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Quaternions, Octonions, and the Quantization of Games

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QUATERNIONS, OCTONIONS, AND THE QUANTIZATION OF GAMES

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ABSTRACT

An abstract of the dissertation of Aden Omar Ahmed for the Doctor of Philosophy in Mathematical Sciences presented April 20, 2009.

Title: Quaternions, Octonions, and the Quantization of Games

We present an effect on classical games that is obtained by replacing the notion of probability distribution with the notions of quantum superposition and measurement. Our particular focus will be on two and three player games where each player has precisely two pure strategic choices.

Games in normal form are represented as "payoff" functions. Game quantization requires the extension of these functions to much larger domains. The main result of this work is the co-ordinatization of these extended functions by either the quaternions or octonions in order to obtain computationally friendly versions of these functions. This computational capability is then exploited to analyze and potentially classify the Nash equilibria in the new extended games with occasionally counter intuitive results.

This thesis is dedicated to my wife Choukri, son Sahal, and daughter Amina. They showed a great deal of patience for long hours during my research. I greatly appreciate Choukri's unconditional and ever-lasting support and Sahal and Amina's patience, without which this work was almost impossible.

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QUANTUM INFORMATION

1.1 Introduction

Ordinary computers rely on vast arrays of tiny transistors, arranged in logic units called *gates*, to represent their information. They typically use the presence or absence of certain amount of electric charge to represent the so-called *bits* $|0\rangle$ and $|1\rangle$ of binary code. The hypothetical quantum computer replaces these bits with entities called *quantum bits* or *qubits*, which are two-state quantum systems. Each qubit is represented by a quantum two-level system like the spin of the electron, the two polarization states of a photon, or two of the energy levels in an atom or ion. For instance, one such energy level would correspond to the $|0\rangle$ state and another distinct level, to the $|1\rangle$ state. However, unlike classical digital states (which are discrete), a qubit can actually be in a superposition of the two discrete states at any given time. Currently, the qubit is the most studied unit of quantum information. High order informational units such as *trits* and *dits* have also been studied in the quantum domain [11].

In general, *Quantum information* is physical information represented by a state of a quantum system, for example, by a qubit in a quantum computer. A novel feature in the quantum world which has no classical analog is *entanglement*, when the states of distinct qubits are no longer completely independent, but are in fact correlated, an effect we will be examining closely. *Quantum communication* occurs when quantum information is transmitted, again as an example, via qubits. Such communication between quantum computers is said to be *mediated* when all messages pass through and are coordinated by a central server.

1.2 Hilbert Spaces and Dirac's Notation

In this section, we introduce the basic definitions and concepts necessary to utilize the mathematical framework of quantum mechanics.

For two elements u and v of an n-dimensional complex vector space $\mathbb{C}^n$, the map $\langle\ ,\ \rangle : \mathbb{C}^n \times \mathbb{C}^n \longmapsto \mathbb{C}$ given by

$$\langle u, v \rangle = \overline{u}^T I v, \tag{1.1}$$

where I is the $n \times n$ identity matrix, defines an inner product in $\mathbb{C}^n$. The associated norm $||\cdot||$ is given by

$$||u|| = \sqrt{\langle u, u \rangle}. \tag{1.2}$$

More generally, consider an *n-dimensional Hilbert space* $\mathcal{H}_n$, that is, a n-dimensional vector space over the field of complex numbers with an inner product and associated norm. For example, an n-dimensional complex space $\mathbb{C}^n$ with the standard inner product above is an n-dimensional Hilbert space. In what follows the inner product in our Hilbert spaces is given in this standard way.

A collection of spanning vectors $\{e_1, e_2, \cdots, e_n\}$ in $\mathcal{H}_n$ is called an orthonormal basis if

$$\langle e_i, e_j \rangle = \begin{cases} 0 & \text{if } i \neq j; \\ 1 & \text{if } i = j. \end{cases}$$

for all $i, j \in \{1, 2, \cdots, n\}$.

Amongst the many possible choices of orthonormal basis vectors in $\mathcal{H}_n$, we select the standard basis vectors. We use the notation of quantum mechanics for vectors in $\mathcal{H}_n$ introduced by Dirac [15] and denote the vectors of the standard basis as *kets*

$$\{|0\rangle, |1\rangle, \cdots, |i\rangle, \cdots, |n-1\rangle\}, \tag{1.3}$$

and their canonical dual vectors as *bras*

$$\{\langle 0|, \langle 1|, \cdots, \langle i|, \cdots, \langle n-1|\}. \tag{1.4}$$

In the more traditional matrix notation used by Heisenberg, each ket vector $|i\rangle$ is represented as a column vector with a 1 in the $i^{th} + 1$ row and 0 in all the others. In symbols

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{pmatrix}, |1\rangle = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \cdots, |i\rangle = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \cdots, |n-1\rangle = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 1 \end{pmatrix}. \tag{1.5}$$

In turn, each bra vector $\langle j|$ is expressed as a row vector with 1 in the $j^{th} + 1$ column and 0 in all the others.

An n-dimensional ket vector $|\psi\rangle$ is thus expressed in the standard basis as a linear combination

$$|\psi\rangle = \alpha_0|1\rangle + \alpha_1|1\rangle + \cdots + \alpha_i|i\rangle + \cdots + \alpha_{n-1}|n-1\rangle, \tag{1.6}$$

where $\alpha_0, \alpha_1, \cdots, \alpha_i, \cdots, \alpha_{n-1}$ are complex numbers.

The inner product of two vectors $|\psi_a\rangle$ and $|\psi_b\rangle$ in $\mathcal{H}_n$ is a complex number. In the Dirac notation, the inner product of $|\psi_a\rangle$ and $|\psi_b\rangle$ is denoted by

$$\langle\psi_a \mid \psi_b\rangle. \tag{1.7}$$

As an example, if $|\psi_a\rangle$ and $|\psi_b\rangle$ are elements of $\mathcal{H}_2$, and

$$|\psi_a\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle \tag{1.8}$$

$$|\psi_b\rangle = \beta_0|0\rangle + \beta_1|1\rangle \tag{1.9}$$

then

$$\langle\psi_a \mid \psi_b\rangle = (\overline{\alpha}_0 \;\; \overline{\alpha}_1)\begin{pmatrix}\beta_0\\ \beta_1\end{pmatrix} = \overline{\alpha}_0\beta_0 + \overline{\alpha}_1\beta_1 \tag{1.10}$$

The norm of a vector $|\psi\rangle \in \mathcal{H}_n$, that is, the square root of the inner product of the vector $|\psi\rangle$ with itself, is thus expressed as

$$||\, |\psi\rangle \,|| = \sqrt{\langle\psi \mid \psi\rangle}. \tag{1.11}$$

1.2.1 Tensor Products

Begin with two vector spaces over the same field F, V that is n-dimensional, and W that is m-dimensional. The *tensor product* $V \otimes W$ of the vector spaces V and W is an nm-dimensional vector space over F and is defined as follows. Let $\{v_1, v_2, \cdots, v_n\}$ be a basis for V and $\{w_1, w_2, \cdots, w_m\}$ a basis for W. Define now nm basis vectors $v_i \otimes w_j$ as formal objects, where $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, m$. The tensor product space $V \otimes W$ has basis

$$\{v_i \otimes w_j \mid i = 1, 2, \cdots, n \text{ and } j = 1, 2, \cdots, m\} \tag{1.12}$$

and consists of the complex linear combinations of the formal basis elements just defined.

The tensor product is bilinear in the basis. That is, for $v = \sum_{i=1}^{n} \alpha_i v_i \in V$ and $w = \sum_{j=1}^{m} \beta_j w_j \in W$,

$$v \otimes w = \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \beta_j v_i \otimes w_j. \tag{1.13}$$

As an example, consider the basis vectors $|0\rangle$ and $|1\rangle$ of $\mathcal{H}_2$. Their tensor product is an element of $\mathcal{H}_4$ and is given in matrix notation by the so-called Kronecker product

$$|0\rangle \otimes |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}. \tag{1.14}$$

We can extend this notation to the *tensor product of two linear operators*, represented as the tensor product of two matrices again given by the Kronecker product. For example, if

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1q} \\ b_{21} & b_{22} & \cdots & b_{2q} \\ \vdots & \vdots & & \vdots \\ b_{p1} & b_{p2} & \cdots & b_{pq} \end{pmatrix} \tag{1.15}$$

are $m \times n$ and $p \times q$ matrices, respectively, then the tensor product of A and B is an $mp \times nq$ matrix and is given by

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix} \tag{1.16}$$

1.3 Axioms of Quantum Mechanics

In this section, we present a brief review of the axioms of quantum mechanics which form the foundations of quantum computation. For more details on the axioms of quantum mechanics, the reader is referred to [43].

1.3.1 Axiom 1: Co-ordinatization

The first axiom states that any *isolated* physical system is associated to a complex vector space with inner product (i.e. a Hilbert space) known as the state space of the system. The system is completely described by its state vectors, which are *projective vectors* in the system's state space.

A classical bit can be in one of two states $|0\rangle$ or $|1\rangle$. In contrast, the state $|\psi\rangle$ of a qubit is mathematically described by a vector in a two-dimensional *projective Hilbert space*, that is, a two-dimensional Hilbert space in which a vector is defined up to non-zero complex scalar multiplication. In the Dirac notation, this state can be represented by

$$|\psi\rangle \equiv \alpha_0|0\rangle + \alpha_1|1\rangle \tag{1.17}$$

where α_0 and α_1 are complex numbers, not all zero and $|0\rangle$, $|1\rangle$ are the standard basis vectors of $\mathbb{C}^2$ called *the computational basis states*. The state $|\psi\rangle$ is called a *superposition* of the basis vectors and is a representative element of an equivalence class of states (describing the same physical state) that differ by multiplication by a nonzero complex scalar, called a *phase*. That is, one regards

$$|\psi\rangle \equiv \lambda|\psi\rangle \tag{1.18}$$

for all nonzero complex numbers λ. The state $|\psi\rangle$ as in Equation (1.17) is said to be *normalized* if

$$|\alpha_0|^2 + |\alpha_1|^2 = 1, \tag{1.19}$$

where $|\alpha_i|$ represents the length of the complex number α_i.

The co-ordinatization of multi-object quantum systems is done by taking the tensor product of the state spaces of the component quantum systems.

Continuing our qubit example, let $|\psi_1\rangle = \alpha|0\rangle + \beta|1\rangle$ and $|\psi_2\rangle = \gamma|0\rangle + \delta|1\rangle$ be states for single qubits. Then this axiom says that a vector describing a joint state $|\psi\rangle$ of the two qubit-system is expressed in the form

$$\begin{aligned} |\psi\rangle &= |\psi_1\rangle \otimes |\psi_2\rangle \\ &= \alpha\gamma|0\rangle \otimes |0\rangle + \alpha\delta|0\rangle \otimes |1\rangle + \beta\gamma|1\rangle \otimes |0\rangle + \beta\delta|1\rangle \otimes |1\rangle, \end{aligned} \tag{1.20}$$

which is expressed in the Dirac notation as

$$\alpha\gamma|00\rangle + \alpha\delta|01\rangle + \beta\gamma|10\rangle + \beta\delta|11\rangle. \tag{1.21}$$

The two-qubit system is thus co-ordinatized by a four-dimensional complex vector space $\mathcal{H}_4$ with standard basis $\{|00\rangle, |10\rangle, |01\rangle, |11\rangle\}$.

1.3.2 Axiom 2: Measurement

Any attempt to extract information from a qubit in a given state requires quantum measurement with respect to some orthogonal, indeed without loss of generality, orthonormal basis. Geometrically, the process of measurement is the projection of the state vector onto one of the orthogonal subspaces spanned by this particular basis of the Hilbert space and the subsequent determination of this image's norm. It is important to note that this axiom also says that quantum measurement changes the state of our quantum object into the projected state observed via measurement. In particular, when the state of a qubit or any quantum state is measured, a particular state corresponding to one of the basis vectors will be observed with a prescribed probability. In principle, a single qubit can store an infinite amount of information, yet when measured in the observational basis say, it appears the classical state $|0\rangle$ or $|1\rangle$ with probabilities that are specified by the quantum states being measured.

Extending our example of Equation (1.17), consider a qubit in state $\alpha_0|0\rangle + \alpha_1|1\rangle$. When we observe or measure the qubit with respect to the observational basis, we will observe the state

$$|0\rangle \text{ with probability } \frac{|\alpha_0|^2}{|\alpha_0|^2 + |\alpha_1|^2} \tag{1.22}$$

and the state

$$|1\rangle \text{ with probability } \frac{|\alpha_0|^2}{|\alpha_0|^2 + |\alpha_1|^2}, \tag{1.23}$$

and this act puts our observed qubit into the observed "pure" state.

In summary, quantized information considers three types of bits: classical bits ($|0\rangle$ or $|1\rangle$), probabilistic bits or *p-bits* which are real convex linear combinations of the states of a classical bit, and quantum bits. The measurement axiom of quantum mechanics says we can only extract a p-bit's worth of information from a qubit.

1.3.3 Axiom 3: Evolution of quantum systems

In quantum theory, the state of a quantum system can change in three distinct ways.

One of these ways, quantum measurement, is described above.

Another, time evolution, is governed by Schrödinger's equation

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2} + V(x,t)\Psi(x,t) = i\hbar\frac{\partial\Psi(x,t)}{\partial t}, \quad (1.24)$$

but is suppressed (by setting it constant) in quantum information theory.

The third way a state may change is via physical processes which, as opposed to the measurement axiom, are typically represented by *unitary operators*, that is, linear transformations which are bijective and inner product preserving. In particular, unitary operators map orthonormal bases to orthonormal bases. In symbols, an operator **U** in a Hilbert space $\mathcal{H}_n$ is unitary if

$$\mathbf{U}\overline{\mathbf{U}}^T = \overline{\mathbf{U}}^T\mathbf{U} = \mathbf{I}, \quad (1.25)$$

where $\overline{\mathbf{U}}^T$ is the conjugate transpose of **U** and **I** is the identity operator. Often, the notation $\mathbf{U}^\dagger$ is used instead of $\overline{\mathbf{U}}^T$. This leads to the so-called *unitary condition* $\overline{\mathbf{U}}^T = \mathbf{U}^{-1}$. An operator is called *special* if its determinant is 1. The set of $n \times n$ special unitary matrices is denoted by $SU(n)$ and is defined as follows:

$$SU(n) = \{\mathbf{A} : \overline{\mathbf{A}}^T\mathbf{A} = \mathbf{I} \text{ and } \det\mathbf{A} = 1\} \quad (1.26)$$

Note that a unitary operator is a *special unitary operator* up to a unitary phase. The state evolution axiom of quantum mechanics [15] states that unitary operators acting on normalized state vectors are the correct physical description of an isolated system evolving in time via physical interactions.

The effect of a unitary transformation on a state vector is essentially a rotation of the vector in the ambient projective Hilbert space. As an example, let $\mathbf{A}$ be a 2×2 special unitary matrix. A typical element $\mathbf{A}$ of $SU(2)$ can be expressed in the form

$$\mathbf{A} = \begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix}, \tag{1.27}$$

where z and w are complex numbers with $z\overline{z} + w\overline{w} = 1$. If we start with a qubit in state $|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$ and have $\mathbf{A}$ act on $|\psi\rangle$, we obtain the new state

$$\begin{aligned} \mathbf{A}|\psi\rangle &= \begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} \\ &= \begin{pmatrix} \alpha_0 z + \alpha_1 w \\ -\alpha_0 \overline{z} + \alpha_1 \overline{w} \end{pmatrix} \end{aligned} \tag{1.28}$$

An interesting special case is the physical operation of "flipping the qubit over". Let

$$\mathbf{A} = \begin{pmatrix} 0 & \eta \\ -\overline{\eta} & 0 \end{pmatrix}, \tag{1.29}$$

where η is a unit complex number. What happens to a qubit in state $|0\rangle$ under the action of $\mathbf{A}$?

$$\mathbf{A}|0\rangle = \begin{pmatrix} 0 & \eta \\ -\overline{\eta} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -\overline{\eta} \end{pmatrix} = -\overline{\eta} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.30}$$

Up to phase, it gets transformed to a qubit in state $|1\rangle$. In turn, up to phase, $\mathbf{A}$ also transforms $|1\rangle$ to $|0\rangle$.

1.4 Entanglement

Entanglement is an important tool in quantum information and a core feature of quantum mechanics which distinguishes a quantum system from its classical counterpart. When two qubits interact, they may become permanently *entangled*, that is, they no longer have individual quantum states, but rather the pair of qubits has a joint quantum state. As described above, this state is represented in the standard basis of $\mathcal{H}_4$ by a complex projective vector of the form

$$\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle, \tag{1.31}$$

where α_{00}, α_{01}, α_{10}, and α_{11} are complex numbers not all zero, and defined up to phase. When we observe the pair of qubits with respect to the standard basis of $\mathcal{H}_4$, the joint state falls into one of the four basis states $|00\rangle$, $|01\rangle$, $|10\rangle$, and $|11\rangle$ with probabilities proportional to $|\alpha_{00}|^2$, $|\alpha_{01}|^2$, $|\alpha_{10}|^2$, $|\alpha_{11}|^2$.

Now consider a special joint state of a pair of qubits in the so-called *maximally entangled* state

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}}, \tag{1.32}$$

that is, when we set

$$\alpha_{01} = \alpha_{10} = 0 \quad \text{and} \quad \alpha_{00} = \alpha_{11} = \frac{1}{\sqrt{2}} \tag{1.33}$$

in Equation (1.31). When we observe the first qubit the two possible outcomes are $|0\rangle$ with probability $1/2$ and $|1\rangle$ with probability $1/2$. When we measure the second qubit, the two possible outcomes are $|0\rangle$ with probability $1/2$ and $|1\rangle$ with probability $1/2$. Note that the two measurements are completely correlated, once we measure one qubit we know with one hundred percent certainty the state the other will be observed in. Note that this phenomenon is not common to this state only. In fact, the same conclusions can be drawn from any pair of qubits in the states

$$\frac{|00\rangle + e^{i\theta}|11\rangle}{\sqrt{2}}, \tag{1.34}$$

or

$$\frac{|01\rangle + e^{i\theta}|10\rangle}{\sqrt{2}}, \tag{1.35}$$

where $\theta \in [0, 2\pi)$. Also note that one obtains the so-called *Bell states* by setting $\theta = 0$ or π in the above equations. These four special states form an orthonormal basis of $\mathcal{H}_4$ and are given by

$$\beta_1 = \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \quad \beta_2 = \frac{|00\rangle - |11\rangle}{\sqrt{2}}, \quad \beta_3 = \frac{|01\rangle + |10\rangle}{\sqrt{2}}, \quad \beta_4 = \frac{|01\rangle - |10\rangle}{\sqrt{2}}. \tag{1.36}$$

A pair of qubits in one of the four Bell states is called an *EPR pair* named after Einstein, Podolsky, and Rosen [2] who were the first to consider the behavior of states such as the Bell states.

Mathematically, if the joint state vector $|\psi\rangle$ of a pair of qubits can be represented as the tensor product

$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \tag{1.37}$$

of the two qubit states $|\psi_1\rangle$ and $|\psi_2\rangle$, we say that the two qubits are *unentangled* (not entangled), that is, the measurement of one qubit is independent of the measurement of the other.

The joint state

$$|00\rangle + |11\rangle \tag{1.38}$$

gives an example of a two-qubit state that is not *tensor-factorable*, that is, there are no complex numbers a, b, c, and d such that

$$|00\rangle + |11\rangle = (a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle)\,. \tag{1.39}$$

Multi-qubit states that are not tensor-factorable are by definition entangled.

1.5 Physical operations on joint states

Consider a pair of qubits in the joint state $|\psi\rangle \otimes |\varphi\rangle$. Suppose we act on the first qubit by a physical operation, represented by $U \in SU(2)$ where

$$U = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}. \tag{1.40}$$

The pair of qubits then transforms into the state $(U|\psi\rangle) \otimes |\varphi\rangle$. If we act on the second qubit by U, it transforms to $|\psi\rangle \otimes (U|\varphi\rangle)$. In general, if the initial state of the pair of qubits is given by

$$\sum_{i=1}^{k} |\psi_i\rangle \otimes |\varphi_i\rangle, \tag{1.41}$$

then an action via U on the first and second qubits will produce the states

$$\sum_{i=1}^{k} (U|\psi_i\rangle) \otimes |\varphi_i\rangle \quad \text{and} \quad \sum_{i=1}^{k} |\psi_i\rangle \otimes (U|\varphi_i\rangle)\,, \tag{1.42}$$

respectively.

As an example, consider two qubits with respect to the standard basis $\{|0\rangle, |1\rangle\}$ of $\mathcal{H}_2$ in the maximally entangled initial state

$$|\psi\rangle = |0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle. \tag{1.43}$$

Suppose we act on the qubits, the first via

$$U_I = \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} \tag{1.44}$$

and the second via

$$U_{II} = \begin{pmatrix} c & d \\ -\overline{d} & \overline{c} \end{pmatrix}, \tag{1.45}$$

respectively. Then the new state can be obtained in two different ways:

Method 1: We act on the first qubit by U_I and the second by U_2 as follows

$$(U_I \otimes U_{II})\,|\psi\rangle = (U_I|0\rangle) \otimes (U_{II}|0\rangle) + (U_I|1\rangle) \otimes (U_{II}|1\rangle)\,,$$

that is

$$(U_I \otimes U_{II})\,|\psi\rangle = \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} c & d \\ -\overline{d} & \overline{c} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} c & d \\ -\overline{d} & \overline{c} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.46}$$

which gives

$$\begin{pmatrix} a \\ -\overline{b} \end{pmatrix} \otimes \begin{pmatrix} c \\ -\overline{d} \end{pmatrix} + \begin{pmatrix} b \\ \overline{a} \end{pmatrix} \otimes \begin{pmatrix} d \\ \overline{c} \end{pmatrix} \tag{1.47}$$

Expanding bilinearly with respect to the standard basis, we obtain

$$(U_I \otimes U_{II})\,|\psi\rangle = (ac + bd)|00\rangle + (-a\overline{d} + b\overline{c})|01\rangle + (-\overline{b}c + \overline{a}d)|10\rangle + (\overline{a}\,\overline{c} + \overline{b}\,\overline{d})|11\rangle. \tag{1.48}$$

Method 2: We first perform the Kronecker product of U_I and U_{II} to obtain

$$U_I \otimes U_{II} = \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} \otimes \begin{pmatrix} c & d \\ -\overline{d} & \overline{c} \end{pmatrix} = \begin{pmatrix} ac & ad & bc & bd \\ -a\overline{d} & a\overline{c} & -b\overline{d} & b\overline{c} \\ -\overline{b}c & -\overline{b}d & \overline{a}c & \overline{a}d \\ \overline{b}\,\overline{d} & -\overline{b}\,\overline{c} & -\overline{a}\,\overline{d} & \overline{a}\,\overline{c} \end{pmatrix}. \tag{1.49}$$

Next we write $|\psi\rangle$ in matrix notation to obtain

$$|\psi\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{1.50}$$

The new state is thus given by

$$\begin{aligned}(U_I \otimes U_{II})\,(|\psi\rangle) &= \begin{pmatrix} ac & ad & bc & bd \\ -a\overline{d} & a\overline{c} & -b\overline{d} & b\overline{c} \\ -\overline{b}c & -\overline{b}d & \overline{a}c & \overline{a}d \\ \overline{b}\,\overline{d} & -\overline{b}\,\overline{c} & -\overline{a}\,\overline{d} & \overline{a}\,\overline{c} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} ac + bd \\ -a\overline{d} + b\overline{c} \\ \overline{a}d - \overline{b}c \\ \overline{a}\,\overline{c} + \overline{b}\,\overline{d} \end{pmatrix} \end{aligned} \tag{1.51}$$

Expanding bilinearly with respect to the standard basis of $\mathbb{R}^4$

$$(ac+bd)\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + (-a\overline{d}+b\overline{c})\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + (\overline{a}d-\overline{b}c)\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + (\overline{a}\,\overline{c}+\overline{b}\,\overline{d})\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \tag{1.52}$$

which gives in the Dirac notation the state

$$(ac + bd)|00\rangle + (-a\overline{d} + b\overline{c})|01\rangle + (-\overline{b}c + \overline{a}d)|10\rangle + (\overline{a}\,\overline{c} + \overline{b}\,\overline{d})|11\rangle. \tag{1.53}$$

Note that the expressions given in (1.48) and (1.53) are the same as expected.

