zero rows or zero on zero sum rows. This Laplacian is row-sum zero and is the graph equivalent to the Riemannian manifold Laplacian. As in the Riemannian case the Laplacian reflects the underlying geometry.

Given a vertex i in a directed graph, the "reachable set" of i is the set of vertices, $R(i)$, with $j \in R(i)$ if there are a series of directed edges from i to j. A "reach" is a maximal reachable set. A "cabal" is a collection of vertices for which the reachable set of the vertex is the entire reach. These features of the graph are all reflected in the Laplacian. Assume that the graph has N vertices. The matrix L is singular $N \times N$ matrix. The vector $1_N = [1, 1, \cdots, 1]$ is in the kernel. More generally, one can show that the dimension of the kernel of L is the number of reaches of the underlying graph (see [28]). One can also show that the eigenvalues of L are all located on the right half plane.

The dynamics on the graph determine the flow of functions $f : V \to \mathbb{R}$, where V is the set of N vertices. The basic linear differential equations have two forms,

$$\frac{dp_k}{dt} = -\sum_{j=1}^{N} L_{kj} p_j, \tag{4}$$

$$\frac{dp_k}{dt} = -\sum_{j=1}^{N} p_j L_{jk}, \tag{5}$$

where $p_k(t)$ is the value at vertex k. The dynamical system described by equation (4) is called consensus and the system described by equation (5) is diffusion. By choosing $-L_{kj}$ we guarantee that the eigenvalues are all in the negative half plane.

One property of the diffusion equation in equation (5) is

$$\frac{d}{dt}\left(\sum_{k=1}^{N} p_k\right) = -\sum_{j,k=1}^{N} p_j L_{jk} = 0,$$

since L_{jk} is row-sum zero. The property is a variant of the conservation of mass. Next, setup a point mass with $p_k = 1$, but all other $p_j = 0$. Equation (5) is

$$\frac{dp_k}{dt} = -\sum_{j=1}^{N} p_j L_{jk} = -L_{kk} \leq 0,$$

so that the point mass will decrease in time. The only case where p_k does not decrease in time is when the node k has no incoming vertices, so that $Q_{kj} = 0$ for all j. This means the kth row of L_{kj} is zero and k is a singleton Cabal. In all other cases the “mass” decreases at k. If there is a vertex $k \to i$, then $Q_{ik} > 0$, so

$$\frac{dp_i}{dt} = -\sum_{j=1}^{N} p_j L_{ji} = -L_{kj} > 0.$$

If the edge $k \to i$ exists, then the value p_i increases. If k is a singleton Cabal, then there are no such edges, and the system is static. In general, the mass diffuses from node k to i and continues to flow opposite to the edge direction until the mass reaches the Cabal. The zero eigenstates of this system consist of equal weights among the Cabal elements.

Let’s look at how a point mass behaves under the consensus equation (4). As before, we set $p_k = 1$ and all other $p_j = 0$. The evolution of p_k is given by

$$\frac{dp_k}{dt} = -\sum_{j=1}^{N} L_{kj} p_j = -L_{kk} \leq 0.$$

Again, the mass decreases unless k has no incoming edges, in which case the derivative is zero. If there is an edge $k \to i$, then

$$\frac{dp_i}{dt} = -\sum_{j=1}^{N} L_{ij} p_j = -L_{ik} > 0.$$

In this case p_i increases, even if k is a singleton Cabal. The “consensus” flows from the Cabal down the edges throughout the entire reach.

The graph in the problem presented in this document is either a cyclic or a line graph and consists of a single reach. The graph structure is simple and does not help understand the dynamical system. Further, our system is second order and so it is not explicitly covered in any of the above references.

1.1.2 Reading this Document

The new research in this document pertains to the system described in Section 2.2. One of the main results is Theorem 2.5. We explore the consequences of this result in Section 2.8.

To introduce basic concepts, Sections 1.2 and 1.3 describe the one-dimensional two and three agent systems. These sections are independent of the rest of the document. Section 1.4 contains a description of the general one-dimensional N-agent system and introduces notation that is used in the remaining sections of the document. Sections 1.5 and 1.6 describe one-dimensional systems that make additional assumptions on the parameters. These sections are not required for Section 2.

Section 2 contains a description of one-dimensional systems with unique weights and is the essence of the thesis. Section 2.1 describes the basic strategy of fixing forward and backward interaction ratios, and varying the agent interaction weights. Section 2.2 describes one of the basic strategies of combining q copies of a sequence of p-agents with random weights. As $q \to \infty$, the eigenvalues become a continuous curve. In Sections 2.3, 2.4 and 2.5, we analyze this curve near the origin. We assume that stable systems are determined by the eigenvalues near the origin, as all other eigenvalues have large negative real parts and so decay more quickly. In Section 2.7, we make a few comments on stability. We close with Section 2.8, which is a collection of simulations that compare the results with various experiments.

1.1.3 Tools

All simulations were written in Matlab and run on Windows 10 with

MATLAB Version: 9.7.0.1190202 (R2019b).

Some computations where done with SAGE,

SageMath version 9.0, Release Date: 2020-01-01

Using Python 3.7.3.

1.2 Simple Coupled Oscillator

In this section we describe a simple example of two agents. We introduce the notation that is used in the remainder of this dissertation.

Given two agents, A_0 and A_1, that lie on $\mathbb{R}^1$, we apply forces so that A_0 and A_1 keep a fixed distance $\Delta > 0$ between them. If x_0 is the location of A_0, and x_1 the location of A_1, the forces will try to enforce the condition,

$$x_1 + \Delta = x_0. \tag{6}$$

Notice that the stable configuration has $x_1 < x_0$. We apply linear forces that depend on x and $\frac{dx}{dt}$. The force is the sum of two terms. The first is a generalization of Hooke's Law. The second term is a dispersion term that acts like a friction term in this simple example. If F_0, F_1 are the forces applied to A_0 and A_1, respectively, then we define

$$F_0 = -K\left(x_0 - (x_1 + \Delta)\right) - B\left(\frac{dx_0}{dt} - \frac{dx_1}{dt}\right) \tag{7}$$

$$F_1 = -K\left((x_1 + \Delta) - x_0\right) - B\left(\frac{dx_1}{dt} - \frac{dx_0}{dt}\right). \tag{8}$$

The equations of motion are simpler with the coordinate transformation,

$$z_k = x_k + k\Delta, \quad k = 0, 1. \tag{9}$$

Using Newton's Law and the coordinate transformation, we get the equations

$$
\begin{aligned}
\frac{d^2 z_0}{dt^2} &= -\frac{K}{m}(z_0 - z_1) - \frac{B}{m}\left(\frac{dz_0}{dt} - \frac{dz_1}{dt}\right) \\
\frac{d^2 z_1}{dt^2} &= -\frac{K}{m}(z_1 - z_0) - \frac{B}{m}\left(\frac{dz_1}{dt} - \frac{dz_0}{dt}\right).
\end{aligned}
$$

We write this in vector form using

$$
z = \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} \quad \dot{z} = \begin{pmatrix} \frac{dz_0}{dt} \\ \frac{dz_1}{dt} \end{pmatrix}. \tag{10}
$$

We can write the equations of motion in matrix form as

$$
\begin{pmatrix} z \\ \dot{z} \end{pmatrix} = \mathbf{M} \begin{pmatrix} z \\ \dot{z} \end{pmatrix}, \tag{11}
$$

where

$$
\mathbf{M} = \begin{pmatrix} \mathbf{0}_2 & \mathbf{I}_2 \\ -\mathbf{L}_x & -\mathbf{L}_v \end{pmatrix}.
$$

We denote by $\mathbf{0}_k$ the $k \times k$ zero matrix and $\mathbf{I}_k$ is the $k \times k$ identity matrix, and

$$
\mathbf{L}_x = \begin{pmatrix} \frac{K}{m} & -\frac{K}{m} \\ -\frac{K}{m} & \frac{K}{m} \end{pmatrix} \tag{12}
$$

$$
\mathbf{L}_v = \begin{pmatrix} \frac{B}{m} & -\frac{B}{m} \\ -\frac{B}{m} & \frac{B}{m} \end{pmatrix}. \tag{13}
$$

The 4×4 matrix $\mathbf{M}$ has the form described in Proposition C.1.1. The eigenvalues and corresponding eigenvectors are found by solving

$$\left(\nu^2 + \nu \mathbf{L}_v + \mathbf{L}_x\right) v = 0. \tag{14}$$

In this simple example the commutator vanishes, $[\mathbf{L}_x, \mathbf{L}_v] = 0$. This means that $\mathbf{L}_x$ and $\mathbf{L}_v$ preserve each other's eigenspaces. In this case they both have two unique eigenvectors given by

$$1_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad u_- = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

1_2 has eigenvalue 0. Plug this into equation (14) to get

$$\left(\nu^2 + \nu \mathbf{L}_v + \mathbf{L}_x\right) 1_2 = \nu^2 1_2 = 0.$$

This means $\nu = 0$ is an eigenvalue with multiplicity 2. The kernel of the matrix M of equation (11) is the span of the two four-dimensional vectors,

$$\begin{pmatrix} 1_2 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1_2 \end{pmatrix}.$$

Notice that the second vector is not an eigenvector, but is in the kernel of M^2, so is in the 0 eigenspace. This two-dimensional eigenspace corresponds to the solution where

- $z_k = a$, for a fixed a. This means $x_k = a - k\Delta$.
- $\dot{z}_k = b$ for a fixed b.

The second eigenvector u_- satisfies

$$\left(\nu^2 + \nu \mathbf{L}_v + \mathbf{L}_x\right) u_- = \nu^2 + \nu \frac{2B}{m} + \frac{2K}{m} = 0. \tag{15}$$

By Proposition B.2.1 the roots are stable if and only if $B > 0$ and $K > 0$. We can write down the eigenvalues,

$$\nu = -\frac{B}{m} \pm \sqrt{(B/m)^2 - 2(K/m)} = -\frac{B}{m} \pm i\sqrt{2(K/m) - (B/m)^2}.$$

If we ignore the trivial solution corresponding to the 0 eigenvalue, we get the general solution,

$$\begin{aligned} Z(t) =& c_0 u_- \exp\left(-\frac{B}{m}t\right) \exp\left(it\sqrt{2(K/m) - (B/m)^2}\right) \\ &+ c_1 u_- \exp\left(-\frac{B}{m}t\right) \exp\left(-it\sqrt{2(K/m) - (B/m)^2}\right) \end{aligned}$$

where c_0 and c_1 are constants that depend on the initial conditions. The system has distinct regions. When $B^2 > 2Km$, then the system is damped and does not oscillate. If $B^2 < 2Km$, then the system oscillates as it damps. But the system damps if $B > 0$.

The starting amplitude is related to the constants c_0, c_1. The phase of the motion is also determined by the constants c_0, c_1. The eigenvector u_- has the two agents moving completely out of phase. For example, let $c_0 = c_1 = \frac{c_b}{2}$ and assume that $B^2 < 2Km$. Use equation 9 to revert to coordinates x_k and we get the solution,

$$\begin{pmatrix} x_0(t) \\ x_1(t) \end{pmatrix} = c_b \begin{pmatrix} 1 \\ -1 \end{pmatrix} \exp\left(-\frac{B}{m}t\right) \cos\left(t\sqrt{2(K/m) - (B/m)^2}\right) + \begin{pmatrix} 0 \\ -\Delta \end{pmatrix}. \tag{16}$$

This is just a damped oscillator. But if we tweak the parameters, we can get other solutions.

1.2.1 Generalizations

The coupling in equation (11) is not Hamiltonian because of the dispersion term controlled by the constant B. But if $B = 0$ then the system is just a coupled oscillator and the equations of motion are derivable from a Lagrangian. However, we shall consider system where the forces F_0 and F_1 are not equal and opposite. To do this we make the coefficients K and B depend on the agent number. Equations (7) and (8) become

$$F_0 = -K^{(0)}\left(x_0 - (x_1 + \Delta)\right) - B^{(0)}\left(\frac{dx_0}{dt} - \frac{dx_1}{dt}\right) \tag{17}$$

$$F_1 = -K^{(1)}\left((x_1 + \Delta) - x_0\right) - B^{(1)}\left(\frac{dx_1}{dt} - \frac{dx_0}{dt}\right). \tag{18}$$

We write the Laplacians as

$$\mathbf{L}_x = \begin{pmatrix} g_x^0 & -g_x^0 \\ -g_x^1 & g_x^1 \end{pmatrix} \qquad \text{where } g_x^{(\alpha)} = \frac{K^{(\alpha)}}{m} \tag{19}$$

$$\mathbf{L}_v = \begin{pmatrix} g_v^0 & -g_v^0 \\ -g_v^1 & g_v^1 \end{pmatrix} \qquad \text{where } g_v^{(\alpha)} = \frac{B^{(\alpha)}}{m}. \tag{20}$$

The Laplacian matrices $\mathbf{L}_x$ and $\mathbf{L}_v$ are still row-sum zero so the 0 eigenvalue solution still exists. The non-zero solution is more interesting. The characteristic polynomial is given by

$$P_2(\nu) = \det \begin{pmatrix} \nu^2 + g_v^{(0)}\nu + g_x^{(0)} & -\left(g_v^{(0)}\nu + g_x^{(0)}\right) \\ -\left(g_v^{(1)}\nu + g_x^{(1)}\right) & \left(\nu^2 + g_v^{(1)}\nu + g_x^{(1)}\right) \end{pmatrix} \tag{21}$$
$$= \nu^2 \left(\nu^2 + (g_v^{(0)} + g_v^{(1)})\nu + (g_x^{(0)} + g_x^{(1)})\right).$$

We have the double root at $\nu = 0$ because the Laplacians $\mathbf{L}_x$ and $\mathbf{L}_v$ are both row-sum zero. The remaining polynomial is Hurwitz stable if and only if

$$T_x = \mathrm{Tr}(\mathbf{L}_x) = g_x^{(0)} + g_x^{(1)} > 0 \tag{22}$$

$$T_v = \mathrm{Tr}(\mathbf{L}_v) = g_v^{(0)} + g_v^{(1)} > 0 \tag{23}$$

where we have defined the two traces T_x and T_v. In any case the additional eigenvalues, are given by

$$\nu = \frac{-T_v \pm \sqrt{T_v^2 - 4T_x}}{2} = \frac{-T_v \pm i\sqrt{4T_x - T_v^2}}{2}. \tag{24}$$

The stability conditions in equations (22) and (23) do not require that all $g_x^{(\alpha)}$ and $g_v^{(\alpha)}$ are positive. For example, we have the following:

Example 1.2.1. *Define a system using the following:*

$$g_x^{(0)} = 4 \qquad g_x^{(1)} = -1$$

$$g_v^{(0)} = 2 \qquad g_v^{(1)} = -1$$

With this choice of parameters L_x and L_v do not commute. The eigenvalues of the system follow from equation (24)*,*

$$\nu = -0.5 \pm 1.66i.$$

The eigenvector corresponding to the eigenvalues are most easily computed as the kernel of equation (21) *with the eigenvalue substituted. The eigenvector corresponding to the non-zero eigenvalue is*

$$u_+ = \begin{pmatrix} 1 \\ 0.35 \end{pmatrix}.$$

Take $\Delta = 1$ *and* $c_1 = c_2 = 1/2$ *to get the following solution,*

$$\begin{pmatrix} x_0(t) \\ x_1(t) \end{pmatrix} = \begin{pmatrix} 1 \\ 0.35 \end{pmatrix} \exp(-0.5t)\cos(1.66t) + \begin{pmatrix} 0 \\ -\Delta \end{pmatrix}. \tag{25}$$

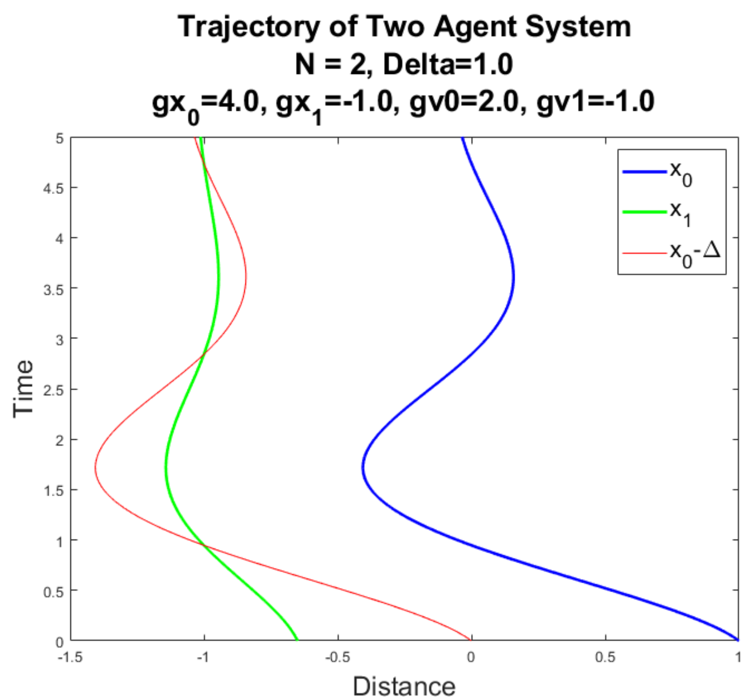


Figure 1: *Two Agent Plot*

The trajectories of $x_0(t)$ and $x_1(t)$ are shown in figure 1. The line in red shows the distance Δ from x_0, which is the stable distance. Agent 0 (in blue) has a larger weight and so reacts faster. Agent 1 (in green) has a negative weight, so pushes away from the stable distance, but the magnitudes $g_x^{(1)}$ and $g_v^{(1)}$ are smaller than the corresponding magnitudes in agent 0, so the forces are less. The two agents move with the same phase, which is quite unlike the standard coupled oscillator. We emphasize, again, that the system is not Hamiltonian, and the forces are not Newtonian.

1.3 Three Coupled Agents On A Circle

We generalize the previous section to include three agents. We have to distinguish between the system with periodic boundary conditions and the system on $\mathbb{R}^1$. The relationship between periodic boundary conditions and the system on a line is subtle. In this section we illustrate the periodic system and present it in a manner similar to systems with a larger number of agents. We only sketch the ideas that are illustrative of later concepts.

The three agents on a ring attempt to keep a distance Δ between them, where Δ is a fixed constant and the circumference is 3Δ. The system is stable and has no interacting forces when

$$x_k = x_0 - \Delta k, \quad k = 0, 1, 2.$$

Periodic boundary conditions mean that

$$x_k + 3\Delta = x_k.$$

The dynamical equations for the system with periodic boundary conditions have the form

$$\begin{aligned} m^{(k)} \frac{d^2 x_k}{dt^2} = & - \mu_{x,-1}^{(k)} \left(x_k - x_{k-1} + \Delta \right) - \mu_{x,1}^{(k)} \left(x_k - x_{k+1} - \Delta \right) \\ & - \mu_{v,-1}^{(k)} \left(\frac{dx_k}{dt} - \frac{dx_{k-1}}{dt} \right) - \mu_{v,1}^{(k)} \left(\frac{dx_k}{dt} - \frac{dx_{k+1}}{dt} \right) \end{aligned} \tag{26}$$

where $k = 0, 1, 2$. Since we assume periodic boundary conditions, all index arithmetic is $\mod (3)$. This means, for example, that $k+3 = k$. The superscript $m^{(k)}$ above the couplings indicate the agent type. This will be more useful when we discuss systems of N agents drawn p agent classes. All agents from a particular agent class have identical couplings.

We change variables using

$$z_k(t) = x_k(t) + \Delta k. \tag{27}$$

The equations become

$$\begin{aligned} \frac{d^2 z_k}{dt^2} = &- \frac{\mu_{x,-1}^{(k)}}{m^{(k)}} (z_k - z_{k-1}) - \frac{\mu_{x,1}^{(k)}}{m^{(k)}} (z_k - z_{k+1}) \\ &- \frac{\mu_{v,-1}^{(k)}}{m^{(k)}} \left(\frac{dz_k}{dt} - \frac{dz_{k-1}}{dt} \right) - \frac{\mu_{v,1}^{(k)}}{m^{(k)}} \left(\frac{dz_k}{dt} - \frac{dz_{k+1}}{dt} \right). \end{aligned} \tag{28}$$

We introduce a new notation that we will use in the remainder of this dissertation. We write these equations in the following form,

$$\frac{d^2 z_k}{dt^2} = - g_x^{(k)} \left(z_k + \rho_{x,1}^{(k)} z_{k+1} + \rho_{x,-1}^{(k)} z_{k-1} \right) - g_v^{(k)} \left(\dot{z}_k + \rho_{v,1}^{(k)} \dot{z}_{k+1} + \rho_{v,-1}^{(k)} \dot{z}_{k-1} \right)$$

where

$$g_x^{(k)} = \frac{\mu_{x,1}^{(k)} + \mu_{x,-1}^{(k)}}{m^{(k)}} \tag{29}$$

$$g_x^{(k)} \rho_{x,1} = -\frac{\mu_{x,1}^{(k)}}{m^{(k)}} \qquad g_x^{(k)} \rho_{x,-1} = -\frac{\mu_{x,-1}^{(k)}}{m^{(k)}} \tag{30}$$

$$g_v^{(k)} = \frac{\mu_{v,1}^{(k)} + \mu_{v,-1}^{(k)}}{m^{(k)}} \tag{31}$$

$$g_v^{(k)} \rho_{v,1} = -\frac{\mu_{v,1}^{(k)}}{m^{(k)}} \qquad g_v^{(k)} \rho_{v,-1} = -\frac{\mu_{v,-1}^{(k)}}{m^{(k)}}. \tag{32}$$

The constants $\rho_{x,k}^{(\alpha)}$ and $\rho_{v,k}^{(\alpha)}$ satisfy

$$1 + \sum_k \rho_{x,k}^{(\alpha)} = 0 \tag{33}$$

$$1 + \sum_k \rho_{v,k}^{(\alpha)} = 0. \tag{34}$$

This notation is easily generalization and is consistent with the graph theoretic formulation of this and related problems. These linear equation can be expressed in matrix form where

$$z = (z_0, z_1, z_2)^T \quad \dot{z} = (\dot{z}_0, \dot{z}_1, \dot{z}_2)^T$$

$$\mathbf{L}_x = \begin{pmatrix} g_x^{(0)} & g_x^{(0)}\rho_{x,1}^{(0)} & g_x^{(0)}\rho_{x,-1}^{(0)} \\ g_x^{(1)}\rho_{x,-1}^{(1)} & g_x^{(1)} & g_x^{(1)}\rho_{x,1}^{(1)} \\ g_x^{(2)}\rho_{x,1}^{(2)} & g_x^{(2)}\rho_{x,-1}^{(2)} & g_x^{(2)} \end{pmatrix}$$

$$\mathbf{L}_v = \begin{pmatrix} g_v^{(0)} & g_v^{(0)}\rho_{v,1}^{(0)} & g_v^{(0)}\rho_{v,-1}^{(0)} \\ g_v^{(1)}\rho_{v,-1}^{(1)} & g_v^{(1)} & g_v^{(1)}\rho_{v,-1}^{(1)} \\ g_v^{(2)}\rho_{v,1}^{(2)} & g_v^{(2)}\rho_{v,-1}^{(2)} & g_v^{(2)} \end{pmatrix}.$$

In matrix form our ODE has the form

$$\frac{d}{dt}\begin{pmatrix} z \\ \dot{z} \end{pmatrix} = \mathbf{M}\begin{pmatrix} z \\ \dot{z} \end{pmatrix} \tag{35}$$

where the matrix $\mathbf{M}$ is

$$\mathbf{M} = \begin{pmatrix} \mathbf{0}_3 & \mathbf{I}_3 \\ -\mathbf{L}_x & -\mathbf{L}_v \end{pmatrix}.$$

We denote by $\mathbf{0}_k$ the $k \times k$ zero matrix and $\mathbf{I}_k$ is the $k \times k$ identity matrix. This system has a $3 \times 2 = 6$ degree characteristic polynomial given by Proposition C.1.1,

$$P(\nu) = \det\left(\nu^2 + \nu\mathbf{L}_v + \mathbf{L}_x\right) = 0 \tag{36}$$

$$= \det\begin{pmatrix} \nu^2 + \nu g_v^{(0)} + g_x^{(0)} & \left(\nu g_v^{(0)}\rho_{v,1}^{(0)} + g_x^{(0)}\rho_{x,1}^{(0)}\right) & \left(\nu g_v^{(0)}\rho_{v,-1}^{(0)} + g_x^{(0)}\rho_{x,-1}^{(0)}\right) \\ \left(\nu g_v^{(1)}\rho_{v,-1}^{(1)} + g_x^{(1)}\rho_{x,-1}^{(1)}\right) & \nu^2 + \nu g_v^{(1)} + g_x^{(1)} & \left(\nu g_v^{(1)}\rho_{v,1}^{(1)} + g_x^{(1)}\rho_{x,1}^{(1)}\right) \\ \left(\nu g_v^{(2)}\rho_{v,1}^{(2)} + g_x^{(2)}\rho_{x,1}^{(2)}\right) & \left(\nu g_v^{(2)}\rho_{v,-1}^{(2)} + g_x^{(2)}\rho_{x,-1}^{(2)}\right) & \nu^2 + \nu g_v^{(2)} + g_x^{(2)} \end{pmatrix}.$$

But the matrices $\mathbf{L}_x$ and $\mathbf{L}_v$ have row-sums zero so there is a double root at the origin. The characteristic polynomial is ν^2 times a fourth degree polynomial. The full polynomial is too complicated to write down but one can write down the first two terms. The following proposition is quite general and only depends on the general form of $\mathbf{M}$.

Proposition 1.3.1. *The polynomial in equation* (36) *is a sixth degree and the first two terms are given by*

$$\nu^6 + \left(g_v^{(0)} + g_v^{(1)} + g_v^{(2)}\right)\nu^5.$$

Proof. The two highest order terms of the characteristic polynomial $P(\nu)$ are given by

$$\nu^6 - \mathrm{Tr}(\mathbf{M})\nu^5 = \nu^6 - \mathrm{Tr}(-\mathbf{L}_v)\nu^5 = \nu^6 + \mathrm{Tr}(\mathbf{L}_v)\nu^5$$

where $\mathbf{M}$ is the matrix defined in equation 35. □

Corollary 1.3.2. *If the system is Hurwitz stable then* $\mathrm{Tr}(\mathbf{L}_v) > 0$.

Proof. This follows from Proposition B.0.1. □

This condition also appeared in the two-agent case in equation (23). The condition in equation (22) is not as straightforward. The characteristic polynomial $P(\nu)$ is too complicated to write down so we will make some additional assumptions. We will assume that $\rho_{x,\pm 1}^{(\alpha)}$ and $\rho_{v,\pm 1}^{(\alpha)}$ are all independent of α. We introduce the parameters β_x and β_v,

$$\beta_x^{(\alpha)} = \rho_{x,1}^{(\alpha)} - \rho_{x,-1}^{(\alpha)} \quad \Rightarrow \rho_{x,1}^{(\alpha)} = -\frac{1-\beta_x^{(\alpha)}}{2} \text{ and } \rho_{x,-1}^{(\alpha)} = -\frac{1+\beta_x^{(\alpha)}}{2}, \tag{37}$$

$$\beta_v^{(\alpha)} = \rho_{v,1}^{(\alpha)} - \rho_{v,-1}^{(\alpha)} \quad \Rightarrow \rho_{v,1}^{(\alpha)} = -\frac{1-\beta_v^{(\alpha)}}{2} \text{ and } \rho_{v,-1}^{(\alpha)} = -\frac{1+\beta_v^{(\alpha)}}{2}. \tag{38}$$

Since $\beta_x^{(\alpha)}$ and $\beta_x^{(\alpha)}$ are independent of α, we denote them by β_x, β_v.

Remark 1.3.3. *The assumption that the $\rho_{x,v,\pm 1}^{(\alpha)}$ are independent of α is a natural assumption for robotic agents. The agents cannot change their overall weight nor can they change the internal engine that applies the forces. However, they can control the ratio of forward and backward pull. Every agent in a group can fix this ratio to a specific value even though the actual forces are dependent on the actual agent.*

With this assumption we have the following:

Proposition 1.3.4. *The term in the characteristic polynomial with lowest degree has the form*

$$\frac{3+\beta_x^2}{4}\left(g_x^{(0)}g_x^{(1)} + g_x^{(1)}g_x^{(2)} + g_x^{(2)}g_x^{(0)}\right)\nu^2.$$

Proof. This is a direct computation. We shall compute a similar quantity in a more general setting. □

Corollary 1.3.5. *If the system is Hurwitz stable then*

$$g_x^{(0)}g_x^{(1)} + g_x^{(1)}g_x^{(2)} + g_x^{(2)}g_x^{(0)} > 0.$$

Proof. This follows from Proposition B.0.1 and Proposition 1.3.4. □

This is the equivalent to (22) in the two agent case.

Example 1.3.6. *To demonstrate the odd behavior of negative weights we take a concrete example with,*

$$
\begin{aligned}
&g_x^{(0)} = 4.0 && g_x^{(1)} = -4.0 && g_x^{(2)} = 4.0 \\
&g_v^{(0)} = 1.0 && g_v^{(1)} = -0.5 && g_v^{(2)} = 0.5 \\
&\rho_{x,1} = -0.25 && \rho_{x,-1} = -0.75 \\
&\rho_{v,1} = -0.25 && \rho_{v,-1} = -0.75
\end{aligned}
$$

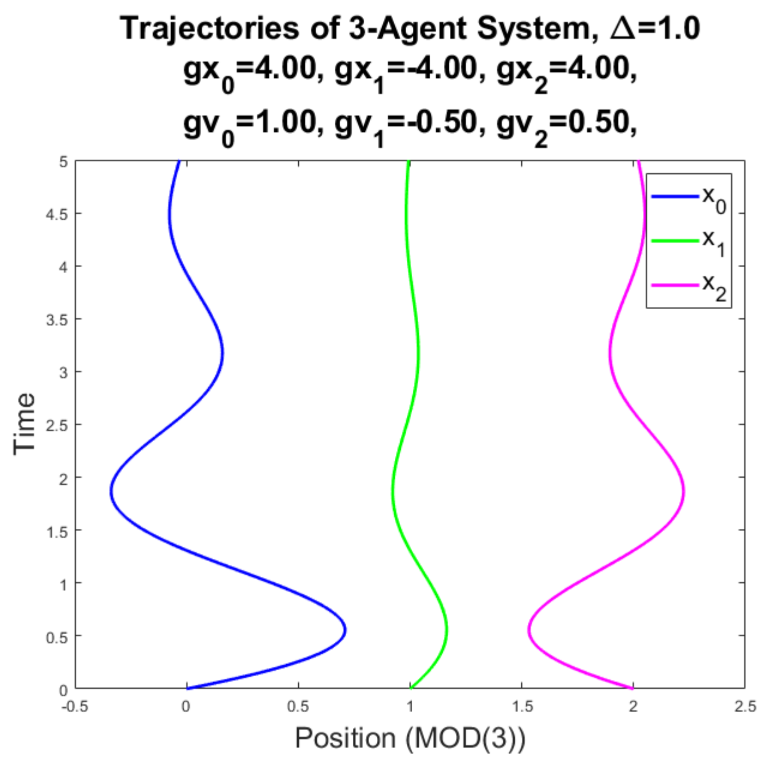


Figure 2: *Three Agent Plot*

The trajectory for the eigenvalue $-0.57 + 2.40i$ *and eigenvector* $[1.000.23, -0.66]^T$ *is shown in figure 2. The weights* $g_x^{(1)}$ *and* $g_v^{(1)}$ *are both negative so this agent is repulsed by both of its neighbors. But the system is still stable.*

Notice that agents 0 *and* 2 *are both attractive. These two agents have phase* *difference of π, as typical for coupled oscillators. Agent 1 is in phase with agent 0.*

1.4 A Sequence of Agents

In this section we present a general framework for a sequence of agents in a one-dimensional space. There is very little known about the general system. In subsequent sections we shall make a series of assumptions to study specific cases.

Consider N agents moving along either $\mathbb{R}^1$ or S^1. We label the agents, $A_0, A_1, \cdots A_{N-1}$ as shown in figure 3. We label their $x-$coordinates by $x_0, x_1, \cdots, x_{N-1}$. The system we shall study consists of N agents on either S^1 or $\mathbb{R}^1$ that attempt to keep a distance Δ between neighbors so that the stable configuration is,

$$x_k = x_0 - k\Delta \quad \text{for } k = 1, 2, \cdots, N-1.$$

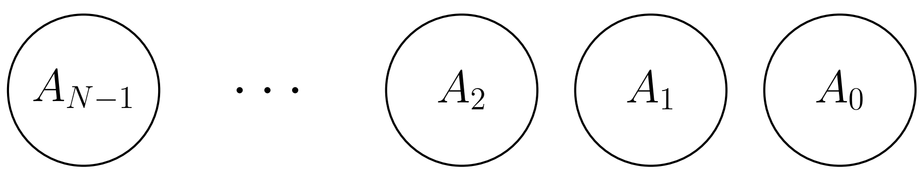


Figure 3: *Sequence of Agents*

We assume a force that depends on the spatial separation with a interaction force similar to Hooke's law. We also have a velocity dependent term that depends on the differences of the velocities. The equations of motion have the form,

$$\ddot{x}_k^{(k)} = -g_x^{(k)} \sum_{j\in N(x_k)} \rho_{x,j}^{(k)} \left(x_{k+j} - x_k - (j-k)\Delta\right) - g_v^{(k)} \sum_{j\in N(x_k)} \rho_{v,j}^{(k)} \left(\dot{x}_{k+j} - \dot{x}_k\right) \tag{39}$$

where $N(x_k)$ are all the agents that interact with agent k. If displayed on a graph, the agents are the vertices, the edges indicate an interaction term such that $N(x_k)$ is the set of neighbors of the vertex k. We shall assume that the N agents consist of agents chosen from q different species. All agents from the same species have identical forces with their neighbors. We will label the species with an index α, so the equations of motion, which is restatement of equation (39), is

$$\ddot{x}_k^{(\alpha)} = -g_x^{(\alpha)} \sum_{j \in N(x_k)} \rho_{x,j}^{(\alpha)} \left(x_{k+j} - x_k - (j-k)\Delta\right) - g_v^{(\alpha)} \sum_{j \in N(x_k)} \rho_{v,j}^{(\alpha)} \left(\dot{x}_{k+j} - \dot{x}_k\right) \tag{40}$$

In the case where each agent is unique then we just have $\alpha = k$ and we revert to equation (39).

The constants that control the dynamics are the following.

- $\rho_{x,j}^{(\alpha)}$ is coupling constant for the force on k applied by $k+j$ computed from the difference of the two position coordinates. The superscript (α) indicates that the constant is the same for all agents of species α. In our current example, each agent is its own species, but we shall restrict this in subsequent examples.

- $\rho_{v,j}^{(\alpha)}$ is coupling constant for the force on k applied by $k+j$ computed from the difference of the two velocities.

- $g_x^{(\alpha)}$ is the overall scale of the spatial forces. By adding this scale, we can insist that $\sum_{j \in N(x_k)} \rho_{x,j}^{(\alpha)} = -1$.

- $g_v^{(\alpha)}$ is the overall scale of the velocity forces. By adding this scale we can insist that $\sum_{j \in N(x_k)} \rho_{v,j}^{(\alpha)} = -1$.

We can simplify equation (40) with the change of variables,

$$z_k^{(\alpha)} = x_k^{(\alpha)} - k\Delta. \tag{41}$$

With this change of variables equation (39) becomes,

$$\ddot{z}_k^{(\alpha)} = -g_x^{(\alpha)} \sum_{j \in N(x_k)} \rho_{x,j}^{(\alpha)} z_{k+j}^{(\alpha+j)} - g_v^{(\alpha)} \sum_{j \in N(x_k)} \rho_{v,j}^{(\alpha)} \dot{z}_{k+j}^{(\alpha+j)} \tag{42}$$

where the constants $\rho_{x,0}^{(\alpha)} = \rho_{v,0}^{(\alpha)} = 1$ so that we are consistent with the normalization of $\rho_{x,j}^{(\alpha)}$ and $\rho_{v,j}^{(\alpha)}$. The constant $\rho_{x,j}^{(\alpha)}$ and $\rho_{x,j}^{(\alpha)}$ control the relative strength of the neighbor interactions and the values $g_x^{(\alpha)}, g_v^{(\alpha)}$ control the overall interaction strength.

This is a linear ODE that has the matrix form,

$$\frac{d}{dt}\begin{pmatrix} z \\ \dot{z} \end{pmatrix} = \begin{pmatrix} \mathbf{0}_N & \mathbf{I}_N \\ -\mathbf{G}_x\mathbf{L}_x & -\mathbf{G}_v\mathbf{L}_v \end{pmatrix}\begin{pmatrix} z \\ \dot{z} \end{pmatrix} = \mathbf{M}\begin{pmatrix} z \\ \dot{z} \end{pmatrix} \tag{43}$$

where $\mathbf{0}_N$ is the $N \times N$ zero matrix, $\mathbf{I}_N$ is the $N \times N$ identity matrix,

$$z = \begin{pmatrix} z_0 & z_1 & \cdots & z_{N-1} \end{pmatrix}^T$$
$$\dot{z} = \begin{pmatrix} \dot{z}_0 & \dot{z}_1 & \cdots & \dot{z}_{N-1} \end{pmatrix}^T$$

$$\mathbf{L}_x = \begin{pmatrix} 1 & \rho_{x,1}^{(0)} & \rho_{x,2}^{(0)} & \cdots & \rho_{x,-1}^{(0)} \\ \rho_{x,-1}^{(1)} & 1 & \rho_{x,1}^{(1)} & \cdots & \rho_{x,-2}^{(1)} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \rho_{x,2}^{(N-2)} & \rho_{x,3}^{(N-2)} & \rho_{x,4}^{(N-2)} & \cdots & \rho_{x,1}^{(N-2)} \\ \rho_{x,1}^{(N-1)} & \rho_{x,2}^{(N-1)} & \rho_{x,3}^{(N-1)} & \cdots & 1 \end{pmatrix} \tag{44}$$

$$\mathbf{L}_v = \begin{pmatrix} 1 & \rho_{v,1}^{(0)} & \rho_{v,2}^{(0)} & \cdots & \rho_{v,-1}^{(0)} \\ \rho_{v,-1}^{(1)} & 1 & \rho_{v,1}^{(1)} & \cdots & \rho_{v,-2}^{(1)} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \rho_{v,2}^{(N-2)} & \rho_{v,3}^{(N-2)} & \rho_{v,4}^{(N-2)} & \cdots & \rho_{v,1}^{(N-2)} \\ \rho_{v,1}^{(N-1)} & \rho_{v,2}^{(N-1)} & \rho_{v,3}^{(N-1)} & \cdots & 1 \end{pmatrix}. \tag{45}$$

$$\mathbf{G}_x = \begin{pmatrix} g_x^{(0)} & 0 & 0 & \cdots & 0 \\ 0 & g_x^{(1)} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & g_x^{(N-1)} \end{pmatrix}, \quad \mathbf{G}_v = \begin{pmatrix} g_v^{(0)} & 0 & 0 & \cdots & 0 \\ 0 & g_v^{(1)} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & g_v^{(N-1)} \end{pmatrix}$$

The matrix $\mathbf{M}$ has the special form given in Proposition C.1.1. The eigenvectors of the matrix $\mathbf{M}$ have the form, $[u, \nu u]^T$ where $u \in \mathbb{R}^N$ and ν is an eigenvalue of $\mathbf{M}$. The eigenvalue ν is a root of the $2N$ polynomial,

$$P(\nu) = \det\left(\nu^2 + \nu \mathbf{G}_v \mathbf{L}_v + \mathbf{G}_x \mathbf{L}_x\right). \tag{46}$$

The following proposition follows immediately from the above,

Proposition 1.4.1. *If $\mathbf{L}_v$ and $\mathbf{L}_x$ are row-sum zero then the following polynomial* *vanishes identically,*

$$R(\nu) = \det\left(\nu \mathbf{G}_v \mathbf{L}_v + \mathbf{G}_x \mathbf{L}_x\right) = 0. \tag{47}$$

Proof. Let $1_N \in \mathbb{R}^N$ be the vector $1_N = (1, 1, 1, \cdots, 1)^T$. It satisfies

$$\mathbf{L}_v 1_N = \mathbf{L}_x 1_N = 0.$$

It follows immediately that,

$$1_N \in \ker(\nu \mathbf{G}_v \mathbf{L}_v + \mathbf{G}_x \mathbf{L}_x) \neq \emptyset.$$

From this it follows that the determinant in equation (47) vanishes. $\square$

Corollary 1.4.2. *The characteristic polynomial $P(\nu)$, in equation (46), has a double root at 0.*

Proof. When computing the terms in $P(\nu)$ you must have a ν^2 in the term or the term comes from $R(\nu)$. This means that the constant and linear terms of the polynomial $P(\nu)$ are the constant and linear terms of the polynomial $R(\nu)$ which vanish identically. $\square$

Remark 1.4.3. *The 0 eigenvalue corresponds to a two-dimension eigenspace but there is only one 0 eigenvalue.*

$$\begin{pmatrix} \mathbf{0}_N & \mathbf{I}_N \\ -\mathbf{G}_x\mathbf{L}_x & -\mathbf{G}_v\mathbf{L}_v \end{pmatrix}^2 \begin{pmatrix} \alpha I \\ \beta I \end{pmatrix} = \begin{pmatrix} \mathbf{0}_N & \mathbf{I}_N \\ -\mathbf{G}_x\mathbf{L}_x & -\mathbf{G}_v\mathbf{L}_v \end{pmatrix} \begin{pmatrix} \beta I \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This eigenspace corresponds to the linear solution,

$$z_k(t) = \alpha t + \beta.$$

Since the polynomial in equation (47) vanishes identically the ν^2 term of the characteristic equation (46) must contain exactly one ν^2 from the $\mathbf{I}_N\nu$ diagonal terms and all the other terms from the constant terms of (46). We shall use this repeatedly in the subsequent chapters. To see how this works we introduce the following 1st degree polynomials,

$$\psi_0^{(\alpha)}(\nu) = \left(\nu g_v^{(\alpha)} + g_x^{(\alpha)}\right) \tag{48}$$

$$\psi_j^{(\alpha)}(\nu) = \left(g_v^{(\alpha)}\rho_{j,v}^{(\alpha)}\nu + g_x^{(\alpha)}\rho_{j,x}^{(\alpha)}\right) \tag{49}$$

The characteristic polynomial in equation (46) becomes,

$$P(\nu) = \det \begin{pmatrix} \nu^2 + \psi_0^{(0)}(\nu) & \psi_1^{(0)}(\nu) & \psi_2^{(0)}(\nu) & \cdots & \psi_{N-1}^{(0)}(\nu) \\ \psi_{-1}^{(1)}(\nu) & \nu^2 + \psi_0^{(1)}(\nu) & \psi_1^{(1)}(\nu) & \cdots & \psi_{N-2}^{(1)}(\nu) \\ \psi_{-2}^{(2)}(\nu) & \psi_{-1}^{(2)}(\nu) & \nu^2 + \psi_0^{(2)}(\nu) & \cdots & \psi_{N-3}^{(2)}(\nu) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \psi_1^{(N-1)}(\nu) & \psi_2^{(N-1)}(\nu) & \psi_3^{(N-1)}(\nu) & \cdots & \nu^2 + \psi_0^{(N-1)}(\nu) \end{pmatrix} \tag{50}$$

where all index arithmetic is $\mod (N)$. This means, for example, that we identify $\rho_{-1}^{(N+1)} = \rho_{N-1}^{(1)}$. we know, by Corollary 1.4.2 that $P(\nu)$ has a double root at $\nu = 0$. Writing down the complete polynomial $P(\nu)$ is, at this point, more than we can manage but we can make a few statements that we summarize in the following proposition.

Proposition 1.4.4. *Write the characteristic polynomial in equation* (50) *as,*

$$P(\nu) = \nu^{2N} + a_{2N-1}\nu^{2N-1} + a_{2N-2}\nu^{2N-2} + \cdots + a_2\nu^2 + a_1\nu + a_0.$$

We have,

$$a_1 = a_0 = 0 \tag{51}$$

$$a_{2N-1} = g_v^{(0)} + g_v^{(1)} + \cdots + g_v^{(N-1)} \tag{52}$$

$$\begin{aligned} a_2 = \sum_{k=0}^{N-1} &\left(g_x^{(k+1)} g_x^{(k+2)} \cdots g_x^{(k+N-1)}\right) \\ \times \det &\begin{pmatrix} 1 & \rho_{x,1}^{(k+1)} & \rho_{x,2}^{(k+1)} & \cdots & \rho_{x,N-2}^{(k+1)} \\ \rho_{x,-1}^{(k+2)} & 1 & \rho_{x,1}^{(k+2)} & \cdots & \rho_{x,N-3}^{(k+2)} \\ \rho_{x,-2}^{(k+3)} & \rho_{x,-1}^{(k+3)} & 1 & \cdots & \rho_{x,N-4}^{(k+3)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \rho_{x,-(N-2)}^{(k-1)} & \rho_{x,-(N-3)}^{(k-1)} & \rho_{x,-(N-4)}^{(k-1)} & \cdots & 1 \end{pmatrix} \end{aligned} \tag{53}$$

Proof. The characteristic polynomial has degree $2N$. Using Corollary 1.4.2, both a_1 and a_0 vanish. To get a term with ν^{2N-1} we must get ν^2 terms from $N-1$ rows. The remaining row must contain a ν term to get $\nu^{2N-2+1} = \nu^{2N-1}$. This means the a_{N-1} term is the ν^{2N-1} term of the polynomial,

$$\nu^{2N-2}\left(\psi_0^{(0)}(\nu) + \psi_0^{(1)}(\nu) + \cdots + \psi_0^{(N-1)}(\nu)\right).$$

Equation (52) follows. To compute the term a_2 we notice that the ν^2 coefficient of $R(\nu)$ (equation (47)) vanishes. The ν^2 terms in $P(\nu)$ must contain a row with ν^2 and all remaining rows in det contain just constants. For each for k we get a ν^2 term from the diagonal element and an $(N-1)\times(N-1)$ determinant of constants. Formula 53 follows. □

Corollary 1.4.5. *If the polynomial in Proposition 1.4.4 is Hurwitz stable then the* *following condition must be true.*

$$g_v^{(0)} + g_v^{(1)} + \cdots + g_v^{(N-1)} > 0.$$

Proof. This follows immediately from Proposition 1.4.4 and Theorem B.0.1. □

Corollary 1.4.6. *Assume that $\rho_{x,k}^{(\alpha)}$ is independent of α for all k and that $\rho_{x,k} \leq 0$ for all k. With these assumptions, if the polynomial in Proposition 1.4.4 is Hurwitz stable then the following condition must be true.*

$$\sum_{k=0}^{N-1} \left(g_x^{(k+1)} g_x^{(k+2)} \cdots g_x^{(k+N-1)} \right) > 0.$$

Proof. In equation (53), each term has a factor,

$$\begin{pmatrix} 1 & \rho_{x,1}^{(k+1)} & \rho_{x,2}^{(k+1)} & \cdots & \rho_{x,N-2}^{(k+1)} \\ \rho_{x,-1}^{(k+2)} & 1 & \rho_{x,1}^{(k+2)} & \cdots & \rho_{x,N-3}^{(k+2)} \\ \rho_{x,-2}^{(k+3)} & \rho_{x,-1}^{(k+3)} & 1 & \cdots & \rho_{x,N-4}^{(k+3)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \rho_{x,-(N-2)}^{(k-1)} & \rho_{x,-(N-3)}^{(k-1)} & \rho_{x,-(N-4)}^{(k-1)} & \cdots & 1 \end{pmatrix}.$$

This matrix $(N-1) \times (N-1)$ is independent of k. By Gershgorin's theorem each eigenvalue is contained in one of the $N-1$ Gershgorin disks that are centered at 1 and have radius given by,

$$|\rho_{x,1}| + |\rho_{x,2}| + \cdots + |\rho_{x,N-1}|,$$

where exactly one of the $\rho_{x,k}$ is removed. Since all the $\rho_{x,k} \leq 0$ we have,

$$|\rho_{x,1}| + |\rho_{x,2}| + \cdots + |\rho_{x,N-1}| = |\rho_{x,1} + \rho_{x,2} + \cdots + \rho_{x,N-1}| \leq 1,$$

since $1 + \rho_{x,1} + \rho_{x,2} + \cdots + \rho_{x,N-1} = 0$. This means that the det factor in a_2 is independent of k and is positive. The Proposition now follows from Theorem B.0.1.

$\square$

1.5 Sequence of Identical Agents

Equation (43) is not easily solved without additional assumptions. In this section we will assume that all the agents are identical so that $g_x^{(\alpha)}, g_v^{(\alpha)}, \rho_{x,j}^{(\alpha)}$ and $\rho_{v,j}^{(\alpha)}$ are all independent of α. With these assumptions the equation (43) takes on a simpler form with,

$$\begin{aligned} \mathbf{G}_x &= g_x \mathbf{I}_N \quad \Rightarrow \mathbf{G}_x \mathbf{L}_x = g_x \mathbf{L}_x, \\ \mathbf{G}_v &= g_v \mathbf{I}_N \quad \Rightarrow \mathbf{G}_x \mathbf{L}_x = g_v \mathbf{L}_v, \end{aligned}$$

We will, also, assume that the system have periodic boundary conditions so the Laplacian's $\mathbf{L}_x$ and $\mathbf{L}_v$ are circulant matrices. In Proposition C.3.4 we see that all circulant matrices commute. This means that the characteristic polynomial has the form,

$$\det \left| \nu^2 + \nu g_v \mathbf{L}_v + g_x \mathbf{L}_x \right| = 0. \tag{54}$$

If all the agents are identical then the eigenvalues of the linear system are solvable. Construct the matrix $\mathbf{M}$,

$$M = \nu^2 + \nu g_v \mathbf{L}_v + g_x \mathbf{L}_x. \tag{55}$$

By Theorem C.4.4 we know the circulant matrices have a set or orthogonal eigenvectors given by $w(\omega_N^m)$ for $m = 0, 1, \cdots N-1$. By Proposition C.1.1 the eigenvectors of $\mathbf{M}$ have the form,

$$\begin{pmatrix} v \\ \nu v \end{pmatrix}, \text{ where } \left(\nu^2 + \nu \mathbf{L}_v + \mathbf{L}_x\right) v = 0.$$

For each $w(\omega_N^m)$ we get two eigenvalues of $\mathbf{M}$ as,

$$\left(\nu^2 + \nu \mathbf{L}_v + \mathbf{L}_x\right) w(\omega_N^m) = \left(\nu^2 + \nu \lambda_{v,m} + \lambda_{x,m}\right) w(\omega_N^m), \tag{56}$$

where,

$$\lambda_{x,m} = g_x \sum_{k=0}^{N-1} \rho_{x,k} \exp\left(\frac{2\pi i}{N} mk\right) \tag{57}$$

$$\lambda_{v,m} = g_v \sum_{k=0}^{N-1} \rho_{v,k} \exp\left(\frac{2\pi i}{N} mk\right) \tag{58}$$

For each m there are 2 solutions of the quadratic,

$$\nu^2 + \nu \lambda_{v,m} + \lambda_{x,m} = 0. \tag{59}$$

Each of these solutions correspond to an eigenvector of $\mathbf{M}$ of the form,

$$\begin{pmatrix} w(\omega_N^m) \\ \nu w(\omega_N^m) \end{pmatrix}.$$

Equation (59) determines the Hurwitz stability of the system (see Definition 1.1.1). Since $\lambda_{v,m}$ and $\lambda_{v,m}$ are complex we use Proposition B.2.3 to derive the stability conditions. In the general circulant case there are $2N+2$ system parameters which make the problem difficult. We turn to a few specific cases that are solved in the literature.

1.5.1 Nearest Neighbor Interaction

If we restrict identical car system to nearest neighbor interactions then we have constants $\rho_{x,\pm 1}, \rho_{v,\pm 1}, g_x$ and g_v for a total of 6 parameters. In this case the system is solvable for asymptotically large N.

The Hurwitz stability of the system is described in the following theorem.

Theorem 1.5.1. *The system described in Section 1.5 with nearest neighbor interactions is stable for large N if and only if the following conditions are satisfied,*

- $g_x > 0$ *and* $g_v > 0$*,*
- $\rho_{x,1} = \rho_{x,-1} = -1/2$*.*

Proof. See [6] Theorem 1. □

For this system there are functions that approximate the trajectories asymptotically for large N. This is summarized in Theorem 2 of [6].

1.5.2 Next-Nearest Neighbor Interaction

We can include next-nearest neighbor interactions in the identical car case. This introduces a significant complexity and makes both the Hurwitz stability and dynamics less tractable. The next-nearest neighbor case starts with the system in Section 1.5 and then sets all $\rho_{x,j}$ and $\rho_{v,j}$ to zero except the following,

$$g_x, g_v, \rho_{x,\pm 1}, \rho_{x,\pm 2}, \rho_{v,\pm 1}, \rho_{v,\pm 2}.$$

This system is described in detail in [16]. Both the stability and dynamics are too complicated to discuss here. The stability is discussed in Remark 4.1 in [16]. There are numerous conditions that must be satisfied to insure stability.

Remark 1.5.2. *Although the stability conditions are numerous and complicated, they are still useful. A practical system should have system constants that are Hurwitz stable in a reasonable neighborhood. This will guarantee Hurwitz stability even as the system parameters change, as they might under different conditions and wear and tear on the agents themselves. As discussed in the introduction, Hurwitz stability is necessary but not sufficient for reasonable dynamics.*

1.6 Sequence of Three Agent Types

This section contains a brief discussion of a sequence of agents with three different agent types. The agent types are alternated as in figure 4. The sequence $White - Green - Red$ is repeated q times and the number of agents is $N = 3q$. This system is described in [3, 2]. We give a brief summary of this work, focusing on the portions that are relevant to Section 2. Baldivieso [3] discusses both nearest neighbor and next nearest neighbor interactions. For the sake of simplicity, we restrict ourselves to nearest neighbor interactions.

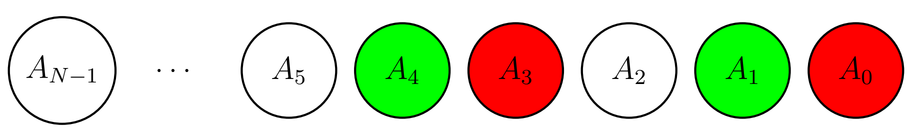


Figure 4: *Sequence With Three Agent Types*

The equations of motion start with equation (40) where $\alpha = 0, 1, 2$. Agents $z^{(\alpha)}_{3k+\alpha}$

all are type α. Equation (42) splits up into equations,

$$\begin{aligned}\ddot{z}^{(\alpha)}_{3k+\alpha} =& -g^{(\alpha)}_x\left(z^{(\alpha)}_{3k+\alpha}+\rho^{(\alpha)}_{x,1}z^{(\alpha+1)}_{3k+\alpha+1}+\rho^{(\alpha)}_{x,-1}z^{(\alpha-1)}_{3k+\alpha-1}\right)\\ &-g^{(\alpha)}_v\left(\dot{z}^{(\alpha)}_{3k+\alpha}+\rho^{(\alpha)}_{v,1}\dot{z}^{(\alpha+1)}_{3k+\alpha+1}+\rho^{(\alpha)}_{v,-1}\dot{z}^{(\alpha-1)}_{3k+\alpha-1}\right)\end{aligned}$$

where $\alpha = 0, 1, 2$ and the index arithmetic on $z_{3k+\alpha+1}$ is $\mod (3)$. The equations of motion of equation (43) have an easier form if we reorder the coordinates z^{α}_k. Group all the α together and the $2N \times 2N$ matrix M splits up into $q \times q$ sub-blocks with the

$$\tilde{\mathbf{M}} = \begin{pmatrix} \mathbf{0}_q & \mathbf{0}_q & \mathbf{0}_q & \mathbf{I}_q & \mathbf{0}_q & \mathbf{0}_q \\ \mathbf{0}_q & \mathbf{0}_q & \mathbf{0}_q & \mathbf{0}_q & \mathbf{I}_q & \mathbf{0}_q \\ \mathbf{0}_q & \mathbf{0}_q & \mathbf{0}_q & \mathbf{0}_q & \mathbf{0}_q & \mathbf{I}_q \\ g^{(0)}_x\mathbf{I}_q & g^{(0)}_x\rho^{(0)}_{x,1}\mathbf{I}_q & g^{(0)}_x\rho^{(0)}_{x,-1}\mathbf{P}_+ & g^{(0)}_v\mathbf{I}_q & g^{(0)}_v\rho^{(0)}_{v,1}\mathbf{I}_q & g^{(0)}_v\rho^{(0)}_{v,-1}\mathbf{P}_+ \\ g^{(1)}_x\rho^{(1)}_{x,-1}\mathbf{I}_q & g^{(1)}_x\mathbf{I}_q & g^{(1)}_x\rho^{(1)}_{x,1}\mathbf{I}_q & g^{(1)}_v\rho^{(1)}_{v,-1}\mathbf{I}_q & g^{(1)}_v\mathbf{I}_q & g^{(1)}_v\rho^{(1)}_{v,1}\mathbf{I}_q \\ g^{(2)}_x\rho^{(1)}_{x,1}\mathbf{P}_- & g^{(2)}_x\rho^{(2)}_{x,-1}\mathbf{I}_q & g^{(2)}_x\mathbf{I}_q & g^{(1)}_v\rho^{(1)}_{v,1}\mathbf{P}_- & g^{(1)}_v\rho^{(1)}_{v,-1}\mathbf{I}_q & g^{(1)}_v\mathbf{I}_q \end{pmatrix} \tag{60}$$

The matrices $\mathbf{0}_q, \mathbf{I}_q, \mathbf{P}_+$ and $\mathbf{P}_-$ are all $q \times q$ circulant matrices. By Proposition C.3.4 all circulant matrices commute and $w(\omega^m_q)$ with $m = 0, 1, \cdots, q-1$ are eigenvectors, defined in equation (158). One can show that the eigenvectors have the form,

$$v = \begin{pmatrix} \epsilon_0 w(\omega^m_q) \\ \epsilon_1 w(\omega^m_q) \\ \epsilon_2 w(\omega^m_q) \\ \epsilon_3 w(\omega^m_q) \\ \epsilon_4 w(\omega^m_q) \\ \epsilon_5 w(\omega^m_q) \end{pmatrix} = \begin{pmatrix} \epsilon_0 \\ \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \end{pmatrix} \otimes w(\omega^m_q).$$

This is an eigenvector if

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ g_x^{(0)} & g_x^{(0)}\rho_{x,1}^{(0)} & g_x^{(0)}\rho_{x,-1}^{(0)}\omega_q^{-m} & g_v^{(0)} & g_v^{(0)}\rho_{v,1}^{(0)} & g_v^{(0)}\rho_{v,-1}^{(0)}\omega_q^{-m} \\ g_x^{(1)}\rho_{x,-1}^{(1)} & g_x^{(1)} & g_x^{(1)}\rho_{x,1}^{(1)} & g_v^{(1)}\rho_{v,-1}^{(1)} & g_v^{(1)} & g_v^{(1)}\rho_{v,1}^{(1)} \\ g_x^{(2)}\rho_{x,1}^{(1)}\omega_q^{m} & g_x^{(2)}\rho_{x,-1}^{(2)} & g_x^{(2)} & g_v^{(1)}\rho_{v,1}^{(1)}\omega_q^{m} & g_v^{(1)}\rho_{v,-1}^{(1)} & g_v^{(1)} \end{pmatrix} \begin{pmatrix} \epsilon_0 \\ \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \end{pmatrix} = 0.$$

Here we used Proposition C.4.6. Using Proposition C.1.1, the eigenvalues are determined by $\epsilon_0, \epsilon_1, \epsilon_2$ where,

$$\begin{pmatrix} \nu^2 + g_v^{(0)}\nu + g_x^{(0)} & g_v^{(0)}\rho_{v,1}^{(0)}\nu + g_x^{(0)}\rho_{x,1}^{(0)} & \left(g_v^{(0)}\rho_{v,-1}^{(0)}\nu + g_x^{(0)}\rho_{x,-1}^{(0)}\right)\omega_q^{-m} \\ g_v^{(1)}\rho_{v,-1}^{(1)}\nu + g_x^{(1)}\rho_{x,-1}^{(1)} & \nu^2 + g_v^{(1)}\nu + g_x^{(1)} & g_v^{(1)}\rho_{v,1}^{(1)}\nu + g_x^{(1)}\rho_{x,1}^{(1)} \\ \left(g_v^{(2)}\rho_{v,1}^{(2)}\nu + g_x^{(2)}\rho_{x,1}^{(2)}\right)\omega_q^{m} & g_v^{(2)}\rho_{v,-1}^{(2)}\nu + g_x^{(2)}\rho_{x,-1}^{(2)} & \nu^2 + g_v^{(2)}\nu + g_x^{(2)} \end{pmatrix} \begin{pmatrix} \epsilon_0 \\ \epsilon_1 \\ \epsilon_2 \end{pmatrix} = 0. \tag{61}$$

Use equations (48) and (49) to write this in the form,

$$\begin{pmatrix} \nu^2 + \psi_0^{(0)}(\nu) & \psi_1^{(0)}(\nu) & \psi_{-1}^{(0)}(\nu)\omega_q^{-m} \\ \psi_{-1}^{(1)}(\nu) & \nu^2 + \psi_0^{(1)}(\nu) & \psi_1^{(1)}(\nu) \\ \psi_1^{(2)}(\nu)\omega_q^{m} & \psi_{-1}^{(2)}(\nu) & \nu^2 + \psi_0^{(2)}(\nu) \end{pmatrix} \begin{pmatrix} \epsilon_0 \\ \epsilon_1 \\ \epsilon_2 \end{pmatrix} = 0. \tag{62}$$

The stability of system in equation (62) is not obvious. Some necessary conditions are derived in [3]. Section 2 extends this work by ascertaining some necessary stability conditions for the case of p agent classes duplicated q times. If q is large then we can take approximate the q values using,

$$\omega_q^m = \exp\left(\frac{2\pi i}{q} m\right) \approx 1 + \frac{2\pi i}{q} m,$$

for small m. We define $\phi(m) = \frac{2\pi}{q}m$ and this approximation is just,

$$e^{i\phi} \approx 1 + i\phi.$$

This is the same as approximating ϕ for values near to 0. This approximation works as long as the discrete locus of important eigenvalues is near to the double eigenvalue at 0. It is possible for the locus to sweep around a curve and cross the imaginary axis for a large m. However, if we consider stable systems where the locus of eigenvalues does not approach the imaginary axis, except at the origin, then this approximation will allow us to approximate both stability and the dynamics of the system.