We now recall a procedure of creating the EPR pair

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}}. \tag{1.54}$$

Suppose we begin with two qubits in state $|0\rangle$, that is, $|\psi_1\rangle = |0\rangle$ and $|\psi_2\rangle = |0\rangle$. Let

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{1.55}$$

Then

$$H|\psi_1\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle + |1\rangle\right). \tag{1.56}$$

Next we take the tensor product of the states $H|\psi_1\rangle$ and $|\psi_2\rangle$ to obtain

$$H|\psi_1\rangle \otimes |\psi_2\rangle = \frac{|00\rangle + |10\rangle}{\sqrt{2}} \tag{1.57}$$

Now define the unitary operator C_{NOT}:

$$C_{NOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{1.58}$$

We apply C_{NOT} to the state $\left(|00\rangle + |10\rangle\right)/\sqrt{2}$ to obtain the desired EPR pair.

$$\begin{aligned} C_{NOT}\left(H|\psi_1\rangle \otimes |\psi_2\rangle\right) &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\ &= \frac{|00\rangle + |11\rangle}{\sqrt{2}}. \end{aligned} \tag{1.59}$$

1.6 Remarks

Combining physics, mathematics and computer science, quantum computing has developed in the past two decades from a visionary idea to one of the most fascinating applications of quantum mechanics. Much of the recent interest in this domain of research was triggered by Peter Shor [51] who showed how a quantum algorithm could exponentially "speed-up" classical computation and factor large numbers into primes much more rapidly (at least in terms of the number of computational steps involved) than any known classical algorithm. Shor's algorithm was soon followed by several other algorithms that aimed to solve combinatorial and algebraic problems. One of these algorithms is the famous Lov Grover's algorithm [24] which uses quantum computers to search an unsorted database faster than a conventional computer. With a present day classical computer, it would take $N/2$ number of searches to find a specific entry in a database with N entries. Grover's algorithm makes it possible to perform the same task in $\sqrt{N}$ searches.

Chapter 2

A FORMALISM FOR QUANTUM GAMES

2.1 Introduction

The theory of games concerns the mathematics of decision making. It provides a general structure within which both cooperation and competition among independent entities may be modeled and gives powerful tools for analyzing these models. Game theory emerged as a mathematical discipline around 1928 when John von Neumann published a fundamental theorem concerning two player zero sum games [54]. Initially proposed for use in economics by John von Neumann and Morgenstern [40], with important contributions by Nash [41], the theory of games has now found a wide variety of applications in many other areas of science including biology, psychology, sociology, computer science, and information theory. Now, however information and computation concern ever smaller 'bits' of information, and we find classical behavior replaced by the rules of quantum mechanics [25]. Appropriately adjusted to the quantum domain, game theory remains a powerful tool for future study.

Already, there are several well known instances in physics where it is useful to think of quantum processes as games, such as in cryptography [19], cloning [56], and computing [23]. Computing is a research area with many opportunities for the application of quantum game theory. Because of the complex logic structures that arise, digital gates can be discussed in terms of games [48]. The rise of quantum computation thus suggests a corresponding development of quantum games. In fact, a quantum computer has been used to demonstrate a quantum game [57] and the progress of quantum computing requires an understanding of a new quantum logic [55]. Further, Meyer [39] has linked important quantum computing algorithms, including the famous Grover algorithm [24], with quantum games.

Quantum game theory also has potential applications in the macroscopic world. Any system that involves interference and correlation is a possible candidate for quantum game theory. Quantum game theory has already been applied to economics [47] [45] [46] and gambling [12] [34] [16].

In this work, we use a new mathematical formalism [8] describing the quantization of games and generalizing the many quantization protocols found in the literature. The study of quantum games is fairly new, arising from a seminal paper of D. Meyer [38]. The ensuing decade has seen an explosion of contributions and controversy over what exactly a quantized game really is and if there is indeed anything new for game theory. The lack of a mathematical formalism for the subject has clouded many of the issues. In this section we recall from [8] such a formalism generalizing the classical notion of mixing strategies, along with proposed resolutions to some of the issues discussed in literature.

2.2 Preliminaries

We begin by recalling a few formal definitions of the game theoristic terms used here, however, for the most of the basic terms and terminology of standard game theory, the reader is referred to [7].

Definition 2.1. *Given a set* $\{1, 2, \cdots, n\}$ *of players, for each player a set* S_i $(i = 1, \cdots, n)$ *of so-called* pure strategies, *and a set* Ω_i $(i = 1, \cdots, n)$ *of* possible outcomes, *a* game G *is a vector-valued function whose domain is the Cartesian product of the* S_i*'s and whose range is the Cartesian product of the* Ω_i*'s. In symbols*

$$\prod_{i=1}^{n} S_i \xrightarrow{G} \prod_{i=1}^{n} \Omega_i \tag{2.2}$$

The function G *is sometimes referred to as the* payoff function.

Definition 2.3. *An* n player, m strategy game *or* $n \times m$ game *is a game with* n *players where each player has access to exactly* m *pure strategies.*

Here a *play* of the game is a choice by each player of a particular strategy s_i the collection of which forms a *strategy profile* $(s_1, \cdots, s_n)$ whose corresponding *outcome profile* is $G(s_1, \cdots, s_n) = (\omega_1, \cdots, \omega_n)$, where the ω_i's represent each player's individual outcome. Note that by assigning a real valued *utility* to each player which quantifies that player's preferences over the various outcomes, we can, without loss of generality, assume that the Ω_i's are all copies of $\mathbb{R}$, the field of real numbers. Also, note that each player's individual outcome ω_i defines that player's individual payoff function. In symbols

$$\omega_i : \Pi S_i \longmapsto \mathbb{R} \tag{2.4}$$

A game is said to be *zero-sum* if the sum of the payoffs to the players is zero no matter what strategies are chosen by the players. That is, a game G is zero-sum if

$$\sum_{i=1}^{n} \omega_i(s_1, s_2, \cdots, s_n) = 0 \;\; \forall \;\; (s_1, s_2, \cdots, s_n) \in \Pi S_i. \tag{2.5}$$

2.2.1 Special Strategies

In game theory one is frequently concerned with the identification of special strategies or strategic profiles. For example, most players would love to identify a strategy that guarantees a maximal utility. As this is not usually possible, a *security strategy*, that is, a strategic choice that guarantees an explicit lower bound to the utility received, is also sought. But, given a fixed profile of opponents' strategies, rational players seek *best reply* strategies.

Definition 2.6. *For a fixed $(n-1)$-tuple of opponents' strategies, a* best reply *is a strategy $s_i^\star \in S_i$ that delivers a utility at least as great, if not greater, than any other strategy $s_i \in S_i$. That is*

$$G(\star, \cdots, \star, s_i^\star, \star, \cdots, \star) \geq G(\star, \cdots, \star, s_i, \star, \cdots, \star) \quad \forall s_i \in S_i \tag{2.7}$$

The situation when every player in the game has chosen such a strategy is of fundamental importance in the theory of games and has a special name.

Definition 2.8. *A* Nash equilibrium (NE), *or a* solution *or just an* equilibrium, *for G is a strategy profile $(s_1, s_2, ..., s_n)$ such that each s_i is a best reply to the $(n-1)$-tuple of opponents' strategies. That is, the strategy profile $(s_1, s_2, \cdots, s_n)$ is a Nash equilibrium if*

$$\forall k, \;\; \omega_k(s_1, s_2, \cdots, s_k, \cdots, s_n) \geq \omega_k(s_1, s_2, \cdots, s_k', \cdots, s_n) \;\; \forall s_k' \tag{2.9}$$

Other ways of expressing this concept include the observation that no player can increase his or her payoff by unilaterally deviating from his or her equilibrium strategy or that at equilibrium a player's opponents are indifferent to that player's strategic choice.

2.2.2 Examples

As an example, consider the Prisoner's Dilemma (PD), a type of non-zero-sum game in which two players may each "cooperate" (C) with or "defect" (D), that is, betray the other player. In this game, as in much of game theory, the players are assumed to be 'rational' in the sense that the only concern of each individual player ("prisoner") is the optimization of his/her own payoff, without any concern for the other player's payoff. This is a two player, two strategy game (a 2×2 or *bimatrix* game) whose payoff function is indicated in Figure 2.1

Here, and in subsequent examples, the numbers in parentheses refer to player one's and player two's payoffs, respectively. The table of Figure 2.1 is sometimes referred to as the *strategic* or *normal form* of the game. It gives a compact description of the game, providing the identities and the strategies of the players as well as their payoff functions.

Here, note that for player 1 the pure strategy D always delivers a higher outcome

		Player II	
		C	D
Player I	C	$(3,3)$	$(0,5)$
	D	$(5,0)$	$(1,1)$

Figure 2.1: Prisoner's Dilemma

than the strategy C (say D *strongly dominates* C) and for player 2 the strategy D strongly dominates C. Hence the pair (D, D) is a (unique) Nash Equilibrium.

Given a game G, a Nash equilibrium may not exist amongst the pure strategy profiles. As an example, consider the game of Simplified Poker, a 2×2 zero-sum game whose payoff function is given in Table 2.1.

	t_1	t_2
s_1	$(5/4, -5/4)$	$(0,0)$
s_2	$(0,0)$	$(5/2, -5/2)$

Table 2.1: Simplified Poker

Here one can easily show that there is no pair of strategies $(s^\star, t^\star)$ such that $s^\star$ is a best reply to $t^\star$ and vice-versa. Hence, the Simplified Poker game has no equilibria in pure strategies.

2.2.3 A Classical Extension of G

Classical Game Theoretic formalism now calls upon the theorist to extend the game G by enlarging the domain and extending the payoff function. Of course, the question of if and how a given function extends is a time honored problem in mathematics and the careful application of the mathematics of extension is what gives the formalism of quantization following its power. Returning to classical game theory, a standard extension of this point is to consider for each player the set of *mixed strategies*, that is, the set of probability distributions over S_i. For a given set X, denote the probability distributions over X by $\Delta(X)$ and note that when X is finite, with k elements say, the set $\Delta(X)$ is just the $k-1$ dimensional simplex $\Delta^{(k-1)}$ over X, i.e., the set of formal real convex linear combinations of elements of X. Of course, we can embed X into $\Delta(X)$ by considering the element x as mapped to the probability distribution which assigns 1 to x and 0 to everything else. For a given game G, denote this embedding of S_i into $\Delta(S_i)$ by e_i.

Now our game G can be extended to a new, larger game G^{mix}, as follows. Given a profile $(p_1, \ldots, p_n)$ of probability distributions over the S_i's, by taking the product distribution we obtain a probability distribution over the product $\prod S_i$. Taking the push out by G of this probability distribution we obtain a probability distribution over the image of G. By following this by the expectation operator we obtain the expected outcome of the *mixed strategy profile* $(p_1, \ldots, p_n)$. Assigning the expected outcome to each mixed strategy profile we obtain the extended game G^{mix}. The discussion above is summarized by the following definition.

Definition 2.10. *Given a game G, the game G^{mix} is a vector valued function whose domain is $\Pi_{i=1}^{n}\Delta(S_i)$ and whose range is $\Pi_{i=1}^{n}\Omega_i$. In symbols*

$$G^{mix} : \prod \Delta(S_i) \to \prod \Omega_i \tag{2.11}$$

The function G^{mix} extends the original function G as $G^{mix} \circ \Pi e_i = G$, that is we have the commutative diagram that appears in Figure 2.2.

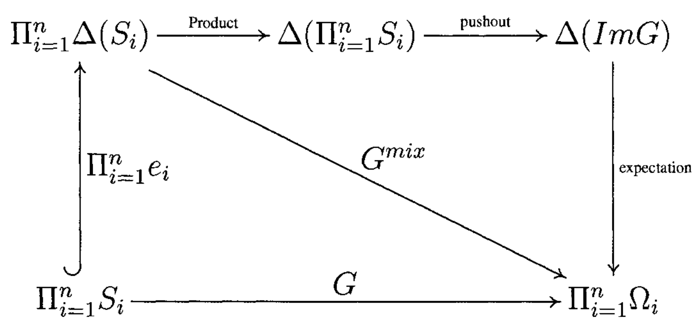

Figure 2.2: Extension of G by G^{mix}

The function G^{mix} is sometimes referred to as the *expected payoff function* and for a given mixed strategy profile $(p_1, p_2, \cdots, p_n)$, the corresponding expected outcome profile is $G^{mix}(p_1, p_2, \cdots, p_n) = (\mathcal{E}_1, \mathcal{E}_2, \cdots, \mathcal{E}_n)$, where the $\mathcal{E}_i$'s represent the players' individual expected outcomes.

Nash's famous theorem says that if the S_i are all finite, then there always exists an equilibrium in G^{mix}. As an application, let us compute the Nash equilibria in G^{mix} where G is the Simplified Poker.

Assume player one plays his pure strategy s_1 with probability p and player two plays her pure strategy t_2 with probability q. Then the expected outcome of the mixed strategy profile (p, q) is given by

$$\begin{aligned}G^{mix}(p,q) &= p(1-q)G(s_1,t_1) + pqG(s_1,t_2) \\ &+ (1-p)(1-q)G(s_2,t_1) + (1-p)qG(s_2,t_2) \\ &= p(1-q)(1.25,-1.25) + pq(0,0) \\ &+ (1-p)(1-q)(0,0) + (1-p)q(2.5,-2.5) \\ &= \left(\mathbf{p}^T A\ \mathbf{q}, \mathbf{q}^T B\ \mathbf{p}\right)\end{aligned} \tag{2.12}$$

where

$$\mathbf{p} = \begin{pmatrix} p \\ 1-p \end{pmatrix}, \quad \mathbf{q} = \begin{pmatrix} 1-q \\ q \end{pmatrix} \tag{2.13}$$

and

$$A = \begin{pmatrix} 5/4 & 0 \\ 0 & 5/2 \end{pmatrix}, \quad B = \begin{pmatrix} -5/4 & 0 \\ 0 & -5/2 \end{pmatrix}. \tag{2.14}$$

The matrices A and B are called player one's and player two's *payoff matrices*, respectively. Let $\mathcal{E}_1(p,q) = \mathbf{p}^T A\mathbf{q}$ and $\mathcal{E}_2(p,q) = \mathbf{q}^T B\mathbf{p}$ be player one's and player two's expected payoffs, respectively. Then

$$\begin{aligned}\mathcal{E}_1(p,q) &= \mathbf{p}^T A\mathbf{q} \\ &= p[1.25(1-q) + 0\cdot q] + (1-p)[0\cdot(1-q) + 2.5q] \\ &= p[1.25(1-q)] + (1-p)[2.5q].\end{aligned} \tag{2.15}$$

This gives three cases to consider. When $1.25(1-q) > 2.5q$, that is, when $q > 1/3$, player one's best reply is to employ s_1 with probability $p = 1$. When $q < 1/3$, his best reply is to play s_2 with probability 1. If $q = 1/3$, player one is indifferent between all his mixed strategies. By a similar analysis, we find that player two has as her best reply the strategy t_2 played with probability $q = 1$ when $p > 2/3$, her best reply is t_1 played with probability 1 when $p < 2/3$, and she remains indifferent between all her mixed strategies when $p = 2/3$. As Simplified Poker has no equilibria amongst the pure strategy profiles, the mixed strategy profile $(p, q) = (2/3, 1/3)$ is the unique equilibrium in G^{mix} with expected payoffs to the players $(5/6, -5/6)$.

Unfortunately, the Nash equilibrium in G^{mix} is called a *mixed strategy equilibrium for* G, when it is not an equilibrium of G at all, the abusive terminology confusing G with its image, ImG. Indeed, this is where much confusion in quantum and classical game theory begins.

2.2.4 Classical Mediated Communication

Before proceeding onto quantization, it is useful to place other classical game theoretical ideas such as classical mediated communication and Aumann's notion of a *correlated* equilibrium [3] into this context. Following [8] one begins by observing that the function from $\prod_{i=1}^{n} \Delta(S_i) \to \Delta(ImG)$ is not necessarily onto. As an example consider any 2×2 game G. As before, if player 1 plays his first pure strategy with probability p, say, and player 2 plays her second pure strategy with probability q, say, the resulting probability distribution over the outcomes of G is given in Table 2.2.

An easy exercise now shows that the element of $\Delta(ImG)$ represented by Table 2.3 is not realizable by any choice of p and q. Classical mediated communication addresses

	t_1	t_2
s_1	$p(1-q)$	pq
s_2	$(1-p)(1-q)$	$(1-p)q$

Table 2.2: A Probability Distribution over ImG

	t_1	t_2
s_1	$1/2$	0
s_2	0	$1/2$

Table 2.3: An Element of $\Delta(ImG)$

this issue. Suppose during pre-play negotiation the players are able to hire a referee for negligible cost. For a given $\rho \in \Delta(ImG)$ the referee is meant to enforce ρ as follows. The referee secretly observes a random event with probability distribution ρ, thus determining an outcome of G. The referee then communicates to each player only his strategic choice which yields the observed outcome.

Note that the players are no longer playing the game G, but in fact a much larger game G_ρ^{com} which is easily described for 2×2 games and whose generalization to games with larger strategic spaces should be clear from the description here. Suppose the strategic space for each player is represented by the pair $S = \{A, B\}$. The strategic spaces for G_ρ^{com} can be represented by the quadruple $T = \{A', B', C', D'\}$ where the strategy C' represents a player always cooperating with the referee, D' represents the strategy where the player always deviates from the referee's instruction (i.e. playing B when he hears A and vice-versa), A' represents cooperating with the referee when A is recommended and deviating otherwise, and B' represents cooperating with the referee when B is recommended and deviating otherwise.

Two important things to note here. First, if both players choose to play C', then the outcome of the new game is exactly the expected outcome of G under ρ. Second, G_ρ^{com} extends the original game G as there are embeddings $f_i : \{A, B\} \rightarrow \{A', B', C', D'\}$ taking A to A' and B to B' such that $G = G_\rho^{com} \circ \prod_{i=1}^{2} f_i$, as in the diagram of Figure 2.3.

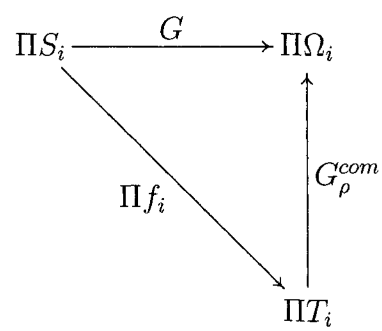

Figure 2.3: Extension of G by G_ρ^{com}

Hence, classical mediated communication gives a *family*, indexed by $\Delta(ImG)$, of extensions of G.

Following Aumann, a *correlated equilibrium for G* occurs whenever (C', C') is a Nash equilibrium in G_ρ^{com}. That is, the players' agreement to follow the referee is *self policing*, meaning that there is no gain to a player from unilaterally deviating from the referee's recommendations. Note again the abusive terminology, the strategic choice for a correlated equilibrium is not a strategic choice for G at all, but rather a strategic choice outside the embedded strategies for G in a larger game. Of course, the use of correlated equilibrium may or may not improve the lot of the players. A classic example of correlated equilibrium improving the players' lot is given by the variant of the 2×2 game of Chicken given in Table 2.4.

An easy exercise shows that (s_2, t_1) and (s_1, t_2) are both pure strategy equilibria

	t_1	t_2
s_1	$(2,2)$	$(0,3)$
s_2	$(3,0)$	$(-1,-1)$

Table 2.4: Chicken

and there is a unique mixed strategy equilibrium where every player plays each of his or her pure strategies with equal probability. This mixed strategy equilibrium pays out 1 to each player. It is also easy to see that even without a referee any real convex linear combination of these three outcomes forms a self policing agreement between the players. For example, the players could jointly observe a fair coin and agree to play the (s_1, t_2) if it falls Heads and (s_2, t_1) if it falls Tails. Note that the expected outcome of this agreement is $\left(\frac{3}{2}, \frac{3}{2}\right)$ which is better than the outcome $(1, 1)$ from the mixed strategy equilibrium. But even better and outside this region is the correlated equilibrium arising from the probability distribution $\frac{1}{3}(2,2) + \frac{1}{3}(0,3) + \frac{1}{3}(3,0)$ yielding the outcome $\left(\frac{5}{3}, \frac{5}{3}\right)$.

An example where mediated communication does not improve the lot of the players is given by Prisoner's Dilemma. One easily checks that due to the strong domination present in each player's strategy set, players always have an incentive to deviate from the referee's instruction if ρ assigns a non-zero probability to any outcome other than the Nash equilibrium (s_2, t_2).

2.3 Bleiler's Formalism for Quantization

We recall from [8] the formalism for quantization developed by Bleiler which generalizes the classical notion of mixed strategies.

Classically, we constructed the probability distributions over the outcomes of a game

G. We now pass to a more general notion of randomization, that of quantum superposition. Begin then with a Hilbert space $\mathcal{H}$ and for now assume that $\mathcal{H}$ is finite dimensional, and that we have a finite set X which is in one-to-one correspondence with an orthogonal basis $\mathcal{B}$ of $\mathcal{H}$.

By a *quantum superposition* of X we mean a complex projective linear combination of elements of X; that is, a representative of an equivalence class of non-zero complex linear combinations where the equivalence between combinations is given by non-zero scalar multiplication. Recall from Chapter 1 that quantum mechanics call this scalar a *phase*. When the context is clear as to the basis to which the set X is identified, denote the set of quantum superpositions for X as $QS(X)$. Of course, it is also possible to define quantum superpositions for infinite sets, but for the purpose here, one need not be so general. Most of what follows can be easily generalized to the infinite case.

For each quantum superposition of X we can obtain a probability distribution over X by assigning to each component the ratio of the square of the length of its coefficient to the square of the length of the combination. For example, the probability distribution produced from $\alpha x + \beta y$ is just

$$\frac{|\alpha|^2}{|\alpha|^2 + |\beta|^2} x + \frac{|\beta|^2}{|\alpha|^2 + |\beta|^2} y \tag{2.16}$$

With respect to the axioms of quantum mechanics, call this function $QS(X) \rightarrow \Delta(X)$ *quantum measurement with respect to* X, and again note that geometrically quantum measurement is defined by projecting a normalized quantum superposition onto the various elements of a normalized basis $\mathcal{B}$. Denote this function by q_X^{meas}, or if the set X is clear from the context, by q^{meas}.

Now given a finite n-player game G, suppose we have a collection $Q_1, \ldots, Q_n$ of non-empty sets and a *protocol*, that is, a function $\mathcal{Q} : \prod Q_i \to QS(\mathrm{ImG})$. Quantum measurement q^{meas}_{ImG} then gives a probability distribution over ImG. Just as in the mixed strategy case we can then form a new game $G^{\mathcal{Q}}$ by applying the expectation operator. This discussion is summarized by the following definition.

Definition 2.17. *A* quantization *of a game G by a protocol $\mathcal{Q}$ is the vector valued function $G^{\mathcal{Q}}$ whose domain is $\prod_{i=1}^{n} Q_i$ and whose range is $\prod_{i=1}^{n} \Omega_i$ such that the diagram of Figure 2.4 commutes*

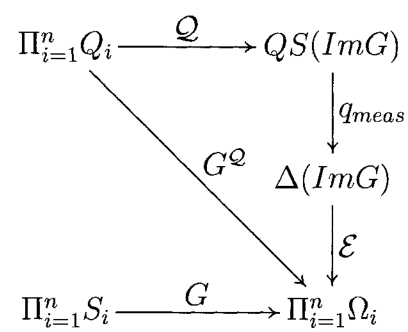

Figure 2.4: Extension of G by $G^{\mathcal{Q}}$

Call the Q_i's sets of *pure quantum strategies* for $G^{\mathcal{Q}}$. Moreover, if there exist embeddings $e_i' \colon S_i \hookrightarrow Q_i$ such that $G^{\mathcal{Q}} \circ \Pi e_i' = G$, call $G^{\mathcal{Q}}$ a *proper quantization* of G. Further, if there exist embeddings $e_i'' \colon \Delta(S_i) \hookrightarrow Q_i$ such that $G^{\mathcal{Q}} \circ \Pi e_i'' \circ \Pi e_i = G$, call $G^{\mathcal{Q}}$ a *complete quantization* of G. These definitions are summed up in the commutative diagram of Figure 2.5, and for proper quantizations the original game is obtained by restricting the quantization to the image of $\prod e_i$. For general extensions, the Game Theory literature refers to this as *recovering* the game G.