2 SEQUENCES WITH DISTINCT WEIGHTS

2.1 Distinct Weights

Section 2 is our main investigation into one-dimensional lattices of distinct agents. To make the problem tractable we add some assumptions. We assume that $\rho_{x,\pm}^{(\alpha)}$ and $\rho_{v,\pm}^{(\alpha)}$ are the same for all agents, but each agent has distinct weights $g_x^{(\alpha)}, g_v^{(\alpha)}$. The physical justification for this assumption is clear. Given a convoy of agents, one can adjust the ratios of the forward to backward x-couplings by changing $\rho_{x,+}^{(\alpha)}$ and $\rho_{x,-}^{(\alpha)}$. But if the agents are all distinct it will be impossible to match $g_x^{(\alpha)}$ in any meaningful way. This same logic also applies $\rho_{v,+}$ and $\rho_{v,-}$. To further simplify the problem, we add the assumption that the forward and backward x-couplings are identical.

Assumptions 2.1.1. *We assume the following,*

- $\rho_{v,+}^{(\alpha)}$ *and* $\rho_{v,-}^{(\alpha)}$ *are independent of* α.
- $\rho_{x,+}^{(\alpha)} = \rho_{x,-}^{(\alpha)} = -\frac{1}{2}$ *for all* α.

The weights $g_x^{(\alpha)}$ and $g_v^{(\alpha)}$ can be unique for each agent. With these assumptions $\mathbf{G}_x\mathbf{L}_x$ and $\mathbf{G}_v\mathbf{L}_v$ of equation (2) are no longer symmetric and they no longer commute. They both are still row-sum zero. These facts are important to what follows.

When we compute the eigenvalues of the equations of motion, the eigenvalues with large negative real parts are transient, as they decay quickly. Thus, it is important to understand the eigenvalues near the complex axis. We know there is an eigenvalue of multiplicity 2 at the origin and we will study the eigenvalues near this point. In the simulations, we show that, indeed, these eigenvalues determine much of the large-scale structure of the agent system.

Solving for the eigenvalues of this general system is difficult, but we have a method to solve for eigenvalues near the origin. In Section 1.6 and [3] there are methods to solve systems with three distinct agent types laid out in sequence. We extend this to a sequence of p agent types. To do this, we set up a one-dimensional lattice of p distinct agents that satisfies the assumptions 2.1, and duplicates this system q times. This repeated system is determined by the eigenvalues of a $p \times p$ matrix but there is a quantity $\exp(\phi i)$ in the matrix where $\phi = 2\pi mk/q$ (see Corollary 2.2.7). If we fix m and let $q \to \infty$ then ϕ becomes a continuous parameter. The eigenvalues become segments of smooth curves. This allows us to expand around the origin and deduce first and second order dynamics (see Theorem 2.5.1).

Throughout this entire computation, we assume periodic boundary conditions. In Section 2.8 we run some simulations to test whether general one-dimensional system satisfy these equations. In the simulations we do not duplicate the p agent sequence q times, but, instead, test the p sequence directly.

2.2 Linear Nearest Neighbor Systems

In this section, we derive some properties of the eigenvalues of the general nearest neighbor system. This discussion follows [20].

We start with a system of p unique agents as described in Section 1.4. Next, form a sequence of pq agents by duplicating the p-sequence q times so that $N = pq$. This N-sequence contains q copies of the original p sequence. If we set $p = 3$ then we get the system of Section 1.6. This was first described in [3] and there is a more detailed discussion in [2]. The schematic for this is given in Figure 5.

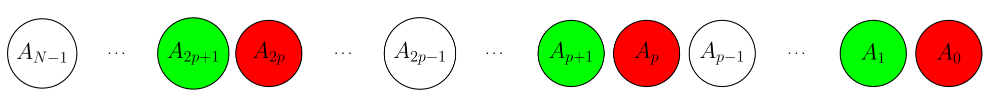


Figure 5: *Sequence With q sub-sequences with p Agent Types*

Each of the p agent types has unique interaction coefficients, so there are q agents that all have identical interaction coefficients. The $N = pq$ agents lie on a

1−dimensional manifold and interact only with their two adjacent neighbors. The agent coordinate for the kth agent is given by x_k, where $k = 0, 1, \cdots N-1$. The agent forces are designed to keep a distance Δ between agents. This system is a specific case of the systems described in Section 1.4. The specific version of equation (39) is given by

$$\begin{aligned}\ddot{x}_k^{(\alpha)} =& - g_x^{(\alpha)} \sum_{j \in N(x_k)} \rho_{x,j}^{(\alpha)} \left(x_j^{(\alpha_j)} - x_k^{(\alpha)} - (j-k)\Delta \right) \\ & - g_v^{(\alpha)} \sum_{j \in N(x_k)} \rho_{v,j}^{(\alpha)} \left(\dot{x}_j^{(\alpha_j)} - \dot{x}_k^{(\alpha)} \right)\end{aligned} \tag{63}$$

We introduced the variable α to indicate agent type. In our case $\alpha = k \mod (p)$. This means, for example, $g_x^{(\alpha)} = g_x^{(\alpha+p)}$. All the α arithmetic in equation (63) is $\mod (p)$.

As in Section 1.4, we change coordinates to z_k, given by

$$z_k = x_k - k\Delta \tag{64}$$

With these assumptions the agents satisfy the following equations of motion:

$$\begin{aligned}\frac{d^2 z_k^{(\alpha)}}{dt^2} =& - g_x^{(\alpha)} \left(z_k^{(\alpha)} + \rho_{x,1}^{(\alpha)} z_{k+1}^{(\alpha+1)} + \rho_{x,-1}^{(\alpha)} z_{k-1}^{(\alpha-1)} \right) \\ & - g_v^{(\alpha)} \left(\dot{z}_k^{(\alpha)} + \rho_{v,1}^{(\alpha)} \dot{z}_{k+1}^{(\alpha+1)} + \rho_{v,-1}^{(\alpha)} \dot{z}_{k-1}^{(\alpha-1)} \right)\end{aligned} \tag{65}$$

We assume periodic boundary conditions and discuss other boundary conditions below. With this assumption, the last car interacts with the first car in the agent sequence. This is the case when the arithmetic in α is $\mod (p)$. In any case, the system has the familiar matrix form of equations (35) or, more generally, (43):

$$\frac{d}{dt}\begin{pmatrix} z \\ \dot{z} \end{pmatrix} = \mathbf{M}\begin{pmatrix} z \\ \dot{z} \end{pmatrix} = \begin{pmatrix} \mathbf{0}_3 & \mathbf{I}_3 \\ -\mathbf{L}_x & -\mathbf{L}_v \end{pmatrix}\begin{pmatrix} z \\ \dot{z} \end{pmatrix} \tag{66}$$

Using Proposition C.1.1, the eigenvalues are the roots of the $2N$ degree characteristic polynomial,

$$\det\left(\nu^2\mathbf{I}_N + \nu\mathbf{L}_v + \mathbf{L}_x\right) = 0. \tag{67}$$

The rows in equations (66) are in k order. Depending on the boundary conditions, the matrix $\mathbf{M}$ is tri-diagonal, except possibly row 0 and row $N-1$. If we have periodic boundary conditions, then there are additional terms on the extreme upper right and lower left of $\mathbf{M}$.

We re-order the rows so all α values are adjacent. In other words, we re-order so that the basis is in the following order:

$$\begin{aligned} k =& 0, p, 2p, \cdots, (q-1)p, 1, 1+p, 1+2p, \cdots, 1+(q-1)p, \\ & \cdots, (p-1), (p-1)+p, \cdots, (p-1)+(q-1)p \end{aligned} \tag{68}$$

where $(p-1)+(q-1)p = N-1$. This is captured by the index mapping function,

$$\sigma(k+mq) = m + kp, \tag{69}$$

where $m = 0, \cdots, p-1$ and $k = 0, \cdots, q-1$. We describe this ordering in more precise language in Appendix D. With this ordering, the $(pq)\times(pq)$ matrix $\mathbf{M}$ breaks up into $p \times p$ sub-blocks. Each of these sub-blocks is a $q \times q$ matrix. To illustrate the transformation, look at the first two rows of $\mathbf{L}_x$,

$$\begin{matrix} g_x^{(0)} & g_x^{(0)}\rho_{x,1}^{(0)} & 0 & \cdots & 0 & g_x^{(0)}\rho_{x,-1}^{(0)} \\ g_x^{(1)}\rho_{x,-1}^{(1)} & g_x^{(1)} & g_x^{(1)}\rho_{x,1}^{(1)} & \cdots & 0 & 0 \end{matrix}$$

With the new re-ordering, the $N \times N$ matrix $\mathbf{L}_x$ becomes,

$$\mathbf{L}_x = \begin{pmatrix} g_x^{(0)}\mathbf{I}_q & g_x^{(0)}\rho_{x,1}^{(0)}\mathbf{I}_q & \mathbf{0}_q & \cdots & \mathbf{0}_q & g_x^{(0)}\rho_{x,-1}^{(0)}\mathbf{P}_+ \\ g_x^{(1)}\rho_{x,-1}^{(1)}\mathbf{I}_q & g_x^{(1)}\mathbf{I}_q & g_x^{(1)}\rho_{x,1}^{(1)}\mathbf{I}_q & \cdots & \mathbf{0}_q & \mathbf{0}_q \\ \mathbf{0}_q & g_x^{(2)}\rho_{x,-1}^{(2)}\mathbf{I}_q & g_x^{(2)}\mathbf{I}_q & \cdots & \mathbf{0}_q & \mathbf{0}_q \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0}_q & \mathbf{0}_q & \mathbf{0}_q & \cdots & g_x^{(p-2)}\mathbf{I}_q & g_x^{(p-2)}\rho_{x,1}^{(p-2)}\mathbf{I}_q \\ g_x^{(p-1)}\rho_{x,1}^{(p-1)}\mathbf{P}_- & \mathbf{0}_q & \mathbf{0}_q & \cdots & g_x^{(p-1)}\rho_{x,-1}^{(p-1)}\mathbf{I}_q & g_x^{(p-1)}\mathbf{I}_q \end{pmatrix}, \tag{70}$$

where $\mathbf{P}_+$ and $\mathbf{P}_-$ are defined in Definition C.2.1, and $\mathbf{P}_+, \mathbf{P}_-, \mathbf{0}_q$ and $\mathbf{I}_q$ are all $q \times q$ matrices. The matrix $\mathbf{L}_v$ is identical except that the x is replaced by v.

To simplify the notation, we introduce the following 1st degree polynomials

$$\psi_0^{(\alpha)}(\nu) = \left(g_v^{(\alpha)}\nu + g_x^{(\alpha)}\right) \tag{71}$$

$$\psi_1^{(\alpha)}(\nu) = g_v^{(\alpha)}\rho_{v,1}^{(\alpha)}\nu + g_x^{(\alpha)}\rho_{x,1}^{(\alpha)} \tag{72}$$

$$\psi_{-1}^{(\alpha)}(\nu) = g_v^{(\alpha)}\rho_{v,-1}^{(\alpha)}\nu + g_x^{(\alpha)}\rho_{x,-1}^{(\alpha)}. \tag{73}$$

For all α and any ν, we have

$$\psi_{-1}^{(\alpha)}(\nu) + \psi_0^{(\alpha)}(\nu) + \psi_1^{(\alpha)}(\nu) = 0. \tag{74}$$

Theorem 2.2.1. *For the periodic system described above, the eigenvectors of the $2N \times 2N$ matrix in equation (66), with eigenvalue ν, are given by the $2N = 2pq$*

vectors,

$$u(m,k) = \begin{pmatrix} v_m \otimes e_{m,k} \\ \nu\left(v_m \otimes e_{m,k}\right) \end{pmatrix} = \begin{pmatrix} e_{m,k} \\ \omega_q^m e_{m,k} \\ \omega_q^{2m} e_{m,k} \\ \vdots \\ \omega_q^{(q-1)m} e_{m,k} \\ \nu e_{m,k} \\ \nu\omega_q^m e_{m,k} \\ \nu\omega_q^{2m} e_{m,k} \\ \vdots \\ \nu\omega_q^{(q-1)m} e_{m,k} \end{pmatrix}, \tag{75}$$

for $k = 0, 1, \cdots 2p-1$, $m \in \{0, \cdots, q-1\}$ *and where* $v_m = w(\omega_q^m)$ *and*

$$e_{m,k} = \begin{pmatrix} e_{m,k}^0 & e_{m,k}^1 & e_{m,k}^2 & \cdots & e_{m,k}^{p-1} \end{pmatrix}^T .$$

For each m *we set* $\phi = \frac{2\pi}{q} m$ *and take the determinant of the matrix* $\mathbf{M}_\phi(\nu)$ *in equation (77). There are* $2p$ *solutions in* ν *which are the eigenvalues. For each of these eigenvalues* ν*, the* $e_{m,k}$ *satisfy*

$$e_{m,k} \in \ker\left(\mathbf{M}_\phi(\nu)\right) , \tag{76}$$

where

$$\mathbf{M}_\phi(\nu) = \begin{pmatrix} \nu^2 + \psi_0^{(0)}(\nu) & \psi_1^{(0)}(\nu) & 0 & \cdots & \psi_{-1}^{(0)}(\nu)e^{-i\phi} \\ \psi_{-1}^{(1)}(\nu) & \nu^2 + \psi_0^{(1)}(\nu) & \psi_1^{(1)}(\nu) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \psi_1^{(p-1)}(\nu)e^{i\phi} & 0 & 0 & \cdots & \nu^2 + \psi_0^{(p-1)}(\nu) \end{pmatrix} . \tag{77}$$

Proof. In section C.4 we show that the vectors $v_m = w(\omega_q^m)$ form an orthogonal basis of eigen-vectors, for $m = 0, 1, \cdots, q-1$, for all circulant matrices. These q vectors diagonalize all the $q \times q$ matrices, $\mathbf{P}_+, \mathbf{P}_-, \mathbf{0}_q$ and $\mathbf{I}_q$.

Apply $\mathbf{M}$, with the reordered $\mathbf{L}_x$ and $\mathbf{L}_v$, to the following $2N$ vector,

$$\begin{pmatrix} e^0 v_m \\ e^1 v_m \\ \vdots \\ e^{p-1} v_m \\ \nu e^0 v_m \\ \nu e^1 v_m \\ \vdots \\ \nu e^{p-1} v_m \end{pmatrix} = \begin{pmatrix} e \otimes v_m \\ \nu \left(e \otimes v_m \right) \end{pmatrix}.$$

By Proposition C.1.1, this vector is an eigenvector if it satisfies

$$\left(\nu^2 + \nu \mathbf{L}_v + \mathbf{L}_x \right) \left(e \otimes v_m \right) = 0. \tag{78}$$

By Proposition C.4.6, we have

$$\begin{aligned} \left(\nu g_v^{(\alpha)} + g_x^{(\alpha)} \right) \mathbf{P}_+ v_m &= \left(\nu g_v^{(\alpha)} + g_x^{(\alpha)} \right) \omega_q^{-m} v_m = \left(\nu g_v^{(\alpha)} + g_x^{(\alpha)} \right) e^{-i\phi} v_m \\ \left(\nu g_v^{(\alpha)} + g_x^{(\alpha)} \right) \mathbf{P}_- v_m &= \left(\nu g_v^{(\alpha)} + g_x^{(\alpha)} \right) \omega_q^{m} v_m = \left(\nu g_v^{(\alpha)} + g_x^{(\alpha)} \right) e^{i\phi} v_m \end{aligned}$$

where $\phi = \frac{2\pi}{q} m$. With these facts, equation (78) becomes $p \times p$ matrix condition equation (77) for e^k.

We have eigenvectors but we must restore the original index order with the inverse to equation (69),

$$\sigma^{-1}(m+kp) = k + mq.$$

With this we get the eigenvectors

$$u(m,k) = \begin{pmatrix} v_m \otimes e_{m,k} \\ \nu\left(v_m \otimes e_{m,k}\right) \end{pmatrix} = \begin{pmatrix} e_{m,k} \\ \omega_q^m e_{m,k} \\ \omega_q^{2m} e_{m,k} \\ \vdots \\ \omega_q^{(p-1)m} e_{m,k} \\ \nu e_{m,k} \\ \nu\omega_q^m e_{m,k} \\ \nu\omega_q^{2m} e_{m,k} \\ \vdots \\ \nu\omega_q^{(p-1)m} e_{m,k} \end{pmatrix}.$$

□

Remark 2.2.2. *The procedure outlined above ignores the case where $\mathbf{M}_\phi(\nu)$ has degenerate eigenvalues. At this point, this is not a problem, as we are mostly interested in the eigenvalues, especially eigenvalues close to the origin. We already know that the origin is a degenerate eigenvalue and we will get two eigenvalue loci emanating from this point. We return to the degeneracy of the eigenvalues when we discuss solutions in Section 2.6.*

Corollary 2.2.3. *The eigenvalues of the system are the roots to the polynomial,*

$$P_\phi(\nu) = \det \begin{pmatrix} \nu^2 + \psi_0^{(0)}(\nu) & \psi_1^{(0)}(\nu) & 0 & \cdots & \psi_{-1}^{(0)}(\nu)e^{-i\phi} \\ \psi_{-1}^{(1)}(\nu) & \nu^2 + \psi_0^{(1)}(\nu) & \psi_1^{(1)}(\nu) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \psi_1^{(p-1)}(\nu)e^{i\phi} & 0 & 0 & \cdots & \nu^2 + \psi_0^{(p-1)}(\nu) \end{pmatrix}. \tag{79}$$

For each value of $\phi = \frac{2\pi i}{q}m$, $m = 0, \cdots q-1$, there are $2p$ roots.

Proposition 2.2.4. *When $\phi = 0$ (e.g., $m = 0$), the constant and linear terms for the polynomial $P_0(\nu)$ both vanish.*

Proof. Neither the linear nor constant terms of the polynomial have ν^2 as a factor. Set $\phi = 0$ and remove the terms with ν^2 as a factor. The resulting polynomial shares linear and constant terms with the following polynomial,

$$\det \begin{pmatrix} \psi_0^{(0)}(\nu) & \psi_1^{(0)}(\nu) & 0 & \cdots & \psi_{-1}^{(0)}(\nu) \\ \psi_{-1}^{(1)}(\nu) & \psi_0^{(1)}(\nu) & \psi_1^{(1)}(\nu) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \psi_1^{(p-1)}(\nu) & 0 & 0 & \cdots & \psi_0^{(p-1)}(\nu) \end{pmatrix}$$

Every row in this matrix sums to zero, by equation (74). This means that the vector, consisting of all 1's, is an eigenvector with eigenvalue 0, and so the determinant vanishes for all ν. The constant and linear terms are contained in this reduced polynomial and so must vanish. $\square$

Remark 2.2.5. *The proof of Proposition 2.2.4 says that the second order term of the polynomial $P_0(\nu)$ must contain exactly one of the diagonal ν^2 terms. The third order term of $P_0(\nu)$ must also contain exactly one of ν^2 terms. These facts will be used below.*

Proposition 2.2.6. *The polynomial $P_\phi(\nu)$ of Corollary 2.2.3 satisfies*

$$P_\phi(\nu) = s(\nu) + (-1)^p r_\phi(\nu),$$

where all the ϕ dependence is in the polynomial

$$r_\phi(\nu) = (1-e^{i\phi})\left(\psi_1^{(0)}(\nu)\psi_1^{(1)}(\nu)\cdots\psi_1^{(p-1)}(\nu)\right)+(1-e^{-i\phi})\left(\psi_{-1}^{(0)}(\nu)\psi_{-1}^{(1)}(\nu)\cdots\psi_{-1}^{(p-1)}(\nu)\right),$$

and $s(\nu)$ has zero constant and linear terms.

Proof. The only terms of the expansion of equation (79) that depend on ϕ contain either $\psi_{-1}^{(0)}(\nu)e^{-i\phi}$ or $\psi_1^{(p-1)}(\nu)e^{i\phi}$, but not both. The determinant is a sum over permutations σ of terms $\mathrm{sgn}(\sigma)M_{0\sigma 0}\cdots M_{p-1\sigma p-1}$. The only non-zero permutations have $\sigma(k) \in \{k-1, k, k+1\}MOD(p)$. The permutations that contain the term $\psi_{-1}^{(0)}(\nu)e^{-i\phi}$ but not $\psi_1^{(p-1)}(\nu)e^{i\phi}$ must have $\sigma(0) = p-1$. The matrix is tri-diagonal, so the value $\sigma(p-1) \in \{0, p-1, p-2\}$ for all σ. But in this case we know $\sigma(p-1) \neq p-1$ and, by assumption, $\sigma(p-1) \neq 0$ (or the term would contain the $e^{i\phi}$ term). Therefore, $\sigma(p-1) = p-2$. By a similar logic, $\sigma(p-2) \in \{p-3, p-2, p-1\}$ but $\sigma(p-2) \neq p-2$ and $\sigma(p-2) \neq p-1$. So we get $\sigma(p-2) = p-3$. Proceed in this way to get the permutation $\sigma = (0, p-1, p-2, \cdots 1)$, which has $\mathrm{sgn}(\sigma) = (-1)^{p-1}$. This corresponds to the term

$$-(-1)^p e^{-i\phi}\psi_{-1}^{(0)}(\nu)\cdots\psi_{-1}^{(p-1)}(\nu).$$

The term that contains $\psi_1^{(p-1)}(\nu)e^{i\phi}$ but not $\psi_{-1}^{(0)}(\nu)e^{-i\phi}$ is computed in a similar way, and seen to be

$$-(-1)^p e^{i\phi}\psi_1^{(0)}(\nu)\cdots\psi_1^{(p-1)}(\nu).$$

We define r_ϕ so that $r_0(\nu) = 0$. From Proposition 2.2.4 it follows immediately that the constant and linear terms of $s(\nu)$ both vanish. □

Corollary 2.2.7. *The polynomial $P_\phi(\nu)$ satisfies*

$$\left.\frac{d^k P_\phi}{d\phi^k}\right|_{\phi=0} = (-1)^{p+1} i^k \left(\psi_1^{(0)}(\nu)\psi_1^{(1)}(\nu)\cdots\psi_1^{(p-1)}(\nu)\right) + (-1)^{p+1}(-i)^k \left(\psi_{-1}^{(0)}(\nu)\psi_{-1}^{(1)}(\nu)\cdots\psi_{-1}^{(p-1)}(\nu)\right)$$

Proof. Since $s(\nu)$ does not depend on ϕ, this follows immediately from Proposition 2.2.6. □

2.3 Characteristic Polynomial Expansion

The roots of the characteristic polynomial of the general system, described in Section 2.2, are given by the roots of a series of p degree polynomials $P_\phi(\nu)$ as described in Corollary 2.2.3. When $\phi = 0$, the characteristic polynomial $P_0(\nu)$ has a root of multiplicity 2 at $\nu = 0$. In this section start with Assumptions (2.1.1), so that $\rho_{x,1}^{(\alpha)} = \rho_{x,-1}^{(\alpha)}$. We will not assume that G_x and G_v commute. In the simulations, we will let g_x^α and g_v^α be independent random variables.

If the system is stable, then roots of the characteristic polynomial all have non-positive real parts. Stable roots with large negative real parts decay quickly so that a stable system is dominated by roots that lie near the imaginary axis. In our system, roots near the origin will dominate the dynamics. Therefore, we will expand around the zero at $\phi = 0, \nu = 0$ to approximate the system dynamics. The details of the expansion are outlined in this section.

We expand $P_\phi(\nu)$ around $\phi = 0, \nu = 0$. If q is large enough, we can approximate the system by using a continuous variable for ϕ. With this approximation, each of the roots at $\phi = 0, \nu = 0$ form continuous zero loci as ϕ varies. These two eigenvalue approximations are continuous maps,

$$\gamma : I \to \mathbb{C}, \tag{80}$$

where $I = (-\epsilon, +\epsilon)$ is some neighborhood of 0. These curves satisfy

$$P_\phi(\gamma(\phi)) = 0. \tag{81}$$

The coefficients of the characteristic polynomial are analytic functions of the real parameter ϕ; a polynomial is an analytic function as well. Expand everything in a Taylor series and use the resulting equations to deduce conditions on the coefficients. Assume that $\gamma(0) = 0$ and write the expansion,

$$\gamma(\phi) = \gamma'(0)\phi + \frac{1}{2}\gamma''(0)\phi^2 + \cdots . \tag{82}$$

Since the coefficients of the polynomial $p_\phi(\nu)$ are real analytic functions of ϕ, we can expand each of them in a Taylor series. The result is an expansion of the form

$$\begin{aligned} P_\phi(\nu) = {} & (a_{00} + a_{01}\phi + \cdots) + (a_{10} + a_{11}\phi + \cdots)\,\nu \\ & + (a_{20} + a_{21}\phi + \cdots)\,\nu^2 + \cdots \end{aligned} \tag{83}$$

where the coefficients $a_{0k}, a_{1k}, \cdots$, arise as the coefficients of the kth derivative of $p_\phi(\nu)$ with respect to ϕ

$$\frac{1}{k!}\left.\frac{d^k P_\phi}{d\phi^k}\right|_{\phi=0} = a_{0k} + a_{1k}\nu + a_{2k}\nu^2 + \cdots . \tag{84}$$

Theorem 2.3.1. *Assume that when $\phi = 0$, $P_0(\nu)$ has a double root at $\nu = 0$, so that*

$a_{00} = a_{10} = 0$. *In this case the second order expansion of the zero locus, near the point* $\phi = 0, \nu = 0$, *gives the coefficients*

$$\gamma'(0) = \frac{-a_{11} \pm \sqrt{a_{11}^2 - 4a_{02}a_{20})}}{2a_{20}}, \tag{85}$$

$$\gamma''(0) = -2\frac{(a_{30}(\gamma'(0))^3 + a_{21}(\gamma'(0))^2 + a_{12}\gamma'(0) + a_{03})}{2a_{20}\gamma'(0) + a_{11}} \tag{86}$$

Proof. The equation $P_\phi(\gamma(\phi)) = 0$ expands to a power series in ϕ. Set $\nu = \gamma(\phi)$ and expand using equation (82). Plug this value of ν into the polynomial of equation (83). Equation (81) says that this expansion in ϕ vanishes identically. When you solve for the derivatives $\gamma^{(m)}(0)$ in terms of a_{jk}, you get equations (85) and (86). This is a straightforward calculation that can be verified using algebraic manipulation software, such as SAGE. □

Notice that there are two solutions to $\gamma'(0)$. The double root at the origin splits into two distinct curves, so that there are two distinct values, $\gamma_+(0)$ and $\gamma_-(0)$.

In the next section, we find the coefficients a_{jk} which are required for our expansion. The reader may find it more digestible to jump to Section 2.5 and refer to Section 2.4 as needed.

2.4 Expansion Coefficients Near the Origin

In this section we compute all the coefficients, a_{ij}, required for the polynomial expansion around the origin, as stated in equations 85 and 86. This section has some specialized and complicated computations which can be skipped on the first reading. The main results are contained in Propositions 2.4.1 and 2.4.2.