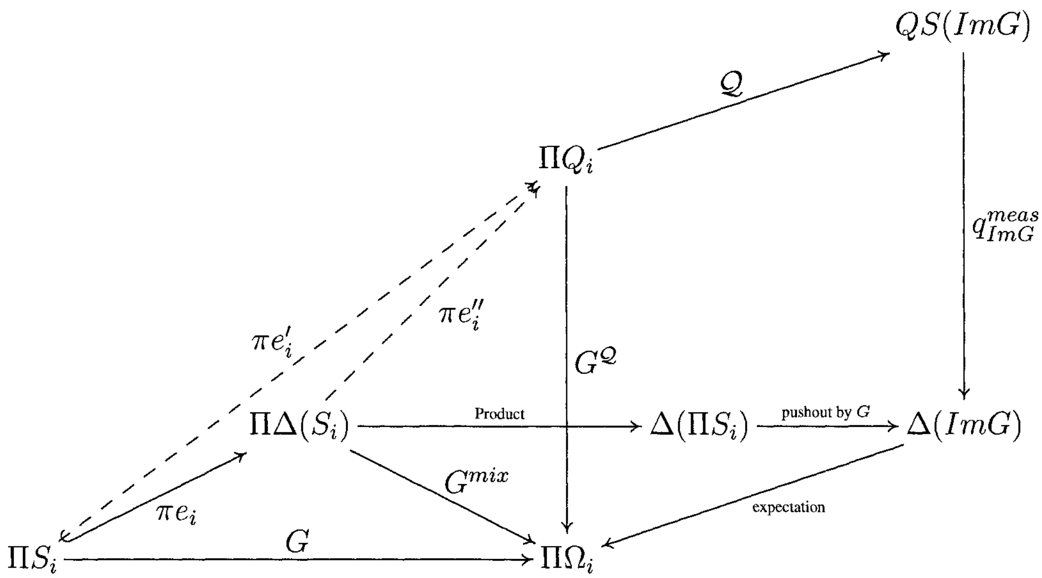

Figure 2.5: A Quantization Formalism

Immediately from the definitions we obtain

Proposition 2.18. *A complete quantization is a proper quantization.*

Furthermore, note that finding a proper quantization of a game G is just the usual problem of extending a function. It is also worth noting here that nothing prohibits us from having a quantized game $G^{\mathcal{Q}}$ play the role of G in the classical situation and by considering the probability distributions over the Q_i, creating a yet larger game $G^{m\mathcal{Q}}$, the *mixed quantization of G with respect to the protocol* $\mathcal{Q}$. For a proper quantization of G, $G^{m\mathcal{Q}}$ is an even larger extension of G. The game $G^{m\mathcal{Q}}$ is described in the commutative diagram of Figure 2.6.

Definition 2.19. *Call the* $\Delta(Q_i)$*'s sets of* mixed quantum strategies.

Note that the quantum strategy sets Q_i need not consist of quantum superpositions.

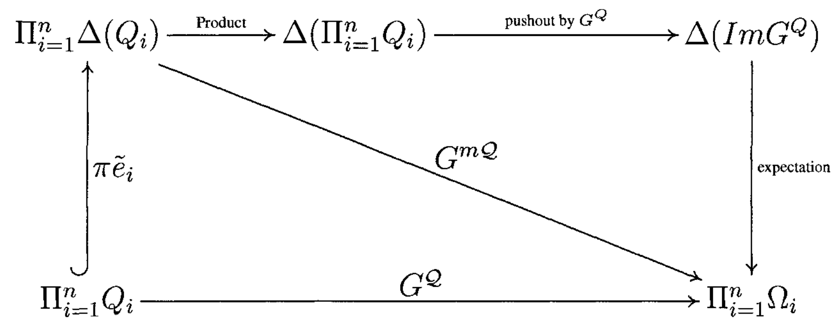

Figure 2.6: Extension of G by G^{mQ}

Indeed, protocols with classical inputs yielding quantum superpositions of the outcomes of certain games have already been posited [14] [29]. Some other specific protocols are discussed in the context of the formalism above next.

As discussed in following sections, the literature gives several protocols for quantizing 1, 2, and occasionally even multi-player games, some improper, some proper but not complete, and some yielding complete quantizations. Yet there has been an ongoing debate in the literature as to what is the ‘correct’ method of quantizing a game. The above formalism shows that this is the wrong question to ask as the formalism clearly indicates that a given game can have several inequivalent quantizations. It also makes clear that comparisons between various quantizations, between quantizations and various classical extensions, and between quantizations and the original game itself often amounts to comparing “apples” to “oranges”. A specific example appears in the following sections.

2.4 Mediated Quantum Communication via an EWL Protocol

In classical mediated communication, players have a referee mediate their game and the communication of their strategic choices. For simplicity, assume the players have but two classical pure strategies to choose from. The communication of each player's strategic choices is implemented by the sending of bits to the players, put into an initial state by the referee. Presumably players then send back their individual bits in the other state (Flipped) or in the original state (Un-Flipped) to indicate the choice of their second or first pure strategy, respectively. The bits are then examined by the referee who then makes the appropriate payoffs.

When the communication between the referee and the players is over quantum channels, Eisert, Wilkens, and Lewenstein (EWL) [18] have proposed family of quantization protocols that individually depend on the initial joint state prepared by the referee. Players and the referee communicate via *qubits*, a two pure state quantum system with a fixed observational basis. In the EWL protocol, the referee uses a new basis corresponding to the actions of (No Flip, No Flip), (No Flip, Flip), (Flip, No Flip), (Flip, Flip) by the players to determine the appropriate payoffs. Players may choose from any physical operation (i.e. the Lie group $SU(2)$) as pure quantum strategies (the Q_i's in the formalism above) or even probabilistic combinations thereof (the ΔQ_i's in the formalism) for their strategic choices. The procedure above describes for each initial state $\mathcal{I}$ a protocol $\mathcal{Q}_\mathcal{I}$ and a quantized and a mixed quantized games $G^{\mathcal{Q}_\mathcal{I}}$ and $G^{m\mathcal{Q}_\mathcal{I}}$ per the formalism above.

If the initial state prepared by the referee is given in the Dirac notation by $|0\rangle \otimes |0\rangle$, then the corresponding EWL protocol is a complete quantization and is in fact equivalent to the classical game G^{mix}. But when the initial state is given by the maximally entangled state $(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle) / \sqrt{2}$, the corresponding EWL protocol not only remains complete, it sets up an onto map from the product of the respective strategy spaces to $\Delta(ImG)$.

When examining these new games for Nash equilibria, a fundamental question arises; are the equilibria in these new quantized games truly new? That is, is the probability distribution that arises from an equilibrium pair in the quantized version of game G different from that arising from a classical correlated equilibrium for G? For two by two games the maximally entangled EWL quantization admits a mixed quantum strategy equilibrium where each player uses the uniform probability distribution over his choice of pure quantum strategy [33]. The resulting probability distribution over the payoffs of G is now again the uniform distribution, assigning an equal probability to each of the four outcomes of G. But for the Prisoner's Dilemma, this distribution does *not* arise from a classical correlated equilibrium as it assigns a non-zero probability to each of the classical non-equilibrium payoffs, and so does not correspond to any classical correlated equilibrium for this game while delivering a superior payoff to the players. An even more remarkable result holds true for the maximally entangled EWL quantization of Simplified Poker [9], where the uniformly mixed quantum equilibrium out-performs the classical equilibrium payoff for player I, yet is still a security strategy for player I against which player II has no recourse.

2.5 Application to the Literature

In this section, we further relate quantum games literature to the Bleiler's quantization formalism.

2.5.1 Meyer's PQ Penny Flip Game

The study of quantum games is quiete new, arising from a seminal paper of D. Meyer [38] published in Physics Review Letters in 1999. In that paper, Meyer put forward an inspiring argument for research on quantum game theory that is worth retelling here. He describes a game that is played by two characters of the popular TV series Star Trek: The Next Generation, Captain Picard and the entity Q. The Starship Enterprise faces a dire emergency, and Picard is preparing for the worst. Suddenly, Q appears on the scene. The entity Q offers to help, provided that Picard can beat him at a simple game involving a coin. In this game, also known as PQ PENNY FLIP (referred to henceforth as PQPF), a coin is placed in a closed box heads up. Neither player can see into the box at any time. The entity Q takes the first turn, then P takes a turn, and then Q gets a final chance. As far as Picard knows, a player can choose to either flip or not flip the coin. When the adversaries together open the box, Q wins if the penny is heads up; if it is tails up Picard wins. This game is an example of a two-player, zero-sum game: What one player gains, the other looses. A classical analysis of the game gives both players an equal probability of success. Aware of this fact, Picard agrees to play but looses to Q, not just once but again and again.

The reason for this is that the player Q takes advantage of a larger set of quantum strategies, namely the set $U(2)$ of 2×2 unitary matrices (Q_1 in the formalism) and Picard is restricted to use the classical strategies *no flip* and *flip* matrices N and F

$$N = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.20}$$

(Q_2 in the formalism). Picard doesn't know that the coin in question is a quantum coin– an object that can be both heads and tails at the same time. The two sides of the coin are identified with the basis vectors of the Hilbert space $\mathcal{H}_2$, namely heads $\equiv |0\rangle$ and tails $\equiv |1\rangle$. The entity Q performs the $U(2)$ element

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{2.21}$$

on the coin. Instead of swapping tails for heads, this quantum move leaves the coin in a superposition of the two states, half heads and half tails as below

$$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} = |\psi\rangle \tag{2.22}$$

On his turn, P responds with a classical flip (F) or does nothing (N). Neither choice, however, alters the coin's superpositioned state as

$$N|\psi\rangle = |\psi\rangle \quad \text{and} \quad F|\psi\rangle = |\psi\rangle. \tag{2.23}$$

The entity Q then performs one more time the quantum move H that unscrambles the superposition, bringing the coin back to heads to win the game as

$$H|\psi\rangle = |0\rangle \tag{2.24}$$

The practical lesson that this game teaches is that quantum theory may offer strategies that at least in some cases can give an insurmontable advantage over classical strategies.

2.5.2 The MW Model

Besides the EWL quantization model, another protocol for quantization of classical games was put forward by Marinatto and Weber [36] and is referred to as the MW Model. This scheme differs from EWL's in the omission of a disentangling operator previous to measurement and through the restriction of the players'strategic choices to probabilistic mixtures of the operators N and F. Each of their protocols requires an initial state upon which the players act, differing initial states yielding different protocols. The resulting quantization is proper when a particular initial state is chosen but for other choices of initial state the quantization may not be. In [36], they applied their protocols to the study of the Battle of the Sexes (BoS) game with pay-off matrix

	t_1	t_2
s_1	(α, β)	(γ, γ)
s_2	(γ, γ)	(β, α)

Table 2.5: Battle of the Sexes

where $\alpha > \beta > \gamma$. They claimed that with the use of a factorisable (not entangled) initial state players cannot improve their expected pay-offs and that the effects of the classical version of the game are reproduced. In the classical version, the game admits three equilibria and the players cannot rationally decide which one to choose. But if the initial state is suitably entangled, Marinatto and Weber [36] showed that the game admits again three equilibria, but a particular unique solution gives more reward.

The EWL and MW protocols have been applied to various classical games such as Prisoner Dilemma [17], the Monty Hall Problem [21], Stag Hunt Game [53], Rock, Scissors and Paper [52], and Chicken [20]. The results show that, in general, the "quantization process" and relations to the background classical problems are not unique. Equilibria can be found but, as in the classical problems, in most cases they are not Pareto optimal[1].

[1]A game result from which no player can improve their pay-off without another player being worse off.

Chapter 3

QUATERNIONIZATION OF TWO PLAYER, TWO STRATEGY QUANTUM GAMES

3.1 Introduction

This chapter concerns quantized versions of generic two player, two strategy games. For a specific quantization protocol originally described by Eisert, Wilkens, and Lewestein [18], Steven Landsburg developed a quaternionic representation of the payoff function, and from this classified all the potential Nash equilibria of such games [33]. Landsburg's construction focused on games with a specific maximally entangled initial state. However, there is an entire circle of this type of maximally entangled states which could be used in these quantizations. Here we reproduce Landsburg's construction and its subsequent results. We then present an extension of Landsburg's construction to games where the initial state is chosen arbitrarily from this circle and for the corresponding quantized games classify the potential Nash equilibria.

3.2 Mediated Classical Communication

Consider a generic two player, two strategy game G whose payoff matrix is indicated by the tableau below

	t_1	t_2
s_1	(X_0, Y_0)	(X_2, Y_2)
s_2	(X_3, Y_3)	(X_1, Y_1)

Table 3.1: A Generic 2x2 Game

where players one and two's pure strategy spaces are given by $S_1 = \{s_1, s_2\}$ and $S_2 = \{t_1, t_2\}$, respectively, and the pairs $(X_t, Y_t) \in \mathbb{R}^2$ represent payoffs to players one and two.

Now consider a generic 2×2 game G where players have a referee mediate their game and the communication of their strategic choices is over classical channels. First, the referee prepares two bits in an initial state. He sends each player one of the bits. The players then send back their individual bits in the other state (Flipped) to indicate a choice of their second strategy (s_2 for player one and t_2 for player two) or in the original state (Un-Flipped) to indicate a choice of their first strategy (s_1 for player one and t_1 for player two). The referee examines the bits and then assigns the appropriate payoffs. So, under mediated classical communication, we can think of G as a two player, two strategy game where both players have the same set of strategies, namely {No Flip, Flip}. Let the actions No Flip and Flip be represented by the $SU(2)$ matrices

$$N = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad F = \begin{pmatrix} 0 & \eta \\ -\overline{\eta} & 0 \end{pmatrix}, \tag{3.1}$$

respectively, where η is a unit complex number. So the two classical pure strategies available to the players are the $SU(2)$ matrices N and F.

As an example, suppose the referee prepares the bits $|0\rangle$ and $|1\rangle$ in the initial state $|0\rangle \otimes |1\rangle$ and sends player one the bit $|0\rangle$ and player two the bit $|1\rangle$. Then depending on the players' actions N and F, the pair of bits ends up in one of the following four states:

$$(N \otimes N)(|0\rangle \otimes |1\rangle) = |0\rangle \otimes |1\rangle \tag{3.2}$$

$$(N \otimes F)(|0\rangle \otimes |1\rangle) = \eta|0\rangle \otimes |0\rangle \equiv |0\rangle \otimes |0\rangle \tag{3.3}$$

$$(F \otimes N)(|0\rangle \otimes |1\rangle) = -\overline{\eta}|1\rangle \otimes |1\rangle \equiv |1\rangle \otimes |1\rangle \tag{3.4}$$

$$(F \otimes F)(|0\rangle \otimes |1\rangle) = -|1\rangle \otimes |0\rangle \equiv |1\rangle \otimes |0\rangle \tag{3.5}$$

3.3 Mediated Quantum Communication

When the communication between the referee and the players is over quantum channels, Eisert, Wilkens, and Lewenstein [18] have proposed specific families of protocols $\mathcal{Q}$ for the quantization of two player, two strategy games. As described in the previous chapter, the EWL protocols require the game G to be played with a referee who communicates between the players through a quantum channel, that is, players and the referee communicate via (quantum) superpositions of the states of classical bits or qubits. Each player is issued a qubit by the referee. The qubits sent to the players are in a joint initial state, that is, an $\mathcal{H}_4$ element of the form

$$\alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle, \tag{3.6}$$

where the coefficients α, β, γ, and δ are complex numbers, not all zero. Note that each choice of the quadruple $(\alpha, \beta, \gamma, \delta)$ gives an initial state $\mathcal{I}$ and this initial state produces in turn a specific protocol $\mathcal{Q}^{\mathcal{I}}$.

Players act on their qubits via elements of $SU(2)$. In particular, the two classical pure strategies available to the players are the two $SU(2)$ elements N and F described above. After the players's actions, the qubits are sent back to the referee who performs quantum measurement on the final state with respect to some specific basis of $QS(ImG)$. Quantum measurement gives a probability distribution over ImG, which can be used to compute the expected payoffs to the players.

3.4 A Specific Initial State $\mathcal{I}$ and Associated Quantized Game $G^{\mathcal{Q}_{\mathcal{I}}}$

We now examine the quantum game arising from the EWL protocol applied to generic two player, two strategy games and corresponding to the initial state

$$\mathcal{I} = \frac{|00\rangle + |11\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}(1 \quad 0 \quad 0 \quad 1)^T. \tag{3.7}$$

As described above, players one and two act on their respective qubits via the general elements of $SU(2)$

$$U_I = \begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix} \text{ and } U_{II} = \begin{pmatrix} P & Q \\ -\overline{Q} & \overline{P} \end{pmatrix}, \tag{3.8}$$

respectively. After the players act the initial state becomes

$$
\begin{aligned}
(U_I \otimes U_{II})(\mathcal{I}) &= \frac{1}{\sqrt{2}} \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix} \otimes \begin{pmatrix} P & Q \\ -\bar{Q} & \bar{P} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{pmatrix} AP & AQ & BP & BQ \\ -A\bar{Q} & A\bar{P} & -B\bar{Q} & B\bar{P} \\ -\bar{B}P & -\bar{B}Q & \bar{A}P & \bar{A}Q \\ \bar{B}\bar{Q} & -\bar{B}\bar{P} & -\bar{A}\bar{Q} & \bar{A}\bar{P} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}
\end{aligned} \tag{3.9}
$$

which gives in the observational basis

$$
\frac{1}{\sqrt{2}} \begin{pmatrix} AP + BQ \\ -A\bar{Q} + B\bar{P} \\ -\bar{B}P + \bar{A}Q \\ \bar{B}\bar{Q} + \bar{A}\bar{P} \end{pmatrix} \begin{matrix} |0\rangle \otimes |0\rangle \\ |0\rangle \otimes |1\rangle \\ |1\rangle \otimes |0\rangle \\ |1\rangle \otimes |1\rangle \end{matrix} \tag{3.10}
$$

or as a linear combination of the observational basis vectors, $\frac{1}{\sqrt{2}}$ times the expression

$$
\begin{aligned}
&(AP + BQ)|0\rangle \otimes |0\rangle + (-A\bar{Q} + B\bar{P})|0\rangle \otimes |1\rangle + \\
&(\bar{A}Q - \bar{B}P)|1\rangle \otimes |0\rangle + (\bar{A}\bar{P} + \bar{B}\bar{Q})|1\rangle \otimes |1\rangle
\end{aligned} \tag{3.11}
$$

Refer to this resulting state as the *game state* of our system.

Note that the referee cannot use the original observational basis to perform measurement and assign payoffs to the players. The referee needs to define a new orthonormal basis whose basis vectors are in a one to one correspondence with the elements of ImG. For this, consider the actions No Flip and Flip, as described above, represented by the $SU(2)$ matrices N and F and perform the following Kronecker products:

$$N \otimes N \equiv NN = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.12}$$

and

$$N \otimes F \equiv NF = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & \eta \\ -\bar{\eta} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \eta & 0 & 0 \\ -\bar{\eta} & 0 & 0 & 0 \\ 0 & 0 & 0 & \eta \\ 0 & 0 & -\bar{\eta} & 0 \end{pmatrix}, \tag{3.13}$$

and

$$F \otimes N \equiv FN = \begin{pmatrix} 0 & \eta \\ -\bar{\eta} & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & \eta & 0 \\ 0 & 0 & 0 & \eta \\ -\bar{\eta} & 0 & 0 & 0 \\ 0 & -\bar{\eta} & 0 & 0 \end{pmatrix}, \tag{3.14}$$

and

$$F \otimes F \equiv FF = \begin{pmatrix} 0 & \eta \\ -\bar{\eta} & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & \eta \\ -\bar{\eta} & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & 0 & \eta^2 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ \bar{\eta}^2 & 0 & 0 & 0 \end{pmatrix}. \tag{3.15}$$

The actions of theses matrices on the entangled qubits yield the following states with respect to the standard basis of $\mathcal{H}_4$

$$
\begin{aligned}
(N \otimes N)(\mathcal{I}) &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \frac{|00\rangle + |11\rangle}{\sqrt{2}}
\end{aligned} \tag{3.16}
$$

and

$$
\begin{aligned}
(N \otimes F)(\mathcal{I}) &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & \eta & 0 \\ 0 & 0 & 0 & \eta \\ -\bar{\eta} & 0 & 0 & 0 \\ 0 & -\bar{\eta} & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -\bar{\eta} \\ \eta \\ 1 \end{pmatrix} = \frac{-\bar{\eta}|01\rangle + \eta|10\rangle}{\sqrt{2}}
\end{aligned} \tag{3.17}
$$

and

$$(F \otimes N)(\mathcal{I}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & \eta & 0 \\ 0 & 0 & 0 & \eta \\ -\bar{\eta} & 0 & 0 & 0 \\ 0 & -\bar{\eta} & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$
$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \eta \\ -\bar{\eta} \\ 0 \end{pmatrix} = \frac{\eta|01\rangle - \bar{\eta}|10\rangle}{\sqrt{2}} \tag{3.18}$$

and

$$(F \otimes F)(\mathcal{I}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & \eta^2 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ \bar{\eta}^2 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$
$$= \frac{1}{\sqrt{2}} \begin{pmatrix} \eta^2 \\ 0 \\ 0 \\ \bar{\eta}^2 \end{pmatrix} = \frac{\eta^2|00\rangle + \bar{\eta}^2|11\rangle}{\sqrt{2}} \tag{3.19}$$

So in the standard basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ of $\mathcal{H}_4$ the game state corresponding to the standard action profiles are given by $\frac{1}{\sqrt{2}}$ times the vectors

$$NN \equiv \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad NF \equiv \begin{pmatrix} 0 \\ -\overline{\eta} \\ \eta \\ 0 \end{pmatrix}, \quad FN \equiv \begin{pmatrix} 0 \\ \eta \\ -\overline{\eta} \\ 0 \end{pmatrix}, \quad FF \equiv \begin{pmatrix} \eta^2 \\ 0 \\ 0 \\ \overline{\eta}^2 \end{pmatrix} \tag{3.20}$$

For the purpose of the EWL protocol [18], these states are to correspond to a physical property observable to the referee. For this, the axioms of quantum mechanics require these states to form an orthonormal basis of the joint state space of the two qubits. The non-trivial orthogonality conditions are thus

$$\begin{aligned} \langle NN, FF\rangle &= \overline{NN}^T \cdot FF \\ &= \frac{1}{\sqrt{2}}(1 \quad 0 \quad 0 \quad 1)\frac{1}{\sqrt{2}}\begin{pmatrix} \eta^2 \\ 0 \\ 0 \\ \overline{\eta}^2 \end{pmatrix} \\ &= \frac{1}{2}\left(\eta^2 + \overline{\eta}^2\right) = 0 \end{aligned} \tag{3.21}$$

and

$$\begin{aligned}
\langle NF, FN\rangle &= \overline{NF}^T \cdot FN \\
&= \frac{1}{\sqrt{2}}(0 \quad -\eta \quad \bar{\eta} \quad 0)\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ \eta \\ -\bar{\eta} \\ 0 \end{pmatrix} \\
&= \frac{1}{2}\left(-\eta^2 - \bar{\eta}^2\right) = 0
\end{aligned} \tag{3.22}$$

Therefore $\eta^8 = 1$. Thus setting

$$\eta = e^{i\frac{\pi}{4}} = \frac{1+i}{\sqrt{2}} \tag{3.23}$$

insures the orthogonality of the game states given by the standard action profiles. Call this family of states the *action basis* of the joint state space and abusively denote these states by NN, NF, FN, and FF.

Note that for this value of η, $\eta^2 = i$ and $\bar{\eta}^2 = -i$. The final forms of the action basis vectors are thus $\frac{1}{\sqrt{2}}$ times the vectors

$$NN \equiv \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad NF \equiv \begin{pmatrix} 0 \\ -\bar{\eta} \\ \eta \\ 0 \end{pmatrix}, \quad FN \equiv \begin{pmatrix} 0 \\ \eta \\ -\bar{\eta} \\ 0 \end{pmatrix}, \quad FF \equiv \begin{pmatrix} i \\ 0 \\ 0 \\ -i \end{pmatrix} \tag{3.24}$$

The basis change matrix of the action basis to the initial observational basis is thus $\frac{1}{\sqrt{2}}$ times the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & i \\ 0 & -\bar{\eta} & \eta & 0 \\ 0 & \eta & -\bar{\eta} & 0 \\ 1 & 0 & 0 & -i \end{pmatrix} \tag{3.25}$$

After normalizing the lengths of the columns (i.e. scaling the matrix by $1/\sqrt{2}$) this matrix is unitary by construction, so $\bar{A}^T = A^{-1}$. So the basis change matrix from the initial observational basis to the action basis is given by $1/\sqrt{2}$ times the matrix

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -\eta & \bar{\eta} & 0 \\ 0 & \bar{\eta} & -\eta & 0 \\ -i & 0 & 0 & i \end{pmatrix} \tag{3.26}$$

Rewriting a generic game state in the action basis gives

$$\frac{1}{\sqrt{2}}A^{-1}\frac{1}{\sqrt{2}} \begin{pmatrix} AP + BQ \\ -A\bar{Q} + B\bar{P} \\ -\bar{B}P + \bar{A}Q \\ \bar{B}\bar{Q} + \bar{A}\bar{P} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} (AP + \bar{A}\bar{P}) + (BQ + \bar{B}\bar{Q}) \\ (\eta A\bar{Q} + \bar{\eta}\bar{A}Q) - (\eta B\bar{P} + \bar{\eta}\bar{B}P) \\ -(\bar{\eta}A\bar{Q} + \eta\bar{A}Q) + (\bar{\eta}B\bar{P} + \eta\bar{B}P) \\ (-iAP + i\bar{A}\bar{P}) + (-iBQ + i\bar{B}\bar{Q}) \end{pmatrix} \tag{3.27}$$

which gives

$$\frac{1}{2}\begin{pmatrix} 2Re\,(AP+BQ) \\ 2Re\left(\eta A\bar{Q}-\eta B\bar{P}\right) \\ -2Re\left(\bar{\eta}A\bar{Q}-\bar{\eta}B\bar{P}\right) \\ -2Re\,(iAP+iBQ) \end{pmatrix} = \begin{pmatrix} Re\,(AP+BQ) \\ Re\left(\eta A\bar{Q}-\eta B\bar{P}\right) \\ -Re\left(\bar{\eta}A\bar{Q}-\bar{\eta}B\bar{P}\right) \\ -Re\,(iAP+iBQ) \end{pmatrix} \tag{3.28}$$

Rewriting the above expression as a linear combination of the action basis vectors gives

$$\begin{aligned} &[Re\,(AP+BQ)]NN + [Re\left(\eta A\bar{Q}-\eta B\bar{P}\right)]NF \\ &+[-Re\left(\bar{\eta}A\bar{Q}-\bar{\eta}B\bar{P}\right)]FN + [-Re\,(iAP+iBQ)]FF \end{aligned} \tag{3.29}$$

Now let

$$\begin{aligned} A &= a_0 + a_1 i, \quad & B &= b_0 + b_1 i, \\ P &= p_0 + p_1 i, \quad & Q &= q_0 + q_1 i. \end{aligned} \tag{3.30}$$

Then

$$\begin{aligned} coeff(NN) &= Re(AP+BQ) \\ &= Re[(a_0+a_1 i)(p_0+p_1 i)+(b_0+b_1 i)(q_0+q_1 i)] \\ &= a_0 p_0 - a_1 p_1 + b_0 q_0 - b_1 q_1 \end{aligned} \tag{3.31}$$

and

$$\begin{aligned} coeff(FF) &= -Re\left(i(AP+BQ)\right) \\ &= -Re\left(i(a_0+a_1i)(p_0+p_1i)+i(b_0+b_1i)(q_0+q_1i)\right) \\ &= a_0p_1+a_1p_0+b_0q_1+b_1q_0 \end{aligned} \tag{3.32}$$

and

$$\begin{aligned} coeff(NF) &= Re\left(\eta(A\bar{Q}-B\bar{P})\right) \\ &= Re[\frac{1+i}{\sqrt{2}}\left((a_0+a_1i)(q_0-q_1i)-(b_0+b_1i)(p_0-p_1i)\right)] \\ &= \frac{1}{\sqrt{2}}\left[(a_0q_0+a_0q_1-a_1q_0+a_1q_1)+(-b_0p_0-b_0p_1+b_1p_0-b_1p_1)\right] \end{aligned} \tag{3.33}$$

and

$$\begin{aligned} coeff(FN) &= -Re\left(\bar{\eta}(A\bar{Q}-B\bar{P})\right) \\ &= -Re[\frac{1-i}{\sqrt{2}}\left((a_0+a_1i)(q_0-q_1i)-(b_0+b_1i)(p_0-p_1i)\right)] \\ &= \frac{1}{\sqrt{2}}\left[(-a_0q_0+a_0q_1-a_1q_0-a_1q_1)+(b_0p_0-b_0p_1+b_1p_0+b_1p_1)\right] \end{aligned} \tag{3.34}$$

The payoffs to players one and two are thus given by

$$G^{\mathcal{Q}_{\mathcal{I}}}(U_I,U_{II}) = \left(P_1(U_I,U_{II}),P_2(U_I,U_{II})\right), \tag{3.35}$$

where

$$\begin{aligned} P_1(U_I, U_{II}) &= [coeff(NN)]^2 X_0 + [coeff(NF)]^2 X_2 \\ &+ [coeff(FN)]^2 X_3 + [coeff(FF)]^2 X_1 \end{aligned} \tag{3.36}$$

and

$$\begin{aligned} P_2(U_I, U_{II}) &= [coeff(NN)]^2 Y_0 + [coeff(NF)]^2 Y_2 \\ &+ [coeff(FN)]^2 Y_3 + [coeff(FF)]^2 Y_1, \end{aligned} \tag{3.37}$$

respectively, and where the X_i's and the Y_i's are the players' individual payoff in the classical game.

3.5 Quaternions

The quaternions, denoted by $\mathbb{H}$, are a 4-dimensional normed division algebra over the real numbers. For more detail on real division algebras, see appendix A. A general quaternion q is of the form

$$q = a + bi + cj + dk, \tag{3.38}$$

where a, b, c, d are real numbers and i, j, and k satisfy Hamilton's relation

$$i^2 = j^2 = k^2 = ijk = -1. \tag{3.39}$$

We can also express a general quaternion in the form

$$q = \alpha + \beta j, \tag{3.40}$$

where α and β are complex numbers. Throughout, we will work with the general quaternions $p = a_0 + a_1 i + a_2 j + a_3 k$ and $q = b_0 + b_1 i + b_2 j + b_2 k$

Definition 3.41. *Addition with quaternions is component wise:*

$$p + q = (a_0 + b_0) + (a_1 + b_1)i + (a_2 + b_2)j + (a_3 + b_3)k. \tag{3.42}$$

Definition 3.43. *Multiplication with quaternions is polynomial subject to Hamilton's relation* $i^2 = j^2 = k^2 = ijk = -1$.

$$\begin{aligned} pq &= (a_0 + a_1 i + a_2 j + a_3 k)(b_0 + b_1 i + b_2 j + b_3 k) \\ &= (a_0 b_0 - a_1 b_1 - a_2 b_2 - a_3 b_3) + (a_0 b_1 + a_1 b_0 + a_2 b_3 - a_3 b_2)i \\ &+ (a_0 b_2 - a_1 b_3 + a_2 b_0 + a_3 b_1)j + (a_0 b_3 + a_1 b_2 - a_2 b_1 + a_3 b_0)k \end{aligned} \tag{3.44}$$

Definition 3.45. *The* quaternionic conjugate *of a quaternion* p *is defined as*

$$\overline{p}^{\mathbb{H}} = a_0 - a_1 i - a_2 j - a_3 k. \tag{3.46}$$

It is straightforward to verify the following properties

1. The product $\overline{p}^{\mathbb{H}} p = a_0^2 + a_1^2 + a_2^2 + a_3^2$ defines the square of a norm $||p||$ for the

quaternion p. That is

$$||p||^2 = \overline{p}^{\mathbb{H}} p = a_0^2 + a_1^2 + a_2^2 + a_3^2. \tag{3.47}$$

2. The norm is multiplicative, that is $||pq|| = ||p|| \cdot ||q||$ for all quaternions p and q.

3. For any nonzero quaternion q,

$$q^{-1} = \frac{\overline{q}^{\mathbb{H}}}{||q||}. \tag{3.48}$$

   This establishes $\mathbb{H} - \{0\}$ as a division algebra.

4. The set of unit quaternions $\mathbb{H}_1 = \{q \mid ||q||^2 = 1\}$ forms a subgroup of $\mathbb{H} - \{0\}$ under quaternionic multiplication and can be thought as the unit 3-sphere $\mathbb{S}^3$ living in $\mathbb{R}^4$.

5. Multiplication with quaternions is not commutative.

6. The distributive laws hold.

In light of all the above properties, the quaternions form a *skew-field*, that is a non-commutative field. In addition, as a real vector space $\mathbb{H}$ can be identified with $\mathbb{R}^4$ via the map

$$(a_0 + a_1 i + a_2 j + a_3 k) \longmapsto \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} \tag{3.49}$$

If p is an element of $\mathbb{H}$, we call $\vec{p}$ the corresponding element of $\mathbb{R}^4$. Furthermore, if $q \in \mathbb{H}$, let denote $\vec{pq}$ the vector in $\mathbb{R}^4$ corresponding to the quaternion pq.

Definition 3.50. *A set* $\{p_0, p_1, p_2, p_3\}$ *of quaternions is called an* orthonormal basis *of* $\mathbb{H}$ *if* $\{\vec{p_0}, \vec{p_1}, \vec{p_2}, \vec{p_3}\}$ *is an orthonormal basis of* $\mathbb{R}^4$. *That is*

$$\vec{p_i} \cdot \vec{p_j} = \begin{cases} 0 & \textit{if } i \neq j \\ 1 & \textit{if } i = j. \end{cases}$$

Throughout, $\{1, i, j, k\}$ is referred to as the *standard basis* of $\mathbb{H}$.

3.6 Landsburg's Representation

Landsburg [33] identifies the unitary matrices U_I and U_{II} with the set of unit quaternions as follows:

$$U_I = \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix} \longleftrightarrow p = A + B\eta j \tag{3.51}$$

and

$$U_{II} = \begin{pmatrix} P & Q \\ -\bar{Q} & \bar{P} \end{pmatrix} \longleftrightarrow q = P - \eta j Q. \tag{3.52}$$

Write the product pq as

$$\begin{aligned}
pq &= (A + B\eta j)\,(P - \eta j Q) \\
&= AP - A\eta j Q + B\eta j P - B\eta j \eta j Q \\
&= AP - \eta A\bar{Q} j + \eta B\bar{P} j - B\eta\bar{\eta} j j Q \\
&= AP - \eta\left(A\bar{Q} - B\bar{P}\right) j + BQ \\
&= [Re(AP + BQ)] + [Im(AP + BQ)]i + [-Re\left(\eta(A\bar{Q} - B\bar{P})\right)]j \\
&\quad + [-Im\left(\eta(A\bar{Q} - B\bar{P})\right)]k \qquad (3.53) \\
&= \pi_0(pq) + \pi_1(pq)i + \pi_2(pq)j + \pi_3(pq)k, \qquad (3.54)
\end{aligned}$$

where the $\pi_t(pq)$ are real numbers. Then

$$\pi_0(pq) = Re(AP + BQ) = coeff(NN), \qquad (3.55)$$

and

$$\pi_1(pq) = Im(AP + BQ) = coeff(FF), \qquad (3.56)$$

and

$$\begin{aligned}
\pi_2(pq) &= -Re\left(\eta(A\bar{Q} - B\bar{P})\right) \\
&= -Re[\frac{1+i}{\sqrt{2}}\left((a_0 + a_1 i)(q_0 - q_1 i) - (b_0 + b_1 i)(p_0 - p_1 i)\right)] \\
&= \frac{1}{\sqrt{2}}\left[(-a_0 q_0 - a_0 q_1 + a_1 q_0 - a_1 q_1) + (b_0 p_0 + b_0 p_1 - b_1 p_0 + b_1 p_1)\right] \\
&= -coeff(NF), \qquad (3.57)
\end{aligned}$$

and

$$\begin{aligned}\pi_3(pq) &= -Im\left(\eta(A\bar{Q} - B\bar{P})\right) \\ &= -Im[\frac{1+i}{\sqrt{2}}((a_0 + a_1 i)(q_0 - q_1 i) - (b_0 + b_1 i)(p_0 - p_1 i))] \\ &= \frac{1}{\sqrt{2}}[(-a_0 q_0 + a_0 q_1 - a_1 q_0 - a_1 q_1) + (b_0 p_0 - b_0 p_1 + b_1 p_0 + b_1 p_1)] \\ &= coeff(FN). \end{aligned} \tag{3.58}$$

Hence, the probability distribution over the outcomes NN, NF, FN, FF in the game described above is given by

$$\begin{aligned} pr(NN) &= [\pi_0(pq)]^2, \quad pr(NF) = [\pi_2(pq)]^2 \\ pr(FN) &= [\pi_3(pq)]^2, \quad pr(FF) = [\pi_1(pq)]^2. \end{aligned} \tag{3.59}$$

This result motivates the following definition:

Definition 3.60. *Let G be the game described in table 3.1. Then the associated* quantization *$G^{\mathcal{Q}_{\mathcal{I}}}$ is the two player game in which each player's strategy space is the unit quaternions, and the payoff functions for players one and two are defined as follows:*

$$P_1(p,q) = \sum_{t=0}^{3}[\pi_t(pq)]^2 X_t \tag{3.61}$$

$$P_2(p,q) = \sum_{t=0}^{3}[\pi_t(pq)]^2 Y_t, \tag{3.62}$$

or in compact form

$$G^{Q_\mathcal{I}}(p,q) = \left(\sum_{t=0}^{3} [\pi_t(pq)]^2 X_t, \ \sum_{t=0}^{3} [\pi_t(pq)]^2 Y_t \right) \tag{3.63}$$

Note that one can improve Landsburg's representation by defining the following identifications of the group $SU(2)$ with the group $\mathbb{S}^3$:

$$U_I = \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix} \longleftrightarrow p = A + B\bar{\eta}j \tag{3.64}$$

and

$$U_{II} = \begin{pmatrix} P & Q \\ -\bar{Q} & \bar{P} \end{pmatrix} \longleftrightarrow q = P - \bar{Q}\bar{\eta}j. \tag{3.65}$$

With these identifications, we find that the coefficients of the game state in the action basis exactly match the coefficients of the unit quaternion pq, namely

$$\begin{aligned} \pi_0(pq) &= coeff(NN); \quad \pi_1(pq) = coeff(FF) \\ \pi_2(pq) &= coeff(FN); \quad \pi_3(pq) = coeff(NF). \end{aligned} \tag{3.66}$$

Now by reindexing the entries of the bimatrix of table 3.1 as in table 3.2

	t_1	t_2
s_1	(X_0, Y_0)	(X_3, Y_3)
s_2	(X_2, Y_2)	(X_1, Y_1)

Table 3.2: A Reindexed Generic 2x2 Game

one can define $G^{Q_\mathcal{I}}$ as in equation (3.63).

3.6.1 Classification of Nash Equilibria

Following Bleiler quantization formalism, one can further extend the game G by a larger game $G^{mQ_{\mathcal{I}}}$ which is the two player game in which each player's strategy space is the probability distribution over the set of unit quaternions $\mathbb{S}^3$.

In this setting, a *mixed quantum strategy* is a probability distribution μ over the set of unit quaternions $\mathbb{S}^3$. The space of mixed quantum strategies is huge and fairly intractable. Landsburg [33] shows that mixed quantum strategies fall naturally into equivalence classes with particularly simple representatives, which we indicate below.

If ν and μ are mixed quantum strategies chosen by Players one and two, we write

$$\mathcal{E}_i(\nu, \mu) = \int_{\mathbb{S}^3 \times \mathbb{S}^3} P_i(p, q)\, d\nu(p)\, d\mu(q). \tag{3.67}$$

for player i's expected payoff.

Definition 3.68. *Two mixed quantum strategies μ and μ' are said to be* equivalent *if and only if*

$$\mathcal{E}_i(\mu, q) = \mathcal{E}_i(\mu', q) \tag{3.69}$$

for all q and for all $i = 1, 2$.

The problem of classifying the equilibria in mixed quantum strategies is simplified by the Landsburg's result:

Theorem 3.70. *[33] Every mixed quantum strategy is equivalent to a mixed quantum strategy supported on at most four points and those four points can be taken to form an orthonormal basis of $\mathbb{R}^4$.*

Without loss of generality, at least one of the players may choose these four points to be the quaternions basis elements $1, i, j, k$. Landsburg then uses the above theorem to show the following:

Theorem 3.71. *[33] If G is the game described in table 3.1, then up to equivalence, every equilibrium in mixed quantum strategies (ν, μ) is one of the following types:*

1. $\nu = \mu = \frac{1}{4}(p+q+r+s)$ *where p, q, r, and s are four orthogonal quaternions.*

2. *Each player's strategy is supported on three of the four quaternions $1, i, j, k$.*

3. *The mixed strategy μ is supported on two orthogonal quaternions $1, \mathbf{v}$; the mixed strategy ν is supported on two orthogonal quaternions $\mathbf{p}, \mathbf{pu}$ and $\alpha(X\mathbf{p} + Y\mathbf{pv}) = X\mathbf{pu} + Y\mathbf{pvu}$ identically in X and Y for some nonzero constant α.*

4. *Each of ν and μ is supported on two orthogonal points, each played with probability $1/2$. Moreover, the supports of ν and μ lie in parallel planes.*

5. *Each player plays a pure strategy from the four point set $\{1, i, j, k\}$.*

3.6.2 New Notations

Writing the expected payoffs to players one and two when they employ mixed quantum strategies can be tedious and time consuming. We discuss in this section notation that makes systematic the computations of players' payoffs.

Suppose that players one and two employ mixed strategies supported on the orthonormal quaternionic bases $\mathcal{B}_1 = \{p_0, p_1, p_2, p_3\}$ and $\mathcal{B}_2 = \{q_0, q_1, q_2, q_3\}$, respectively. If player 1 chooses

$$\nu = a_0 p_0 + a_1 p_1 + a_2 p_2 + a_3 p_3 \tag{3.72}$$

and player 2 chooses

$$\mu = b_0 q_0 + b_1 q_1 + b_2 q_2 + b_3 q_3. \tag{3.73}$$

Then the expected payoff function of player 1, for example, is calculated as follows

$$\begin{aligned}\mathcal{E}_1(\nu, \mu) &= \sum_{m,n=0}^{3} a_m b_n P_1(p_m, q_n) \\ &= \sum_{m,n=0}^{3} a_m b_n \left(\sum_{t=0}^{3} [\pi_t(p_m q_n)]^2 X_t \right) \\ &= \sum_{m,n,t=0}^{3} a_m b_n [\pi_t(p_m q_n)]^2 X_t \end{aligned} \tag{3.74}$$

This notation leads to the following definition of the mixed quantized game $G^{m\mathcal{Q}_\mathcal{I}}$:

Definition 3.75. *Let G be the game described in table 3.2. Then the associated* mixed quantization $G^{m\mathcal{Q}_\mathcal{I}}$ *is the two player game in which each player's strategy space is the set of equivalence classes of probability distributions over the set of unit quaternions given by Definition 3.68 and represented as in Theorem 3.70. The expected payoff functions for players one and two in this game are defined as follows:*

$$\mathcal{E}_1(\nu, \mu) = \mathbf{a}^T \mathcal{M}_1 \mathbf{b}; \quad \mathcal{E}_2(\nu, \mu) = \mathbf{b}^T \mathcal{M}_2 \mathbf{a}, \tag{3.76}$$

or in compact form

$$G^{m\mathcal{Q}_{\mathcal{I}}}(\nu, \mu) = \left(\mathbf{a}^T \mathcal{M}_1 \mathbf{b} \ , \ \mathbf{b}^T \mathcal{M}_2 \mathbf{a}\right), \tag{3.77}$$

where

$$\mathbf{a} = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{pmatrix} \tag{3.78}$$

and

$$\mathcal{M}_1 = \begin{pmatrix} \sum_{t=0}^{3} \pi_t^2(p_0 q_0) X_t & \sum_{t=0}^{3} \pi_t^2(p_0 q_1) X_t & \sum_{t=0}^{3} \pi_t^2(p_0 q_2) X_t & \sum_{t=0}^{3} \pi_t^2(p_0 q_3) X_t \\ \sum_{t=0}^{3} \pi_t^2(p_1 q_0) X_t & \sum_{t=0}^{3} \pi_t^2(p_1 q_1) X_t & \sum_{t=0}^{3} \pi_t^2(p_1 q_2) X_t & \sum_{t=0}^{3} \pi_t^2(p_1 q_3) X_t \\ \sum_{t=0}^{3} \pi_t^2(p_2 q_0) X_t & \sum_{t=0}^{3} \pi_t^2(p_2 q_1) X_t & \sum_{t=0}^{3} \pi_t^2(p_2 q_2) X_t & \sum_{t=0}^{3} \pi_t^2(p_2 q_3) X_t \\ \sum_{t=0}^{3} \pi_t^2(p_3 q_0) X_t & \sum_{t=0}^{3} \pi_t^2(p_3 q_1) X_t & \sum_{t=0}^{3} \pi_t^2(p_3 q_2) X_t & \sum_{t=0}^{3} \pi_t^2(p_3 q_3) X_t \end{pmatrix}, \tag{3.79}$$

$$\mathcal{M}_2 = \begin{pmatrix} \sum_{t=0}^{3} \pi_t^2(p_0 q_0) Y_t & \sum_{t=0}^{3} \pi_t^2(p_0 q_1) Y_t & \sum_{t=0}^{3} \pi_t^2(p_0 q_2) Y_t & \sum_{t=0}^{3} \pi_t^2(p_0 q_3) Y_t \\ \sum_{t=0}^{3} \pi_t^2(p_1 q_0) Y_t & \sum_{t=0}^{3} \pi_t^2(p_1 q_1) Y_t & \sum_{t=0}^{3} \pi_t^2(p_1 q_2) Y_t & \sum_{t=0}^{3} \pi_t^2(p_1 q_3) Y_t \\ \sum_{t=0}^{3} \pi_t^2(p_2 q_0) Y_t & \sum_{t=0}^{3} \pi_t^2(p_2 q_1) Y_t & \sum_{t=0}^{3} \pi_t^2(p_2 q_2) Y_t & \sum_{t=0}^{3} \pi_t^2(p_2 q_3) Y_t \\ \sum_{t=0}^{3} \pi_t^2(p_3 q_0) Y_t & \sum_{t=0}^{3} \pi_t^2(p_3 q_1) Y_t & \sum_{t=0}^{3} \pi_t^2(p_3 q_2) Y_t & \sum_{t=0}^{3} \pi_t^2(p_3 q_3) Y_t \end{pmatrix} \tag{3.80}$$

and where the real numbers X_t and Y_t are taken from table 3.2. Call the matrices $\mathcal{M}_1$ and $\mathcal{M}_2$ the quantum payoff matrices.