The polynomial $P_\phi(\nu)$ has analytic coefficients so that we can expand the coefficients in a power series. The result is an expansion of the form,

$$\begin{aligned} P_\phi(\nu) = & (a_{00} + a_{01}\phi + \cdots) + (a_{10} + a_{11}\phi + \cdots)\,\nu \\ & + (a_{20} + a_{21}\phi + \cdots)\,\nu^2 + \cdots , \end{aligned} \tag{87}$$

where the coefficients $a_{0k}, a_{1k}, \cdots$, arise as the coefficients of the kth derivative of $p_\phi(\nu)$ with respect to ϕ,

$$\frac{1}{k!}\left.\frac{d^k P_\phi}{d\phi^k}\right|_{\phi=0} = a_{0k} + a_{1k}\nu + a_{2k}\nu^2 + \cdots . \tag{88}$$

Using Proposition 2.2.6 we can now take derivative with respect to ϕ and the term $s(\nu)$ vanishes. We get

$$\begin{aligned} \left.\frac{d^k P_\phi}{d\phi^k}\right|_{\phi=0} = -(i)^k(-1)^p \Big(& \psi_1^{(0)}(\nu)\psi_1^{(1)}(\nu)\cdots\psi_1^{(p-1)}(\nu) \\ & +(-1)^k\psi_{-1}^{(0)}(\nu)\psi_{-1}^{(1)}(\nu)\cdots\psi_{-1}^{(p-1)}(\nu)\Big) \end{aligned} \tag{89}$$

Define β_v by

$$\rho_{v,1} = -\frac{1-\beta_v}{2} \Rightarrow \rho_{v,-1} = -\frac{1+\beta_v}{2}. \tag{90}$$

In our case $\beta_v \in [-1, +1]$. From this we get

$$\beta_v = \rho_{v,1} - \rho_{v,-1} = 1 + 2\rho_{v,1} = -\left(1 + 2\rho_{v,-1}\right) \tag{91}$$

With this definition and the assumption $\rho_{x,1} = \rho_{x,-1} = -\frac{1}{2}$, we have,

$$\psi_1^{(\alpha)}(\nu) = -\frac{1}{2}\left(g_x^{(\alpha)} + g_v^{(\alpha)}(1-\beta_v)\nu\right) \tag{92}$$

$$\psi_{-1}^{(\alpha)}(\nu) = -\frac{1}{2}\left(g_x^{(\alpha)} + g_v^{(\alpha)}(1+\beta_v)\nu\right) \tag{93}$$

Now we have

$$\begin{aligned}
&\psi_{\pm1}^{(0)}(\nu)\psi_{\pm1}^{(1)}(\nu)\cdots\psi_{\pm1}^{(p-1)}(\nu) \\
&\quad= (-1)^p(1/2)^p(g_x^{(0)} + g_v^{(0)}(1\pm\beta_v)\nu)\cdots(g_x^{(p-1)} + g_v^{(p-1)}(1\pm\beta_v)\nu) \\
&\quad= (-1)^p(1/2)^p\prod_{j=0}^{p-1} g_x^{(j)}\left(1 + \sum_{k=0}^{p-1}\frac{g_v^{(k)}}{g_x^{(k)}}(1\pm\beta_v)\nu\right. \\
&\qquad\qquad \left. + \sum_{0=k_1<k_2}^{p-1}\frac{g_v^{(k_1)}g_v^{(k_2)}}{g_x^{(k_1)}g_x^{(k_2)}}(1\pm\beta_v)^2\nu^2 + \cdots\right) \\
&\quad= (-1)^p(1/2)^p\left(\prod_{j=0}^{p-1} g_x^{(j)}\right)\left(1 + \sum_{k=0}^{p-1}\frac{g_v^{(k)}}{g_x^{(k)}}(1\pm\beta_v)\nu\right. \\
&\qquad\qquad \left. +\frac{1}{2}\sum_{\substack{k_1,k_2\\k_1\neq k_2}}^{p-1}\frac{g_v^{(k_1)}g_v^{(k_2)}}{g_x^{(k_1)}g_x^{(k_2)}}(1\pm\beta_v)^2\nu^2 + \cdots\right)
\end{aligned}$$

Proposition 2.4.1. *The first derivative gives us*

$$a_{01} = 0, \tag{94}$$

$$\begin{aligned} a_{11} &= \frac{i}{2^{p-1}} \left(\prod_{j=0}^{p-1} g_x^{(j)} \right) \left(\sum_{k=0}^{p-1} \frac{g_v^{(k)}}{g_x^{(k)}} \right) \beta_v \\ &= \frac{2ip}{2^p} \left(\prod_{j=0}^{p-1} g_x^{(j)} \right) \mathrm{Avg} \left(\frac{g_v^{(k)}}{g_x^{(k)}} \right) \beta_v, \end{aligned} \tag{95}$$

$$\begin{aligned} a_{21} &= \frac{2i}{2^p} \left(\prod_{j=0}^{p-1} g_x^{(j)} \right) \left(\sum_{\substack{k_1,k_2 \\ k_1 \neq k_2}}^{p-1} \frac{g_v^{(k_1)} g_v^{(k_2)}}{g_x^{(k_1)} g_x^{(k_2)}} \right) \beta_v \\ &= \frac{2i}{2^p} \left(\prod_{j=0}^{p-1} g_x^{(j)} \right) \left(\sum_{k_1=0}^{p-1} \frac{g_v^{(k_1)}}{g_x^{(k_1)}} \sum_{k_2=0}^{p-1} \frac{g_v^{(k_2)}}{g_x^{(k_2)}} - \sum_{k=0}^{p-1} \frac{g_v^{(k)} g_v^{(k)}}{g_x^{(k)} g_x^{(k)}} \right) \beta_v \\ &= \frac{ip^2}{2^{p-1}} \left(\prod_{j=0}^{p-1} g_x^{(j)} \right) \left(\mathrm{Avg} \left(\frac{g_v^{(k_1)}}{g_x^{(k_1)}} \right) \mathrm{Avg} \left(\frac{g_v^{(k_2)}}{g_x^{(k_2)}} \right) - \frac{1}{p} \mathrm{Avg} \left(\frac{g_v^{(k)} g_v^{(k)}}{g_x^{(k)} g_x^{(k)}} \right) \right) \beta_v \end{aligned} \tag{96}$$

The second derivative gives us

$$a_{02} = \frac{1}{2^p} \left(\prod_{j=0}^{p-1} g_x^{(j)} \right), \tag{97}$$

$$a_{12} = \frac{1}{2^p} \left(\prod_{j=0}^{p-1} g_x^{(j)} \right) \sum_{k=0}^{p-1} \frac{g_v^{(k)}}{g_x^{(k)}} = \frac{1}{2^p} \left(\prod_{j=0}^{p-1} g_x^{(j)} \right) p \, \mathrm{Avg} \left(\frac{g_v^{(k)}}{g_x^{(k)}} \right) \tag{98}$$

The third derivative gives us

$$a_{03} = 0. \tag{99}$$

Proof. Using equation (89) with $k = 1$ gives

$$\begin{aligned}
\left.\frac{dP_\phi}{d\phi}\right|_{\phi=0} =& - i(-1)^p \left(\psi_1^{(0)}(\nu)\cdots\psi_1^{(p-1)}(\nu) - \psi_{-1}^{(0)}(\nu)\cdots\psi_{-1}^{(p-1)}(\nu)\right) \\
=& - i(1/2)^p \left(\prod_{j=0}^{p-1} g_x^{(j)}\right)\left((1-\beta_v - 1 - \beta_v)\nu \sum_{k=0}^{p-1} \frac{g_v^{(k)}}{g_x^{(k)}}\right. \\
& \left. + \left((1-\beta_v)^2 - (1+\beta_v)^2\right)\nu^2 \frac{1}{2} \sum_{\substack{k_1,k_2 \\ k_1\neq k_2}}^{p-1} \frac{g_v^{(k_1)} g_v^{(k_2)}}{g_x^{(k_1)} g_x^{(k_2)}} + \cdots\right) \\
=& i(1/2)^{p-1}\left(\prod_{j=0}^{p-1} g_x^{(j)}\right)\left(\nu\beta_v \sum_{k=0}^{p-1}\frac{g_v^{(k)}}{g_x^{(k)}} + \nu^2\beta_v \sum_{\substack{k_1,k_2 \\ k_1\neq k_2}}^{p-1} \frac{g_v^{(k_1)} g_v^{(k_2)}}{g_x^{(k_1)} g_x^{(k_2)}} + \cdots\right)
\end{aligned}$$

The formulas for a_{01}, a_{11}, a_{21} all follow from equation (88). The second derivative is

$$\begin{aligned}
\left.\frac{1}{2}\frac{d^2P_\phi}{d\phi^2}\right|_{\phi=0} &= (-1)^p(1/2)\left(\psi_1^{(0)}(\nu)\cdots\psi_1^{(p-1)}(\nu) + \psi_{-1}^{(0)}(\nu)\cdots\psi_{-1}^{(p-1)}(\nu)\right) \\
&= (1/2)^{p+1}\left(\prod_{j=0}^{p-1} g_x^{(j)}\right)\left(2 + \sum_{k=0}^{p-1}\frac{g_v^{(k)}}{g_x^{(k)}}(1-\beta_v+1+\beta_v) + \cdots\right)
\end{aligned}$$

The formulas for a_{02}, a_{12}, a_{22} all follow from equation (88). The third derivative is

$$\begin{aligned}
\left.\frac{1}{3!}\frac{d^3P_\phi}{d\phi^3}\right|_{\phi=0} &= i(1/6)(-1)^p\left(\psi_1^{(0)}(\nu)\cdots\psi_1^{(p-1)}(\nu) - \psi_{-1}^{(0)}(\nu)\cdots\psi_{-1}^{(p-1)}(\nu)\right) \\
&= i(1/6)\frac{1}{2^p}\left(\prod_{j=0}^{p-1} g_x^{(j)}\right)\left(0 + \nu\sum_{k=0}^{p-1}\frac{g_v^{(k)}}{g_x^{(k)}}(1-\beta_v-1-\beta_v) + \cdots\right)
\end{aligned}$$

The formula for a_{03} follows from equation (88). □

Proposition 2.4.2. *The coefficients $a_{00}, a_{10}, a_{20}, a_{30}$ are given by*

$$a_{00} = 0, \tag{100}$$

$$a_{10} = 0, \tag{101}$$

$$a_{20} = \frac{p}{2^{p-1}} \prod_{j=0}^{p-1} g_x^{(j)} \left(\sum_{k=0}^{p-1} \frac{1}{g_x^{(k)}} \right) = \frac{2p^2}{2^p} \left(\prod_{j=0}^{p-1} g_x^{(j)} \right) \operatorname{Avg} \left(\frac{1}{g_x^{(k)}} \right) \tag{102}$$

$$\begin{aligned} a_{30} = {} & \frac{2p}{2^p} \left(\prod_{j=0}^{p-1} g_x^{(j)} \right) \left(\left(\sum_{k=0}^{p-1} \frac{1}{g_x^{(k)}} \right) \left(\sum_{j=0}^{p-1} \frac{g_v^{(j)}}{g_x^{(j)}} \right) - \sum_{k=0}^{p-1} \frac{g_v^{(k)}}{g_x^{(k)} g_x^{(k)}} \right) \\ & + \frac{2\beta_v}{2^p} \left(\prod_{j=0}^{p-1} g_x^{(j)} \right) \left(\sum_{k=0}^{p-1} \sum_{j=1}^{p-1} \frac{g_v^{(k+j)}}{g_x^{(k)} g_x^{(k+j)}} (p - 2j) \right). \end{aligned} \tag{103}$$

Proof. The coefficients are determined by the polynomial $P_\phi(\nu)$ with $\phi = 0$. By Proposition 2.2.6,

$$P_0(\nu) = s(\nu) + (-1)^p r_0(\nu) = s(\nu).$$

The polynomial $s(\nu)$ has zero constant and linear terms by Proposition 2.2.6, so a_{00}, a_{10} are both zero.

The derivations of a_{20} and a_{30} both rely on Remark 2.2.5. To compute a_{20} you gather the second order terms, ν^2. When computing the determinant $P_0(\nu)$ you must have exactly one ν^2 factor from the diagonal terms. None of the remaining terms depend on ν. The derivation for a_{20} follows in a similar way to a_{30}. We present the details for a_{30} and leave a_{20} to the reader.

The polynomial ν^3 coefficient must include exactly one ν^2 from a diagonal element. The contribution from the ν^2 term in the(k, k) diagonal element is computed using the co-factor of this element. If we use arithmetic $\mod (p)$, then the co-factor has rows and columns with the index values

$$k + 1, k + 2, \cdots, k + (p - 1) \mod (p).$$

Using Proposition C.5.1, the ν^3 term is the ν^3 term in the following polynomial:

$$\nu^2 \Big(\psi_{-1}^{(k+1)}(\nu) \cdots \psi_{-1}^{(k+p-2)}(\nu)\psi_{-1}^{(k+p-1)}(\nu) + \psi_{-1}^{(k+1)}(\nu) \cdots \psi_{-1}^{(k+p-2)}(\nu)\psi_{1}^{(k+p-1)}(\nu) + \cdots$$
$$+\psi_{-1}^{(k+1)}(\nu)\psi_{1}^{(k+2)}(\nu) \cdots \psi_{1}^{(k+p-1)}(\nu) + \psi_{1}^{(k+1)}(\nu)\psi_{1}^{(k+2)}(\nu) \cdots \psi_{1}^{(k+p-1)}(\nu)\Big). \tag{104}$$

Each summand contains $p-1$ factors in the product. We write out all the contributions to ν^3 by writing the first summand on the first row, followed by the contribution of the second summand on the second row until we get to the p'th row. The result is the ν^3 term,

$$\begin{aligned}
&\frac{\nu^3(-1)^{p-1}}{(-2)^{p-2}} \left(\prod_{j=0}^{p-1} g_x^{(j)} \right) \\
&\qquad \left(\frac{g_v^{(k+1)}}{g_x^{(k)} g_x^{(k+1)}} \rho_{v,-1} + \frac{g_v^{(k+2)}}{g_x^{(k)} g_x^{(k+2)}} \rho_{v,-1} + \cdots + \frac{g_v^{(k+p-1)}}{g_x^{(k)} g_x^{(k+p-1)}} \rho_{v,-1} \right. \\
&\qquad + \frac{g_v^{(k+1)}}{g_x^{(k)} g_x^{(k+1)}} \rho_{v,-1} + \frac{g_v^{(k+2)}}{g_x^{(k)} g_x^{(k+2)}} \rho_{v,-1} + \cdots + \frac{g_v^{(k+p-1)}}{g_x^{(k)} g_x^{(k+p-1)}} \rho_{v,1} \\
&\qquad + \frac{g_v^{(k+1)}}{g_x^{(k)} g_x^{(k+1)}} \rho_{v,-1} + \frac{g_v^{(k+2)}}{g_x^{(k)} g_x^{(k+2)}} \rho_{v,-1} + \cdots + \frac{g_v^{(k+p-1)}}{g_x^{(k)} g_x^{(k+p-1)}} \rho_{v,1} \\
&\qquad + \cdots \\
&\qquad + \frac{g_v^{(k+1)}}{g_x^{(k)} g_x^{(k+1)}} \rho_{v,-1} + \frac{g_v^{(k+2)}}{g_x^{(k)} g_x^{(k+2)}} \rho_{v,1} + \cdots + \frac{g_v^{(k+p-1)}}{g_x^{(k)} g_x^{(k+p-1)}} \rho_{v,1} \\
&\qquad \left. + \frac{g_v^{(k+1)}}{g_x^{(k)} g_x^{(k+1)}} \rho_{v,1} + \frac{g_v^{(k+2)}}{g_x^{(k)} g_x^{(k+2)}} \rho_{v,1} + \cdots + \frac{g_v^{(k+p-1)}}{g_x^{(k)} g_x^{(k+p-1)}} \rho_{v,1} \right).
\end{aligned}$$

For each $j = 1, 2, \cdots, p-1$, we add up the terms with $\frac{g_v^{(k+j)}}{g_x^{(k)} g_x^{(k+j)}}$. These are the vertical columns in the previous equation. Some of the terms have factor $\rho_{v,1}$ and some have $\rho_{v,-1}$. The ν^3 term then becomes

$$\frac{\nu^3(-1)^{p-1}}{(-2)^{p-2}}\left(\prod_{j=0}^{p-1} g_x^{(j)}\right)\sum_{j=1}^{p-1}\frac{g_v^{(k+j)}}{g_x^{(k)}g_x^{(k+j)}}\left(\rho_{v,-1}(p-j)+\rho_{v,1}j\right)$$
$$=\frac{\nu^3}{2^{p-1}}\left(\prod_{j=0}^{p-1} g_x^{(j)}\right)\sum_{j=1}^{p-1}\frac{g_v^{(k+j)}}{g_x^{(k)}g_x^{(k+j)}}\left((1+\beta_v)(p-j)+(1-\beta_v)j\right)$$
$$=\frac{\nu^3}{2^{p-1}}\left(\prod_{j=0}^{p-1} g_x^{(j)}\right)\sum_{j=1}^{p-1}\frac{g_v^{(k+j)}}{g_x^{(k)}g_x^{(k+j)}}\left(p+(p-2j)\beta_v\right).$$

To get the coefficient a_{30} we sum over k,

$$\begin{aligned}
a_{30} &= \frac{1}{2^{p-1}}\left(\prod_{j=0}^{p-1} g_x^{(j)}\right)\sum_{k=0}^{p-1}\sum_{j=1}^{p-1}\frac{g_v^{(k+j)}}{g_x^{(k)}g_x^{(k+j)}}\left(p+\beta_v(p-2j)\right)\\
&= \frac{1}{2^{p-1}}\left(\prod_{j=0}^{p-1} g_x^{(j)}\right)\left(\sum_{k=0}^{p-1}\sum_{j=0}^{p-1}\frac{g_v^{(k+j)}}{g_x^{(k)}g_x^{(k+j)}}\left(p+\beta_v(p-2j)\right)-\sum_{k=0}^{p-1}\frac{g_v^{(k)}}{g_x^{(k)}g_x^{(k)}}\left(p+\beta_v p\right)\right)\\
&= \frac{1}{2^{p-1}}\left(\prod_{j=0}^{p-1} g_x^{(j)}\right)\left(\sum_{k=0}^{p-1}\sum_{m=0}^{p-1}\frac{g_v^{(m)}}{g_x^{(k)}g_x^{(m)}}\left(p+\beta_v(p-2(m-k))\right)\right.\\
&\qquad\left.-p(1+\beta_v)\sum_{k=0}^{p-1}\frac{g_v^{(k)}}{g_x^{(k)}g_x^{(k)}}\right).
\end{aligned}$$

The coefficient a_{30} is the sum over all k,

$$a_{30}=\frac{2}{2^p}\left(\prod_{j=0}^{p-1} g_x^{(j)}\right)\sum_{k=0}^{p-1}\sum_{\substack{j=0\\ j\neq k}}^{p-1}\frac{g_v^{(j)}}{g_x^{(k)}g_x^{(j)}}\left(p+\beta_v(p-2j)\right).$$

□

2.5 Characteristic Polynomial Near 0

The characteristic equation of the system is given in Corollary 2.2.3. Although the parameter $\phi=\frac{2\pi}{q}m$ is discrete, we approximate it as a real parameter. The Taylor expansion of this equation in the variable ϕ, described in Section 2.3, results in an approximation for the roots that lie near the double root at $\nu = 0$. The main result is given in Theorem 2.5.1. See Section 2.4 for a detailed derivation of the coefficients needed in the expansion.

Theorem 2.5.1. *With the assumptions described at the start of Section 2.2 the characteristic polynomial $P_\phi(\nu)$ has a double zero when $\phi = 0$. As $N \to \infty$, the set of zeros near the origin approach two curves γ that pass through the origin. The derivatives of γ are given by*

$$\gamma'_\pm(0) = \frac{i}{p} c_{1,\pm} \tag{105}$$

$$\gamma''_\pm(0) = -\frac{2}{p^2} c_{2,\pm} \ , \tag{106}$$

where the constants $c_{1,\pm}$ and $c_{2,\pm}$ are given by

$$c_{1,\pm} = \frac{-\beta_v \operatorname{Avg}\left(\frac{g_v^{(k)}}{g_x^{(k)}}\right) \pm \sqrt{2 \operatorname{Avg}\left(\frac{1}{g_x^{(k)}}\right) + \beta_v^2 \operatorname{Avg}\left(\frac{g_v^{(k)}}{g_x^{(k)}}\right)^2}}{2 \operatorname{Avg}\left(\frac{1}{g_x^{(k)}}\right)} \tag{107}$$

$$\begin{aligned} c_{2,\pm} = &\pm \big(2FV^3\beta_v^4 + 2AMV^2\beta_v^3 - 2CV^3\beta_v^3 + 3FMV\beta_v^2 \\ &\quad - M^2V^2\beta_v + 2AM^2\beta_v - 3MVC\beta_v\big) / 4M^3\Gamma_R \\ &+ \big(-2FV^2\beta_v^3 - 2MVA\beta_v^2 + 2V^2C\beta_v^2 - FM\beta_v \\ &\quad + M^2V + CM\big) / 4M^3, \end{aligned} \tag{108}$$

and where we've defined the following,

$$M = \mathrm{Avg}\left(\frac{1}{g_x^{(k)}}\right) \qquad V = \mathrm{Avg}\left(\frac{g_v^{(k)}}{g_x^{(k)}}\right) \tag{109}$$

$$R = \mathrm{Avg}\left(\frac{g_v^{(k)}}{g_x^{(k)} g_x^{(k)}}\right) \qquad S = \mathrm{Avg}\left(\frac{g_v^{(k)} g_v^{(k)}}{g_x^{(k)} g_x^{(k)}}\right) \tag{110}$$

$$F = \frac{1}{p^2}\sum_{k=0}^{p-1}\sum_{j=1}^{p-1}\frac{g_v^{(k+j)}}{g_x^{(k)} g_x^{(k+j)}}(p-2j) \tag{111}$$

$$C = R - MV \qquad A = S - V^2. \tag{112}$$

This means that the zero loci, near the double root, is approximated by the two curves (see equation (82)*)*

$$\begin{aligned} \gamma_+(\phi) &= ic_{1,+}\left(\frac{\phi}{p}\right) - c_{2,+}\left(\frac{\phi}{p}\right)^2 + \cdots \\ \gamma_-(\phi) &= ic_{1,-}\left(\frac{\phi}{p}\right) - c_{2,-}\left(\frac{\phi}{p}\right)^2 + \cdots . \end{aligned} \tag{113}$$

These are parabolic approximations to the two continuous eigenvalue loci emanating from the origin.

Proof. To find $\gamma'_\pm(0)$ we start with equation (85). This equation has the $\pm$ but we leave this off and identify the $\pm$ curves at the end of the proof. Use Proposition 2.4.2

equation (102) and Proposition 2.4.1 equations (97) and (95). We get

$$\begin{aligned}
\gamma'_\pm(0) =& \frac{-a_{11} \pm \sqrt{a_{11}^2 - 4a_{02}a_{20})}}{2a_{20}},\\
=& \frac{-\frac{2i}{2^p}\left(\prod_{j=0}^{p-1} g_x^{(j)}\right)\left(\sum_{k=0}^{p-1} \frac{g_v^{(k)}}{g_x^{(k)}}\right)\beta_v}{\frac{4p^2}{2^p}\left(\prod_{j=0}^{p-1} g_x^{(j)}\right)\operatorname{Avg}\left(\frac{1}{g_x^{(k)}}\right)}\\
&\pm \frac{i\frac{\left(\prod_{j=0}^{p-1} g_x^{(j)}\right)}{2^p}\sqrt{4\beta_v^2 p^2 \operatorname{Avg}\left(\frac{g_v^{(k)}}{g_x^{(k)}}\right)^2 + 8p^2 \operatorname{Avg}\left(\frac{1}{g_x^{(k)}}\right)}}{\frac{4p^2}{2^p}\left(\prod_{j=0}^{p-1} g_x^{(j)}\right)\operatorname{Avg}\left(\frac{1}{g_x^{(k)}}\right)}\\
=& i\frac{-\operatorname{Avg}\left(\frac{g_v^{(k)}}{g_x^{(k)}}\right)\beta_v \pm \sqrt{2\operatorname{Avg}\left(\frac{1}{g_x^{(k)}}\right) + \beta_v^2 \operatorname{Avg}\left(\frac{g_v^{(k)}}{g_x^{(k)}}\right)^2}}{2p\operatorname{Avg}\left(\frac{1}{g_x^{(k)}}\right)}.
\end{aligned}$$

Equation (107) follows.

To find $\gamma''(0)$, start with Theorem 2.3.1. Set

$$\gamma''(0) = -2\frac{\tilde{N}_p}{\tilde{D}_p},$$

where

$$\begin{aligned}
\tilde{D}_p &= 2a_{20}\gamma'(0) + a_{11} = \left(-a_{11} \pm \sqrt{a_{11}^2 - 4a_{02}a_{20}}\right) + a_{11}\\
&= \pm\sqrt{a_{11}^2 - 4a_{02}a_{20}}\\
&= \pm\frac{2ip}{2^p}\left(\prod_{j=0}^{p-1} g_x^{(j)}\right)\sqrt{2\operatorname{Avg}\left(\frac{1}{g_x^{(k)}}\right) + \beta_v^2 \operatorname{Avg}\left(\frac{g_v^{(k)}}{g_x^{(k)}}\right)^2}\\
&= \pm 2ip\mathfrak{P}_x\sqrt{2\operatorname{Avg}\left(\frac{1}{g_x^{(k)}}\right) + \beta_v^2 \operatorname{Avg}\left(\frac{g_v^{(k)}}{g_x^{(k)}}\right)^2},
\end{aligned}$$

where we have defined the constant

$$\mathfrak{P}_x = \frac{1}{2^p}\left(\prod_{j=0}^{p-1} g_x^{(j)}\right). \tag{114}$$

Except for the sign, this is the product of the forward coupling constants.

The numerator is given by

$$\tilde{N}_p = \big(a_{30}(\gamma'(0))^3 + a_{21}(\gamma'(0))^2 + a_{12}\gamma'(0) + a_{03}\big) \tag{115}$$

The required a_{ij} are the following,

$$a_{30} = \mathfrak{P}_x p^2 \left(2pMV - 2R + 2\beta_v F\right)$$
$$a_{21} = \mathfrak{P}_x ip \left(2pV^2 - 2S\right)\beta_v$$
$$a_{12} = \mathfrak{P}_x pV$$
$$a_{03} = 0.$$

Each of these coefficients has a factor of $\mathfrak{P}_x$. which will cancel the factor appearing in $\tilde{D}_p$. In equation (115) there are factors of $\gamma'(0)^k$. Set

$$\gamma'(0) = \frac{i\Gamma_N}{2Mp}, \tag{116}$$

where

$$\Gamma_N = -\beta_v V + s\sqrt{2M + \beta_v^2 V^2}. \tag{117}$$

The numerator has the form

$$\tilde{N}_p = \left(a_{30}(\gamma'(0))^3 + a_{21}(\gamma'(0))^2 + a_{12}\gamma'(0) + a_{03}\right)$$
$$= \frac{i\mathfrak{P}_x}{p(2M)^3}\left(-(2pMV - 2R + 2F\beta_v)\Gamma_N^3 \right.$$
$$\left. -2M(2pV^2 - 2V_2)\beta_v\Gamma_N^2 + (2M)^2(pV)\Gamma_N\right)$$

The factor $i\mathfrak{P}_x$ cancels in numerator and denominator. We move the $(2M)^3$ to the denominator. We re-write the $\gamma''(0)$ as

$$\gamma''(0) = \frac{-N_p}{D_p},$$

where

$$N_p = (-2pMV + 2R - 2F\beta_v)\Gamma_N^3 + 2M(-2pV^2 + 2V_2)\beta_v\Gamma_N^2 + (2M)^2(pV)\Gamma_N \tag{118}$$

$$D_p = \pm\, 2(2M)^3p^2\sqrt{2M + \beta^2V^2}. \tag{119}$$

With these definitions we have

$$\gamma''(0) = -2\frac{N_p}{D_p}. \tag{120}$$

We insert the expression of equation (118) into SAGE to perform the manipulation (see PolyNumerExpand.sage listing). The result is

```
Nexp =
  8*F*V^3*betav^4 + 8*M*S*V^2*betav^3
- 8*R*V^3*betav^3 + 12*F*M*V*betav^2
```

```
+ 8*M^2*S*betav - 12*M*R*V*betav
- 8*sqrt(V^2*betav^2 + 2*M)*F*V^2*betav^3
- 8*sqrt(V^2*betav^2 + 2*M)*M*S*V*betav^2
+ 8*sqrt(V^2*betav^2 + 2*M)*R*V^2*betav^2
- 4*sqrt(V^2*betav^2 + 2*M)*F*M*betav
+ 4*sqrt(V^2*betav^2 + 2*M)*M*R
```

There is an implicit $\pm$ in front of every sqrt statement. Denote the radical portion of $\gamma'(0)$ by Γ_R so that

$$\Gamma_R = \sqrt{2M + V^2\beta_v^2}. \tag{121}$$

Assemble the results. There is a factor of 4 and another factor of 2 that cancel from the numerator and denominator. We get

$$\begin{aligned}\gamma''(0) =& -2\frac{N_p}{D_p} \\ =& -\frac{1}{p^2 2M^3\Gamma_R}\left(\pm\left(2FV^3\beta_v^4 + 2MV^2S\beta_v^3 - 2V^3R\beta_v^3\right.\right. \\ &\left. +3FMV\beta_v^2 + 2M^2S\beta_v - 3MVR\beta_v\right) \\ &\left.+\left(-2FV^2\beta_v^3 - 2MVS\beta_v^2 + 2V^2R\beta_v^2 - FM\beta_v + MR\right)\Gamma_R\right)\end{aligned}$$

We have

$$\begin{aligned}c_{2,\pm} =& \frac{1}{4M^3\Gamma_R}\left(\pm\left(2FV^3\beta_v^4 + 2MV^2S\beta_v^3 - 2V^3R\beta_v^3\right.\right. \\ &\left. +3FMV\beta_v^2 + 2M^2S\beta_v - 3MVR\beta_v\right) \\ &\left.+\left(-2FV^2\beta_v^3 - 2MVS\beta_v^2 + 2V^2R\beta_v^2 - FM\beta_v + MR\right)\Gamma_R\right).\end{aligned} \tag{122}$$

The solution $c_{2,+}$ corresponds to $c_{1,+}$ and $c_{2,-}$ corresponds to $c_{1,-}$, so there are two parabolic approximations at the origin.

Finally, we can replace R and S with the variables C and A to get,

$$
\begin{aligned}
c_{2,\pm} =& \frac{1}{4M^3\Gamma_R}\big(\pm\big(2FV^3\beta_v^4 + 2AMV^2\beta_v^3 - 2CV^3\beta_v^3 \\
&\qquad\qquad +3FMV\beta_v^2 - M^2V^2\beta_v + 2AM^2\beta_v - 3MVC\beta_v\big) \\
&+ \big(-2FV^2\beta_v^3 - 2MVA\beta_v^2 + 2V^2C\beta_v^2 - FM\beta_v + M^2V + CM\big)\Gamma_R\big).
\end{aligned}
$$

$\square$

Remark 2.5.2. *In the theorem, we have averages over k and averages over α. Because of the system setup, these two averages are the same. The average over α is the average over the p agent types and the average over k is the average over all $N = pq$ agents. However, this is just a collection of q copies of the p agents and so these two averages are the same. This means that*

$$
\operatorname{Avg}\left(\frac{1}{g_x^{(k)}}\right) = \operatorname{Avg}\left(\frac{1}{g_x^{(\alpha)}}\right)
$$

$$\vdots$$

Only F must be taken over the smaller population of p elements so that the filter can be applied correctly.

Remark 2.5.3. *The value $c_{1,+}$ is the velocity in the positive x-axis direction. In the second Cantos paper [5] the value $c_{1,+}$ is in the direction of increasing agents which is the negative of the meaning in this work.*

Corollary 2.5.4. *Assume we have the system described in Theorem 2.5.1. If $\beta_v = 0$*

then we have

$$c_{1,\pm} = \pm\sqrt{\frac{1}{2\,\mathrm{Avg}\left(\frac{1}{g_x^{(k)}}\right)}} \tag{123}$$

$$c_{2,\pm} = \pm\frac{M^2V + CM}{4M^3} = \pm\frac{\mathrm{Avg}\left(\frac{g_v^{(k)}}{g_x^{(k)}g_x^{(k)}}\right)}{\left(2\,\mathrm{Avg}\left(\frac{1}{g_x^{(k)}}\right)\right)^2}. \tag{124}$$

Corollary 2.5.4 is the main result in [20].

2.5.1 Random Variables

The averages in equation (109) are the expectation values of the two discrete random variables

$$\mathbb{M} = \frac{1}{g_x^{(\alpha)}} \tag{125}$$

$$\mathbb{V} = \frac{g_v^{(\alpha)}}{g_x^{(\alpha)}}, \tag{126}$$

where $\alpha = 0, 1, \cdots p-1$. In terms of these two random variables we have

$$M = E\left[\mathbb{M}\right] \qquad V = E\left[\mathbb{V}\right]$$

$$R = E\left[\mathbb{M}\mathbb{V}\right] \qquad S = E\left[\mathbb{V}^2\right],$$

where $E\left[.\right]$ denotes the expectation value. More natural than R is the covariance,

$$C = E\left[\left(\mathbb{M} - E[\mathbb{M}]\right)\left(\mathbb{V} - E[\mathbb{V}]\right)\right] = R - MV. \tag{127}$$

More natural than S is the variance of $\mathbb{V}$,

$$A = E\left[(\mathbb{V} - E[\mathbb{V}])^2\right] = S - V^2. \tag{128}$$

2.5.2 Other Eigenvalues

The expansion of Theorem 2.5.1 assumes a smooth eigenvalue locus around the origin. This approximation means that N must be large. These are not the only eigenvalues of the system. Theorem 2.2.1 outlines the method to find all the eigenvalues. For each $m = 0, \cdots, q-1$, we set $\phi = \frac{2\pi}{q}m$ and then construct $\mathbf{M}_\phi(\nu)$. The determinant of this has $2p$ roots which are the $2p$ eigenvalues corresponding to this m.

By Proposition 2.2.4, if $\phi = 0$ then $\mathbf{M}_0(\nu)$ has $2p$ roots and a double root at the origin. Figure 6 contains a plot of the $2p = 8$ roots of $\mathbf{M}_0(\nu)$ indicated by the large X. To count the 8 roots, you must count the double root at the origin twice. As we vary m, and so ϕ, the roots move along a smooth curve. The result is a series of eigenvalue loci that emanate from the 8 eigenvalues of $\mathbf{M}_0(\nu)$. The two curves emanating from the origin are the two curves that we approximate in this section. One of the curves corresponds to the two parameters $c_{1,+}, c_{2,+}$ and the second curve corresponds to the two parameters $c_{1,-}, c_{2,-}$. Both of these curves go through the origin.

To estimate the long-term behavior of the system, we can ignore eigenvalues with large negative real part. The eigenvectors with these eigenvalues will decay rapidly so they have little impact on large time results. The curves emanating from the origin are the most important determinant for large time behavior.

2.5.3 Immediate Consequences

The formulations for $c_{1,\pm}$ and $c_{2,\pm}$ lead to a few immediate consequences.

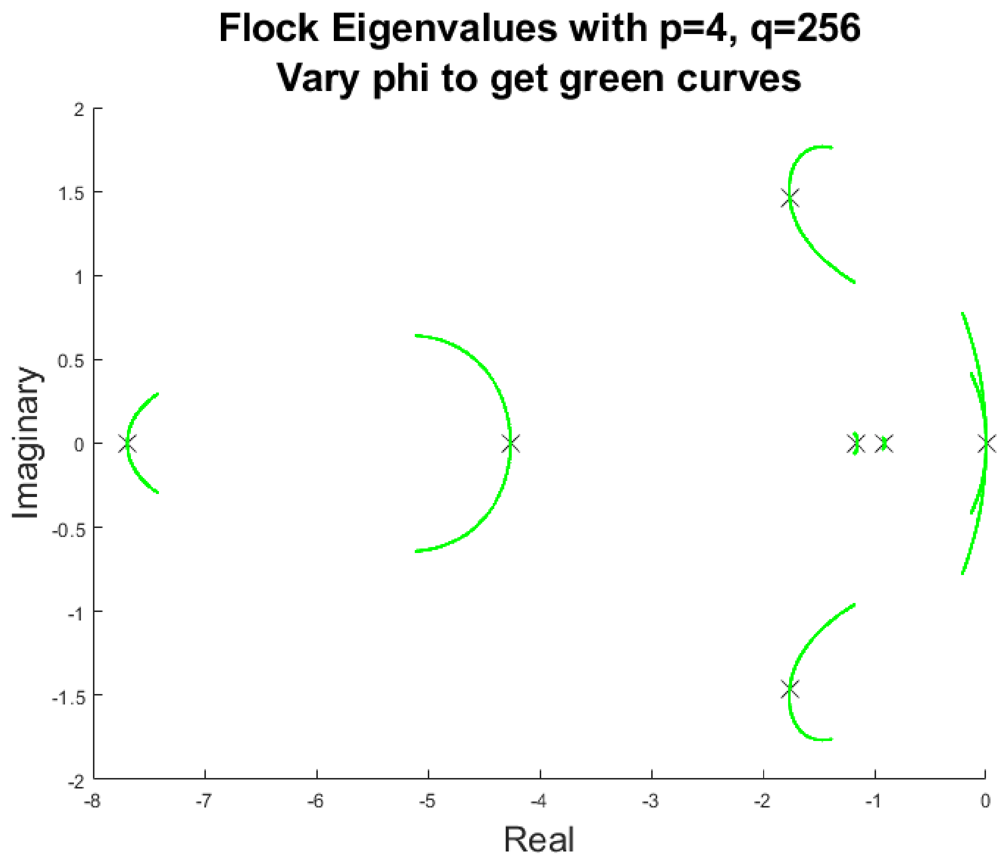


Figure 6: *Plot showing eigenvalues of flock with $p = 4$. The matrix $\mathbf{M}_0(\nu)$ has $2p = 8$ eigenvalues shown by the X's. The green lines are the eigenvalues for $\mathbf{M}_\phi$ as ϕ changes. The plots only contain half of the allowable values, so the curves are clear. Notice there are two curves of different curvatures emanating from the origin.*

Corollary 2.5.5. *For the values $c_{1,+}$ and $c_{1,-}$ in Theorem 2.5.1 we have*

$$c_{1,+} \geq 0$$

$$c_{1,-} \leq 0.$$

Proof. We know that $2\,\mathrm{Avg}\left(\frac{1}{g_x^{(\alpha)}}\right) > 0$. This means that

$$\left|\beta_v \,\mathrm{Avg}\left(\frac{1}{g_x^{(\alpha)}}\right)\right| < \sqrt{2\,\mathrm{Avg}\left(\frac{1}{g_x^{(\alpha)}}\right) + \beta_v^2 \,\mathrm{Avg}\left(\frac{g_v^{(\alpha)}}{g_x^{(\alpha)}}\right)^2}.$$

The results follow. $\square$

Corollary 2.5.6. *For the values $c_{1,+}$ and $c_{1,-}$ in Theorem 2.5.1 we have*

$$\beta_v = 0 \Rightarrow |c_{1,+}| = |c_{1,-}|$$

$$\beta_v > 0 \Rightarrow |c_{1,+}| < |c_{1,-}|$$

$$\beta_v < 0 \Rightarrow |c_{1,+}| > |c_{1,-}|.$$

Proof. As we stated in Corollary 2.5.5,

$$\left|\beta_v \operatorname{Avg}\left(\frac{1}{g_x^{(\alpha)}}\right)\right| < \sqrt{2 \operatorname{Avg}\left(\frac{1}{g_x^{(\alpha)}}\right) + \beta_v^2 \operatorname{Avg}\left(\frac{g_v^{(\alpha)}}{g_x^{(\alpha)}}\right)^2}.$$

The results follow. $\square$

Corollary 2.5.7. *Denote the β_v dependence by $c_{1,\pm}(\beta_v)$. For the values $c_{1,+}$ and $c_{1,-}$ in Theorem 2.5.1 we have*

$$c_{1,+}(-\beta_v) = -c_{1,-}(\beta_v).$$

This means the two solutions of $c_{1,+}, c_{1,-}$ at β_v are the negatives of the solutions $c_{1,-}, c_{1,+}$ at $-\beta_v$.

Proof.

$$
\begin{aligned}
c_{1,+}(-\beta) &= \frac{\beta_v \operatorname{Avg}\left(\frac{g_v^{(k)}}{g_x^{(k)}}\right) + \sqrt{2\operatorname{Avg}\left(\frac{1}{g_x^{(k)}}\right) + \beta_v^2 \operatorname{Avg}\left(\frac{g_v^{(k)}}{g_x^{(k)}}\right)^2}}{2\operatorname{Avg}\left(\frac{1}{g_x^{(k)}}\right)} \\
&= -\left(\frac{-\beta_v \operatorname{Avg}\left(\frac{g_v^{(k)}}{g_x^{(k)}}\right) - \sqrt{2\operatorname{Avg}\left(\frac{1}{g_x^{(k)}}\right) + \beta_v^2 \operatorname{Avg}\left(\frac{g_v^{(k)}}{g_x^{(k)}}\right)^2}}{2\operatorname{Avg}\left(\frac{1}{g_x^{(k)}}\right)}\right) \\
&= -c_{1,-}(\beta_v).
\end{aligned}
$$

□

Similar statements for the values $c_{2,\pm}$ require an additional condition.

Corollary 2.5.8. *Denote the β_v dependence by $c_{2,\pm}(\beta_v)$. If $\beta_v F = 0$ then we have,*

$$
c_{2,+}(-\beta_v) = c_{2,-}(\beta_v).
$$

Proof. The denominator of both $c_{2,+}$ and $c_{2,-}$ are independent of β_v and F. Set $\beta_v F = 0$ and compute,

$$
\begin{aligned}
c_{2,+}(-\beta_v) =& \frac{1}{4M^3\Gamma_R}\Big(\big(2AMV^2(-\beta_v)^3 - 2CV^3(-\beta_v)^3 \\
&\qquad -M^2V^2(-\beta_v) + 2AM^2(-\beta_v) - 3MVC(-\beta_v)\big) \\
&+ \big(-2MVA(-\beta_v)^2 + 2V^2C(-\beta_v)^2 + M^2V + CM\big)\Gamma_R\Big) \\
=& \, c_{2,-}(\beta_v).
\end{aligned}
$$

□

Remark 2.5.9. *For every β_v there are two distinct second order eigenvalue approximations given by $c_{1,+}(\beta_v), c_{2,+}(\beta_v)$ and $c_{1,-}(\beta_v), c_{2,-}(\beta_v)$. These two second order* *approximations are shown in Figure 6. The Corollaries above show that when F is small, the two curves for β_v are the same two curves for $-\beta_v$ except that the curve determined by $c_{1,+}(\beta_v), c_{2,+}(\beta_v)$ is $c_{1,-}(-\beta_v), c_{2,-}(-\beta_v)$.*

2.5.4 Conclusions

Theorem 2.5.1 gives a prediction for the eigenvalues near the origin when the matrix M has periodic boundary conditions. We will see, in Section 2.8, that this requirement is subtle and somewhat problematic. However, despite shortcomings, the theorem is quite useful for predicting several features of large-scale flock behavior.

Theorem 2.5.1 was proved by chaining q sub-sequences together to get a sequence of length $N = pq$. What is surprising is that Theorem 2.5.1 works well even when $q = 1$. In simulations, Theorem 2.5.1 seems to work well even when the p agents follow some pattern, like a ramp. This seems contradictory, but one can understand this heuristically. If the p agents are chosen at random, then the dynamics of the p random agents duplicated q times should be close to the dynamics of $N = pq$ random agents. Indeed, this issue seems less important than the issue of boundary conditions. We shall discuss these issues further in Section 2.8.

2.6 Solutions

We can use Theorem 2.5.1 to construct some practical systems. Let's review our assumptions.

Assumptions 2.6.1. *The linear nearest neighbor system described in Section 2.2 is called realistic $x-$symmetric if it satisfies the following axioms:*

1. $\rho_{x,+}^{(\alpha)} = \rho_{x,-}^{(\alpha)} = -\frac{1}{2}$, *for all* α,
2. $\rho_{v,+}^{(\alpha)}, \rho_{v,-}^{(\alpha)} \leq 0$ *and both quantities are independent of* α,

3. $g_x^{\alpha)} > 0$ *for all* α,

4. $g_v^{\alpha)} > 0$ *for all* α,

5. The system is stable and all of the eigenvalues that are close to the imaginary axis are close to the locus described in Theorem 2.5.1.

The values $g_x^{\alpha)}$ and $g_v^{\alpha)}$ might be different for each of the p distinct agents. Assumption 2.6.1 item 1 is required for stability, as seen in Theorem 2.7.1. Assumption 2.6.1 items 2, 3 and 4 are related to stability, although we've seen in Sections 1.2.1 and 1.3, that they are not necessary for stability.

Assumption 2.6.1 item 5 is the most problematic. We must assume that all eigenvalues other than those described by Theorem 2.5.1 have large negative real parts, so that they decay quickly. We can construct unstable systems that violate this assumption, but these systems are of little practical use. Assumption 2.6.1 item 5 is further justified by the simulations in Section 2.8. With this assumption the dispersion relation becomes tractable.

To put together a solution with initial conditions that satisfies the assumptions 2.6.1, we assemble a solution out of the eigenvectors listed in Theorem 2.2.1. Start by setting m to an integer in the range $0, \cdots, q-1$. With this m, construct the matrix $M_\phi(\nu)$ of equation (77), where $\phi = \frac{2\pi}{q}m$. For each m, we solve for the $2p$ roots of $\det(M_\phi(\nu))$. For each root ν_{mn}, we construct an element of the kernel of $M_\phi(\nu)$, which we denote by $e_{m,n}$.

The result is an eigenvector of the form

$$u(m,n) = \begin{pmatrix} v_m \otimes e_{m,n} \\ \nu_{mn}\left(v_m \otimes e_{m,n}\right) \end{pmatrix},$$

where ν_{mn} is the eigenvalue corresponding to the eigenvector $u(m,n)$ and v_m is an N-vector with coordinates ω_q^{mj} (see Section C.4). We are looking for a solution of the form

$$z_k(t) = \sum_{m=0}^{q-1} \sum_{n=0}^{2p-1} a_{mn} \exp\left(t\nu_{mn}\right) \left(v_m \otimes e_{m,n}\right)_k,$$

where a_{mn} are real numbers that depend on the initial conditions. If the system is known at $t = 0$ then the initial conditions are given by

$$z_k(0) = \sum_{m=0}^{q-1} \sum_{n=0}^{2p-1} a_{mn}(v_m \otimes e_{m,n}), \tag{129}$$

$$\dot{z}_k(0) = \sum_{m=0}^{q-1} \sum_{n=0}^{2p-1} a_{mn}\nu_{mn}(v_m \otimes e_{m,n}). \tag{130}$$

Equation (130) is the bottom half of the eigenvector. These two equations give $2pq$ equations to determine the $N = 2pq$ coefficients a_{mn}.

The term corresponding to a fixed m and n is

$$a_{mn} \exp\left(t\nu_{mn}\right) \left(v_m \otimes e_{m,n}\right)_k = a_{mn} \begin{pmatrix} e_{m,n} \\ \omega_q^m e_{m,n} \\ \omega_q^{2m} e_{m,n} \\ \vdots \\ \omega_q^{(q-1)m} e_{m,n} \end{pmatrix} \exp\left(t\nu_{mn}\right).$$

With Assumptions 2.6.1 we can use the eigenvalue formula of Theorem 2.5.1. The states corresponding to these eigenvalues have the form,

$$\begin{pmatrix} e^0 \\ \vdots \\ e^{p-1} \\ \omega_q^m e^0 \\ \vdots \\ \omega_q^m e^{p-1} \\ \vdots \\ \omega_q^{(q-1)m} e^0 \\ \vdots \\ \omega_q^{(q-1)m} e^{p-1} \end{pmatrix} \exp\left(itc_1\left(\frac{\phi}{p}\right)\right)\exp\left(-tc_2\left(\frac{\phi}{p}\right)^2\right) \tag{131}$$

If we ignore the $c_{2,\pm}$ damping term, the flock pattern given by this eigenvector advances $(-p)$ agents (e.g., p agents to the left) in time δ, where δ satisfies,

$$\begin{aligned} e^k\omega_q^{nm}\exp\left(i(t_0+\delta)c_1\left(\frac{\phi}{p}\right)\right) &= \omega_q^{(n-1)m}e^k\exp\left(it_0c_1\left(\frac{\phi}{p}\right)\right) \\ \Rightarrow \exp\left(i\delta c_1\left(\frac{\phi}{p}\right)\right) &= \omega_q^{-m} = \exp\left(-\frac{2\pi i}{q}m\right) \\ \Rightarrow \exp\left(\frac{2\pi i}{N}c_1\delta m\right) &= \exp\left(-\frac{2\pi i}{N}pm\right) \end{aligned}$$

when k and n are fixed. So if we ignore the $c_{2,\pm}$ term, the flock pattern repeats at a time δ when the following holds for some integer j.

$$\begin{aligned} (-c_1)\delta m &= pm + jN \\ \Rightarrow \delta &= \frac{p}{-c_1} + \frac{jN}{c_1 m}. \end{aligned}$$

The repeat time in this direction has $c_1 < 0$ and the pattern first repeats when $j = 0$ so that

$$\delta_1 = \frac{p}{-c_1} = \frac{p}{|c_1|}.$$

During this interval the pattern has moved $-p$ agents. The phase velocity, in agents per unit time, is given by,

$$|c_1| = \frac{p}{\delta_1}. \tag{132}$$

This is the phase velocity for this eigenstate. All the eigenvectors approximated by Theorem 2.5.1 are determined by the first two terms of the two curves in equation (113). This means that all the eigenstates have phase velocity either $c_{1,+}$ or $c_{1,-}$. By Corollary 2.5.5 the phase velocities point in opposite directions.

We would normally expect the term

$$u(m,n)\exp\left(itc_1\frac{\phi}{p}\right) = u(m,n)\exp\left(\frac{2\pi i}{N}mc_1t\right)$$

to repeat whenever $t \to t + \delta$ where $\delta = \frac{N}{c_1}$. But the flock eigenvector in equation (131) repeats far more frequently. To understand this, recall that a pure wave has a factor

$$\exp\left(i(kx - \omega t)\right) = \exp\left(2\pi i(\frac{x}{\lambda} - ft)\right).$$

A system with no dispersion has $f\lambda = c_1$ for all wavelengths (e.g., k and ω are linearly related). If we ignore the c_2 term, our system satisfies this for multiples of a wavelength equal to p agents. We assume that p is small relative to N and that we can assume this relationship holds for all multiples of a wavelength of 1 agent. We add this as an assumption.

Assumptions 2.6.2. *We assume that the details of flock behavior in small times can* *be ignored and the phase velocity of the flock for the eigenvector $u(m,n)$ is c_1.*

If the flock contains all identical agents, then you can show that this approximation is valid and you can get an explicit bound on the error [5]. In our case, we must add this as an assumption.

Using this, we expect that for large times, all flock eigenvectors have phase velocity $c_{1,+}$ or $c_{1,-}$. This means that the trajectory for agent k will have the form

$$z_k(t) = f_+\left(t + \frac{k}{c_{1,+}}\right) + f_-\left(t + \frac{k}{c_{1,-}}\right). \tag{133}$$

In this expression, the velocity $c_{1,+}$ moves towards the positive $x-$axis, which is the negative agent number, and the velocity $c_{1,-}$ moves in the negative $x-$axis, which is the positive index number. To this equation, we also have to add the damping term controlled by $c_{2,\pm}$.

Another feature of our system is that there are two distinct phase velocities that are opposite in direction but have different magnitudes.

Remark 2.6.3. *In short times the wave can change shape, but it re-assembles itself after moving through exactly p agents so that it looks like the original packet. The only change in the waveform is the attenuation of the damping term $c_{2,\pm}$.*

2.6.1 Boundary Conditions

In this section we describe a typical boundary condition and derive some additional conditions. In Section 2.2 we placed agent 0 at the origin and spaced the other agents sequentially along the negative axis so that agent $N-1$ had the most negative coordinate. We consider a system that starts at rest at $t < -\epsilon$. As $t > -\epsilon$, agent 0 accelerates until it reaches a velocity v_0. The other agents follow agent 0 as it moves along the positive x-axis at velocity v_0. To define precise boundary conditions, we start with a simple definition.

Definition 2.6.4. *A smooth velocity ramp is a function $\psi_\epsilon(t)$ that has the following properties,*

- ψ_ϵ *is smooth,*
- $\psi_\epsilon(t) = 0$ *when* $t < -\epsilon$,
- $\frac{d\psi_\epsilon}{dt} = 1$ *when* $t > \epsilon$.

The function ψ_ϵ models a smooth transition from zero velocity to v_0. Now we proceed to describe the boundary conditions.

Definition 2.6.5. *A one-dimensional array of agents satisfies the regular boundary condition if the tail agent is disconnected from agent* 0 *and* z_{N-1} *satisfies*

$$\frac{d^2 z_{N-1}}{dt^2} = -g_x^{(p-1)}\gamma_x \left(z_{N-1} - z_{N-2}\right) - g_v^{(p-1)}\gamma_v \left(\dot{z}_{N-1} - \dot{z}_{N-2}\right),$$

where γ_x *and* γ_v *satisfy*

$$\gamma_x = \gamma_v = 1.$$

In the simulations we often use the following definition.

Definition 2.6.6. *A one-dimensional flock satisfies the constant velocity boundary conditions if it satisfies the 'regular boundary conditions' of Definition 2.6.5 and, agent* 0 *moves to the right according to*

$$x_0(t) = z_0(t) = \psi_\epsilon(t) v_0$$

where v_0 *is a constant.*

One method to solve for this boundary condition is to solve for the agents,

$$y_k(t) = \frac{d^2 z_k}{dt^2}.$$

Because the ODE is linear, we know that $y_k(t)$ satisfies the same ODE as the $z_k(t)$, but it has different boundary conditions. In particular, the value $y_0(t)$ satisfies

$$y_0(t) = \frac{d^2\psi_\epsilon}{dt^2} = p(t)v_0. \tag{134}$$

The properties of $p(t)$ are summarized in the following Proposition.

Proposition 2.6.7. *If* $p(t) = \frac{d^2\psi_\epsilon}{dt^2}$ *then we have*

- p *is smooth,*
- $p(t) = 0$ *when* $t < -\epsilon$*,*
- $p(t) = 0$ *when* $t > \epsilon$*,*
- $\int\limits_{-\infty}^{\infty} p(s)ds = 1$*.*

Proof. These follow immediately from Definition 2.6.4. □

If the system is stable, then for the conditions at the tail, we shall use 'open tail' boundary condition that is defined as follows.

Definition 2.6.8. *An agent sequence* $\{y_k\}$ *satisfies open tail conditions if,*

$$\left.\frac{\partial y_k}{\partial k}\right|_{k=N-1} = 0.$$

See Remark 2.6.11 for the acoustic analog to the open tail condition defined in Definition 2.6.8.

By solving for the trajectory $y_N(t)$ one can integrate twice to get $z_{N-1}(t)$. We characterize the flock by measuring the distance $d(t) = z_0(t) - z_{N-1}(t)$. As agent 0 moves to the right the distance $d(t)$ grows. It reaches its first maximum at T_1 and the maximum is given by $A_1 = d(T_1)$. At T_2 the distance $d(t)$ reaches a local minimum given by $A_2 = d(T_2)$. We proceed through k_0 extremal points.

Using a heuristic argument we can prove the following Theorem which we use to predict some large scale flock behavior. This behavior is tested in the simulations of Section 2.8.

Theorem 2.6.9. *Assume that the system $z_k(t)$ is stable and satisfies the 'regular' boundary conditions and that also satisfy the "constant velocity boundary conditions" of Definition 2.6.6. Also assume that*

$$z_k(t) = f_+\left(t + \frac{k}{c_{1,+}}\right) + f_-\left(t + \frac{k}{c_{1,-}}\right).$$

For large N there is a fixed K_0 so that the trajectory of the last agent $z_N(t)$ can be approximated by the following properties.

- $T_k = \frac{(N-1)}{-c_{1,-}} + (k-1)\left(\frac{(N-1)}{c_{1,+}} - \frac{(N-1)}{c_{1,-}}\right)$ *for* $k = 1, \cdots K_0$,
- $A_k = \left(\frac{c_{1,+}}{c_{1,-}}\right)^{k-1} \frac{(N-1)v_0}{c_{1,-}}$ *for* $k = 1, \cdots K_0$,

where T_k is the time to the kth extremal distance $z_0 - z_N$ and A_k is the difference $z_{N-1}(T_k) - z_0(T_k)$. When converting back to x_k this will be the distance from the stable point.

Proof. See Appendix F for further discussion. The proof follows the lines in Cantos [5]. □

Remark 2.6.10. *Theorem 2.6.9 gives a prediction of system stability for systems with the constant velocity boundary conditions. The formula for A_k implies that*

$$\left|\frac{A_{k+1}}{A_k}\right| = \left|\frac{c_{1,+}}{c_{1,-}}\right|.$$

If the ratio is less than 1 *we expect a stable system and if the ratio is greater than* 1 *then we expect an unstable system.*

The utility of predicting stability using Theorem 2.6.9 and Remark 2.6.10 will become clear in the simulations in Section 2.8. See, in particular, Figure 17.

Remark 2.6.11. *The equations of motion for longitudinal sound waves is a PDE system that is analogous to our system. This PDE system is described in Appendices G and H. In acoustics, standing waves in an organ pipe are derived assuming certain boundary conditions. If $D(x,t)$ is the displacement of a disk of air in the pipe oriented along the x-axis, then*

$$P_s = \Delta P = -B\frac{\partial D}{\partial x}$$

where $P(x,t)$ is the pressure. This is described in detail in equation (188). *At the closed end of the pipe the displacement is zero, so $D(x_c,t) = 0$. At the open end of the pipe the change in pressure vanishes so we have*

$$\left.\frac{\partial D}{\partial x}\right|_{x_o} = 0.$$

The 'open' boundary conditions of Definition 2.6.8 are the same as the open end of an organ pipe.

The constant velocity boundary conditions of Definition 2.6.6 state that the velocity of agent 0 *does not change. In the acoustics analog this condition means the displacement velocity does not change. The only way this can happen in the acoustics* *case is when $D = 0$. The acoustic wave has a preferred reference frame of the ambient air and the equations of motion determine the motion of the over-pressure P_s and the displacement D. In the agent case the entire caravan can move at a constant velocity and still obey the equations of motion.*

2.7 Stability

From this we can prove a necessary condition for stability.

Theorem 2.7.1. *If, for a general (linear) nearest neighbor system,*

$$\prod_i \rho_{x,1}^{(i)} - \prod_i \rho_{x,-1}^{(i)} \neq 0$$

the system is unstable in one sense or another.

Proof. By [3] (specifically, see Appendix in [3]), the constant term of $\left.\frac{dP_\phi}{d\phi}\right|_{\phi=0}$ must vanish. By Corollary 2.2.7 the derivative of the constant term is,

$$(-1)^{p+1} i \left(\rho_{x,1}^{(0)} \rho_{x,1}^{(1)}(\nu) \cdots \rho_{x,1}^{(p-1)} - \rho_{x,-1}^{(0)} \rho_{x,-1}^{(1)} \cdots \rho_{x,-1}^{(p-1)} \right)$$

The $\rho_{x,1}^{(\alpha)}, \rho_{x,-1}^{(\alpha)}$ are all real so the theorem follows. □

In this general case it is difficult to come up with sufficient conditions for stability. If we simplify the problem a bit there is more that can be said.

One nice feature of the symmetric case is that one can prove stability in a restrictive sense using the following fact.

Proposition 2.7.2. *If* $\mathbf{M}$ *is a diagonalizable matrix with eigenvalues in the left half complex plane and* $\mathbf{G}$ *is a positive definite matrix then* $\mathbf{GM}$ *has eigenvalues in the left half complex plane.*

Proof. Variants of this Proposition are known. We include the proof for completeness. If $\mathbf{G}$ is a positive definite matrix then there is a non-singular square root $\mathbf{G}^{1/2}$. For any vector X there is a $\mathbf{Y}$ with $\mathbf{X} = \mathbf{G}^{1/2}Y$. So, for any $\mathbf{X}$ we have,

$$\begin{aligned}
\langle \mathbf{GMY}, \mathbf{Y} \rangle = \langle \mathbf{G}^{1/2}\mathbf{G}^{1/2} M \mathbf{Y}, \mathbf{Y} \rangle &= \langle \mathbf{G}^{1/2}\mathbf{MY}, \mathbf{G}^{1/2}\mathbf{Y} \rangle \\
&= \langle \mathbf{G}^{1/2}\mathbf{MG}^{-1/2}\mathbf{G}^{1/2}\mathbf{Y}, \mathbf{G}^{1/2}\mathbf{Y} \rangle \\
&= \langle \mathbf{G}^{1/2}\mathbf{MG}^{-1/2}\mathbf{X}, \mathbf{X} \rangle.
\end{aligned}$$

The eigenvalues of $\mathbf{G}^{1/2}\mathbf{MG}^{-1/2}$ are the same as $\mathbf{M}$. So, for any $\mathbf{Y}$ we have,

$$\Re\left(\langle \mathbf{GMY}, \mathbf{Y} \rangle\right) \leq 0.$$

$\square$

Corollary 2.7.3. *Let $\mathbf{L}_x$ and $\mathbf{L}_v$ be two circulant $N \times N$ Laplacians matrices and assume that following matrix has all roots in the left half complex plane,*

$$\begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{L}_x & -\mathbf{L}_v \end{pmatrix}, \tag{135}$$

Let $\mathbf{G}_x = \mathbf{G}_v = \mathbf{G}$ be a diagonal positive matrix. Then the roots of the characteristic polynomial of,

$$\begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{GL}_x & -\mathbf{GL}_v \end{pmatrix}, \tag{136}$$

all lie in the left half complex plane. There is a double root at 0*. If the matrix in equation* (135) *has all non-zero eigenvalues in the open left complex plane, then the same is true for* (136)*.*

Proof. Stability follows from the following,

$$\begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{G}\mathbf{L}_x & -\mathbf{G}\mathbf{L}_v \end{pmatrix} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{G} \end{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{L}_x & -\mathbf{L}_v \end{pmatrix}$$

By Proposition 2.7.2 the roots of the characteristic polynomial of this matrix lie in the negative half complex plane.

$\mathbf{L}_x$ and $\mathbf{L}_v$ are both row-sum zero so the vectors $[1_N, 0]^T$ and $[0, 1_N]^T$ span a 2-dimensional eigenspace corresponding to the eigenvalue 0, where 1_N is the N-dimensional vector consisting of all 1's. □

Remark 2.7.4. *With the assumptions in Corollary 2.7.3 we have,*

$$[\mathbf{G}_x\mathbf{L}_x, \mathbf{G}_v\mathbf{L}_v] = [\mathbf{G}\mathbf{L}_x, \mathbf{G}\mathbf{L}_v] = 0.$$

This commutator no longer vanishes when $\mathbf{G}_x \neq \mathbf{G}_v$.

For the nearest neighbor system, described in Section 2.2, there are a few more things one can say. For example, we have the following.