Note that if $\mathcal{B}_1 = \mathcal{B}_2 = \{1, i, j, k\}$, the quantum payoff matrices $\mathcal{M}_1$ and $\mathcal{M}_2$ reduce to

$$\mathcal{M}_1 = \begin{pmatrix} X_0 & X_1 & X_2 & X_3 \\ X_1 & X_0 & X_3 & X_2 \\ X_2 & X_3 & X_0 & X_1 \\ X_3 & X_2 & X_1 & X_0 \end{pmatrix}, \quad \mathcal{M}_2 = \begin{pmatrix} Y_0 & Y_1 & Y_2 & Y_3 \\ Y_1 & Y_0 & Y_3 & Y_2 \\ Y_2 & Y_3 & Y_0 & Y_1 \\ Y_3 & Y_2 & Y_1 & Y_0 \end{pmatrix}. \tag{3.81}$$

When the support of the mixed quantum strategies ν and μ is the standard basis of $\mathbb{H}$, we can express the players' expected payoff as *frequencies* against *returns* as follows.

$$\begin{aligned}\mathcal{E}_1(\nu,\mu) &= a_0(b_0X_0 + b_1X_1 + b_2X_2 + b_3X_3) + a_1(b_0X_1 + b_1X_0 + b_2X_3 + b_3X_2)\\ &\quad + a_2(b_0X_2 + b_1X_3 + b_2X_0 + b_3X_1) + a_3(b_0X_3 + b_1X_2 + b_2X_1 + b_3X_0)\\ &= a_0A_0 + a_1A_1 + a_2A_2 + a_3A_3 \qquad (3.82)\end{aligned}$$

and

$$\begin{aligned}\mathcal{E}_2(\nu,\mu) &= b_0(a_0Y_0 + a_1Y_1 + a_2Y_2 + a_3Y_3) + b_1(a_0Y_1 + a_1Y_0 + a_2Y_3 + a_3Y_2)\\ &\quad + b_2(a_0Y_2 + a_1Y_3 + a_2Y_0 + a_3Y_1) + b_3(a_0Y_3 + a_1Y_2 + a_2Y_1 + a_3Y_0)\\ &= b_0B_0 + b_1B_1 + b_2B_2 + b_3B_3 \qquad (3.83)\end{aligned}$$

Call the a_i's and b_i's the *frequencies* and the A_i's and B_i's the *returns*. Using these equations, we can now classify the equilibria in $G^{mQ_{\mathcal{I}}}$. The key idea for this is to note that a best response must "concentrate" the frequency on the largest returns.

For example, consider the case where all the A_i are equal and maximal. Then in equilibrium, Player one will choose the uniform probability distribution over the set $\{1, i, j, k\}$, and similarly for Player Two. Note, this is an equilibrium of type 1.

We finish this section with a summary of Landsburg's construction.

Exploiting the identification of the group $SU(2)$ with the group of unit quaternions, and after identifying each pure quantum strategy for each player with a suitable unit quaternion p or q, Landsburg shows that the probability distribution over the four possible outcomes when the players use these strategies is then given by the squares of the coefficients of the unit quaternion pq [33]. This corresponds in Bleiler's formalism [8] to a map L from ΠQ_i to $\Delta(ImG^{\mathcal{Q}})$ as shown in figure 3.1. Using this map when mixed quantum strategies are played produces an additional map $L_\star$ from $\Delta(\Pi Q_i)$ to $\Delta(ImG^{\mathcal{Q}})$. These maps give Landsburg the computational capability to recognize and classify the potential Nash equilibria of $G^{\mathcal{Q}_\mathcal{I}}$ and $G^{m\mathcal{Q}_\mathcal{I}}$.

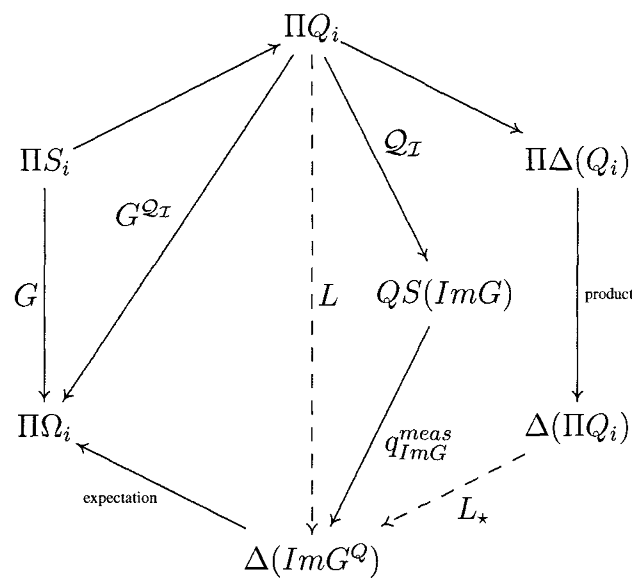

Figure 3.1: Landsburg's Maps L and $L_\star$

3.7 Generalizing the Landsburg Representation

We extend Landsburg's construction to 2x2 games where the initial state is chosen arbitrarily from a circle of maximally entangled states with equal superpositions.

3.7.1 Two Qubit Maximally Entangled States

Using the Dirac notation we consider now a system of two qubits in a four-dimensional complex vector space $\mathcal{H}_4$. We choose as a basis the vectors $|0\rangle \otimes |0\rangle = |00\rangle$, $|0\rangle \otimes |1\rangle = |01\rangle$, $|1\rangle \otimes |0\rangle = |10\rangle$, and $|1\rangle \otimes |1\rangle = |11\rangle$. A general two-qubit joint state is thus expressed in the form

$$\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle, \tag{3.84}$$

where α_{00}, α_{01}, α_{10}, and α_{11} are complex numbers, not all zero, and defined up to phase, that is a non-zero complex multiple.

Now consider a special state of a two-qubit system when

$$\alpha_{01} = \alpha_{10} = 0 \quad and \quad |\alpha_{00}| = |\alpha_{11}|. \tag{3.85}$$

This state is called a *maximally entangled state with equal superpositions* and has form

$$\alpha_{00}|00\rangle + \alpha_{11}|11\rangle. \tag{3.86}$$

Factoring for α_{00}, we obtain

$$\alpha_{00}\left(|00\rangle + \frac{\alpha_{11}}{\alpha_{00}}|11\rangle\right) \tag{3.87}$$

Since two states are the same up to a phase, we obtain

$$\begin{aligned}\alpha_{00}|00\rangle + \alpha_{11}|11\rangle &= \alpha_{00}\left(|00\rangle + \frac{\alpha_{11}}{\alpha_{00}}|11\rangle\right)\\ &\equiv |00\rangle + \frac{\alpha_{11}}{\alpha_{00}}|11\rangle \end{aligned} \tag{3.88}$$

Note that $|\frac{\alpha_{11}}{\alpha_{00}}| = \frac{|\alpha_{11}|}{|\alpha_{00}|} = 1$. Therefore, $\frac{\alpha_{11}}{\alpha_{00}}$ is a unit complex number. Hence, when normalized, a general two-qubit maximally entangled state with equal superpositions has form

$$\frac{|00\rangle + \nu|11\rangle}{\sqrt{2}}, \tag{3.89}$$

where ν is a unit complex number, $\nu = e^{i\theta}$ with $0 \leq \theta < 2\pi$, say. This gives a circle of maximally entangled states

$$I_\theta = \frac{|00\rangle + e^{i\theta}|11\rangle}{\sqrt{2}}, \tag{3.90}$$

where θ is an element of the half open interval $[0, 2\pi)$. If the two-qubit system is in the state I_θ, when we observe the first qubit the two possible outcomes are $|0\rangle$ with probability 1/2 and $|1\rangle$ with probability 1/2. When we measure the second qubit, the two possible outcomes are $|0\rangle$ with probability 1/2 and $|1\rangle$ with probability 1/2. The two measurements are completely correlated, once we measure one qubit we know with one hundred percent certainty the state the other will be observed in.

3.7.2 Initial State $\mathcal{I}_\theta$ and Associated Quantized Game $G^{Q\mathcal{I}_\theta}$

Consider two qubits with respect to the observational basis $\{H, T\}$, where

$$H \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle \text{ and } T \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle \tag{3.91}$$

in the initial state

$$\mathcal{I}_\theta = (H \otimes H + \nu T \otimes T)/\sqrt{2} \tag{3.92}$$

$$= \frac{|00\rangle + e^{i\theta}|11\rangle}{\sqrt{2}} \tag{3.93}$$

The players operate on their respective qubits, the first via

$$U_I = \begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix} \tag{3.94}$$

and the second via

$$U_{II} = \begin{pmatrix} P & Q \\ -\overline{Q} & \overline{P} \end{pmatrix}, \tag{3.95}$$

respectively.

Ignoring the normalization constant $1/\sqrt{2}$, after the players act the initial state becomes

$$\begin{pmatrix} A \\ -\overline{B} \end{pmatrix} \otimes \begin{pmatrix} P \\ -\overline{Q} \end{pmatrix} + \nu \begin{pmatrix} B \\ \overline{A} \end{pmatrix} \otimes \begin{pmatrix} Q \\ \overline{P} \end{pmatrix} \tag{3.96}$$

Expanding bilinearly we obtain

$$\left[A\begin{pmatrix}1\\0\end{pmatrix}-\overline{B}\begin{pmatrix}0\\1\end{pmatrix}\right]\otimes\left[P\begin{pmatrix}1\\0\end{pmatrix}-\overline{Q}\begin{pmatrix}0\\1\end{pmatrix}\right]$$
$$+\nu\left[B\begin{pmatrix}1\\0\end{pmatrix}+\overline{A}\begin{pmatrix}0\\1\end{pmatrix}\right]\otimes\left[Q\begin{pmatrix}1\\0\end{pmatrix}+\overline{P}\begin{pmatrix}0\\1\end{pmatrix}\right], \tag{3.97}$$

or

$$[AP+\nu BQ]\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}+\left[-A\overline{Q}+\nu B\overline{P}\right]\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix} \tag{3.98}$$
$$+\left[-\overline{B}P+\nu\overline{A}Q\right]\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}+\left[\,\overline{BQ}+\nu\overline{AP}\,\right]\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}, \tag{3.99}$$

and with respect to the observational basis

$$(AP+\nu BQ)\,H\otimes H+\left(-\overline{A}Q+\nu B\overline{P}\,\right)H\otimes T$$
$$+\left(-\overline{B}P+\nu\overline{A}Q\right)T\otimes H+\left(\,\overline{BQ}+\nu\overline{AP}\,\right)T\otimes T \tag{3.100}$$

Refer to this resulting state as the *game state* of our system.

Consider next the actions *no flip* and *flip* represented by the *SU(2)* matrices

$$N=\begin{pmatrix}1&0\\0&1\end{pmatrix}\quad \textit{and}\quad F=\begin{pmatrix}0&\eta\\-\overline{\eta}&0\end{pmatrix} \tag{3.101}$$

respectively, where η is a unit complex number to be determined shortly. Note that $F(H) = -\overline{\eta}T$ and $F(T) = \eta H$. Then the so called *standard action profiles* of the players consist of NN, NF, FN, FF. Evaluating the game state expansion for these action profiles we get that the corresponding values of A, B, P, Q are given in the table below

	A	B	P	Q
NN	1	0	1	0
NF	1	0	0	η
FN	0	η	1	0
FF	0	η	0	η

Table 3.3: Evaluation of the Game State Expansion of the Action Profiles

So in the joint observational basis $\{HH, HT, TH, TT\}$ we obtain that the game states corresponding to the standard action profiles are given by

$$NN \equiv \begin{pmatrix} 1 \\ 0 \\ 0 \\ \nu \end{pmatrix}, \; NF \equiv \begin{pmatrix} 0 \\ -\overline{\eta} \\ \nu\eta \\ 0 \end{pmatrix}, \; FN \equiv \begin{pmatrix} 0 \\ \nu\eta \\ -\overline{\eta} \\ 0 \end{pmatrix}, \; FF \equiv \begin{pmatrix} \nu\eta^2 \\ 0 \\ 0 \\ \overline{\eta}^2 \end{pmatrix}. \tag{3.102}$$

For the purpose of the EWL protocols, these states are to correspond to a physical property observable to the referee. For this, the axioms of quantum mechanics require these states to form an orthogonal basis of the joint state space of the two qubits. The non-trivial orthogonality conditions are thus

$$\langle NN, FF \rangle = \overline{\nu}\,\overline{\eta}^2 + \nu\eta^2 = 0, \;\; \langle NF, FN \rangle = -\overline{\nu}\,\overline{\eta}^2 - \nu\eta^2 = 0 \tag{3.103}$$

Therefore $\eta^4 = i^2\,\overline{\nu}^2$. Thus setting $\eta = \eta_\theta = e^{i\frac{\pi-2\theta}{4}}$ insures the orthogonality of the game states given by the standard action profiles. Call this family of states the *action basis* of the joint state space and abusively denote these states by NN, NF, FN, and FF.

The basis change matrix of the action basis to the initial observational basis is thus

$$A_\theta = \begin{pmatrix} 1 & 0 & 0 & i \\ 0 & -e^{i\frac{2\theta-\pi}{4}} & e^{i\frac{2\theta+\pi}{4}} & 0 \\ 0 & e^{i\frac{2\theta+\pi}{4}} & -e^{i\frac{2\theta-\pi}{4}} & 0 \\ e^{i\theta} & 0 & 0 & e^{i\frac{2\theta-\pi}{2}} \end{pmatrix} \tag{3.104}$$

After normalizing the lengths of the columns (i.e. scaling the matrix by $1/\sqrt{2}$) this matrix is unitary by construction, so $\overline{A_\theta}^T = A_\theta^{-1}$. So the basis change matrix from the initial observational basis to the action basis is given by $1/\sqrt{2}$ times the matrix

$$A_\theta^{-1} = \begin{pmatrix} 1 & 0 & 0 & e^{-i\theta} \\ 0 & -e^{-i\frac{\pi-2\theta}{4}} & e^{-i\frac{2\theta+\pi}{4}} & 0 \\ 0 & e^{-i\frac{2\theta+\pi}{4}} & -e^{-i\frac{\pi+2\theta}{4}} & 0 \\ -i & 0 & 0 & e^{i\frac{\pi-2\theta}{2}} \end{pmatrix} \tag{3.105}$$

Rewriting a generic game state in the action basis gives (up to normalization)

$$A_\theta^{-1} \begin{pmatrix} AP + e^{i\theta}\overline{BQ} \\ AQ - e^{i\theta}\overline{BP} \\ BP - e^{i\theta}\overline{AQ} \\ BQ + e^{i\theta}\overline{AP} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & e^{-i\theta} \\ 0 & -e^{-i\frac{\pi-2\theta}{4}} & e^{-i\frac{2\theta+\pi}{4}} & 0 \\ 0 & e^{-i\frac{2\theta+\pi}{4}} & -e^{-i\frac{\pi+2\theta}{4}} & 0 \\ -i & 0 & 0 & e^{i\frac{\pi-2\theta}{2}} \end{pmatrix} \begin{pmatrix} AP + e^{i\theta}\overline{BQ} \\ AQ - e^{i\theta}\overline{BP} \\ BP - e^{i\theta}\overline{AQ} \\ BQ + e^{i\theta}\overline{AP} \end{pmatrix} \tag{3.106}$$

which gives

$$\begin{pmatrix} (AP+\overline{AP})+(e^{i\theta}BQ+e^{-i\theta}\overline{BQ}) \\ \left(e^{i\frac{\pi-2\theta}{4}}A\overline{Q}+e^{-i\frac{\pi-2\theta}{4}}\overline{A}Q\right)-\left(e^{i\frac{\pi+2\theta}{4}}B\overline{P}+e^{-i\frac{2\theta+\pi}{4}}\overline{B}P\right) \\ -\left(e^{i\frac{\pi+2\theta}{4}}\overline{A}Q+e^{-i\frac{2\theta+\pi}{4}}A\overline{Q}\right)+\left(e^{i\frac{\pi-2\theta}{4}}\overline{B}P+e^{-i\frac{\pi-2\theta}{4}}B\overline{P}\right) \\ (-iAP-\overline{iAP})+(-ie^{i\theta}BQ+ie^{-i\theta}\overline{BQ}) \end{pmatrix} \begin{matrix} NN \\ NF \\ FN \\ FF \end{matrix} \tag{3.107}$$

or

$$\begin{pmatrix} 2Re(AP)+2Re(e^{i\theta}BQ) \\ 2Re(e^{i\frac{\pi-2\theta}{4}}A\overline{Q})-2Re(e^{i\frac{\pi+2\theta}{4}}B\overline{P}) \\ -2Re(e^{i\frac{\pi+2\theta}{4}}\overline{A}Q)+2Re(e^{i\frac{\pi-2\theta}{4}}\overline{B}P) \\ -2Re(iAP)-2Re(ie^{i\theta}BQ) \end{pmatrix} \begin{matrix} NN \\ NF \\ FN \\ FF \end{matrix} \tag{3.108}$$

It is straightforward to verify all the following identities

- $Re(e^{i\frac{\pi+2\theta}{4}}\overline{A}Q)=Im(e^{i\frac{\pi-2\theta}{4}}A\overline{Q})$
- $Re(e^{i\frac{\pi-2\theta}{4}}\overline{B}P)=Im(e^{i\frac{\pi+2\theta}{4}}B\overline{P})$
- $-Re(iAP)=Im(AP)$
- $-Re(ie^{i\theta}BQ)=Im(e^{i\theta}BQ)$
- $Re(z)+Re(w)=Re(z+w)$
- $Im(z)+Im(w)=Im(z+w)$
- $zj=j\bar{z}$ and $zk=k\bar{z}$ for all complex numbers z.

Using the above identities and rewriting a generic game state in the action basis gives

(up to normalization)

$$\begin{pmatrix} Re(AP + e^{i\theta}BQ) \\ Re\left(e^{i\frac{\pi}{4}}(e^{-i\frac{\theta}{2}}A\overline{Q} - e^{i\frac{\theta}{2}}B\overline{P})\right) \\ -Im\left(e^{i\frac{\pi}{4}}(e^{-i\frac{\theta}{2}}A\overline{Q} - e^{i\frac{\theta}{2}}B\overline{P})\right) \\ Im(AP + e^{i\theta}BQ) \end{pmatrix} \begin{matrix} NN \\ NF \\ FN \\ FF \end{matrix} \tag{3.109}$$

Hence, up to normalization, the referee observing the game state in the action basis sees each pure action state with probability given by

$$pr(NN) = \left[Re(AP + e^{i\theta}BQ)\right]^2 \tag{3.110}$$

$$pr(NF) = \left[Re\left(e^{i\frac{\pi}{4}}(e^{-i\frac{\theta}{2}}A\overline{Q} - e^{i\frac{\theta}{2}}B\overline{P})\right)\right]^2 \tag{3.111}$$

$$pr(FN) = \left[-Im\left(e^{i\frac{\pi}{4}}(e^{-i\frac{\theta}{2}}A\overline{Q} - e^{i\frac{\theta}{2}}B\overline{P})\right)\right]^2 \tag{3.112}$$

$$pr(FF) = \left[Im(AP + e^{i\theta}BQ)\right]^2 \tag{3.113}$$

Note that Landsburg's representation is the case where $\theta = 0$.

3.7.3 Unit Quaternions as Quantum Strategies

Now consider the identifications of the group $SU(2)$ with the group $\mathbb{S}^3$, considered as the unit quaternions equipped with quaternionic multiplication, via the maps

$$\begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix} \mapsto p_\theta = A + B\overline{\eta_\theta}e^{i\frac{\theta}{2}}j \tag{3.114}$$

and

$$\begin{pmatrix} P & Q \\ -\overline{Q} & \overline{P} \end{pmatrix} \mapsto q_\theta = P - \overline{Q}\overline{\eta_\theta}e^{-i\frac{\theta}{2}}j \tag{3.115}$$

It is straightforward to check that p_θ and q_θ are unit quaternions.

Proposition 3.116. *The maps given in (3.114) and (3.115) are group isomorphisms for all* $\theta \in [0, 2\pi)$.

Proof. We will show that the map given in (3.114) is a group isomorphism. The proof of (3.115) is similar and omitted. Call the map given in (3.114) φ_θ and let

$$M_1 = \begin{pmatrix} A_1 & B_1 \\ -\overline{B}_1 & \overline{A}_1 \end{pmatrix} \text{ and } M_2 = \begin{pmatrix} A_2 & B_2 \\ -\overline{B}_2 & \overline{A}_2 \end{pmatrix}$$

be elements of $SU(2)$. Then

$$\begin{aligned}
\varphi_\theta(M_1M_2) &= \varphi_\theta \begin{pmatrix} A_1A_2 - B_1\overline{B}_2 & A_1B_2 + \overline{A}_2B_1 \\ -\overline{A}_1\,\overline{B}_2 - \overline{A}_2\,\overline{B}_1 & \overline{A}_1\,\overline{A}_2 - \overline{B}_1B_2 \end{pmatrix} \\
&= \left(A_1A_2 - B_1\overline{B}_2\right) + \left(A_1B_2 + \overline{A}_2B_1\right)\overline{\eta_\theta}e^{i\frac{\theta}{2}}j \\
&= \left(A_1 + B_1\overline{\eta_\theta}e^{i\frac{\theta}{2}}j\right)\left(A_2 + B_2\overline{\eta_\theta}e^{i\frac{\theta}{2}}j\right) \\
&= \varphi_\theta(M_1)\varphi_\theta(M_2)
\end{aligned}$$

Note that $Ker_{\varphi_\theta} = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\}$. Hence φ_θ is one-to-one.

For each unit quaternion $p = a_0 + a_1i + a_2j + a_3k$, there corresponds a special unitary matrix $M = \begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix}$ such that $p = \varphi_\theta(M)$. It is sufficient to choose $A = a_0 + a_1 i$ and $B = \frac{1}{\sqrt{2}}\left[(a_2 - a_3) + (a_2 + a_3)i\right] e^{-i\theta}$. Therefore, φ_θ is onto. Accordingly, φ_θ is a group isomorphism from $SU(2)$ onto $\mathbb{S}^3$.

□

Now suppose that player 1 chooses the unit quaternion p_θ as defined in (3.114) and player 2 chooses the unit quaternion q_θ as defined in (3.115). Write the product $p_\theta q_\theta$ as

$$\begin{aligned}
p_\theta q_\theta &= \left(A + B\overline{\eta_\theta} e^{i\frac{\theta}{2}} j\right)\left(P - \overline{Q}\overline{\eta_\theta} e^{-i\frac{\theta}{2}} j\right) \qquad (3.117)\\
&= AP - A\overline{Q}\overline{\eta} e^{-i\frac{\theta}{2}} j + B\overline{\eta} e^{i\frac{\theta}{2}} jP - B\overline{\eta_\theta} e^{i\frac{\theta}{2}} j\overline{Q}\overline{\eta_\theta} e^{-i\frac{\theta}{2}} j\\
&= \left(AP + e^{i\theta} BQ\right) - \left(e^{-i\frac{\pi}{4}}(e^{-i\frac{\theta}{2}} A\overline{Q} - e^{i\frac{\theta}{2}} B\overline{P})\right) e^{i\frac{\theta}{2}} j\\
&= \left[Re\left(AP + e^{i\theta} BQ\right)\right] \cdot 1 + \left[Im\left(AP + e^{i\theta} BQ\right)\right] \cdot i\\
&- \left[Re\left(e^{-i\frac{\pi}{4}}(e^{-i\frac{\theta}{2}} A\overline{Q} - e^{i\frac{\theta}{2}} B\overline{P})\right)\right] \cdot e^{i\frac{\theta}{2}} j - \left[Im\left(e^{-i\frac{\pi}{4}}(e^{-i\frac{\theta}{2}} A\overline{Q} - e^{i\frac{\theta}{2}} B\overline{P})\right)\right] \cdot e^{i\frac{\theta}{2}} k\\
&= \pi_0(p_\theta q_\theta) \cdot 1 + \pi_1(p_\theta q_\theta) \cdot i + \pi_2(p_\theta q_\theta) \cdot e^{i\frac{\theta}{2}} j + \pi_3(p_\theta q_\theta) \cdot e^{i\frac{\theta}{2}} k
\end{aligned}$$

where the $\pi_t(p_\theta q_\theta)$ are real numbers. Then we are led to the following theorem

Theorem 3.118. *The probability distribution over the outcomes NN, NF, FN, FF in the game described in table 3.2 is given by*

$$\begin{aligned}
pr(NN) &= [\pi_0(p_\theta q_\theta)]^2, \quad pr(NF) = [\pi_3(p_\theta q_\theta)]^2\\
pr(FN) &= [\pi_2(p_\theta q_\theta)]^2, \quad pr(FF) = [\pi_1(p_\theta q_\theta)]^2 \qquad (3.119)
\end{aligned}$$

Proof. Comparing the final stage of the game state expressed in the action basis (equation 3.109) and the coefficients of the unit quaternion $p_\theta q_\theta$ we observe that $\pi_0(p_\theta q_\theta) =$

$coeff(NN)$ and $\pi_1(p_\theta q_\theta) = coeff(FF)$. It remains to show that $\pi_2(p_\theta q_\theta) = coeff(FN)$ and $\pi_3(p_\theta q\theta) = coeff(NF)$. For this, consider the products

$$e^{i\frac{\pi}{4}}z = \frac{\sqrt{2}}{2}(1+i)(z_0+z_1 i) = \frac{\sqrt{2}}{2}\left[(z_0-z_1)+(z_0+z_1)i\right] \tag{3.120}$$

$$e^{-i\frac{\pi}{4}}z = \frac{\sqrt{2}}{2}(1-i)(z_0+z_1 i) = \frac{\sqrt{2}}{2}\left[(z_0+z_1)+(-z_0+z_1)i\right], \tag{3.121}$$

where $z = z_0 + z_1 i$ is a complex number. Therefore, $Re\left(e^{i\frac{\pi}{4}}z\right) = -Im\left(e^{-i\frac{\pi}{4}}z\right)$ and $Im\left(e^{i\frac{\pi}{4}}z\right) = Re\left(e^{-i\frac{\pi}{4}}z\right)$. Thus

$$\begin{aligned} coeff(NF) &= Re\left(e^{i\frac{\pi}{4}}(e^{-i\frac{\theta}{2}}A\overline{Q} - e^{i\frac{\theta}{2}}B\overline{P})\right) \\ &= -Im\left(e^{-i\frac{\pi}{4}}(e^{-i\frac{\theta}{2}}A\overline{Q} - e^{i\frac{\theta}{2}}B\overline{P})\right) \\ &= \pi_3(p_\theta q\theta) \end{aligned}$$

and

$$\begin{aligned} coeff(FN) &= Im\left(e^{i\frac{\pi}{4}}(e^{-i\frac{\theta}{2}}A\overline{Q} - e^{i\frac{\theta}{2}}B\overline{P})\right) \\ &= Re\left(e^{-i\frac{\pi}{4}}(e^{-i\frac{\theta}{2}}A\overline{Q} - e^{i\frac{\theta}{2}}B\overline{P})\right) \\ &= \pi_2(p_\theta q_\theta) \end{aligned}$$

□

Note that $\mathcal{B}_\theta = \{1, i, e^{i\frac{\theta}{2}}j, e^{i\frac{\theta}{2}}k\}$ is an orthonormal basis of $\mathbb{H}$. The basis change matrix of the basis $\mathcal{B}_\theta$ to the standard basis of $\mathbb{H}$ is

$$\mathcal{A}_\theta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\theta/2) & -\sin(\theta/2) \\ 0 & 0 & \sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \tag{3.122}$$

This matrix is unitary by construction, so $\mathcal{A}_\theta^{-1} = \overline{\mathcal{A}}_\theta^T$. So the basis change matrix from the standard basis to the basis $\mathcal{B}_\theta$ is given by

$$\mathcal{A}_\theta^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\theta/2) & \sin(\theta/2) \\ 0 & 0 & -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \tag{3.123}$$

For a general unit quaternion p given with respect to the standard basis, abusively denote $\mathcal{A}_\theta^{-1}(\vec{p}\,)$ the corresponding unit quaternion with respect to the basis $\mathcal{B}_\theta$. Then for all unit quaternions p and q expressed with respect to the standard basis, we have the following result.

Proposition 3.124. $\mathcal{A}_\theta^{-1}(\vec{pq}) = \mathcal{A}_\theta^{-1}(\,\vec{p}\,) \cdot \mathcal{A}_\theta^{-1}(\,\vec{q}\,)$. *That is, the expression of* pq *with respect to the basis* $\mathcal{B}_\theta$ *is the same as the product of the expressions of* p *and* q *with respect to the basis* $\mathcal{B}_\theta$.

In fact more is true about the map $\mathcal{A}_\theta^{-1}$. It is an automorphism of the group $\mathbb{S}^3$ of the unit quaternions, considered under quaternionic multiplication. The proof is straightforward calculations.

Now let p and q be unit quaternions given with respect to the standard basis. Write the product pq as

$$pq = \pi_0(pq) \cdot 1 + \pi_1(pq) \cdot i + \pi_2(pq) \cdot j + \pi_3(pq) \cdot k, \tag{3.125}$$

where the $\pi_t(pq)$ are real numbers. Then the probability distribution over the outcomes NN, NF, FN, FF is given by

$$\begin{aligned} pr(NN) &= [\pi_0(\mathcal{A}_\theta^{-1}(pq))]^2, \quad pr(NF) = [\pi_3(\mathcal{A}_\theta^{-1}(pq))]^2 \\ pr(FN) &= [\pi_2(\mathcal{A}_\theta^{-1}(pq))]^2, \quad pr(FF) = [\pi_1(\mathcal{A}_\theta^{-1}(pq))]^2 \end{aligned} \tag{3.126}$$

This result leads to the following definition:

Definition 3.127. *Let G be the game described in table 3.2. Then the associated* quantization $G^{\mathcal{Q}_{\mathcal{I}_\theta}}$ *is the two player game in which each player's strategy space is the unit quaternions, and the payoff functions for players 1 and 2 are defined as follows:*

$$P_1(p,q) = \sum_{t=0}^{3} [\pi_t(\mathcal{A}_\theta^{-1}(pq))]^2 X_t \tag{3.128}$$

$$P_2(p,q) = \sum_{t=0}^{3} [\pi_t(\mathcal{A}_\theta^{-1}(pq))]^2 Y_t \tag{3.129}$$

Note that by reparameterizing the strategic spaces with the unit quaternions in $G^{\mathcal{Q}_{\mathcal{I}_\theta}}$, we obtain Landsburg's abstract game. Indeed, if S_1 and S_2 are players one and two strategic spaces, respectively, Landsburg makes the identifications

$$S_1 \cong S_2 \cong SU(2) \cong \mathbb{S}^3 \tag{3.130}$$

and shows that if $p, q \in \mathbb{S}^3$, then the payoff is given by pq. Similarly, we made the above identifications of the strategic spaces and showed that, in $G^{\mathcal{Q}_{\mathcal{I}_\theta}}$, if players one and two employ the unit quaternions p and q, respectively, then the payoff function is given by the unit quaternion $\mathcal{A}_\theta^{-1}(pq) = \mathcal{A}_\theta^{-1}(p)\mathcal{A}_\theta^{-1}(q)$, where $\mathcal{A}_\theta^{-1}$ is the basis change matrix from the standard basis of $\mathbb{H}$ to the basis $\{1, i, e^{i\frac{\theta}{2}}j, e^{i\frac{\theta}{2}}k\}$. Now set

$$\mathcal{A}_\theta^{-1}(p) = \widetilde{p} \text{ and } \mathcal{A}_\theta^{-1}(q) = \widetilde{q}, \tag{3.131}$$

then,

$$p = \mathcal{A}_\theta(\widetilde{p}\,) \quad \text{and} \quad q = \mathcal{A}_\theta(\widetilde{q}\,). \tag{3.132}$$

So the payoffs are related by

$$pq = \mathcal{A}_\theta(\widetilde{p}\,\widetilde{q}\,) \tag{3.133}$$

and thus as abstract games, the $G^{\mathcal{Q}_{\mathcal{I}_\theta}}$'s and $G^{\mathcal{Q}_{\mathcal{I}_0}}$ are the same, and each satisfies Definition 3.60. Hence, Theorems 3.70 and 3.71 apply to the games $G^{\mathcal{Q}_{\mathcal{I}_\theta}}$ and $G^{m\mathcal{Q}_{\mathcal{I}_\theta}}$, and classify the potential Nash equilibria therein.

3.7.4 A Motivational Example

As a motivation for the next chapter, we show that the mixed quantum strategies $\mu* = \frac{1}{4}(1 + i + j + k)$ and $\nu* = \frac{1}{4}(1 + i + j + k)$ for players one and two, respectively are best replies to each other thereby giving a Nash equilibrium in $G^{m\mathcal{Q}_{\mathcal{I}_\theta}}$ for all $\theta \in [0, 2\pi)$ and irrespective of the classical payoffs for the players.

For this, take player one's strategy as given and suppose player two responds with a pure quantum strategy represented by the unit quaternion $q = q_0 + q_1 i + q_2 j + q_3 k$.

Then player two's expected payoff is given by

$$
\begin{aligned}
\mathcal{E}_2(\mu*, q) &= \frac{1}{4}\left[P_2(1,q) + P_2(i,q) + P_2(j,q) + P_2(k,q)\right] \\
&= \frac{1}{4}[\sum_{k=0}^{3} \pi_k^2(\mathcal{A}_\theta^{-1}(1q))Y_k + \sum_{k=0}^{3} \pi_k^2(\mathcal{A}_\theta^{-1}(iq))Y_k \\
&+ \sum_{k=0}^{3} \pi_k^2(\mathcal{A}_\theta^{-1}(jq))Y_k + \sum_{k=0}^{3} \pi_k^2(\mathcal{A}_\theta^{-1}(kq))Y_k] \\
&= \frac{1}{4}\{[q_0^2 Y_0 + q_1^2 Y_1 + (q_2 \cos\frac{\theta}{2} - q_3 \sin\frac{\theta}{2})^2 Y_2 + (q_2 \sin\frac{\theta}{2} + q_3 \cos\frac{\theta}{2})^2 Y_3] \\
&+ [q_1^2 Y_0 + q_0^2 Y_1 + (-q_3 \cos\frac{\theta}{2} - q_2 \sin\frac{\theta}{2})^2 Y_2 + (-q_3 \sin\frac{\theta}{2} + q_2 \cos\frac{\theta}{2})^2 Y_3] \\
&+ [q_2^2 Y_0 + q_3^2 Y_1 + (q_0 \cos\frac{\theta}{2} + q_1 \sin\frac{\theta}{2})^2 Y_2 + (q_0 \sin\frac{\theta}{2} - q_1 \cos\frac{\theta}{2})^2 Y_3] \\
&+ [q_3^2 Y_0 + q_2^2 Y_1 + (q_1 \cos\frac{\theta}{2} - q_0 \sin\frac{\theta}{2})^2 Y_2 + (q_2 \sin\frac{\theta}{2} + q_3 \cos\frac{\theta}{2})^2 Y_3]\} \\
&= \frac{1}{4}\left(Y_0 + Y_1 + Y_2 + Y_3\right) \qquad (3.134)
\end{aligned}
$$

Hence, no matter what pure quantum strategy player two chooses, she gets the average of her classical payoffs. Thus, she is indifferent between all her pure quantum strategies.

Now assume that player two responds with a mixed quantum strategy represented by a probability distribution ν over $\mathbb{S}^3$. Then player two's expected payoff is given by

$$\begin{aligned}
\mathcal{E}_2(\mu*,\nu) &= \int_{\mathbb{S}^3\times\mathbb{S}^3} P_2(p,q)d(\mu*\times\nu)(p,q) \\
&= \int_{\mathbb{S}^3}\left[\int_{\mathbb{S}^3} P_2(p,q)d\mu*(p)\right]d\nu(q) \\
&= \int_{\mathbb{S}^3}\left[\frac{1}{4}(Y_0+Y_1+Y_2+Y_3)\right]d\nu(q) \quad \text{By (3.134)} \\
&= \frac{1}{4}(Y_0+Y_1+Y_2+Y_3)\nu\left(\mathbb{S}^3\right) \\
&= \frac{1}{4}(Y_0+Y_1+Y_2+Y_3)
\end{aligned}$$

Hence, player two is indifferent between all her mixed quantum strategies.

Similarly, one verifies that if player two employs $\nu*$, then $\mu*$ is an optimal response for player one. Therefore $(\mu*,\nu*)$ is a Nash equilibrium in $G^{mQ_{\mathcal{I}_\theta}}$ with expected payoff to the players given by

$$\frac{1}{4}\left(\sum_{k=0}^{3}X_k,\ \sum_{k=0}^{3}Y_k\right).$$

Note that this result holds for all $\theta\in[0,2\pi)$.

Chapter 4

OCTONIONIZATION OF THREE PLAYER, TWO STRATEGY QUANTUM GAMES

4.1 Introduction

The main goal of this work is to identify the possible equilibria in quantized versions of generic three player, two strategy games. To this end, we use an octonionic representation of the pay-off function for such games recently developed by Ahmed, Bleiler, and Khan [1]. This representation is a parallel development of S. Landsburg's quaternionic representation of the pay-off function of quantized versions of generic two player, two strategy games [33], as described in chapter 3. The octonionic representation is important for the fundamental understanding of the relevant quantum probabilities and also for the relative ease in working with octonionic arithmetic as opposed to multi-variant tensors. Moreover, this construction provides a fresh computational framework and gives us the potential to classify all the possible Nash equilibria in these quantized games. While the full classification remains a goal of future research, our representation has already established the existence of certain Nash equilibria in these quantized games. Here we reproduce the Ahmed-Bleiler-Khan construction and its subsequent results.

4.2 Preliminaries

We consider a generic three player, two strategy game G whose payoff function is indicated by the figure below where players one, two, and three's pure strategy spaces are

		II: t_1	II: t_2
I	s_1	(X_0, Y_0, Z_0)	(X_6, Y_6, Z_6)
I	s_2	(X_4, Y_4, Z_4)	(X_3, Y_3, Z_3)

Player III chooses r_1

		II: t_1	II: t_2
I	s_1	(X_7, Y_7, Z_7)	(X_2, Y_2, Z_2)
I	s_2	(X_5, Y_5, Z_5)	(X_1, Y_1, Z_1)

Player III chooses r_2

Figure 4.1: A Generic Three Player, Two Strategy Game

given by $S_1 = \{s_1, s_2\}$, $S_2 = \{t_1, t_2\}$, and $S_3 = \{r_1, r_2\}$, respectively, and the triples $(X_i, Y_i, Z_i) \in \mathbb{R}^3$ represent the payoffs for a given strategic profile to players one, two, and three, respectively. As an example, if players one, two, and three employ the pure strategies s_1, t_2, and r_1, respectively, then the payoff to the players is given by

$$G(s_1, t_2, r_1) = (X_6, Y_6, Z_6). \tag{4.1}$$

The choice of the indices in the above tables will be justified in a later section.

Our game will be quantized in a manner similar to that given by Eisert et al [18] and Landsburg [33] for generic two player, two strategy games as described in chapter 3. Each player communicates his strategic choice to a referee via a qubit in superposition.

The referee initially sends to the three players qubits in the maximally entangled state

$$\mathcal{I} = \frac{|000\rangle + |111\rangle}{\sqrt{2}}. \tag{4.2}$$

The two classical pure strategies available to the players are, as before, flip and no flip denoted by F and N, respectively, and represented by the $SU(2)$ matrices

$$F = \begin{pmatrix} 0 & \eta \\ -\bar{\eta} & 0 \end{pmatrix}, \ N = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4.3}$$

where η is a unit complex number chosen so that the eight outcome states of our three player game form an orthogonal basis of the state space $\mathbb{C}^8$ with the *observational basis*

$$\mathcal{B}_{ob} = \{|000\rangle, |001\rangle, |010\rangle, |011\rangle, |100\rangle, |101\rangle, |110\rangle, |111\rangle\}. \tag{4.4}$$

These eight outcome states are denoted by *NNN, NNF, NFN, NFF, FNN, FNF, FFN, FFF*, where any of these triples of N and F represent the pure strategy choices of players one, two, and three, respectively. Now a direct calculation appearing in section 4.4.2 shows that $\eta^6 = 1$, so setting

$$\eta = \frac{1}{2} + \frac{\sqrt{3}}{2}i \tag{4.5}$$

insures orthogonality of our eight outcome states.

A pure quantum strategy for each player is again represented by an element of $SU(2)$, the group of two by two special unitary matrices. Recall that the group $SU(2)$

is isomorphic to the group of unit quaternions, $SP(1)$, which in turn can be thought as the unit three-sphere $\mathbb{S}^3$ living in $\mathbb{R}^4$. This gives a quaternionic co-ordinatization of the quantized games' strategy spaces.

We develop an *octonionization* of our maximally entangled three player, two strategy quantum game described above by identifying each strategic choice of player one, two, or three with a unit octonion s, t, or u respectively, where each of s, t, or u lies in a particular, possibly different copy of the unit quaternions embedded in the octonionic real division algebra. The probability distribution over the eight possible outcomes is then shown to be determined by an expression involving the associated triple product $(st)u$ of the strategies s, t, and u. The associated nature of this product is in fact natural as the octonions are in general non-associative. As in Landsburg's case, these identifications above and the resulting probability distribution over the outcome allows us to examine the effect on the payoffs to each player of the game when players use mixtures, superpositions, or mixed superpositions of the pure strategies.

4.3 Octonions

The octonions $\mathbb{O}$ are a non-associative, non-commutative, 8-dimensional, normed division algebra over the real numbers. For more detail on division algebras, see Appendix A. The octonions are not obtained from the set of quaternions $\mathbb{H}$ the way we obtain the set of complex numbers $\mathbb{C}$ from $\mathbb{R}^2$ as

$$\mathbb{C} = \mathbb{R} + \mathbb{R}i \tag{4.6}$$

or the way we obtain $\mathbb{H}$ from $\mathbb{C}^2$ as

$$\mathbb{H} = \mathbb{C} + \mathbb{C}j. \tag{4.7}$$

However, they are spanned by the real number 1 and 7 basic distinct square roots of -1 denoted by $i_1, i_2, i_3, i_4, i_5, i_6$, and i_7. A general octonion o can be represented in the form

$$o = a_0 + a_1 i_1 + a_2 i_2 + a_3 i_3 + a_4 i_4 + a_5 i_5 + a_6 i_6 + a_7 i_7, \tag{4.8}$$

where the a_j's are real numbers and the i_j's have the property that $i_j^2 = -1$. Addition with octonions is component wise as in $\mathbb{R}^8$.

Now, given any two basic distinct square roots of -1 i_r and i_s, there is a third i_t, so that these three basic distinct square roots of -1 satisfy Hamilton's relation

$$i_r^2 = i_s^2 = i_t^2 = i_r i_s i_t = -1. \tag{4.9}$$

Thus any pair of basic distinct square roots of -1 determines a quaternionic subalgebra. Up to order there are exactly seven such choices. Therefore, there are seven "natural" quaternionic subalgebras all together. Call these the *standard quaternionic subalgebras*. Any pair of such quaternionic subalgebras intersect in a common copy of the complex numbers. Now if we consider the seven basic square roots of -1 as "points" and the seven standard quaternionic subalgebras as "lines" these points are incident to, the octonionic algebra satisfies the following two axioms of projective geometry.

- Axiom 1: Two points determine a line.

- Axiom 2: Two lines determine a point.

Not surprisingly, then octonionic multiplication of the seven basic square roots of -1 is modeled along the 7 point, 7 line projective plane shown in Figure 4.2, the so-called *Fano plane*.

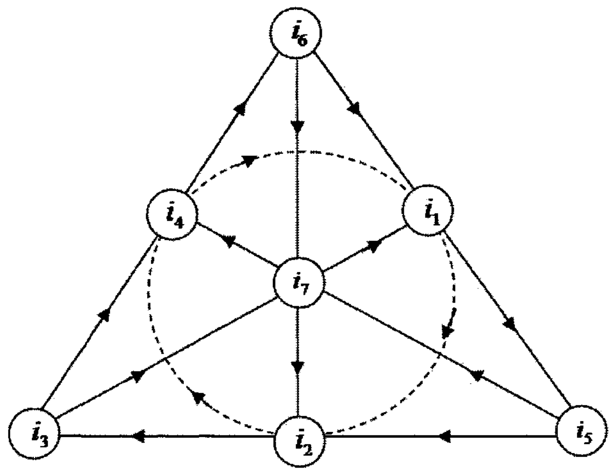

Figure 4.2: An edge oriented Fano plane.

One can utilize an edge oriented Fano plane to model octonionic multiplication of the basic octonionic square roots of -1. From this, it will follow that multiplication of general octonions is polynomial subject to the relation $i_j^2 = i_k^2 = i_l^2 = i_j i_k i_l = -1$ if the i_j, i_k, i_l are cyclically ordered as in the edge oriented Fano plane of Figure 4.2.

Definition 4.10. *The octonionic conjugate of an octonion o as in (4.8) is defined as*

$$\overline{o}^{\mathbb{O}} = a_0 - \sum_{j=1}^{7} a_j i_j. \tag{4.11}$$

One easily verifies the following facts about the octonions:

1. The product $\overline{o}^{\mathbb{O}}o = \sum_{j=0}^{7} a_j^2$ defines the square of a norm $||o||$ of the octonion o. That is

$$||o||^2 = \overline{o}^{\mathbb{O}}o = \sum_{j=0}^{7} a_j^2. \tag{4.12}$$

2. The norm is multiplicative, that is $||o_1 o_2|| = ||o_1||\ ||o_2||$ for all octonions o_1 and o_2.

3. For any nonzero octonion o,

$$o^{-1} = \frac{\overline{o}^{\mathbb{O}}}{||o||}. \tag{4.13}$$

4. Multiplication with octonions is not in general commutative.

5. Multiplication with octonions is not in general associative.

6. The distributive laws hold.

A unit octonion has length 1. The set of unit octonions $\{o \in \mathbb{O} \mid ||o||^2 = 1\}$ can be thought as the unit 7-sphere $\mathbb{S}^7$ living in $\mathbb{R}^8$.

Amongst the 7 standard quaternionic subalgebras of $\mathbb{O}$, we are interested in three copies with a common embedded copy of the complex numbers $\mathbb{C}$. For this we choose the quaternionic subalgebras

$$\mathbb{H}_I = \{a_0 + a_1 i_1 + b_0 i_2 + b_1 i_4\} \tag{4.14}$$

$$\mathbb{H}_{II} = \{p_0 + p_1 i_1 + q_0 i_5 + q_1 i_6\} \tag{4.15}$$

$$\mathbb{H}_{III} = \{e_0 + e_1 i_1 + f_0 i_2 + f_1 i_4\} \tag{4.16}$$

which meet in the complex subalgebra $\{x_0 + x_1 i_1\}$. We focus our attention on the unit $\mathbb{S}^3$'s in each of these four dimensional copies of $\mathbb{H}$ and consider each such $\mathbb{S}^3$ as a "longitude" of the unit octonions which form a seven dimensional sphere $\mathbb{S}^7 \subset \mathbb{O}$.

4.4 Octonionic Representation

We now identify a pure quantum strategy available to each player with a particular unit octonion.

Recall that the elements of the group $SU(2)$ are 2×2 complex matrices and can be written in the form

$$\begin{pmatrix} x & y \\ -\bar{y} & \bar{x} \end{pmatrix}, \tag{4.17}$$

where x and y are complex numbers subject to $x\bar{x} + y\bar{y} = 1$. If player one chooses the pure quantum strategy corresponding to

$$U_I = \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix} \tag{4.18}$$

where $A = a_0 + a_1 i$ and $B = b_0 + b_1 i$, identify this strategy with the unit octonion given by

$$\begin{aligned} s_{00} &\equiv A + B\bar{\eta} i_4 \\ &= a_0 + a_1 i_1 + (b_0 + b_1 i_1)\left(\frac{1}{2} - \frac{\sqrt{3}}{2} i_1\right) i_4 \\ &= a_0 + a_1 i_1 + \left(\frac{\sqrt{3}}{2} b_0 - \frac{1}{2} b_1\right) i_2 + \left(\frac{1}{2} b_0 + \frac{\sqrt{3}}{2} b_1\right) i_4 \end{aligned} \tag{4.19}$$

The subscript of the unit octonion s is used to track sign changes on the two first real coefficients in the expression for s, namely, a_0, a_1. A positive sign will be represented by 0 and a sign change to a negative in the expression for s will be represented by 1. This notation will be used below to extract the appropriate coefficient for the probability distribution that will determine the expected payoffs for our quantized and mixed quantized games.