Proposition 2.7.5. *If the nearest neighbor, described in Section 2.2, has $\rho_{x,1} < 0$ and $\rho_{x,-1} < 0$ and if this system is stable then the following conditions must hold.*

- $g_v^{(0)} + g_v^{(1)} + \cdots + g_v^{(N-1)} > 0$,
- $\sum\limits_{k=0}^{N-1} \left(g_x^{(k+1)} g_x^{(k+2)} \cdots g_x^{(k+N-1)} \right) > 0$,

where the superscript arithmetic is $\mod (N)$.

Proof. This is a special case of Proposition 1.4.4. The determinant,

$$\det\begin{pmatrix} 1 & \rho_{x,1} & 0 & \cdots & 0 \\ \rho_{x,-1} & 1 & \rho_{x,1} & \cdots & 0 \\ 0 & \rho_{x,-1} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

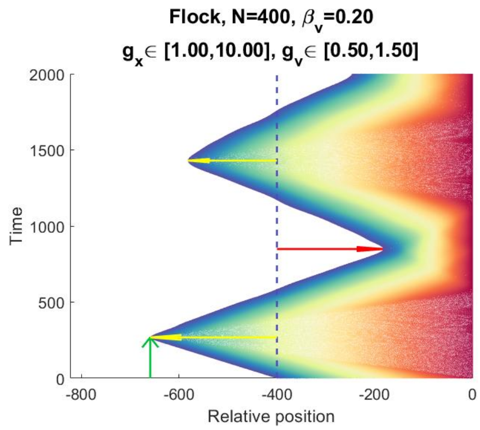
is independent of k. By Proposition C.5.1 this determinant is

$$(-\rho_{x,-1})^{N-1} + (-\rho_{x,1})(-\rho_{x,-1})^{N-2} + \cdots + (-\rho_{x,1})^{N-2}(-\rho_{x,-1}) + (-\rho_{x,1})^{N-2}.$$

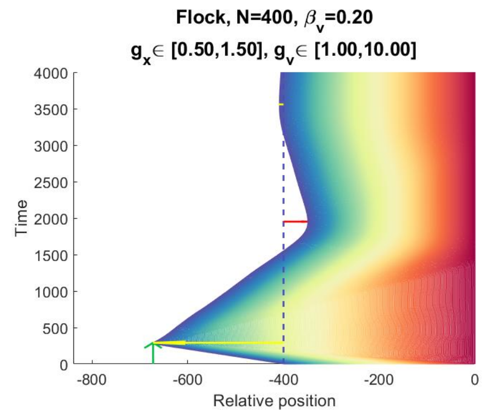
Since $\rho_{x,1}, \rho_{x,-1} \leq 0$ this term is strictly positive. The Proposition follows. $\square$

2.8 Simulations

To illustrate flock behavior, it is useful to plot agent differences. Plot time on the y-axis and plot agent k at $x-$coordinate $x_k - x_0$, for $k = 0, 1, \cdots, N-1$. The initial state has x_k placed along the negative $x-$axis, so these differences are usually negative. Two examples of these plots are given in Figure 7 where each agent is shaded a different color, so the flock structure is more apparent. Agent 0 is moving to the right, but we are plotting $x_0 - x_0 = 0$ so the trajectory is along the y-axis. The trajectory for agent $N-1$ moves to the left as the difference $x_{N-1} - x_0$ grows more negative. The first yellow arrow shows the point where $x_{N-1} - x_0$ is the most negative. This is A_1 described in Theorem 2.6.9. The signal has now reached agent $N-1$ and it starts to move toward agent 0. The distance $|x_{N-1} - x_0|$ reaches a minimum at A_2 and the distance then starts to grow again. This distance is indicated by the red arrow.



(a) *Plot showing flock behavior with various measurements indicated*



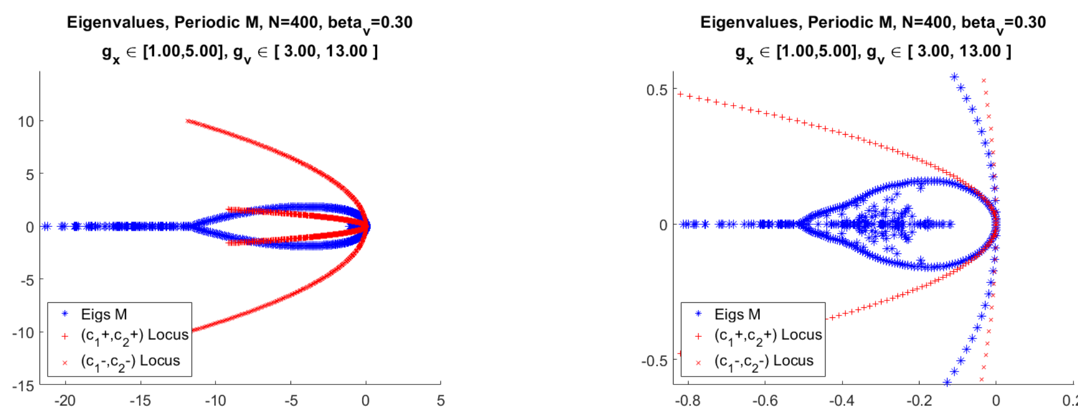
(b) *Flock with a different set of parameters. This flock is damped more quickly.*

Figure 7: *Plots showing flock behavior with various measurements indicated. The green vertical arrow shows the time T_1. The yellow and red arrows show max and min separations respectively. In this example, the longest yellow arrow is A_1. The red arrow is A_2 and the smaller yellow arrow is A_3.*

The notation $g_x^{(\alpha)} \in [a, b]$ indicates that the $g_x^{(\alpha)}$ are selected from a uniform distribution on $[a, b]$. In most of the examples we use the uniform distribution as it is easy to analyze and has strict bounds on the values of the random variables.

Remark 2.8.1. *There is a bit of legerdemain in the simulations. When we set up the problem in Section 2.2 we assumed q copies of p unique agents. The approximation assumed that q is large. In the simulations we just take N unique agents. In a sense, we are setting $q = 1$. But the results of the simulations are close to the predictions. A long distribution of N agents, chosen from a single distribution, could be divided arbitrarily into q groups of p. We would expect the dynamics of this system to be very close to the case where $p = \frac{N}{q}$ unique agents were duplicated q times. This is probably true for large N. However, the example in Figure 18 has two distributions and the fit there is excellent as well. The exact domain of applicability of the theory is not yet known.*

Next, we inspect the eigenvalues for a particular system. Theorem 2.5.1 gives a pair of second order approximations to the eigenvalues of **M**. In Figure 8 we compare these second order approximations, shown in red, to the actual eigenvalues of the matrix **M**, shown as blue stars. The two second order curves have different markers but are both in red. The matrix **M** in Figure 8 has periodic boundary conditions. The second order approximation is a close fit near the origin, which is the region that captures the large scale flock behavior. The system is stable, as in Definition 1.1.1, as the eigenvalues are all in the negative half complex plane and the estimates of Theorem 2.5.1 predict this.



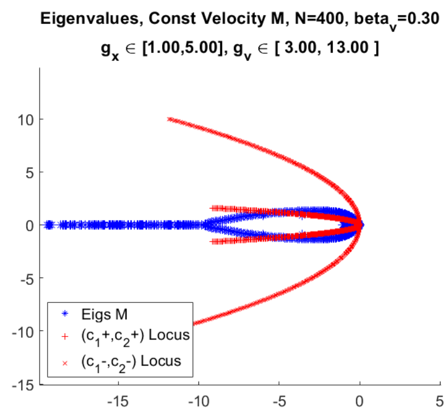
(a) *Plot showing eigenvalues of* **M** *and the two second order approximations.*

(b) *This is a magnification of the plot on the left.*

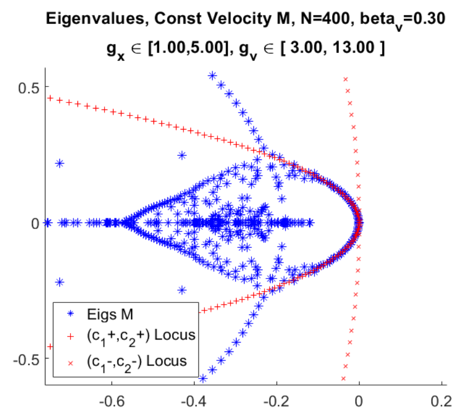
Figure 8: *In this plot* **M** *has periodic boundary conditions. The approximation for the eigenvalues is quite close near the origin and both loci are clearly visible. Eigenvalues far from the origin are not well approximated by the second order curves of Theorem 2.5.1.*

In Figure 9 we compare the approximation to the same system except with "constant velocity" boundary conditions, defined in 2.6.6. In this case the $(c_{1,+}, c_{2,+})$ approximation is quite good but the locus for the $(c_{1,-}, c_{2,-})$ approximation is not apparent in the scatter plot of the actual eigenvalues.

Unfortunately, the actual eigenvalues of **M**, with constant velocity boundary conditions, do not precisely match the predicted locus at the origin. Figure 10 shows the



(a) *Plot showing eigenvalues of* $\mathbf{M}$ *and the two second order approximations.*



(b) *This is a magnification of the plot on the left.*

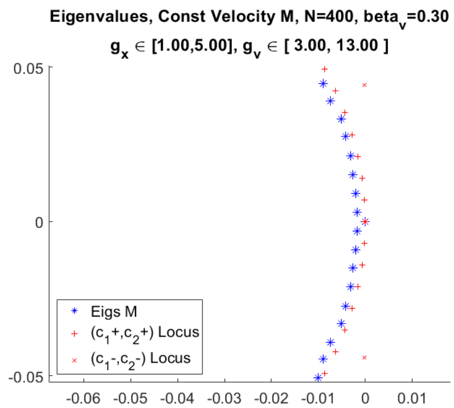
Figure 9: *In this plot we compare the eigenvalues of* $\mathbf{M}$ *to the two second order approximations. This is the same system as Figure 8 except* $\mathbf{M}$ *satisfies "constant velocity" boundary conditions. In this case, the* $(c_{1,-}, c_{2,-})$ *locus is missing from* $\mathbf{M}$.

eigenvalues for $\mathbf{M}$ are magnified around the origin. Note that Laplacians with this boundary condition are row-sum zero so the origin is still an eigenvalue. The actual Eigenvalue locus of $\mathbf{M}$ is offset slightly to the left of the predicted locus. Figure 10b shows the results of the full simulation of this constant velocity system. The system is stable, which is is consistent with the stability of the eigenvalues.

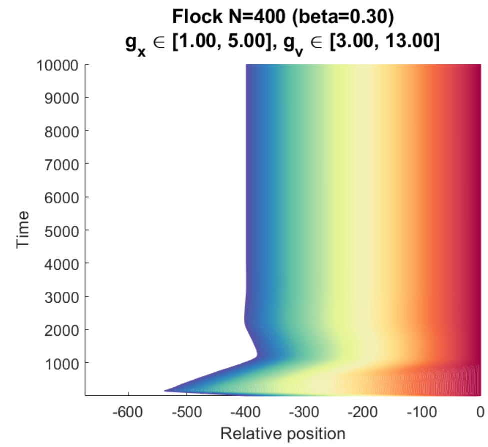
Figure 11 is a plot of the same system as Figure 10 except that we have changed β_v,

$$\beta_v \to -\beta_v.$$

In this case the locus of eigenvalues of $\mathbf{M}$ are slightly to the right of the second order approximation. Consistent with this, the flock is unstable, as shown in the simulation in Figure 11b. Notice, also, that the two second order approximations $(c_{1,+}, c_{2,+})$ and $(c_{1,-}, c_{2,-})$ have switched places, as we expect from Remark 2.5.9 (in this example $F = 0.09$). So the estimates of Theorem 2.5.1 predict stability and we observe instability. The "constant velocity" boundary conditions introduce instability



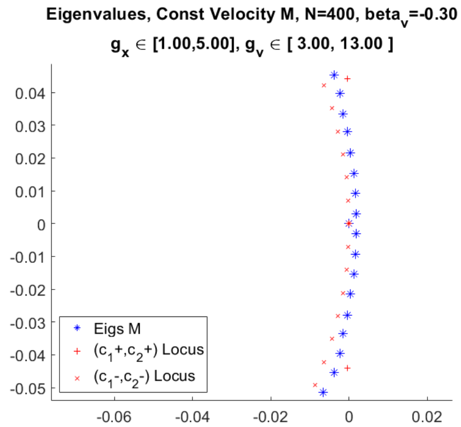
(a) *Plot showing eigenvalues of* $\mathbf{M}$ *and the two second order approximations. The eigenvalues of* $\mathbf{M}$ *are slightly offset to the left compared with the estimate. The eigenvalues are all stable.*



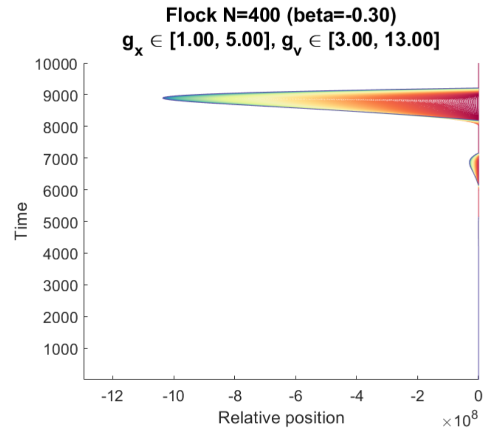
(b) *Plot showing the actual flock behavior for* $\mathbf{M}$ *with constant velocity boundary conditions. The trajectory appears stable, which is consistent with the eigenvalues shown on the left.*

Figure 10: *This plot is a deeper magnification of the system in Figure 9 and so* $\mathbf{M}$ *satisfies "constant velocity" boundary conditions.*

for $\beta_v = -0.30$.



(a) *Plot showing eigenvalues of* **M** *and second order approximation. There are several eigenvalues of* **M** *that are in the right have complex plane, so the system is not stable.*



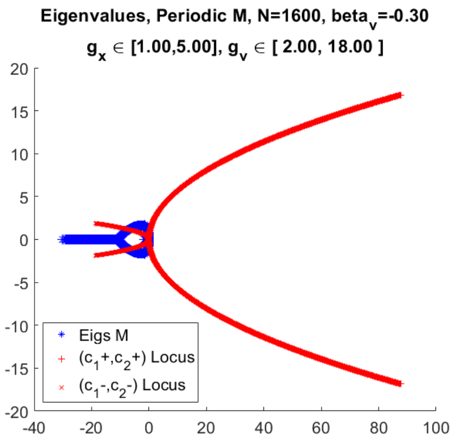
(b) *Plot showing the actual flock behavior for* **M** *with constant velocity boundary conditions. The trajectory appears unstable, which is consistent with the eigenvalues shown on the left.*

Figure 11: *The system in this figure is the same as in Figure 9 except that $\beta_v < 0$. In this case the eigenvalues of* **M** *are offset to the right of the estimation locus. The result is that several of the eigenvalues are negative and the system is unstable.*

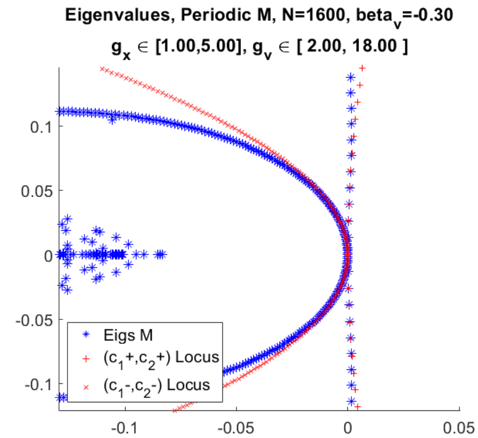
Figure 12 shows the eigenvalues for a system with a different set of parameters. The periodic system is unstable as several of the eigenvalues are in the right-hand plane. It appears that the theoretic locus in the right half plane does not reflect actual eigenvalues of **M**. However, in Figure 12b we see that there is a small bend into the positive half plane. It could be that our second order approximation is correct and that higher order terms are required to pick up the trajectory when it curves back into the negative half complex plane.

Figure 13 shows the eigenvalues for the same system except that **M** has "constant velocity" boundary conditions. In this case the eigenvalues are all stable and the curve with the unstable eigenvalues is absent.

Figure 14 shows the flock behavior of the system in Figure 13. Figure 14a shows

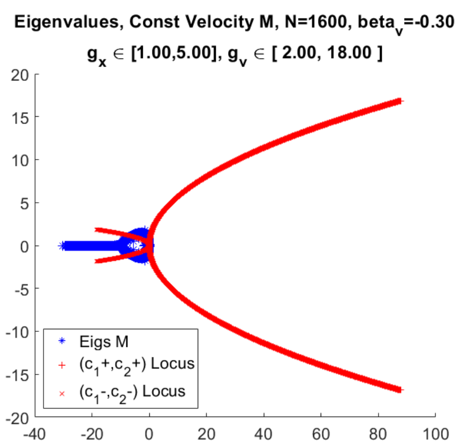


(a) *Plot showing eigenvalues of* $\mathbf{M}$ *and second order approximation. The system satisfies periodic boundary conditions and is not stable.*

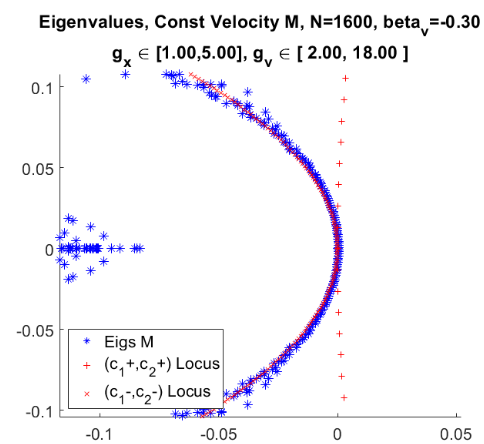


(b) *This is a magnification of the plot on the left. The actual eigenvalues of* $\mathbf{M}$ *bend into the positive complex half-plane before they proceed into the negative half plane.*

Figure 12: *Example comparing eigenvalues of* $\mathbf{M}$ *to the two estimate loci. The system shown in this figure satisfies periodic boundary condition.*



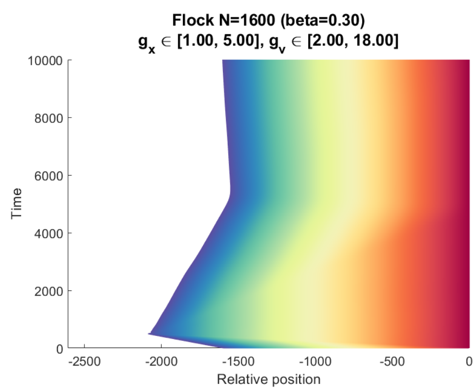
(a) *Plot showing eigenvalues of* $\mathbf{M}$ *and second order approximation. This system satisfies periodic boundary conditions and is not stable.*



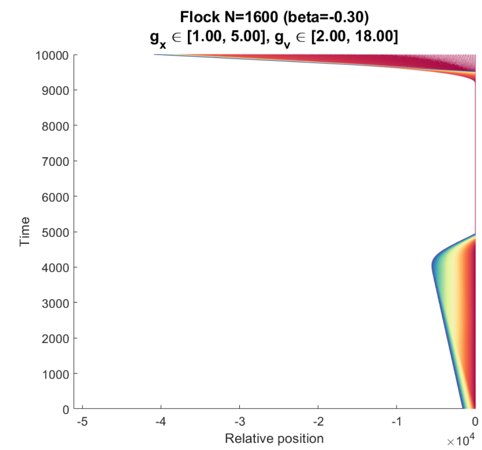
(b) *This is a magnification of the plot on the left.*

Figure 13: *These plots show the eigenvalues of the system in Figure 12 except that* $\mathbf{M}$ *has "constant velocity" boundary condition. The locus determined by* $(c_{1,+}, c_{2,+})$ *does not appear in the actual eigenvalues of* $\mathbf{M}$.

that the flock is stable, which agrees with the eigenvalue plots of the "constant velocity" system. In Figure 14b we show the flock behavior for the same system except that $\beta_v \to -\beta_v$. As we saw in our first example, the system is now unstable. In this case $F = 0.06$, so the periodic system for $-\beta_v$ should just swap the two eigenvalue loci. But, as in our first example, the stability of the "constant velocity" system is not predicted by our second order estimate.



(a) *Plot showing the flock behavior with constant velocity boundary conditions with* $\beta_v \geq 0$.

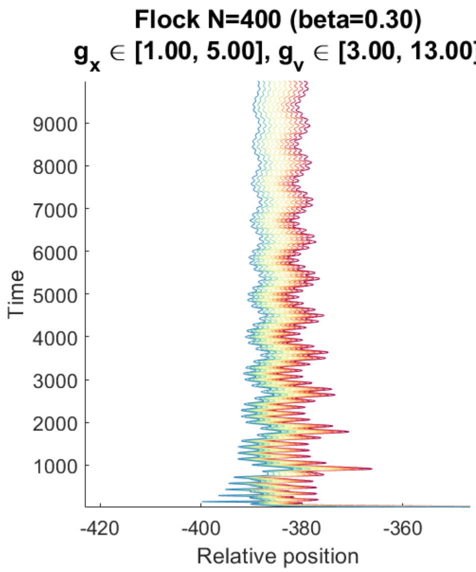


(b) *Plot showing the flock behavior with constant velocity boundary conditions with* $\beta_v \leq 0$.

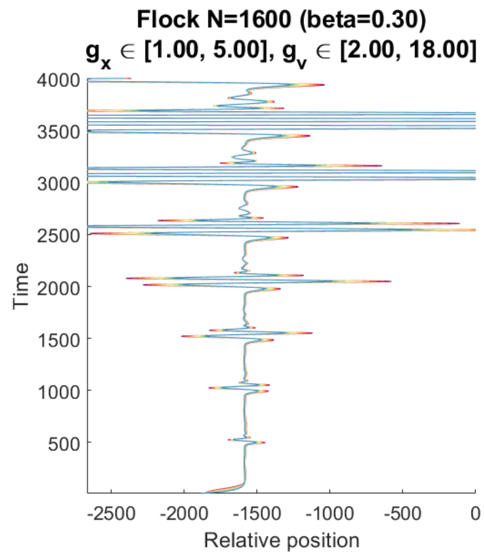
Figure 14: *This system is the same as in Figure 12 except that* $\mathbf{M}$ *has "constant velocity" boundary conditions. These plots show the flock behavior for* $\mathbf{M}$. *The plot on the left is stable and the plot on the right is not. The plot on the right shows the same system except* $\beta_v \to \beta_v$. *In this case the flock is unstable.*

Figure 15 shows the flock behavior for the systems with periodic boundary conditions. To test stability, we set the initial position of agent 0 to a large value and then let the system develop with periodic boundary conditions. We examine the 'ringing' of the system and look for instabilities. Figure 15a shows the flock dynamics corresponding to the system in Figure 12. This system appears stable, consistent with the eigenvalue plot. Figure 15b shows the flock dynamics corresponding to the system in Figure 12. This later system had unstable eigenvalues that are predicted by Theorem

2.5.1 and observed in the figure. These simulations show that the second order predictions in Theorem 2.5.1 do correctly predict the behavior of the periodic system, but that the periodic system does not behave in the same way as the “constant velocity” system.



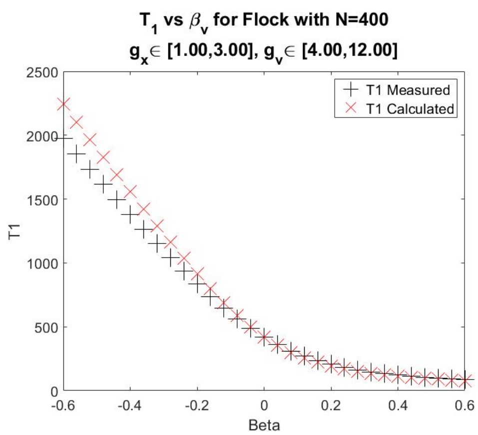
(a) *Plot showing the flock behavior with periodic boundary conditions. This is the same system as Figure 8.*



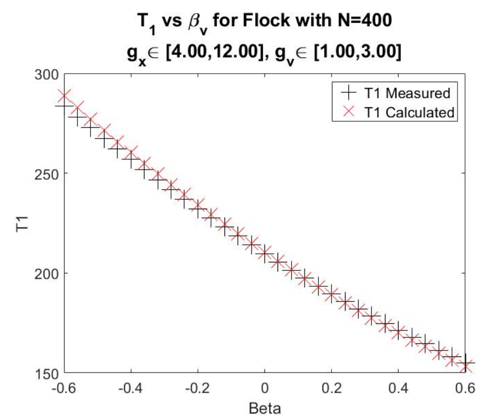
(b) *Plot showing the flock behavior with periodic boundary conditions. This is the same system as Figure 12.*

Figure 15: *These plots show the flock behavior when* **M** *has periodic boundary conditions. A single delta function is applied, and the system is left to “ring”. These plots only show the last 20 elements in the flock tail. The system in Figure 8 is stable and the system in Figure 12 is unstable.*

The approximation of Theorem 2.5.1 should predict the large-scale flock behavior. With the constant velocity initial conditions described in Section 2.6, the time T_1 that it takes the system to reach the point of greatest distance A_1 is predicted by Theorem 2.6.9. In Figure 16 we plot the predicted and measured values of T_1 for different values of β_v. The two plots in the figure show this comparison for two distinct choices of the parameters.



(a) *Plot showing predicted T_1 vs computed*



(b) *Plot showing predicted T_1 vs computed*

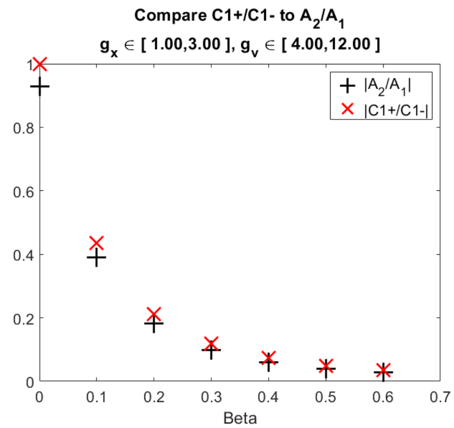
Figure 16: *Plots comparing computed vs predicted T_1 for two sets of parameters.*

In a well-behaved system, the amplitudes will decrease as $t \to \infty$. Using Theorem 2.6.9, the ratio of adjacent amplitudes is given by

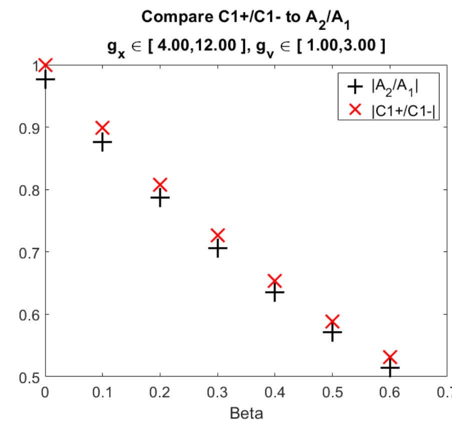
$$A_2/A_1 = \frac{c_{1,+}}{c_{1,-}}. \tag{137}$$

Notice that the ratio is negative, consistent with the change in polarity of A_2 and A_1. In Figure 17, we plot the measured values A_2/A_1 for various β_v and compare it to the values estimates $c_{1,+}/c_{1,-}$. The agreement for $\beta_v \geq 0$ is good. As stated in Remark 2.6.10, this is an indicator for stability in the constant velocity case.

We conclude with the realistic system discussed in the introduction. With this



(a) *Plot showing predicted $|A_2/A_1|$ vs computed*



(b) *Plot showing predicted $|A_2/A_1|$ vs computed*

Figure 17: *Plots comparing computed $|c_{1,+}/c_{1,-}|$ estimates to the measured values $|A_2/A_1|$.*

example we demonstrate that the tools presented in this dissertation can be used to analyze more complicated and realistic problems. We make some rough estimates in this next section. An automotive engineer could refine these numbers with more realistic estimates. We model a convoy of N trucks traveling on the highway. The convoy attempts to keep a fixed spacing between trucks and the trucks are all different. As the convoy travels, lighter cars might, inadvertently, enter the convoy creating a 1-dimensional convoy with very different agents. To use our model, we must estimate the agent weights g_x^k and g_v^k. The weights for agent k are force coupling divided by the mass of the agent. The mass of an 18-wheel truck is somewhere between 14 and 40 thousand kilograms and the coupling force is determined by the torque of the engine. To make things simpler we shall assume the force divided by the mass produces a given acceleration and we can estimate this acceleration. For example, a truck can accelerate from 0 to 60 mph = 26.8 m/sec in 1 to 5 minutes. So, we take our truck weights g_x^k in the range,

$$g_x^k \in [26.8/60, 26.8/300], \quad \text{when } k \text{ is a truck.}$$

We insert cars into the convoy by randomly replacing 10% of the agents with lighter cars. Cars, typically, have higher power to mass and so have larger weights. We take a collection of cars that accelerate from 0 to 60 in a range of 6 to 20 seconds, so that for car agents,

$$g_x^k \in [26.8/6, 26.8/20], \quad k \text{ is a car.}$$

To guarantee stability we take $\mathbf{G}_v = \alpha \mathbf{G}_x$ where $\alpha = 10.0$. Increasing α, as we've seen, will increase the damping. U.S. 18-wheel trucks are typically around 23 meters long. The convoy attempts to keep a bumper-to-bumper distance of $2 \times 23 = 46$ meters between the agents.

The simulation results are shown in figure 18. The convoy of 400 starts with a bumper-to-bumper spacing of 46 meters. Add to this the length of the truck and the stable convoy has an approximate length of $46(N-1) + 23N$, which is just over 27 km long. The first truck suddenly increases its speed 10 meters/second, and it takes 1095 seconds for the signal to reach the last truck. The time duration is long because the weights are small (e.g., the trucks accelerate slowly). The distance from the leader to the tail lengthens to 37.5 km = 94 m/truck before the tail starts to catch up. This is a bumper-to-bumper distance of $94 - 23 = 71$ m. The damping is not critical, the tail overshoots the optimal distance, and the entire convoy shrinks to 18 km = 45 m/truck before expanding again. This is a bumper-to-bumper distance of $45 - 23 = 22$ m. This simulation assumed $\rho_{v,1} = \rho_{v,-1}$.