Similarly, if player two chooses a quantum strategy corresponding to

$$U_{II} = \begin{pmatrix} P & Q \\ -\bar{P} & \bar{Q} \end{pmatrix} \tag{4.20}$$

where $P = p_0 + p_1 i$, $Q = q_0 + q_1 i$, consider the unit octonion given by

$$\begin{aligned} t_{00} &\equiv P + Q\bar{\eta} i_6 \\ &= p_0 + p_1 i_1 + (q_0 + q_1 i_1)\left(\frac{1}{2} - \frac{\sqrt{3}}{2} i_1\right) i_6 \\ &= p_0 + p_1 i_1 + \left(\frac{\sqrt{3}}{2} q_0 - \frac{1}{2} q_1\right) i_5 + \left(\frac{1}{2} q_0 + \frac{\sqrt{3}}{2} q_1\right) i_6 \end{aligned} \tag{4.21}$$

And if player three chooses the quantum strategy corresponding to

$$U_{III} = \begin{pmatrix} E & F \\ -\bar{F} & \bar{E} \end{pmatrix}, \tag{4.22}$$

where $E = e_0 + e_1 i$ and $F = f_0 + f_1 i$, consider the unit octonion given by

$$\begin{aligned} u_{00} &\equiv E + F\bar{\eta} i_7 \\ &= e_0 + e_1 i_1 + (f_0 + f_1 i_1)\left(\frac{1}{2} - \frac{\sqrt{3}}{2} i_1\right) i_7 \\ &= e_0 + e_1 i_1 + \left(\frac{\sqrt{3}}{2} f_0 - \frac{1}{2} f_1\right) i_3 + \left(\frac{1}{2} f_0 + \frac{\sqrt{3}}{2} f_1\right) i_7 \end{aligned} \tag{4.23}$$

Using the above identifications, we note the following

$$NNN \equiv (1 \cdot 1) \cdot 1 = 1 \tag{4.24}$$

$$NNF \equiv (1 \cdot 1) \cdot (\eta\bar{\eta} i_7) = i_7 \tag{4.25}$$

$$NFN \equiv [1 \cdot (\eta\bar{\eta} i_6)] \cdot 1 = i_6 \tag{4.26}$$

$$NFF \equiv [1 \cdot (\eta\bar{\eta} i_6)] \cdot (\eta\bar{\eta} i_7) = i_6 i_7 = i_2 \tag{4.27}$$

$$FNN \equiv [(\eta\bar{\eta} i_4) \cdot 1] \cdot 1 = i_4 \tag{4.28}$$

$$FNF \equiv [(\eta\bar{\eta} i_4) \cdot 1] \cdot (\eta\bar{\eta} i_7) = i_4 i_7 = -i_5 \tag{4.29}$$

$$FFN \equiv [(\eta\bar{\eta} i_4) \cdot (\eta\bar{\eta} i_6)] \cdot 1 = i_4 i_6 = i_3 \tag{4.30}$$

$$FFF \equiv [(\eta\bar{\eta} i_4) \cdot (\eta\bar{\eta} i_6)] \cdot (\eta\bar{\eta} i_7) = (i_4 i_6) i_7 = i_3 i_7 = i_1 \tag{4.31}$$

which is the origin of our choice of indices in Figure 4.1.

Here is the main theorem of this chapter.

Theorem 4.32. *If in the maximally entangled three players, two strategy quantum game, player one employs the pure quantum strategy U_I, player two the pure quantum strategy U_{II}, and player three the pure quantum strategy U_{III}, then the probability distribution* *over the set of outcomes is given by*

$$pr(NNN) = \left[\pi_0\left(\frac{(s_{10}t_{10})\,u_{01} + (s_{01}t_{10})\,u_{01}}{2}\right)\right]^2 + \left[\pi_0\left(\frac{(s_{10}t_{10})\,u_{01} - (s_{01}t_{10})\,u_{01}}{2}\right)\right]^2 \tag{4.33}$$

$$pr(FFF) = \left[\pi_1\left(\frac{(s_{10}t_{10})\,u_{01} + (s_{01}t_{10})\,u_{01}}{2}\right)\right]^2 + \left[\pi_1\left(\frac{(s_{10}t_{10})\,u_{01} - (s_{01}t_{10})\,u_{01}}{2}\right)\right]^2 \tag{4.34}$$

$$pr(FFN) = \left[\pi_3\left(\frac{(s_{10}t_{10})\,u_{01} + (s_{01}t_{10})\,u_{01}}{2}\right)\right]^2 + \left[\pi_3\left(\frac{(s_{10}t_{10})\,u_{01} - (s_{01}t_{10})\,u_{01}}{2}\right)\right]^2 \tag{4.35}$$

$$pr(NNF) = \left[\pi_7\left(\frac{(s_{10}t_{10})\,u_{01} + (s_{01}t_{10})\,u_{01}}{2}\right)\right]^2 + \left[\pi_7\left(\frac{(s_{10}t_{10})\,u_{01} - (s_{01}t_{10})\,u_{01}}{2}\right)\right]^2 \tag{4.36}$$

$$pr(NFF) = \left[\pi_2\left(\frac{(s_{10}t_{00})\,u_{00} + (s_{01}t_{00})\,u_{00}}{2}\right)\right]^2 + \left[\pi_2\left(\frac{(s_{10}t_{00})\,u_{00} - (s_{01}t_{00})\,u_{00}}{2}\right)\right]^2 \tag{4.37}$$

$$pr(FNN) = \left[\pi_4\left(\frac{(s_{10}t_{00})\,u_{00} + (s_{01}t_{00})\,u_{00}}{2}\right)\right]^2 + \left[\pi_4\left(\frac{(s_{10}t_{00})\,u_{00} - (s_{01}t_{00})\,u_{00}}{2}\right)\right]^2 \qquad (4.38)$$

$$pr(FNF) = \left[\pi_5\left(\frac{(s_{10}t_{00})\,u_{00} + (s_{01}t_{00})\,u_{00}}{2}\right)\right]^2 + \left[\pi_5\left(\frac{(s_{10}t_{00})\,u_{00} - (s_{01}t_{00})\,u_{00}}{2}\right)\right]^2 \qquad (4.39)$$

$$pr(NFN) = \left[\pi_6\left(\frac{(s_{10}t_{00})\,u_{00} + (s_{01}t_{00})\,u_{00}}{2}\right)\right]^2 + \left[\pi_6\left(\frac{(s_{10}t_{00})\,u_{00} - (s_{01}t_{00})\,u_{00}}{2}\right)\right]^2 \qquad (4.40)$$

where $\pi_j(o)$ *denotes the projection of the octonion* o *onto the subspace of* $\mathbb{O}$ *spanned by the vector basis element* i_j *with the convention that* $i_0 = 1$.

The proof of theorem 4.32 is technical and is based on the ideas developed in the following sections.

4.4.1 The Game State

Consider three qubits with respect to the initial observational basis $\{H, T\}$, where

$$H \equiv |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad T \equiv |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad (4.41)$$

and in the initial state

$$\mathcal{I} = \frac{|000\rangle + |111\rangle}{\sqrt{2}} \tag{4.42}$$

The players operate on their respective qubits, the first via U_I, the second via U_{II}, and the third via U_{III}, respectively.

Ignoring the normalization constant $1/\sqrt{2}$, after the players act the initial state becomes

$$\begin{pmatrix} A \\ -\bar{B} \end{pmatrix} \otimes \begin{pmatrix} P \\ -\bar{Q} \end{pmatrix} \otimes \begin{pmatrix} E \\ -\bar{F} \end{pmatrix} + \begin{pmatrix} B \\ \bar{A} \end{pmatrix} \otimes \begin{pmatrix} Q \\ \bar{P} \end{pmatrix} \otimes \begin{pmatrix} F \\ \bar{E} \end{pmatrix} \tag{4.43}$$

Expanding bilinearly we obtain

$$\begin{aligned}
&(APE + BQF)\, HHH + \left(BQ\bar{E} - AP\bar{F}\right) HHT \\
&+ \left(B\bar{P}F - A\bar{Q}E\right) HTH + \left(A\bar{Q}\bar{F} + B\bar{P}\bar{E}\right) HTT \\
&+ \left(\bar{A}QF - \bar{B}PE\right) THH + \left(\bar{B}P\bar{F} + \bar{A}Q\bar{E}\right) THT \\
&+ \left(\bar{B}\bar{Q}E + \bar{A}\bar{P}F\right) TTH + \left(\bar{A}\bar{P}\bar{E} - \bar{B}\bar{Q}\bar{F}\right) TTT,
\end{aligned} \tag{4.44}$$

and as a vector in $\mathbb{C}^8$

$$\begin{pmatrix} APE + BQF \\ BQ\bar{E} - AP\bar{F} \\ B\bar{P}F - A\bar{Q}E \\ A\bar{Q}\bar{F} + B\bar{P}\bar{E} \\ \bar{A}QF - \bar{B}PE \\ \bar{B}P\bar{F} + \bar{A}Q\bar{E} \\ \bar{B}\bar{Q}E + \bar{A}\bar{P}F \\ \bar{A}\bar{P}\bar{E} - \bar{B}\bar{Q}\bar{F} \end{pmatrix} \begin{matrix} HHH \\ HHT \\ HTH \\ HTT \\ THH \\ THT \\ TTH \\ TTT \end{matrix} \tag{4.45}$$

Call the vector in (4.45) the *game state* of our system.

4.4.2 Orthogonality and Change of Basis

We consider now the actions *flip* and *no flip*. Recall from section 4.2 that the action *no flip* is represented by the $SU(2)$ matrix

$$N = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4.46}$$

and the action *flip* by

$$F = \begin{pmatrix} 0 & \eta \\ -\bar{\eta} & 0 \end{pmatrix}, \tag{4.47}$$

where η is a unit complex number to be determined shortly. Note that $F(H) = -\bar{\eta}T$ and $F(T) = \eta H$. Then the so-called *standard actions profiles* of the players consist of $NNN, NNF, NFN, NFF, FNN, FNF, FFN, FFF$. Evaluating the game state expansion for these action profiles we obtain that the corresponding values of A, B, P, Q, E, and F are given in the table below

	A	B	P	Q	E	F
NNN	1	0	1	0	1	0
NNF	1	0	1	0	0	η
NFN	1	0	0	η	1	0
NFF	1	0	0	η	0	η
FNN	0	η	1	0	1	0
FNF	0	η	1	0	0	η
FFN	0	η	0	η	1	0
FFF	0	η	0	η	0	η

So in the joint observational basis

$$\{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\} \tag{4.48}$$

we obtain that the game states corresponding to the standard action profiles are given by

$$NNN \equiv \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad NNF = \begin{pmatrix} 0 \\ -\bar{\eta} \\ 0 \\ 0 \\ 0 \\ 0 \\ \eta \\ 0 \end{pmatrix}, \quad NFN = \begin{pmatrix} 0 \\ 0 \\ -\bar{\eta} \\ 0 \\ 0 \\ \eta \\ 0 \\ 0 \end{pmatrix}, \quad NFF = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \bar{\eta}^2 \\ \eta^2 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$FFF = \begin{pmatrix} \eta^3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\bar{\eta}^3 \end{pmatrix}, \quad FFN = \begin{pmatrix} 0 \\ \eta^2 \\ 0 \\ 0 \\ 0 \\ 0 \\ \bar{\eta}^2 \\ 0 \end{pmatrix}, \quad FNN = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \eta \\ -\bar{\eta} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad FNF = \begin{pmatrix} 0 \\ 0 \\ \eta^2 \\ 0 \\ 0 \\ \bar{\eta}^2 \\ 0 \\ 0 \end{pmatrix}$$

These states are to correspond to a physical property observable to the referee. For this, the axioms of quantum mechanics require these states to form an orthonormal basis of the joint state space of the three qubits. The non-trivial orthogonality conditions are thus

$$\langle NNN, FFF\rangle = \bar{\eta}^3 - \eta^3 = 0, \tag{4.49}$$

$$\langle NNF, FFN\rangle = -\bar{\eta}^3 + \eta^3 = 0, \tag{4.50}$$

$$\langle NFN, FNF\rangle = -\bar{\eta}^3 + \eta^3 = 0, \tag{4.51}$$

$$\langle NFF, FNN\rangle = \bar{\eta}^3 - \eta^3 = 0. \tag{4.52}$$

Therefore $\eta^6 = 1$. Thus setting $\eta = e^{i\frac{\pi}{3}} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ insures orthogonality of the game states given by the standard action profiles. Call this family of states the *action basis* of the joint state space and abusively denote these states by NNN, NNF, NFN, NFF, FNN, FNF, FFN, and FFF.

The basis change matrix of the action basis to the initial observational basis is thus

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \eta^3 \\ 0 & -\bar{\eta} & 0 & 0 & 0 & 0 & \eta^2 & 0 \\ 0 & 0 & -\bar{\eta} & 0 & 0 & \eta^2 & 0 & 0 \\ 0 & 0 & 0 & \bar{\eta}^2 & \eta & 0 & 0 & 0 \\ 0 & 0 & 0 & \eta^2 & -\bar{\eta} & 0 & 0 & 0 \\ 0 & 0 & \eta & 0 & 0 & \bar{\eta}^2 & 0 & 0 \\ 0 & \eta & 0 & 0 & 0 & 0 & \bar{\eta}^2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & -\bar{\eta}^3 \end{pmatrix}. \tag{4.53}$$

After normalizing the lengths of the columns (i.e. scaling the matrix by $1/\sqrt{2}$), this matrix is unitary by construction so $\bar{A}^T = A^{-1}$. The basis change matrix from the initial observational basis to the action basis is thus given by $1/\sqrt{2}$ times the matrix

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -\eta & 0 & 0 & 0 & 0 & \bar{\eta} & 0 \\ 0 & 0 & -\eta & 0 & 0 & \bar{\eta} & 0 & 0 \\ 0 & 0 & 0 & \eta^2 & \bar{\eta}^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{\eta} & -\eta & 0 & 0 & 0 \\ 0 & 0 & \bar{\eta}^2 & 0 & 0 & \eta^2 & 0 & 0 \\ 0 & \bar{\eta}^2 & 0 & 0 & 0 & 0 & \eta^2 & 0 \\ \bar{\eta}^3 & 0 & 0 & 0 & 0 & 0 & 0 & -\eta^3 \end{pmatrix}. \tag{4.54}$$

Rewriting a generic game state in the action basis gives (up to normalization)

$$A^{-1} \begin{pmatrix} APE + BQF \\ BQ\bar{E} - AP\bar{F} \\ B\bar{P}F - A\bar{Q}E \\ A\bar{Q}\bar{F} + B\bar{P}\bar{E} \\ \bar{A}QF - \bar{B}PE \\ \bar{B}P\bar{F} + \bar{A}Q\bar{E} \\ \bar{B}\bar{Q}E + \bar{A}\bar{P}F \\ \bar{A}\bar{P}\bar{E} - \bar{B}\bar{Q}\bar{F} \end{pmatrix} = \begin{pmatrix} (APE + \bar{A}\bar{P}\bar{E}) + (BQF - \bar{B}\bar{Q}\bar{F}) \\ (\eta AP\bar{F} + \bar{\eta}\bar{A}\bar{P}F) + (-\eta BQ\bar{E} + \bar{\eta}\bar{B}\bar{Q}E) \\ (\eta A\bar{Q}E + \bar{\eta}\bar{A}Q\bar{E}) + (-\eta B\bar{P}F + \bar{\eta}\bar{B}P\bar{F}) \\ (\eta^2 A\bar{Q}\bar{F} + \bar{\eta}^2\bar{A}QF) + (\eta^2 B\bar{P}\bar{E} - \bar{\eta}^2\bar{B}PE) \\ (\bar{\eta}A\bar{Q}\bar{F} - \eta\bar{A}QF) + (\bar{\eta}B\bar{P}\bar{E} + \eta\bar{B}PE) \\ (-\bar{\eta}^2 A\bar{Q}E + \eta^2\bar{A}Q\bar{E}) + (\bar{\eta}^2 B\bar{P}F + \eta^2\bar{B}P\bar{F}) \\ (-\bar{\eta}^2 AP\bar{F} + \eta^2\bar{A}\bar{P}F) + (\bar{\eta}^2 BQ\bar{E} + \eta^2\bar{B}\bar{Q}E) \\ (\bar{\eta}^3 APE - \eta^3\bar{A}\bar{P}\bar{E}) + (\bar{\eta}^3 BQF + \eta^3\bar{B}\bar{Q}\bar{F}) \end{pmatrix} \tag{4.55}$$

Note that for our particular value of η, namely $\frac{1}{2} + \frac{\sqrt{3}}{2}i = e^{\frac{\pi}{3}i}$, we obtain

$$\eta^2 = -\bar{\eta}, \quad \bar{\eta}^2 = -\eta, \quad \text{and} \quad \eta^3 = \bar{\eta}^3 = -1. \tag{4.56}$$

Hence our game state becomes

$$\begin{pmatrix} 2Re\,(APE) + 2Im\,(BQF)\,i \\ 2Re\,(\eta AP\bar{F}) - 2Im\,(\eta BQ\bar{E})\,i \\ 2Re\,(\eta A\bar{Q}E) - 2Im\,(\eta B\bar{P}F)\,i \\ -2Re\,(\eta\bar{A}QF) + 2Im\,(\eta\bar{B}PE)\,i \\ -2Im\,(\eta\bar{A}QF)\,i + 2Re\,(\eta\bar{B}PE) \\ 2Im\,(\eta A\bar{Q}E)\,i - 2Re\,(\eta B\bar{P}F) \\ 2Im\,(\eta AP\bar{F})\,i - 2Re\,(\eta BQ\bar{E}) \\ -2Im\,(APE)\,i - 2Re\,(BQF) \end{pmatrix} \tag{4.57}$$

4.4.3 Probability Distribution over $\mathbf{ImG}$

As per the axioms of quantum mechanics, up to normalization, the referee observing the game state in the action basis sees each pure action state according to the following probability distribution.

$$pr(NNN) = [Re(APE)]^2 + [Im(BQF)]^2 \tag{4.58}$$

$$pr(NNF) = [Re(\eta AP\bar{F})]^2 + [Im(\eta BQ\bar{E})]^2 \tag{4.59}$$

$$pr(NFN) = [Re(\eta A\bar{Q}E)]^2 + [Im(\eta B\bar{P}F)]^2 \tag{4.60}$$

$$pr(NFF) = [Re(\eta \bar{A}QF)]^2 + [Im(\eta \bar{B}PE)]^2 \tag{4.61}$$

$$pr(FNN) = [Re(\eta \bar{B}PE)]^2 + [Im(\eta \bar{A}QF)]^2 \tag{4.62}$$

$$pr(FNF) = [Re(\eta A\bar{P}F)]^2 + [Im(\eta A\bar{Q}E)]^2 \tag{4.63}$$

$$pr(FFN) = [Re(\eta BQ\bar{E})]^2 + [Im(\eta AP\bar{F})]^2 \tag{4.64}$$

$$pr(FFF) = [Re(BQF)]^2 + [Im(APE)]^2 \tag{4.65}$$

Now let

$$A = a_0 + a_1 i, \quad B = b_0 + b_1 i \tag{4.66}$$

$$P = p_0 + q_1 i, \quad Q = q_0 + q_1 i \tag{4.67}$$

$$E = e_0 + e_1 i, \quad F = f_0 + f_1 i. \tag{4.68}$$

Then the above probability distribution over the pure action states *NNN, NNF, NFN, NFF, FNN, FNF, FFN, FFF* can be expressed as

$$\begin{aligned} pr(NNN) = {} & (b_0 q_0 f_1 + b_0 q_1 f_0 + b_1 q_0 f_0 - b_1 q_1 f_1)^2 \\ & + (a_0 p_0 e_0 - a_0 p_1 e_1 - a_1 p_0 e_1 - a_1 p_1 e_0)^2, \end{aligned} \tag{4.69}$$

and

$$\begin{aligned} pr(NNF) = {} & [\frac{1}{2}(-b_0q_0e_1 + b_0q_1e_0 + b_1q_0e_0 + b_1q_1e_1) \\ & + \frac{\sqrt{3}}{2}(b_0q_0e_0 + b_0q_1e_1 + b_1q_0e_1 - b_1q_1e_0)]^2 \\ & + [\frac{1}{2}(a_0p_0f_0 + a_0p_1f_1 + a_1p_0f_1 - a_1p_1f_1) \\ & + \frac{\sqrt{3}}{2}(a_0p_0f_1 - a_0p_1f_0 - a_1p_0f_0 - a_1p_1f_1)]^2, \end{aligned} \tag{4.70}$$

and

$$\begin{aligned} pr(NFN) = {} & [\frac{1}{2}(a_0q_0e_0 + a_0q_1e_1 - a_1q_0e_1 + a_1q_1e_0) \\ & + \frac{\sqrt{3}}{2}(-a_0q_0e_1 + a_0q_1e_0 - a_1q_0e_0 - a_1q_1e_1)]^2 \\ & + [\frac{1}{2}(b_0p_0f_1 - b_0p_1f_0 + b_1p_0f_0 + b_1p_1f_1) \\ & + \frac{\sqrt{3}}{2}(b_0p_0f_0 + b_0p_1f_1 - b_1p_0f_1 + b_1p_1f_0)]^2, \end{aligned} \tag{4.71}$$

and

$$\begin{aligned} pr(NFF) = {} & [\frac{1}{2}(a_0q_0f_0 - a_0q_1f_1 + a_1q_0f_1 + a_1q_1f_0) \\ & + \frac{\sqrt{3}}{2}(-a_0q_0f_1 - a_0q_1f_0 + a_1q_0f_0 - a_1q_1f_1)]^2 \\ & + [\frac{1}{2}(b_0p_0e_1 + b_0p_1e_0 - b_1p_0e_0 + b_1p_1e_1) \\ & + \frac{\sqrt{3}}{2}(b_0p_0e_0 - b_0p_1e_1 + b_1p_0f_1 + b_1p_1e_0)]^2, \end{aligned} \tag{4.72}$$

and

$$
\begin{aligned}
pr(FNN) = [&\frac{1}{2}(a_0q_0f_1 + a_0q_1f_0 - a_1q_0f_0 + a_1q_1f_1) \\
&+ \frac{\sqrt{3}}{2}(a_0q_0f_0 - a_0q_1f_1 + a_1q_0f_1 + a_1q_1f_0)]^2 \\
&+ [\frac{1}{2}(b_0p_0e_0 - b_0p_1e_1 + b_1p_0f_1 + b_1p_1e_0) \\
&+ \frac{\sqrt{3}}{2}(-b_0p_0e_1 - b_0p_1e_0 + b_1p_0e_0 - b_1p_1e_1)]^2,
\end{aligned} \tag{4.73}
$$

and

$$
\begin{aligned}
pr(FNF) = [&\frac{1}{2}(a_0q_0e_1 - a_0q_1e_0 + a_1q_0e_0 + a_1q_1e_1) \\
&+ \frac{\sqrt{3}}{2}(a_0q_0e_0 + a_0q_1e_1 - a_1q_0e_1 + a_1q_1e_0)]^2 \\
&+ [\frac{1}{2}(b_0p_0f_0 + b_0p_1f_1 - b_1p_0f_1 + b_1p_1f_0) \\
&+ \frac{\sqrt{3}}{2}(-b_0p_0f_1 + b_0p_1f_0 - b_1p_0f_0 - b_1p_1f_1)]^2,
\end{aligned} \tag{4.74}
$$

and

$$
\begin{aligned}
pr(FFN) = [&\frac{1}{2}(b_0q_0e_0 + b_0q_1e_1 + b_1q_0e_1 - b_1q_1e_0) \\
&+ \frac{\sqrt{3}}{2}(b_0q_0e_1 - b_0q_1e_0 - b_1q_0e_0 - b_1q_1e_1)]^2 \\
&+ [\frac{1}{2}(-a_0p_0f_1 + a_0p_1f_0 + a_1p_0f_0 + a_1p_1f_1) \\
&+ \frac{\sqrt{3}}{2}(a_0p_0f_0 + a_0p_1f_1 + a_1p_0f_1 - a_1p_1f_0)]^2,
\end{aligned} \tag{4.75}
$$

and

$$pr(FFF) = (b_0q_0f_0 - b_0q_1f_1 - b_1q_0f_1 - b_1q_1f_0)^2 + (a_0p_0e_1 + a_0p_1e_0 + a_1p_0e_0 - a_1p_1e_1)^2 \quad (4.76)$$