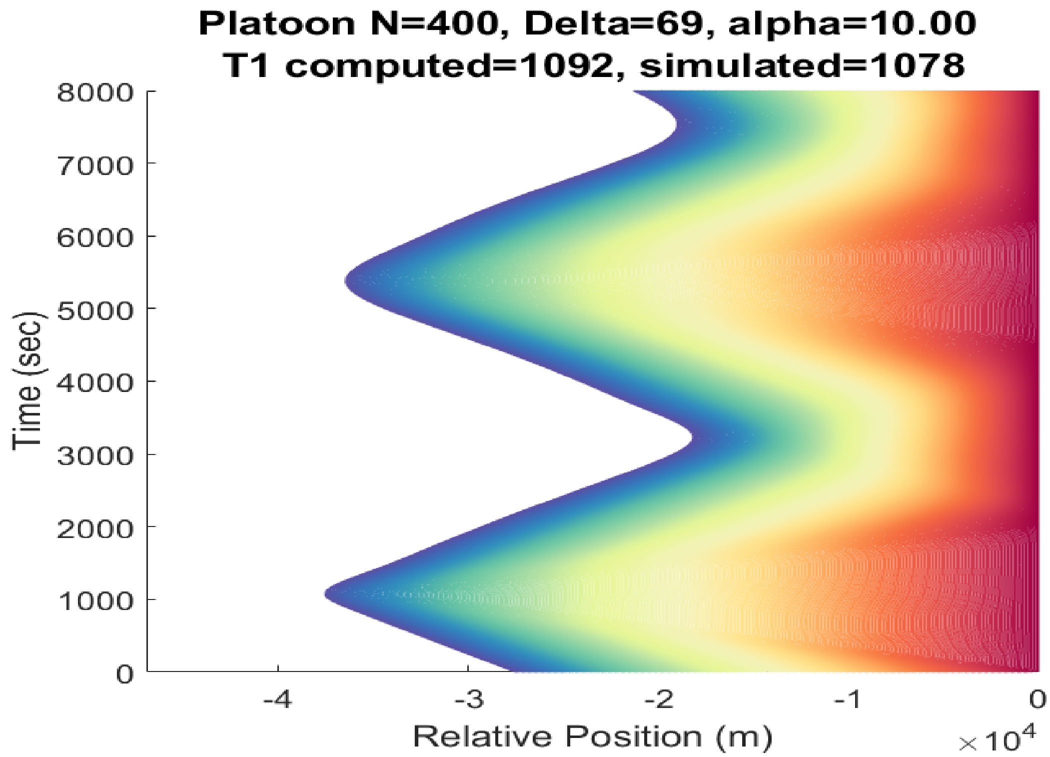


Figure 18: *Simulation of truck convoy with 400 agents.*

2.9 Conclusions

In our research we studied one-dimensional flocks with linear nearest-neighbor couplings, where each agent has its own coupling weights. Systems with varying couplings have not been covered adequately in the literature, despite the fact that agent dependent couplings are an essential part of any real-world system. A realistic flock will have agents of different masses and with different propulsion systems. For example, a convoy of traffic may contain a combination of cars, SUVs, and trucks that have different weights and engines. Letting each agent have a unique coupling weight complicates the system quite substantially and very little is known about these systems, even in the one-dimensional linear nearest-neighbor case.

Our system has an eigenvalue at 0 with multiplicity 2. In Theorem 2.5.1, we derived a pair of discrete quadratics, each with vertex at the origin, that approximate the eigenvalues near the origin. Derivation of this Theorem requires some of the Laplacian properties described in Section 1.1.1. The result of the theorem is an estimate for the eigenvalues of the system near the origin. If the system is stable, then all the eigenvalues, except the origin, are in the open left half complex plane and eigenvalues with large negative real part will decay quickly in time. If we assume that all eigenvalues near the imaginary axis are near the origin, then Theorem 2.5.1 should approximate all the eigenvalues necessary to describe the flock behavior at large times.

To test Theorem 2.5.1, we ran a variety of Matlab simulations. Figures 8 and 12 demonstrate that the quadratics approximate the eigenvalues near the origin with reasonable accuracy for systems with periodic boundary conditions. Figure 15 shows that this second-order approximation predicts stability for the periodic system. However, for systems with "constant velocity" initial conditions, the situation is not so simple. Figures 11 and 14 show examples where the stability of the "constant velocity" system is different from the periodic system.

However, when the "constant velocity" system is stable, the large-scale flock behavior is determined by the eigenvalues that are close to the origin. In this case, Theorem 2.5.1 predicts flock behavior at large times reasonably well. Figures 16 and 17 show that some of the features of the flock are captured quite accurately by the second order predictions. Indeed, the successive amplitude ratio A_2/A_1, described by Theorem 2.6.9, is a decent indicator of the stability of the "constant velocity" system. But how this relates to the eigenvalues is still a bit of a mystery. Why do the eigenvalues shift when we move to "constant velocity" boundary conditions? When $\beta_v \geq 0$, the eigenvalues shift to the left and so become more stable. When $\beta_v \leq 0$, the eigenvalues seem to shift to the right and the system becomes unstable. This is consistent with Theorem 2.6.9 but it is not captured by our quadratic approximations.

The reasons for this are not yet known.

2.9.1 Future Work

Our results provide a step towards a more complete understanding of the one-dimensional linear nearest-neighbor system. For this to be generally useful to the engineering community, we need a better understanding of how different boundary conditions relate to the stability of the system. One unresolved mystery in the "constant velocity" case is how the stability prediction of Theorem 2.5.1 relates to stability prediction of the amplitude ratios of Theorem 2.6.9. More generally, how do boundary conditions affect stabilty. We described some stability conditions in Section 2.7, but these conditions do not capture the specifics of any boundary conditions.

Another possible area of research is the general problem of systems on a graph with second order time derivatives. In Section 1.1.1 we outlined some of the issues for first order Laplacians on a general graph. It would be interesting to know if there is some equivalent to the approximation of Theorem 2.5.1 in the case of second order systems on a graph.

REFERENCES

[1] N.W. Ashcroft and N.D. Mermin. *Solid State Physics*. Saunders College, Philadelphia, 1976.

[2] Pablo E. Baldivieso. *Necessary Conditions for Stability of Vehicle Formations*. PhD thesis, Portland State University, 2019.

[3] Pablo E. Baldivieso and J.J.P. Veerman. Stability conditions for coupled autonomous vehicles formations. *IEEE Transactions on Control of Network Systems*, in press.

[4] Yury S. Barkovsky. Letures on the routh-hurwitz problem. *arXiv:0802.1805v1*, 2008.

[5] C.E. Cantos, D.K Hammond, and J.J.P. Veerman. Transients in the synchronization of assymmetrically coupled oscillator arrays. *The European Physical Journal Special Topics*, 225:1115–1125, 2016.

[6] C.E. Cantos, J.J.P. Veerman, and D.K. Hammond. Signal velocity in oscillator arrays. *The European Physical Journal Special Topics*, 225:1199–1210, 2016.

[7] Robert E. Chandler, Robert Herman, and Elliott W. Montroll. Traffic dynamics: Studies in car following. *Operations Research*, 6:165–184, 1957.

[8] F. R. K. Chung. *Spectral Graph Theory*. American Mathematical Society, 1997.

[9] Freeman J. Dyson. The dynamics of a disordered linear chain. *Phys. Rev.*, 92:1331–1338, Dec 1953.

[10] Richard Feynman. *Lectures in Physics*. Springer-Verlag, 2017.

[11] Steven Garrett. *Understanding Acoustics*. Springer-Verlag, 2017.

[12] G. C. Gaunaurd and G. C. Everstine. Propagation of energy pulses in absorbing/viscous material media. *Naval Surface Warfare Center*, 2000.

[13] Chris Godsil and Gordon Royle. *Algebraic Graph Theory*. Springer, 2001.

[14] Claudius Gros. *Complex and Adaptive Dynamical Systems: A Primer, Fourth Edition*. Springer-Verlag, 2015.

[15] He Hao and Prabir Barooah. Stability and robustness of large platoons of vehicle with double-integrator models and nearest neighbor interaction. *International Journal of Robust and Nonlinear Control*, 23, 12 2013.

[16] J. Herbrych, A. Chazirakis, N. Christakis, and J.J.P. Veerman. Dynamics of locally coupled agents with next nearest neighbor interaction. *Differential Equations and Dynamical Systems*, 2017.

[17] B.S. Kerner. *The Physics of Traffic*. Springer-Verlag, 2004.

[18] S. E. Li, Y. Zheng, K. Li, Y. Wu, J. K. Hedrick, F. Gao, and H. Zhang. Dynamical modeling and distributed control of connected and automated vehicles: Challenges and opportunities. *IEEE Intelligent Transportation Systems Magazine*, 9(3):46–58, 2017.

[19] Fu Lin, Makan Fardad, and Mihailo R. Jovanovic. Optimal control of vehicular formations with nearest neighbor interactions. *IEEE Transactions on Automatic Control*, 57(9):2203–2218, 2012.

[20] Robert G. Lyons and J.J.P. Veerman. Linear nearest neighbor flocks with all distinct agents. *The European Physical Journal B*, 94(8):174, Aug 2021.

[21] Uwe Mackenroth. *Robust Control Systems*. Springer-Verlag, 2004.

[22] Richard H. Middleton and Julio H. Braslavsky. String instability in classes of linear time invariant formation control with limited communication range. *IEEE Transactions on Automatic Control*, 55(7):1519–1530, 2010.

[23] M. Newman. *Networks*. OUP Oxford, 2018.

[24] Amirreza Rahmani, Meng Ji, Mehran Mesbahi, and Magnus Egerstedt. Controllability of multi-agent systems from a graph-theoretic perspective. *SIAM Journal on Control and Optimization*, 48, 01 2009.

[25] Wei Ren and Randal Beard. *Distributed Consensus in Multi-vehicle Cooperative Control*. Springer-Verlag, 2008.

[26] Esther Chepngetich Ropl, Okoya Michael Oduor, and Owino Maurice Oduor. Solution of third order viscous wave equation using finite difference method. *Nonlinear Analysis and Differential Equations*, 7(1):53–68, 2019.

[27] J. J. P. Veerman, G. Lafferriere, J. S. Caughman, and A. Williams. Flocks and formations. *Journal of Statistical Physics*, 121(3):901–936, 2005.

[28] J.J.P. Veerman and Robert G. Lyons. A primer on laplacian dynamics in directed graphs. *Nonlinear Phenomena in Complex Systems*, 23:196–206, 07 2020.

Appendix A GENERAL FORMULAS

In the signal processing literature, the following shorthand is often used.

$$\omega_N = \exp\left(\frac{2\pi j}{N}\right) \tag{138}$$

Proposition A.0.1.

$$\sum_{k=0}^{N-1} \omega_N^{mk} = \left\{ \begin{array}{ll} N & \textit{when } m = 0 \\ 0 & \textit{when } m \neq 0 \end{array} \right\} \tag{139}$$

$$\sum_{k=1}^{N-1} \cos\left(\frac{2\pi}{N} mk\right) = -1 \qquad \textit{when } m \neq 0, \tag{140}$$

$$\sum_{k=1}^{N-1} \sin\left(\frac{2\pi}{N} mk\right) = 0 \qquad \textit{when } m \neq 0. \tag{141}$$

Proof.

$$\sum_{k=0}^{N-1} \omega_N^{mk} = \sum_{k=0}^{N-1} \exp\left(\frac{2\pi i}{N} mk\right) = \frac{1 - \exp\left(\frac{2\pi i}{N} mN\right)}{1 - \exp\left(\frac{2\pi i}{N} m\right)} = 0.$$

Setting the real (e.g., cos) and imaginary (sin) parts to zero results in the formulas. □

Appendix B ROUTH HURWITZ STABILITY

We shall not prove the Routh-Hurwitz stability criteria, but we will describe the recipe. For a complete proof see [4]. We do prove the following condition that is a condition necessary for stability.

Theorem B.0.1. *If $p(z)$ is degree n real stable polynomial and has lead coefficient $a_n > 0$, then all the other coefficients a_k are positive.*

Proof. Let $-r, -\alpha + i\beta, -\alpha - i\beta$ be 3 stable roots of $p(z)$, where $r, \alpha, \beta \in \mathbb{R}$. and r, α are positive. All stable roots have one of these forms. The polynomial has a factor,

$$(z + r)(z + \alpha - i\beta)(z + \alpha + i\beta) = (z + r)(z^2 + 2\alpha z + \alpha^2 + \beta^2).$$

This cubic has positive coefficients. $p(z)$ is a product of terms similar to these (multiplied by a_n). If $\alpha, r > 0$ then none of the a_k are zero. $\square$

B.1 Routh–Hurwitz Recipe

The Routh-Hurwitz criteria determines how many roots lie in the right half plane. We start with a polynomial of the form,

$$p(t) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0, \tag{142}$$

where $a_0 \neq 0$ and $a_n > 0$.

We write out the table for $n = 5$ in order to demonstrate how the 0's percolate through the table however we keep the indexes general to help with the general case.

The Routh-Hurwitz criteria is specified in the following theorem.

Theorem B.1.1. *None of the roots of the polynomial in equation* (142) *lie in the*

t^n	a_n	a_{n-2}	$a_{n-4} = a_1$	0
t^{n-1}	a_{n-1}	a_{n-3}	$a_{n-5} = a_0$	0
t^{n-2}	b_1	b_2	0	0
t^{n-3}	c_1	c_2	0	0
t^{n-4}	d_1	0	0	0
t^{n-5}	e_1	0	0	0

Table 1: *Routh-Hurwitz table*

right half plane if and only if all the coefficients a_i are positive and all the coefficients in the first column vanish.

The following criteria must be handled by special cases

- All rows in the Routh matrix must not be identically zero.
- The elements in the first column must not vanish.

The coefficients are defined by,

$$
\begin{aligned}
b_1 &= \frac{(a_{n-1}a_{n-2} - a_n a_{n-3})}{a_{n-1}} \\
b_2 &= \frac{(a_{n-1}a_{n-4} - a_n a_{n-5})}{a_{n-1}} \\
b_3 &= 0.
\end{aligned}
$$

The coefficients of c_k are derived from p_k and b_k,

$$
\begin{aligned}
c_1 &= \frac{(b_1 p_{n-3} - p_{n-1} b_2)}{b_1} \\
c_2 &= \frac{(b_1 p_{n-5} - p_{n-1} b_3)}{b_1} \\
c_3 &= 0.
\end{aligned}
$$

The coefficients of d_k are derived from b_k and c_k,

$$d_1 = \frac{(c_1 b_2 - b_1 c_2)}{c_1}$$
$$d_2 = 0.$$

The coefficients of e_k are derived from c_k and d_k,

$$e_1 = \frac{1}{d_1}(d_1 c_2 - c_1 d_2)$$
$$e_2 = 0.$$

B.2 Routh–Hurwitz Examples

In this section we apply Routh-Hurwitz to various polynomials used in the main text.

Proposition B.2.1. *Given the quadratic with $a_1, a_0 \in \mathbb{R}$,*

$$p_2(z) = z^2 + a_1 z + a_0. \tag{143}$$

The quadratic polynomial is Routh-Hurwitz stable (e.g., has roots in the right half complex plane) if and only if a_1 and a_0 are positive.

Proof. The proposition follows directly from the expression of the solutions. One can, also, use Routh-Hurwitz where the table is given by, The Routh-Hurwitz condition is

z^2	1	a_0
z^1	a_1	0
z^0	$b_1 = a_0$	0

Table 2: *Routh-Hurwitz table for quadratic*

$a_1 > 0$ and $a_2 > 0$. $\square$

Proposition B.2.2.

$$p_4(z) = a_4 z^4 + a_3 z^3 + a_2 z^2 + a_1 z + a_0 \tag{144}$$

where $a_0 - a_4$ are real constants and $a_4 > 0$. The polynomial has roots in the left half plane if and only if

$$a_k \geq 0 \tag{145}$$

$$a_2 a_3 - a_1 a_4 \geq 0 \tag{146}$$

$$a_1 a_2 a_3 - a_1^2 a_4 - a_3^2 a_0 \geq 0 \tag{147}$$

If conditions (147) *and* (145) *are true then* (146) *must be true.*

Proof. We know that $a_k \geq 0$ for all k. The Routh-Hurwitz table is the following,

z^4	a_4	a_2	a_0	0
z^3	a_3	a_1	0	0
z^2	b_1	b_2	0	0
z^1	c_1	0	0	0
z^0	d_1	0	0	0

Table 3: *Routh-Hurwitz table*

where we have,

$$\begin{aligned}
b_1 &= \frac{1}{a_3}\left(a_3 a_2 - a_4 a_1\right), \\
b_2 &= \frac{1}{a_3}\left(a_3 a_0\right) = a_0, \\
c_1 &= \frac{1}{b_1}\left(b_1 a_1 - b_2 a_3\right) = \frac{1}{b_1}\left(b_1 a_1 - a_0 a_3\right) \\
d_1 &= \frac{1}{c_1}\left(c_1 b_2\right) = b_2.
\end{aligned}$$

The only unique conditions are $b_1 \geq 0$ and $c_1 \geq 0$. The first is equivalent to,

$$0 \leq a_3 b_1 = a_2 a_3 - a_1 a_3.$$

The second is equivalent to

$$\begin{aligned} 0 \leq & b_1 a_3 c_1 = (a_3 b_1 a_1 - b_2 a_3^2) \\ & = a_2 a_3 a_1 - a_1 a_4 a_1 - a_0 a_3^2 \end{aligned}$$

If all the roots are stable and if condition 147 is true then we have,

$$a_1 (a_2 a_3 - a_1 a_4) \geq a_3^2 a_0 \geq 0,$$

Since $a_1, a_0 \geq 0$ we have,

$$a_2 a_3 - a_1 a_4 \geq 0.$$

But this is just condition 146. $\square$

Proposition B.2.3.

$$p_{c2}(z) = z^2 + w_1 z + w_0 \tag{148}$$

where w_0, w_1 are complex constants. The polynomial is Hurwitz stable if and only if,

$$\Re(w_1) > 0, \tag{149}$$

$$2\Re(w_0) + |w_1|^2 > 0, \tag{150}$$

$$\Re(w_0 w_1^*) > 0, \tag{151}$$

$$\Re(w_0)\Re(w_1)^2 + \Re(w_1)\Im(w_0)\Im(w_1) - \Im(w_0)^2 > 0 \tag{152}$$

There is one additional Routh-Hurwitz constraint that is implied by the above equations, as in equation (146),

$$\Re(w_0)\Re(w_1) + \Re(w_1)^3 + \Re(w_1)\Im(w_1)^2 - \Im(w_0)\Im(w_1) > 0. \tag{153}$$

Proof. We construct a real polynomial of order z^4.

$$\begin{aligned}
p_{R4}(z) &= \left(z^2 + w_1 z + w_0\right)\left(z^2 + w_1^* z + w_0^*\right) \\
&= z^4 + (w_1 + w_1^*)\, z^3 + \left(w_0 + w_0^* + |w_1|^2\right) z^2 + (w_1 w_0^* + w_1^* w_0)\, z + |w_0|^2 \\
&= z^4 + 2\Re(w_1) z^3 + \left(2\Re(w_0) + |w_1|^2\right) z^2 + 2\Re(w_1 w_0^*) z + |w_0|^2
\end{aligned}$$

The stability conditions of equation (145) are just,

$$\begin{aligned}
\Re(w_1) &> 0 \\
2\Re(w_0) + |w_1|^2 &> 0 \\
\Re(w_1 w_0^*) &> 0
\end{aligned}$$

The condition of equation (147) is,

$$
\begin{aligned}
0 \leq & a_1 a_2 a_3 - a_1^2 a_4 - a_3^2 a_0 \\
= & 2\Re(w_1 w_0^*) \left(2\Re(w_0) + |w_1|^2\right) 2\Re(w_1) - 4\Re(w_0^* w_1)^2 - 4\Re(w_1)^2 |w_0|^2 \\
= & 8\Re(w_0)\Re(w_1)(\Re(w_0)\Re(w_1) + \Im(w_0)\Im(w_1)) \\
& + 4\Re(w_1)(\Re(w_1)^2 + \Im(w_1)^2)(\Re(w_0)\Re(w_1) + \Im(w_0)\Im(w_1)) \\
& - 4(\Re(w_0)\Re(w_1) + \Im(w_0)\Im(w_1))^2 - 4(\Re(w_0)^2 + \Im(w_0)^2)\Re(w_1)^2 \\
= & (\Im(w_1)\Im(w_1) + \Re(w_1)\Re(w_1)) \\
& \times \left(\Re(w_0)\Re(w_1)^2 + \Re(w_1)\Im(w_0)\Im(w_1) - \Im(w_0)^2\right)
\end{aligned}
$$

Since $w_1 \neq 0$, equation (152) follows. $\square$

Appendix C MATRICES

In this section we discuss a few properties of matrices that are required in the main text. In this section we will let V be an N dimensional vector space with basis $e_0, e_1, \cdots e_{N-1}$. The matrices are linear operators on V.

C.1 Special Matrices

The following matrix is typical for second order linear ODE systems.

Proposition C.1.1. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be* $N \times N$ *matrices and* $\mathbf{I}_N$ *be the identity matrix. The eigenvectors of the matrix* M*,*

$$\mathbf{M} = \begin{bmatrix} \mathbf{0} & \mathbf{I}_N \\ \mathbf{A} & \mathbf{B} \end{bmatrix} \text{ are vectors of the form, } \begin{bmatrix} v \\ \lambda v \end{bmatrix}.$$

where λ *is the corresponding eigenvalue and* v *is an* N*-vector that satisfies,*

$$\left(\lambda^2 \mathbf{I}_N - \lambda \mathbf{B} - \mathbf{A}\right) v = 0. \tag{154}$$

Conversely, if λ *and* v *satisfy equation* (154) *then* $\begin{bmatrix} v & \lambda v \end{bmatrix}^T$ *is an eigenvector* $\mathbf{M}$ *with eigenvalue* λ*.*

Proof. If $\begin{bmatrix} v & w \end{bmatrix}^T$ is an eigenvector with eigenvalue λ then we have,

$$\begin{bmatrix} \mathbf{0} & \mathbf{I}_N \\ \mathbf{A} & \mathbf{B} \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} = \lambda \begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} w \\ (\mathbf{A}v + \mathbf{B}w) \end{bmatrix} \;\Rightarrow\; w = \lambda v.$$

From this we also get,

$$\left(\lambda^2 \mathbf{I}_N - \lambda \mathbf{B} - \mathbf{A}\right) v = 0.$$

$\square$

C.2 Shift Operator

Let V be an N dimensional vector space with basis $e_0, e_1, \cdots e_{N-1}$.

Definition C.2.1. *The positive shift operator is defined by,*

$$\mathbf{P}_+(e_k) = e_{k+1} \text{ for } k = 0, \cdots N-2$$
$$\mathbf{P}_+(e_{N-1}) = e_0$$

The negative shift operator is defined by,

$$\mathbf{P}_-(e_{k+1}) = e_k \text{ for } k = 0, \cdots N-2$$
$$\mathbf{P}_-(e_0) = e_{N-1}$$

Proposition C.2.2. *The operators $\mathbf{P}_+$ and $\mathbf{P}_-$ are invertible and*

- $\mathbf{P}_+^{-1} = \mathbf{P}_-$,
- $\mathbf{P}_-^{N-1} = \mathbf{P}_+$ *and* $\mathbf{P}_+^{N-1} = \mathbf{P}_-$,
- $\mathbf{P}_+^N = \mathbf{P}_-^N = I_N$,
- $[\mathbf{P}_+, \mathbf{P}_-] = 0$.

Proof. These all follow directly from Definition C.2.1. $\square$

In the standard basis the shift operators are given by the $N \times N$ matrices,

$$\mathbf{P}_+ = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \mathbf{P}_- = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \tag{155}$$

C.3 Circulant Matrices

Circulant matrices arise in many sequence ODE's where periodic boundary condition are used. In several of our examples the Laplacian is a circulant matrix.

Definition C.3.1. *Let v be a vector in an N-dimensional vector space. A circulant matrix is a matrix formed from v as follows,*

$$\mathbf{C}_v = \begin{bmatrix} v^T \\ (P_+ v)^T \\ (P_+^2 v)^T \\ \vdots \\ (P_+^{N-1} v)^T \end{bmatrix} = \begin{bmatrix} v^0 & v^1 & v^2 & \cdots & v^{N-1} \\ v^{N-1} & v^0 & v^1 & \cdots & v^{N-2} \\ v^{N-2} & v^{N-1} & v^0 & \cdots & v^{N-3} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ v^1 & v^2 & v^3 & \cdots & v^0 \end{bmatrix}$$

The matrices $\mathbf{P}_+$ and $\mathbf{P}_-$ are circulant matrices with,

$$\mathbf{C}_{e_{N-1}} = \mathbf{P}_+$$
$$\mathbf{C}_{e_2} = \mathbf{P}_-$$

Any circulant matrix $\mathbf{C}_v$ is the sum of powers of $\mathbf{P}_+$,

$$\mathbf{C}_v = \sum_{k=0}^{N-1} v^k \mathbf{P}_-^k \tag{156}$$

Proposition C.3.2. *The circulant matrices form an N dimensional vector subspace of $GL(V)$. This means that for any vectors v, w and scalar a we have,*

$$\mathbf{C}_{v+w} = \mathbf{C}_v + \mathbf{C}_w$$
$$\mathbf{C}_{av} = a\mathbf{C}_v.$$

Proof. This follows immediately from Definition C.3.1. □

Proposition C.3.3. *The $N \times N$ dimensional circulant matrices form an N-dimensional sub-algebra of $GL(V)$ where the product obeys the formula,*

$$\mathbf{C}_v \mathbf{C}_w = \mathbf{C}_{v*w}.$$

Proof. We shall use $MOD(N)$ arithmetic in the index calculations that follow.

$$\begin{aligned}
\mathbf{C}_v \mathbf{C}_w &= \sum_{k=0}^{N-1} \sum_{m=0}^{N-1} v^k \mathbf{P}_-^k w^m \mathbf{P}_-^m \\
&= \sum_{m=0}^{N-1} \sum_{l=0}^{N-1} v^{l-m} w^m \mathbf{P}_-^l \\
&= \sum_{l=0}^{N-1} (v * w)^l \mathbf{P}_-^l = \mathbf{C}_{v*w}
\end{aligned}$$

where $l = k + m$. □

Proposition C.3.4. *Any two $N \times N$ circulant matrices commute.*

Proof.

$$[\mathbf{C}(v), \mathbf{C}(w)] = \sum_{k,j=0}^{N-1} \left[v^j \mathbf{P}_-^j, w^k \mathbf{P}_-^k\right] = \sum_{k,j=0}^{N-1} v^j w^k \left[\mathbf{P}_-^j, \mathbf{P}_-^k\right] = 0.$$

□

C.4 Eigenvalues of Circulant Matrices

Let $\mathbf{C}(v)$ be the $N \times N$ circulant matrix determined by the $N-$vector v. As in equation (138) we define,

$$\omega_N = \exp\left(\frac{2\pi i}{N}\right) \tag{157}$$

Define the N vectors, for each of $u = 0, 1, \cdots N-1$, by

$$w(\omega_N^m) = \begin{bmatrix} 1 \\ \omega_N^m \\ \omega_N^{2m} \\ \vdots \\ \omega_N^{(N-1)m} \end{bmatrix} \tag{158}$$

These will constitute an orthogonal set of eigenvectors for any circulant matrix.

The spectra of circulant matrices are related to the Discrete Fourier Transform, which we define next.

Definition C.4.1. *Let v^k be an N-vector in an N-dimensional vector space V. The Discrete Fourier Transform (DFT) of v^k is the N-vector, $\mathscr{D}\{v\}$, where*

$$\mathscr{D}\{v\}^m = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} v^k \omega_N^{-km}$$

The mapping $v \to \mathscr{D}\{v\}$ is a linear map. A short computation proves the following.

Proposition C.4.2. *The inverse of the Discrete Fourier Transform is* $\mathscr{D}^{-1}\{v\}$*,*

$$\mathscr{D}^{-1}\{v\}^m = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} v^k \omega_N^{km}$$

The DFT is the discrete version of the Fourier Transform and is vital to the study of discrete signals. The relationship to circulant matrices is given by the following.

Proposition C.4.3. *The vector* $w(\omega_N^m)$ *is an eigenvector of* $\mathbf{C}(v)$*. The eigenvalue of* $\mathbf{C}(v)$ *corresponding to* $w(\omega_N^m)$ *are given by,*

$$\sqrt{N}\mathscr{D}^{-1}\{v\}[u] = \sum_{k=0}^{N-1} v^k \exp\left(\frac{2\pi i}{N} mk\right).$$

The vectors $w(\omega_N^m)$*, for* $m = 0, \cdots, N-1$ *form a complete set of eigenvectors.*

Proof.

$$\begin{aligned}
\mathbf{C}(v)w(\omega_N^m) &= \begin{bmatrix} \sum_{k=0}^{N-1} v^k \omega_N^{mk} \\ \sum_{k=0}^{N-1} v^{k-1} \omega_N^{mk} \\ \vdots \\ \sum_{k=0}^{N-1} v^{k-(N-1)} \omega_N^{mk} \end{bmatrix} = \begin{bmatrix} \sum_{k=0}^{N-1} v^k \omega_N^{mk} \\ \sum_{k=0}^{N-1} v^k \omega_N^{m(k+1)} \\ \vdots \\ \sum_{k=0}^{N-1} v^k \omega_N^{m(k+(N-1))} \end{bmatrix} = \begin{bmatrix} 1 \sum_{k=0}^{N-1} v^k \omega_N^{mk} \\ \omega_N^u \sum_{k=0}^{N-1} v^k \omega_N^{mk} \\ \vdots \\ \omega_N^{u(N-1)} \sum_{k=0}^{N-1} v^k \omega_N^{mk} \end{bmatrix} \\
&= \left(\sum_{k=0}^{N-1} v^k \omega_N^{mk}\right) w(\omega_N^m).
\end{aligned}$$

where all arithmetic is $\mod (N)$. □

Theorem C.4.4. *Let* $\mathbf{C}(v)$ *be a circulant matrix. The vectors* $\frac{1}{\sqrt{N}} w(\omega_N^m)$ *for* $m = 0, 1, \cdots N-1$ *form* N *orthonormal eigenvectors of* $\mathbf{C}(v)$*. The eigenvalue of* $w(\omega_N^m)$

is,

$$\sum_{k=0}^{N-1} v^k \exp\left(\frac{2\pi i}{N} mk\right).$$

Proof. Using Proposition C.4.3 we only need to show the vectors $w(\omega_N^m)$ are orthonormal. We compute the inner product,

$$\langle w(\omega_N^u), w(\omega_N^v)\rangle = \sum_{k=0}^{N-1} \omega_N^{uk}\omega_N^{-vk} = \sum_{k=0}^{N-1} \omega_N^{(u-v)k} = \left\{ \begin{array}{ll} N & \text{when } m = 0 \\ 0 & \text{when } m \neq 0 \end{array} \right\}$$

This follows from Proposition 139. □

Remark C.4.5. *If $\mathbf{C}(v)$ is a circulant matrix that is tri-diagonal then v has the form,*

$$v = \left[v^0, v^1, 0, \cdots, 0, v^{N-1}\right]$$

The eigenvalues of this matrix are given by Proposition C.4.3, and we denote them by λ_k,

$$\lambda_k = \sum_{j=0}^{N-1} v^j \omega_N^{kj} = v^{N-1}\omega_N^{-k} + v^0 + v^1\omega_N^k$$

Using Euler's Formula,

$$\lambda_k = v^0 + \left(v^1 + v^{N-1}\right) cos\left(\frac{2\pi}{N}k\right) + i\left(v^1 - v^{N-1}\right) sin\left(\frac{2\pi}{N}k\right) \tag{159}$$

If the matrix $\mathbf{C}(v)$ is also row-sum zero then we have,

$$v^{-1} + v^0 + v^{-1} = 0.$$

In this case the eigenvalues are given by,

$$\lambda_k = v^0 \left(1 - cos\left(\frac{2\pi}{N}k\right)\right) + i\left(v^1 - v^{N-1}\right) sin\left(\frac{2\pi}{N}k\right). \tag{160}$$

If $v^0 < 0$ then this entire series of eigenvalues lies in the left half complex plane. This means that the eigenvalues of the matrix $\mathbf{C}(v)$ are stable.

The following follows immediately from the definitions.

Proposition C.4.6.

$$\mathbf{P}_+ w(\omega_N^m) = \omega_N^{-m} w(\omega_N^m) \tag{161}$$

$$\mathbf{P}_- w(\omega_N^m) = \omega_N^{m} w(\omega_N^m) \tag{162}$$

C.5 Almost Circulant

In this section we state a prove a Proposition that appears in [20].

Proposition C.5.1. *Let $\mathbf{D}_n$ be the determinant,*

$$\mathbf{D}_n = \begin{vmatrix} (c_1+d_1) & -c_1 & 0 & 0 & \cdots & 0 & 0 \\ -d_2 & (d_2+c_2) & -c_2 & 0 & \cdots & 0 & 0 \\ 0 & -d_3 & (d_3+c_3) & -c_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & (d_{n-1}+c_{n-1}) & -c_{n-1} \\ 0 & 0 & 0 & 0 & \cdots & -d_n & (d_n+c_n) \end{vmatrix}$$

then the determinant is given by,

$$\mathbf{D}_n = (d_1 \cdots d_n) + (d_1 \cdots d_{n-1} c_n) + \cdots + (d_1 c_2 \cdots c_{n-1} c_n) + (c_1 \cdots c_n).$$

Proof. We proceed by induction and use the general form for determinants of tri-diagonal matrices. The formula is easily derived from the definition of the determinant and has the form,

$$\mathbf{D}_n(1,n) = (c_1 + d_1)\mathbf{D}_{n-1}(2,n) - c_1 d_2 \mathbf{D}_{n-2}(3,n), \tag{163}$$

where $\mathbf{D}_{n-1}(k_1,k_2)$ is the determinant of the $k_2 - k_1 + 1$ square matrix with row and column indices between k_1 and k_2 inclusive. The cases $n = 3$ and $n = 4$ are easily computed directly. The case for general n can be proved by induction using equation (163). $\square$

Appendix D COORDINATE TRANSFORM

Let V be a vector space of dimension N. Define the set of index values,

$$\mathcal{N} = \{0, 1, \cdots, N-1\}$$

Let $\sigma \in S_N$ be a permutation of the index values $\mathcal{N}$. For each permutation there is a natural representation as a linear transform $\mathbf{S}_\sigma : V \to V$ defined by,

$$\mathbf{S}_\sigma \begin{pmatrix} v^0 \\ v^1 \\ v^2 \\ \vdots \\ v^{N-1} \end{pmatrix} = \begin{pmatrix} v^{\sigma 0} \\ v^{\sigma 1} \\ v^{\sigma 2} \\ \vdots \\ v^{\sigma(N-1)} \end{pmatrix}.$$

This is the standard permutation representation. This satisfies, $\mathbf{S}_{\sigma\rho} = \mathbf{S}_\sigma \mathbf{S}_\rho$. We only need the special case of this,

$$\mathbf{S}_\sigma^{-1} = \mathbf{S}_{\sigma^{-1}}.$$

Our interest is a one specific permutation when $N = pq$, that is given by

$$\sigma(k + mq) = m + kp, \tag{164}$$

where $k = 0, 1, \cdots, (q-1)$ and $m = 1, 2, \cdots, (p-1)$. Apply this permutation to the natural index order and we get

$$[\sigma 0, \sigma 1, \sigma 2, \cdots \sigma(N-1)] = [0, p, 2p, \cdots, N-1],$$

which is the same as the ordering in equation (68). If v is a $q-$vector and w a $p-$vector then the tensor product can be written as a $N = pq$ vector as,

$$v \otimes w = \left[v^0 w^0, v^0 w^1, \cdots v^0 w^{p-1}, v^1 w^0, v^1 w^1, \ldots, v^{q-1} w^{p-1}\right]^T$$

We have,

$$\mathbf{S}_\sigma (v \otimes w) = w \otimes v.$$

Proposition D.0.1. *If A is $q \times q$ and B is $p \times p$ then the permutation in equation* (164) *has the property that,*

$$\mathbf{S}_\sigma (A \otimes B) \mathbf{S}_{\sigma^{-1}} = B \otimes A.$$

Proof. For any $p-$vector w and $q-$vector v then we have,

$$\left(\mathbf{S}_\sigma (A \otimes B) \mathbf{S}_{\sigma^{-1}}\right)(w \otimes v) = \mathbf{S}_\sigma (A \otimes B) \mathbf{S}_{\sigma^{-1}} \left(\mathbf{S}_\sigma (v \otimes w)\right) = \mathbf{S}_\sigma (Av \otimes Bw)$$
$$= (B \otimes A)(w \otimes v)$$

□

Appendix E IDENTICAL AGENT CALCULATION IN CANTOS

Section 1.5 contains a review of an N identical agents with nearest neighbor interactions, as described in Cantos [5, 6]. In this section we compare our results to this previous work. Cantos defines the following,

$$I_{x,k} = -g_x \sum_{j \in N} \rho_{x,j} j^k = -g_x \left(\rho_{x,+} + \rho_{x,-}(-1)^k \right), \tag{165}$$

$$I_{v,k} = -g_x \sum_{j \in N} \rho_{x,j} j^k = -g_x \left(\rho_{x,+} + \rho_{x,-}(-1)^k \right), \tag{166}$$

Cantos also defines,

$$a = \frac{I_{v,1}^2}{4} + \frac{I_{x,2}}{2} = \frac{g_v^2 \beta_v^2}{4} + \frac{-g_x(-1)}{2} = \frac{g_x^2}{4} \left(\beta_v^2 \frac{g_v^2}{g_x^2} + 2\frac{1}{g_x} \right)$$

Comparing this to equation (121) we get,

$$\sqrt{a} = \pm \frac{g_x}{2} \Gamma_R. \tag{167}$$

According to Proposition 4 in [6] the first order term in the polynomial expansion is given by,

$$i\phi \left(\frac{I_{v,1}}{2} \pm a^{1/2} \right) = \frac{1}{2(1/g_x)} \left(-\beta_v \frac{g_v}{g_x} \pm \sqrt{\beta_v^2 \frac{g_v^2}{g_x^2} + 2\frac{1}{g_x}} \right) \tag{168}$$

This is the same form as equation (107). Note that in equation (107) we have,

$$\frac{\phi}{p} = \frac{2\pi m}{qp} = \frac{2\pi m}{N},$$

which is the meaning of ϕ in Cantos. The second order term in Cantos is

$$-\phi^2\frac{-I_{v,2}}{4} - \pm\frac{1}{2a^{1/2}}\left(\frac{I_{v,1}I_{v,2}}{4} + \frac{I_{x,3}}{6}\right)$$
$$= -\phi^2\left(\frac{g_v}{4} \pm \frac{1}{g_x\Gamma_R}\left(\frac{-g_v^2\beta_v}{4}\right)\right) = -\phi^2\frac{1}{4(1/g_x)}\left(g_v/g_x \pm \frac{1}{\Gamma_R}\left(-(g_v/g_x)^2\beta_v\right)\right)$$

To compare this to Theorem 2.5.1 notice that when the agents are identical, we have,

$$M = \frac{1}{g_x} \qquad V = \frac{g_v}{g_x} \tag{169}$$

$$C = A = F = 0. \tag{170}$$

With this notation the second order term in Cantos is,

$$-\phi^2\frac{1}{4M}\left(V \pm \frac{-V^2\beta_v}{\Gamma_R}\right) \tag{171}$$

We see the results in Section 2.5 agree with Cantos when equations (169) and (170) are used.

Remark E.0.1. *Given a one-dimensional nearest neighbor system one can compute M and V. Now use the values,*

$$g_x = \frac{1}{M} \qquad g_v = \frac{V}{M}$$

Using these values, you can use the results in Cantos to approximate the solutions. If the variances of the distributions $g_x^{(\alpha)}$ and $g_v^{(\alpha)}$ are not too large this will give a rough approximation to the large-scale system dynamics.

Appendix F TWO DISTINCT PHASE VELOCITIES

In this section we will assume that a one-dimensional flock has two phase velocities with opposite signs, and derive some consequences. In this section the values c_+ and c_- correspond to the values $c_{1,+}$ and $c_{1,-}$. They are shortened for convenience. In Assumptions F.0.1 we specify the system and then derive some system properties.

Assumptions F.0.1. *Assume we have a one dimensional flock with equations of motion that are linear second order ordinary differential equations. Assume that all eigenvectors have one of two phase velocities, given by c_+ and c_- and assume that c_+ and c_- are two real numbers that satisfy,*

$$c_+ > 0$$
$$c_- < 0.$$

This means that a general solution for a particular agent has the form,

$$z_k(t) = f_+ \left(t + \frac{k}{c_+} \right) + f_- \left(t + \frac{k}{c_-} \right). \tag{172}$$

We assume that z_k satisfies the constant velocity boundary conditions of Definition 2.6.6. We define a system y_k that obeys the same ODE and satisfies

$$y_k(t) = \frac{d^2 z_k}{dt^2}. \tag{173}$$

From this we can prove the following.

Proposition F.0.2. *Assume the system satisfies the assumptions described in F.0.1. Let $y_k(t)$ satisfy the following boundary conditions,*

1. *Agent* 0 *satisfies* $y_0(t) = p(t)$ *(see equation* (134)*).*

2. Agent $N-1$ satisfies $\frac{\partial y_k}{\partial k}\big|_{k=N-1} = 0$ (see Definition 2.6.8).

The agents satisfy,

$$\begin{aligned} y_j(t) =& p\left(t + \frac{j}{c_-}\right) \\ &+ \sum_{k=1}^{\infty} \left(\frac{c_+}{c_-}\right)^k \left(p\left(t + \frac{j}{c_-} - kP\right) v_0 - p\left(t + \frac{j}{c_+} - kP\right) v_0\right) \end{aligned} \tag{174}$$

where we have defined

$$P = (N-1)\left(\frac{1}{c_+} - \frac{1}{c_-}\right) \tag{175}$$

Proof. We apply the boundary conditions to

$$y_k(t) = f_+(t + k/c_+) + f_-(t + k/c_-) \tag{176}$$

Condition 1 in Proposition F.0.2 is

$$y_0(t) = f_+(t) + f_-(t) = p(t)v_0. \tag{177}$$

We are assuming that f_+ and f_- are differentiable so condition 2 Proposition F.0.2 is

$$\frac{\partial y_k}{\partial k}\bigg|_{N-1} = \frac{1}{c_+} f_+'(t + (N-1)/c_+) + \frac{1}{c_-} f_-'(t + (N-1)/c_-) = 0.$$

We can integrate this equation and re-arrange terms to get the condition

$$f_+(s) = -\left(\frac{c_+}{c_-}\right) f_-(s - (N-1)(1/c_+ - 1/c_-)) = -\left(\frac{c_+}{c_-}\right) f_-(s - P). \tag{178}$$

Insert this into equation (177) to get the recursive relationship

$$f_-(t) = p(t)v_0 - f_+(t) = p(t)v_0 + \left(\frac{c_+}{c_-}\right) f_-(s-P). \tag{179}$$

The function $p(t)v_0$ has compact support (see Proposition 2.6.7) so after a finite number of steps we get

$$f_-(t) = \sum_{k=0}^{\infty} \left(\frac{c_+}{c_-}\right)^k p\left(t - kP\right) v_0 \tag{180}$$

By equation (177) we get

$$f_+(t) = p(t)v_0 - f_-(t) = -\sum_{k=1}^{\infty} \left(\frac{c_+}{c_-}\right)^k p\left(t - kP\right) v_0 \tag{181}$$

Insert equations (180) and (181) into equation (176) to get

$$\begin{aligned} y_j(t) &= -\sum_{k=1}^{\infty} \left(\frac{c_+}{c_-}\right)^k p\left(t + \frac{j}{c_+} - kP\right) v_0 + \sum_{k=0}^{\infty} \left(\frac{c_+}{c_-}\right)^k p\left(t + \frac{j}{c_-} - kP\right) v_0 \\ &= p\left(t + \frac{j}{c_-}\right) v_0 + \sum_{k=1}^{\infty} \left(\frac{c_+}{c_-}\right)^k \left(p\left(t + \frac{j}{c_-} - kP\right) v_0 - p\left(t + \frac{j}{c_+} - kP\right) v_0\right) \end{aligned}$$

Equation (174) follows from this. □

Corollary F.0.3. *With the system of Proposition F.0.2 we have following trajectory for the tail agent $N-1$,*

$$y_{N-1}(t) = \left(\frac{c_- - c_+}{c_-}\right) \sum_{k=0}^{\infty} \left(\frac{c_+}{c_-}\right)^k p\left(t + \frac{(N-1)}{c_-} - kP\right) v_0$$

Proof. Set $j = N$ in equation (174) to get

$$\begin{aligned} y_{N-1}(t) =& p\left(t + \frac{(N-1)}{c_-}\right) v_0 \\ &+ \sum_{k=1}^{\infty} \left(\frac{c_+}{c_-}\right)^k \left(p\left(t + \frac{N-1}{c_-} - kP\right) - p\left(t + \frac{N-1}{c_+} - kP\right)\right) v_0 \end{aligned}$$

Use the following,

$$\begin{aligned} p\left(t + \frac{(N-1)}{c_-}\right) v_0 - \sum_{k=1}^{\infty} \left(\frac{c_+}{c_-}\right)^k \left(p\left(t + \frac{(N-1)}{c_+} - kP\right) v_0\right) \\ =& p\left(t + \frac{N-1}{c_-}\right) v_0 - \left(\frac{c_+}{c_-}\right) p\left(t + \frac{(N-1)}{c_-}\right) v_0 \\ & - \left(\frac{c_+}{c_-}\right) \sum_{k=1}^{\infty} \left(\frac{c_+}{c_-}\right)^k \left(p\left(t + \frac{(N-1)}{c_-} - kP\right) v_0\right) \end{aligned}$$

We combine both sums using

$$1 + \left(\frac{c_+}{c_-}\right) = \frac{c_- + c_+}{c_-}.$$

The result follows. ☐

Remark F.0.4. *The signs are a bit confusing. Corollary F.0.3 says that $y_N(t)$ is zero until,*

$$t + \frac{(N-1)}{c_-} = t - \frac{(N-1)}{|c_-|}$$

is with ϵ of 0*, so $t \approx \frac{N-1}{|c_-|}$. At this point y_N starts moving to the right as,*

$$\frac{c_- - c_+}{c_-} = \frac{-|c_-| - |c_+|}{-|c_-|} = \frac{|c_-| + |c_+|}{|c_-|} > 0.$$

From this we can prove the following

Theorem F.0.5. *Assume that the system $z_k(t)$ is stable and satisfies the 'regular'* *boundary conditions that also satisfies the "constant velocity boundary conditions" of Definition 2.6.6. Also assume also that*

$$z_k(t) = f_+\left(t + \frac{k}{c_{1,+}}\right) + f_-\left(t + \frac{k}{c_{1,-}}\right).$$

For large N and some constant K_0 the trajectory of the last agent $z_N(t)$ can be approximated by the following properties.

- $T_k = \frac{(N-1)}{-c_-} + (k-1)\left(\frac{(N-1)}{c_+} - \frac{(N-1)}{c_-}\right)$ *for* $k = 1, \cdots K_0$,
- $A_k = \left(\frac{c_+}{c_-}\right)^{k-1} \frac{(N-1)v_0}{c_-}$ *for* $k = 1, \cdots K_0$,

where T_k is the time to the kth extremal distance $z_0 - z_N$ and A_k is the difference $z_{N-1}(T_k) - z_0(T_k)$. When converting back to x_k this will be the distance from the stable point.

Proof. The system z_k satisfies,

$$y_k(t) = \frac{d^2 z_k}{dt} = \frac{d\dot{z}_k}{dt}.$$

We integrate the formula for $y_N(t)$ in Corollary F.0.3. From Proposition 2.6.7 we know that $p(t)$ has support in $[-\epsilon, \epsilon]$ so the velocity $\dot{z}_N(t)$ only changes in neighborhoods of the times

$$T_k = -\frac{(N-1)}{c_-} + kP = \frac{(N-1)}{|c_-|} + kP.$$

In the neighborhood of T_k the velocity changes by

$$\Delta\dot{z} = \int_{T_k-\epsilon}^{T_k+\epsilon} y_N(t)dt = \left(\frac{c_- - c_+}{c_-}\right)\left(\frac{c_+}{c_-}\right)^{k-1} v_0.$$

For the time $t < T_1 - \epsilon$, agent $N-1$ is not moving and agent y_0 is moving with velocity v_0. So after T_1 the distance is very close to,

$$A_1 = -T_1 v_0 = \frac{(N-1)v_0}{c_-}.$$

After time $T_1 + \epsilon$ agent $N-1$ has velocity

$$-v_0 + \left(\frac{c_- - c_+}{c_-}\right) v_0 = -\left(\frac{c_+}{c_-}\right) > 0.$$

This proceeds for P time and we get

$$A_2 = A_1 + \left(-\frac{c_+}{c_-}\right) P v_0 = \left(\frac{c_+}{c_-}\right) \frac{(N-1)}{c_-} v_0,$$

which is now positive. The result now follows from induction. The induction to find the velocity u_k between $T_k + \epsilon$ and $T_{k+1} - \epsilon$ is

$$u_k = -\left(\frac{c_+}{c_-}\right)^{k-1} v_0 + \left(\frac{c_- - c_+}{c_-}\right)\left(\frac{c_+}{c_-}\right)^{k-1} v_0 = -\left(\frac{c_+}{c_-}\right)^{k} v_0.$$

The induction step for the value A_k is

$$A_{k+1} = A_k + u_k \left(T_{k+1} - T_k\right) = \left(\frac{c_+}{c_-}\right)^{k-1} \left(\frac{1}{c_-} - \frac{c_+}{c_-}\left(\frac{1}{c_+} - \frac{1}{c_-}\right)\right)(N-1)v_0.$$

$\square$

Appendix G SOUND WAVES

In this section we derive the PDE for a sound wave in a pipe with constant cross-section. This section follows the ideas in [10] which covers the case of small perturbations of both density and pressure. See [11] for details of the more general case. The pipe is assumed to have a constant area A, and the result is a $1-$dimensional PDE.

In this section we ignore viscosity, even though this is important for the system in Sections 2.5 and H. A sound wave is an increase in density and pressure over the ambient atmosphere. Let P_0 and ρ_0 be the ambient pressure and air density respectively. A sound wave produces a very small increase to pressure and density given by,

$$P = P_0 + P_s \tag{182}$$

$$\rho = \rho_0 + \rho_s. \tag{183}$$

We assume that P is a function of ρ near the point ρ_0 so that $P_0 = \lambda(\rho_0)$. For small over-pressures ρ_s we have,

$$\begin{aligned} P(\rho) = \lambda(\rho) &= \lambda(\rho_0 + \rho_s) = \lambda(\rho_0) + \lambda'(\rho_0)\rho_s \\ &= P_0 + P_s. \end{aligned}$$

This means that the over-pressure is given by

$$P_s = \lambda'(\rho_0)\rho_s. \tag{184}$$

Let $D(x,t)$ be the distance the air at x is offset due to the sound wave. Conservation of mass means that the air in $[x, x+\Delta x] \times A$ is the same as in

$$[x + D(x,t), x + \Delta x + D(x+\Delta x, t)] \times A.$$

To get the mass we must multiply by the density and we get the conversation of mass.

$$\begin{aligned}
\rho_0 \left(x + \Delta x - x\right) A = \rho_0 \Delta x A \\
&= (\rho_0 + \rho_s)\left(x + \Delta x + D(x+\Delta x, t) - x - D(x,t)\right) A \\
&= (\rho_0 + \rho_s)\left(\Delta x + D(x,t) + \frac{\partial D}{\partial x}\Delta x - D(x,t)\right) A \\
&= (\rho_0 + \rho_s)\left(\Delta x + \frac{\partial D}{\partial x}\Delta x\right) A
\end{aligned}$$

From this we get the following conservation of mass equation,

$$\rho_s = -(\rho_0 + \rho_s)\frac{\partial D}{\partial x} = -\rho_0 \frac{\partial D}{\partial x}. \tag{185}$$

This last equation follows because, for a sound wave,

$$\rho_s <<< \rho_0.$$

We use Newton's Laws and write down the equations of motion for a slice of air. The air in the region $[x, x+\Delta x] \times A$ is the volume of a thin slice of air in the sound pipe that moves to $[x + D(x,t), x + \Delta x + D(x+\Delta x, t)] \times A$. The total force on the air slice is the pressure difference times the area A. So Newton's Law is

$$(P(x) - P(x+\Delta x))A = \rho_0 \Delta x A \frac{\partial^2 D}{\partial t^2} \tag{186}$$

Now we expand, using Taylor Series,

$$P(x) - P(x) - \frac{\partial P}{\partial x}\Delta x A = -\frac{\partial P_s}{\partial x}\Delta x A = \rho_0 \Delta x A \frac{\partial^2 D}{\partial t^2}$$

So Newton's Law is just,

$$\frac{\partial P_s}{\partial x} = -\rho_0 \frac{\partial^2 D}{\partial t^2} \tag{187}$$

We use the equation of state in equation (184), conservation of mass in (185) and Newton's Law's in (187) to derive our PDE. First substitute equation (184) into equation (185) to get,

$$P_s = \rho_s \lambda'(\rho_0) = -\rho_0 \lambda'(\rho_0) \frac{\partial D}{\partial x}. \tag{188}$$

We start with equation (187) and use equation (188) to get

$$\rho_0 \frac{\partial^2 D}{\partial t^2} = \rho_0 \lambda'(\rho_0) \frac{\partial^2 D}{\partial x^2}.$$

We write this as,

$$\frac{1}{\lambda'(\rho_0)} \frac{\partial^2 D}{\partial t^2} - \frac{\partial^2 D}{\partial x^2} = 0. \tag{189}$$

This is a wave equation with the speed of sound given by

$$v_s^2 = \lambda'(\rho_0). \tag{190}$$

Appendix H PDE ANALOG

In this section we present a heuristic argument that connects our discrete agent system to a PDE that has the form in equation (197). We start with the basic agent ODE,

$$\begin{aligned}\frac{d^2 z_k}{dt^2} =& - g_{x,N}\left(z_k + \rho_{x,1} z_{k+1} + \rho_{x,-1} z_{k-1}\right) \\ & - g_{v,N}\left(\dot{z}_k + \rho_{v,1}\dot{z}_{k+1} + \rho_{v,-1}\dot{z}_{k-1}\right) \end{aligned} \tag{191}$$

We restrict out discussions to systems where the constants $g_{x,N}, g_{v,N}, \rho_{x,\pm}, \rho_{v,\pm}$ are all independent of the agent. The N agents attempt to keep a spacing of Δ. What happens as $N \to \infty$. We have added a subscript N to the constants g_x, g_v, as these values will scale as $N \to \infty$.

We want the agents to be spread over a compact interval of length L so as $\Delta \to 0$ we have

$$N\Delta = L. \tag{192}$$

You could consider systems that are spread over an infinite length, but we shall not do so. As N grows the number of agents per unit length, grows. As a result, the total mass of the system grows. We'd like keep the total mass of the system constant as we take the limit. If m_N is the mass of each agent in the N-agent system, then this requirement is

$$N m_N = \mu L \Rightarrow m_N = \frac{\mu L}{N} = \mu\Delta,$$

where μ is the mass per unit length of the system. Equation (191) does not have an explicit mass. Instead it has the factors $g_{x,N}$ and $g_{v,N}$. If we write the harmonic oscillator in the form of equation (191) then we would set $g_{v,N} = 0$ and $g_{x,N} = \frac{K_h}{m_a}$, where K_h is "Hooke's Constant". Let's take sound as the analogous system in the continuous domain. In this case the system "stiffness" is just Young's modulus. If Y is Young's modulus we have

$$Y = \frac{F}{(\Delta L/L)}$$

where F is the force applied to stretch or compress the system from L to $L + \Delta L$. We relate this to the Hooke's constant of each agent as follows,

$$Y = \frac{F}{(\Delta L/L)} = \frac{k_H L \Delta L/N}{\Delta L} = k_H \Delta.$$

Young's modulus will remain constant as we take the limit so, as the spacing gets smaller, we require a smaller agent displacement to achieve the same force. This means Hooke's constant must increase. We put all this together and g_x should scale as

$$g_{x,N} = \frac{k_H}{m_N} = \frac{1}{\Delta^2}\frac{Y}{\mu} \tag{193}$$

and the values $\frac{Y}{\mu}$ is a constant. A similar argument applies to g_v and we shall just write

$$g_{v,N} = \frac{1}{\Delta^2}\frac{Y_v}{\mu} \tag{194}$$

The basic computation turning the agent equation into a PDE approximation is the following

$$\begin{aligned}
z_k + \rho_{x,1} z_{k+1} + \rho_{x,-1} z_{k-1} &= z_k - \frac{1}{2} z_{k+1} - \frac{1}{2} z_{k-1} + \left(\rho_{x,1} + \frac{1}{2}\right) z_{k+1} + \left(\rho_{x,-1} + \frac{1}{2}\right) z_{k-1} \\
&= -\frac{\Delta}{2}\left(\frac{z_{k+1} - z_k}{\Delta} - \frac{z_k - z_{k-1}}{\Delta}\right) + \frac{\beta_v}{2}\left(z_{k+1} - z_{k-1}\right) \\
&= -\frac{\Delta^2}{2}\left(\frac{\frac{z_{k+1}-z_k}{\Delta} - \frac{z_k - z_{k-1}}{\Delta}}{\Delta}\right) + \frac{\beta_v \Delta}{2}\left(\frac{z_{k+1} - z_{k-1}}{\Delta}\right)
\end{aligned}$$

We want to take a limit as $N \to \infty$ and $\Delta \to 0$. This term approaches,

$$-\frac{\Delta^2}{2}\frac{\partial^2 u}{\partial x^2} + \frac{\beta_x \Delta}{2}\frac{\partial u}{\partial x} \tag{195}$$

The terms in $\dot{z}$ are handled in an identical way except that the terms become

$$-\frac{\Delta^2}{2}\frac{\partial^3 u}{\partial x^2 \partial t} + \frac{\beta_v \Delta}{2}\frac{\partial^2 u}{\partial x \partial t} \tag{196}$$

To formulate the PDE we start with equation (191) and use our difference formulas. As $N \to \infty$ we get

$$\begin{aligned} u_{tt} &= \frac{g_{x,N}\Delta^2}{2}u_{xx} + \frac{g_{x,N}\Delta^2}{2}u_{xxt} + \frac{\beta_x \Delta}{2}u_x + \frac{\beta_v \Delta}{2}u_{xt} \\ &= \frac{Y}{2\mu}u_{xx} + \frac{Y_v}{2\mu}u_{xxt} + \frac{\beta_x \Delta}{2}u_x + \frac{\beta_v \Delta}{2}u_{xt} \end{aligned}$$

The scaling of the β_x, β_v terms is not as obvious. We will ignore this difficulty and write the PDE as,

$$u_{tt} = \frac{G_x}{2}u_{xx} + \frac{G_v}{2}u_{xxt} + B_x u_x + B_v u_{xt}, \tag{197}$$

where G_x, G_v, B_x, B_v are all constants.

If we set $B_x = B_v = 0$ then we get

$$u_{tt} = \frac{G_x}{2}u_{xx} + \frac{G_v}{2}u_{xxt}. \tag{198}$$

This is the equation of sound in a viscous fluid. Section G derives the basic equations of motion for the non-viscous case. There is some literature on the viscous case [12, 26] but we shall not discuss the physics any further. We shall also, not go into the solution details except to point out one specific solution. Assume that

$$u(x,t) = e^{i(kx-\omega t)}.$$

Plug this into equation (198) and we get the dispersion relation

$$\begin{aligned}
(-i\omega)^2 u &= \frac{G_x}{2}(ik)^2 u + \frac{G_v}{2}(ik)^2(-i\omega)u \\
\Rightarrow & \omega^2 + i\frac{G_v}{2}k^2\omega - \frac{G_x}{2}k^2 = 0. \\
\Rightarrow & \omega = \frac{-iG_v k^2}{4} \pm k\sqrt{\frac{G_x}{2} - \left(\frac{G_v k}{4}\right)^2}
\end{aligned}$$

We write out solution as

$$u(x,t) = \exp\left(\frac{-G_v k^2}{4} t\right) \exp\left(i(kx \pm \omega_i(k)t)\right), \text{ where} \tag{199}$$

$$\omega_i(k) = k\sqrt{\frac{G_x}{2} - \left(\frac{G_v k}{4}\right)^2} \tag{200}$$

Equations (199) and (200) are analogous to the solutions in Corollary 2.5.4. The equations in this section assumed that g_x and g_v are independent of agent which means that G_x and G_v are just constants. This means that we have

$$\begin{aligned}
\text{Avg}\left(\frac{1}{g_x^{(k)}}\right) &= \frac{1}{g_x} \Rightarrow c_{1,\pm} = \pm\sqrt{\frac{g_x}{2}}, \\
\text{Avg}\left(\frac{g_v^{(k)}}{g_x^{(k)} g_x^{(k)}}\right) &= \frac{g_v}{g_x^2}. \Rightarrow c_{2,\pm} = \frac{g_v}{4}.
\end{aligned}$$

The continuous solution makes the association

$$k \to \frac{\phi}{p} = \frac{2\pi}{pq}m = \frac{2\pi}{N}m.$$

The solution of equations (199) and (200) correspond to the solutions of Corollary 2.5.4 except, there is an extra term of the form $-\left(\frac{G_v k}{4}\right)^2$ in equation (200). Although this is not in our quadratic approximation, it may be in the higher order solutions of [6, 5].

The more general problem, dealt with in Theorem 2.5.1, would require that G_x and G_v depend on x. The solution to this more complicated PDE is beyond the scope of this research.