4.4.4 Proof of Theorem 4.32

To prove theorem 4.32, we need to reconcile the expressions of the probability distribution in section 4.4.3 with the octonionic formulae appearing in theorem 4.32. Recall that these octonionic formulae arose from the identification of the players' pure quantum strategy sets with quaternionic subalgebras of the unit octonions via

$$\begin{aligned} s_{00} \equiv A + B\bar{\eta}i_4 &= a_0 + a_1i_1 + (b_0 + b_1i_1)\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i_1\right)i_4 \\ &= a_0 + a_1i_1 + \left(\frac{\sqrt{3}}{2}b_0 - \frac{1}{2}b_1\right)i_2 + \left(\frac{1}{2}b_0 + \frac{\sqrt{3}}{2}b_1\right)i_4, \end{aligned} \quad (4.77)$$

and

$$\begin{aligned} t_{00} \equiv P + Q\bar{\eta}i_6 &= p_0 + p_1i_1 + (q_0 + q_1i_1)\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i_1\right)i_6 \\ &= p_0 + p_1i_1 + \left(\frac{\sqrt{3}}{2}q_0 - \frac{1}{2}q_1\right)i_5 + \left(\frac{1}{2}q_0 + \frac{\sqrt{3}}{2}q_1\right)i_6, \end{aligned} \quad (4.78)$$

and

$$u_{00} \equiv E + F\bar{\eta} i_7 = e_0 + e_1 i_1 + (f_0 + f_1 i_1)\left(\frac{1}{2} - \frac{\sqrt{3}}{2} i_1\right) i_7$$
$$= e_0 + e_1 i_1 + \left(\frac{\sqrt{3}}{2} f_0 - \frac{1}{2} f_1\right) i_3 + \left(\frac{1}{2} f_0 + \frac{\sqrt{3}}{2} f_1\right) i_7 \tag{4.79}$$

In addition, we utilize for our proof the following unit octonions which are derived directly from the unit octonions s_{00}, t_{00}, and u_{00}, namely

$$s_{10} = -a_0 + a_1 i_1 + \left(\frac{\sqrt{3}}{2} b_0 - \frac{1}{2} b_1\right) i_2 + \left(\frac{1}{2} b_0 + \frac{\sqrt{3}}{2} b_1\right) i_4 \tag{4.80}$$
$$s_{01} = a_0 - a_1 i_1 + \left(\frac{\sqrt{3}}{2} b_0 - \frac{1}{2} b_1\right) i_2 + \left(\frac{1}{2} b_0 + \frac{\sqrt{3}}{2} b_1\right) i_4 \tag{4.81}$$
$$t_{10} = -p_0 + p_1 i_1 + \left(\frac{\sqrt{3}}{2} q_0 - \frac{1}{2} q_1\right) i_5 + \left(\frac{1}{2} q_0 + \frac{\sqrt{3}}{2} q_1\right) i_6 \tag{4.82}$$
$$u_{01} = e_0 - e_1 i_1 + \left(\frac{\sqrt{3}}{2} f_0 - \frac{1}{2} f_1\right) i_3 + \left(\frac{1}{2} f_0 + \frac{\sqrt{3}}{2} f_1\right) i_7 \tag{4.83}$$

For any octonion o, denote by $\pi_k(o)$ the projection of the octonion o onto the subspace of $\mathbb{O}$ spanned by the vector basis element i_k where we set $i_0 = 1$. Note that

$$\pi_0\left[\frac{(s_{10}t_{10})u_{01} + (s_{01}t_{10})u_{01}}{2}\right] = -b_0 q_0 f_1 - b_0 q_1 f_0 - b_1 q_0 f_0 + b_1 q_1 f_1 \tag{4.84}$$

and

$$\pi_0\left[\frac{(s_{10}t_{10})u_{01} - (s_{01}t_{10})u_{01}}{2}\right] = a_0 p_0 e_0 - a_0 p_1 e_1 - a_1 p_0 e_1 - a_1 p_1 e_0 \tag{4.85}$$

Thus

$$pr(NNN) = \left(\pi_0\left[\frac{(s_{10}t_{10})u_{01} + (s_{01}t_{10})u_{01}}{2}\right]\right)^2 + \left(\pi_0\left[\frac{(s_{10}t_{10})u_{01} - (s_{01}t_{10})u_{01}}{2}\right]\right)^2 \tag{4.86}$$

Similarly, we make the following reconciliations:
To obtain the probability of observing the state FFF, note that

$$\pi_1\left[\frac{(s_{10}t_{10})u_{01} + (s_{01}t_{10})u_{01}}{2}\right] = -b_0q_0f_0 + b_0q_1f_1 + b_1q_0f_1 + b_1q_1f_0 \tag{4.87}$$

and

$$\pi_1\left[\frac{(s_{10}t_{10})u_{01} - (s_{01}t_{10})u_{01}}{2}\right] = -a_0p_0e_1 - a_0p_1e_0 - a_1p_0e_0 + a_1p_1e_1 \tag{4.88}$$

Thus

$$pr(FFF) = \left(\pi_1\left[\frac{(s_{10}t_{10})u_{01} + (s_{01}t_{10})u_{01}}{2}\right]\right)^2 + \left(\pi_1\left[\frac{(s_{10}t_{10})u_{01} - (s_{01}t_{10})u_{01}}{2}\right]\right)^2 \tag{4.89}$$

To obtain the probability of observing the state NNF, note that

$$\pi_7\left[\frac{(s_{10}t_{10})u_{01} + (s_{01}t_{10})u_{01}}{2}\right] = \frac{1}{2}(-b_0q_0e_1 + b_0q_1e_0 + b_1q_0e_0 + b_1q_1e_1) + \frac{\sqrt{3}}{2}(b_0q_0e_0 + b_0q_1e_1 + b_1q_0e_1 - b_1q_1e_0) \tag{4.90}$$

and

$$\pi_7\left[\frac{(s_{10}t_{10})u_{01} - (s_{01}t_{10})u_{01}}{2}\right] = \frac{1}{2}(a_0p_0f_0 + a_0p_1f_1 + a_1p_0f_1 - a_1p_1f_0) + \frac{\sqrt{3}}{2}(a_0p_0f_1 - a_0p_1f_0 - a_1p_0f_0 - a_1p_1f_1) \tag{4.91}$$

Thus

$$pr(NNF) = \left(\pi_7\left[\frac{(s_{10}t_{10})u_{01} + (s_{01}t_{10})u_{01}}{2}\right]\right)^2 + \left(\pi_7\left[\frac{(s_{10}t_{10})u_{01} - (s_{01}t_{10})u_{01}}{2}\right]\right)^2 \tag{4.92}$$

To obtain the probability of observing the state FFN, note that

$$\pi_3\left[\frac{(s_{10}t_{10})u_{01} + (s_{01}t_{10})u_{01}}{2}\right] = \frac{1}{2}\left(-b_0q_0e_0 - b_0q_1e_1 - b_1q_0e_1 + b_1q_1e_0\right) + \frac{\sqrt{3}}{2}\left(-b_0q_0e_1 + b_0q_1e_0 + b_1q_0e_0 + b_1q_1e_1\right) \tag{4.93}$$

and

$$\pi_3\left[\frac{(s_{10}t_{10})u_{01} - (s_{01}t_{10})u_{01}}{2}\right] = \frac{1}{2}\left(-a_0p_0f_1 + a_0p_1f_0 + a_1p_0f_0 + a_1p_1f_1\right) + \frac{\sqrt{3}}{2}\left(a_0p_0f_0 + a_0p_1f_1 + a_1p_0f_1 - a_1p_1f_0\right) \tag{4.94}$$

Thus

$$pr(FFN) = \left(\pi_3\left[\frac{(s_{10}t_{10})u_{01} + (s_{01}t_{10})u_{01}}{2}\right]\right)^2 + \left(\pi_3\left[\frac{(s_{10}t_{10})u_{01} - (s_{01}t_{10})u_{01}}{2}\right]\right)^2 \tag{4.95}$$

To obtain the probability of observing the state NFF, note that

$$\pi_2\left[\frac{(s_{10}t_{00})u_{00} + (s_{01}t_{00})u_{00}}{2}\right] = \frac{1}{2}\left(a_0q_0f_0 - a_0q_1f_1 + a_1q_0f_1 + a_1q_1f_0\right) + \frac{\sqrt{3}}{2}\left(-a_0q_0f_1 - a_0q_1f_0 + a_1q_0f_0 - a_1q_1f_1\right) \tag{4.96}$$

and

$$\pi_2\left[\frac{(s_{10}t_{00})u_{00}-(s_{01}t_{00})u_{00}}{2}\right]=\frac{1}{2}\left(b_0p_0e_1+b_0p_1e_0-b_1p_0e_0+b_1p_1e_1\right)+\frac{\sqrt{3}}{2}\left(b_0p_0e_0-b_0p_1e_1+b_1p_0f_1+b_1p_1e_0\right) \tag{4.97}$$

Thus

$$pr(NFF)=\left(\pi_2\left[\frac{(s_{10}t_{00})u_{00}+(s_{01}t_{00})u_{00}}{2}\right]\right)^2+\left(\pi_2\left[\frac{(s_{10}t_{00})u_{00}-(s_{01}t_{00})u_{00}}{2}\right]\right)^2 \tag{4.98}$$

To obtain the probability of observing the state FNN, note that

$$\pi_4\left[\frac{(s_{10}t_{00})u_{00}+(s_{01}t_{00})u_{00}}{2}\right]=\frac{1}{2}\left(b_0p_0e_0-b_0p_1e_1+b_1p_0f_1+b_1p_1e_0\right)+\frac{\sqrt{3}}{2}\left(-b_0p_0e_1-b_0p_1e_0+b_1p_0e_0-b_1p_1e_1\right) \tag{4.99}$$

and

$$\pi_4\left[\frac{(s_{10}t_{00})u_{00}-(s_{01}t_{00})u_{00}}{2}\right]=\frac{1}{2}\left(-a_0q_0f_1-a_0q_1f_0+a_1q_0f_0-a_1q_1f_1\right)+\frac{\sqrt{3}}{2}\left(-a_0q_0f_0+a_0q_1f_1-a_1q_0f_1-a_1q_1f_0\right) \tag{4.100}$$

Thus

$$pr(FNN)=\left(\pi_4\left[\frac{(s_{10}t_{00})u_{00}+(s_{01}t_{00})u_{00}}{2}\right]\right)^2+\left(\pi_4\left[\frac{(s_{10}t_{00})u_{00}-(s_{01}t_{00})u_{00}}{2}\right]\right)^2 \tag{4.101}$$

To obtain the probability of observing the state FNF, note that

$$\pi_5\left[\frac{(s_{10}t_{00})u_{00}+(s_{01}t_{00})u_{00}}{2}\right]=\frac{1}{2}\left(b_0p_0f_0+b_0p_1f_1-b_1p_0f_1+b_1p_1f_0\right)+\frac{\sqrt{3}}{2}\left(-b_0p_0f_1+b_0p_1f_0-b_1p_0f_0-b_1p_1f_1\right) \tag{4.102}$$

and

$$\pi_5\left[\frac{(s_{10}t_{00})u_{00}-(s_{01}t_{00})u_{00}}{2}\right]=\frac{1}{2}\left(-a_0q_0e_1+a_0q_1e_0-a_1q_0e_0-a_1q_1e_1\right)+\frac{\sqrt{3}}{2}\left(-a_0q_0e_0-a_0q_1e_1+a_1q_0e_1-a_1q_1e_0\right) \tag{4.103}$$

Thus

$$\begin{aligned} pr(FNF) &= \left(\pi_5\left[\frac{(s_{10}t_{00})u_{00}+(s_{01}t_{00})u_{00}}{2}\right]\right)^2 \\ &+ \left(\pi_5\left[\frac{(s_{10}t_{00})u_{00}-(s_{01}t_{00})u_{00}}{2}\right]\right)^2 \end{aligned} \tag{4.104}$$

To obtain the probability of observing the state NFN, note that

$$\pi_6\left[\frac{(s_{10}t_{00})u_{00}+(s_{01}t_{00})u_{00}}{2}\right]=\frac{1}{2}\left(b_0p_0f_0+b_0p_1f_1-b_1p_0f_1+b_1p_1f_0\right)+\frac{\sqrt{3}}{2}\left(-b_0p_0f_1+b_0p_1f_0-b_1p_0f_0-b_1p_1f_1\right) \tag{4.105}$$

and

$$\pi_6\left[\frac{(s_{10}t_{00})u_{00}-(s_{01}t_{00})u_{00}}{2}\right]=\frac{1}{2}\left(-a_0q_0e_1+a_0q_1e_0-a_1q_0e_0-a_1q_1e_1\right)+\frac{\sqrt{3}}{2}\left(-a_0q_0e_0-a_0q_1e_1+a_1q_0e_1-a_1q_1e_0\right) \tag{4.106}$$

Thus

$$pr(NFN) = \left(\pi_6\left[\frac{(s_{10}t_{00})u_{00} + (s_{01}t_{00})u_{00}}{2}\right]\right)^2 + \left(\pi_6\left[\frac{(s_{10}t_{00})u_{00} - (s_{01}t_{00})u_{00}}{2}\right]\right)^2 \tag{4.107}$$

Theorem 4.32 admits the following useful corollaries

Corollary 4.108. *If two of the players employ pure quantum strategies which are represented by canonical octonionic basis elements and the third player employs a pure quantum strategy represented by a unit octonion, that is, if the players employ a pure quantum strategic profile (s, t, u) of the form*

$$(s_0 + s_1 i_1 + s_2 i_2 + s_3 i_4,\ i_l,\ i_m), \tag{4.109}$$

$$(i_k,\ t_0 + t_1 i_1 + t_2 i_5 + t_3 i_6,\ i_m), \tag{4.110}$$

or

$$(i_k,\ i_l,\ u_0 + u_1 i_1 + u_2 i_3 + u_3 i_7), \tag{4.111}$$

where $k \in \{0,1,2,4\}, l \in \{0,1,5,6\}$, and $m \in \{0,1,3,7\}$, then the conclusion of Theorem 4.32 reduces to

$$\begin{array}{ll} pr(NNN) = [\pi_0((st)u)]^2 & pr(FFF) = [\pi_1((st)u)]^2 \\ pr(FFN) = [\pi_3((st)u)]^2 & pr(NNF) = [\pi_7((st)u)]^2 \\ pr(NFF) = [\pi_2((st)u)]^2 & pr(FNN) = [\pi_4((st)u)]^2 \\ pr(FNF) = [\pi_5((st)u)]^2 & pr(NFN) = [\pi_6((st)u)]^2 \end{array} \tag{4.112}$$

and the payoff to player α is given by

$$P_\alpha(s,t,u) = \sum_{j=0}^{7}[\pi_j((st)u)]W_j, \tag{4.113}$$

where W is X for player 1, Y for player 2, and Z for player 3.

The proof appears in Appendix B.

Immediately from Corollary 4.108, we obtain

Corollary 4.114. *If each player employs a pure quantum strategy which is represented by a canonical octonionic basis element, that is, if s is any element of the set $\{1, i_1, i_2, i_4\}$, t any element of the set $\{1, i_1, i_5, i_6\}$, and u any element of the set $\{1, i_1, i_3, i_7\}$, then the conclusion of Theorem 4.32 reduces to Equation 4.112 and the payoff to player k is given by*

$$P_k(s,t,u) = \sum_{l=0}^{7}[\pi_l((st)u)]^2 W_l. \tag{4.115}$$

The proof appears in Appendix B.

4.5 A Special Discrete Distribution

Consider a generic three player, two strategy game with the following payoff matrix

III	Z_0	Z_7	Z_6	Z_2	Z_4	Z_5	Z_3	Z_1
II	Y_0	Y_7	Y_6	Y_2	Y_4	Y_5	Y_3	Y_1
I	X_0	X_7	X_6	X_2	X_4	X_5	X_3	X_1
Players	NNN	NNF	NFN	NFF	FNN	FNF	FFN	FFF
	1	i_7	i_6	i_2	i_4	$-i_5$	i_3	i_1

where the X_t's, Y_t's, and Z_t's are all real numbers.

Definition 4.116. *Define a* discrete distribution *as a mixed strategy that is supported on a finite number of points.*

One such distributions is the *special* discrete distribution where each player plays his pure strategy corresponding to a single octonionic basis element with probability $\frac{1}{4}$. In particular, for player I, this is the mixed quantum strategy

$$\sigma = \frac{1}{4} + \frac{1}{4}i_1 + \frac{1}{4}i_2 + \frac{1}{4}i_4, \tag{4.117}$$

for player II

$$\tau = \frac{1}{4} + \frac{1}{4}i_1 + \frac{1}{4}i_5 + \frac{1}{4}i_6, \tag{4.118}$$

and for player III

$$\upsilon = \frac{1}{4} + \frac{1}{4}i_1 + \frac{1}{4}i_3 + \frac{1}{4}i_7. \tag{4.119}$$

Suppose that players I, II, and III employ the special discrete distributions σ, τ, and υ, respectively. Then the expected payoff to player k is given by

$$\begin{aligned}
\mathcal{E}_k(\sigma,\tau,\upsilon) &= \frac{1}{64}\sum_{l,m,n} P_k(i_l,\ i_m,\ i_n) \\
&= \frac{1}{64}[P_k(1,1,1) + P_k(1,1,i_1) + P_k(1,1,i_3) + P_k(1,1,i_7) + P_k(1,i_1,1) \\
&+ P_k(1,i_1,i_1) + P_k(1,i_1,i_3) + P_k(1,i_1,i_7) + P_k(1,i_5,1) + P_k(1,i_5,i_1) \\
&+ P_k(1,i_5,i_3) + P_k(1,i_5,i_7) + P_k(1,i_6,1) + P_k(1,i_6,i_1) + P_k(1,i_6,i_3) \\
&+ P_k(1,i_6,i_7) + P_k(i_1,1,1) + P_k(i_1,1,i_1) + P_k(i_1,1,i_3) + P_k(i_1,1,i_7) \\
&+ P_k(i_1,i_1,1) + P_k(i_1,i_1,i_1) + P_k(i_1,i_1,i_3) + P_k(i_1,i_1,i_7) + P_k(i_1,i_5,1) \\
&+ P_k(i_1,i_5,i_1) + P_k(i_1,i_5,i_3) + P_k(i_1,i_5,i_7) + P_k(i_1,i_6,1) + P_k(i_1,i_6,i_1) \\
&+ P_k(i_1,i_6,i_3) + P_k(i_1,i_6,i_7) + P_k(i_2,1,1) + P_k(i_2,1,i_1) + P_k(i_2,1,i_3) \\
&+ P_k(i_2,1,i_7) + P_k(i_2,i_1,1) + P_k(i_2,i_1,i_1) + P_k(i_2,i_1,i_3) + P_k(i_2,i_1,i_7) \\
&+ P_k(i_2,i_5,1) + P_k(i_2,i_5,i_1) + P_k(i_2,i_5,i_3) + P_k(i_2,i_5,i_7) + P_k(i_2,i_6,1) \\
&+ P_k(i_2,i_6,i_1) + P_k(i_2,i_6,i_3) + P_k(i_2,i_6,i_7) + P_k(i_4,1,1) + P_k(i_4,1,i_1) \\
&+ P_k(i_4,1,i_3) + P_k(i_4,1,i_7) + P_k(i_4,i_1,1) + P_k(i_4,i_1,i_1) + P_k(i_4,i_1,i_3) \\
&+ P_k(i_4,i_1,i_7) + P_k(i_4,i_5,1) + P_k(i_4,i_5,i_1) + P_k(i_4,i_5,i_3) + P_k(i_4,i_5,i_7) \\
&+ P_k(i_4,i_6,1) + P_k(i_4,i_6,i_1) + P_k(i_4,i_6,i_3) + P_k(i_4,i_6,i_7)\ ]
\end{aligned} \tag{4.120}$$

Using Corollary 4.114, we then obtain

$$\begin{aligned}
P_k(1,1,1) &= P_k(1,i_1,i_1) = P_k(i_1,1,i_1) = P_k(i_1,i_1,1) \\
&= P_k(i_2,i_5,i_3) = P_k(i_2,i_6,i_7) = P_k(i_4,i_6,i_3) = P_k(i_4,i_5,i_7)
\end{aligned} \tag{4.121}$$

and

$$\begin{aligned}
P_k(i_1,1,1) &= P_k(1,1,i_1) = P_k(1,i_1,1) = P_k(i_1,i_1,i_1) \\
&= P_k(i_2,i_5,i_7) = P_k(i_2,i_6,i_3) = P_k(i_4,i_6,i_7) = P_k(i_4,i_5,i_3)
\end{aligned} \tag{4.122}$$

and

$$P_k(i_2,1,1) = P_k(1,i_5,i_3) = P_k(1,i_6,i_7) = P_k(i_1,i_5,i_7)$$
$$= P_k(i_1,i_6,i_3) = P_k(i_4,1,i_1) = P_k(i_2,i_1,i_1) = P_k(i_4,i_1,1) \quad (4.123)$$

and

$$P_k(1,1,i_3) = P_k(1,i_1,i_7) = P_k(i_1,1,i_7) = P_k(i_1,i_1,i_3)$$
$$= P_k(i_2,i_5,1) = P_k(i_2,i_6,i_1) = P_k(i_4,i_5,i_1) = P_k(i_4,i_6,1) \quad (4.124)$$

and

$$P_k(i_4,1,1) = P_k(1,i_5,i_7) = P_k(1,i_6,i_3) = P_k(i_1,i_5,i_3)$$
$$= P_k(i_1,i_6,i_7) = P_k(i_2,1,i_1) = P_k(i_2,i_1,1) = P_k(i_4,i_1,i_1) \quad (4.125)$$

and

$$P_k(1,i_5,1) = P_k(1,i_6,i_1) = P_k(i_1,i_5,i_1) = P_k(i_1,i_6,1)$$
$$= P_k(i_2,1,i_3) = P_k(i_2,i_1,i_7) = P_k(i_4,1,i_7) = P_k(i_4,i_1,i_3) \quad (4.126)$$

and

$$P_k(1,i_6,1) = P_k(1,i_5,i_1) = P_k(i_1,i_5,1) = P_k(i_1,i_6,i_1)$$
$$= P_k(i_2,1,i_7) = P_k(i_2,i_1,i_3) = P_k(i_4,1,i_3) = P_k(i_4,i_1,i_7) \quad (4.127)$$

and

$$P_k(1,1,i_7) = P_k(1,i_1,i_3) = P_k(i_1,1,i_3) = P_k(i_1,i_1,i_7)$$
$$= P_k(i_2,i_5,i_1) = P_k(i_2,i_6,1) = P_k(i_4,i_5,1) = P_k(i_4,i_6,i_3) \quad (4.128)$$

Therefore, the expected payoff to player k can be rewritten as

$$\begin{aligned}\mathcal{E}_k(\sigma,\tau,\upsilon) &= \frac{1}{64}[8P_k(1,1,1) + 8P_k(i_1,1,1) + 8P_k(i_2,1,1) + 8P_k(1,1,i_3) \\ &+ 8P_k(i_4,1,1) + 8P_k(1,i_5,1) + 8P_k(1,i_6,1) + 8P_k(1,1,i_7)] \\ &= \frac{W_0+W_1+W_2+W_3+W_4+W_5+W_6+W_7}{8}\end{aligned} \tag{4.129}$$

Note that this is the average of the classical individual payoffs for player k.

Theorem 4.130. *The strategic profile (σ,τ,υ) is a Nash equilibrium.*

Proof. Take players one and three's strategies as given in (4.117) and (4.119), respectively. Then player two can respond via a pure quantum strategy or via a mixed quantum strategy.

Case 1: Player two employs a pure quantum strategy, that is, $\tau = t = t_0 + t_1 i_1 + t_2 i_5 + t_3 i_6$, where the t_k's are real numbers with $t_0^2+t_1^2+t_2^2+t_3^2 = 1$. Then the expected payoff to player 2 is given by

$$\begin{aligned}16P_2(\sigma,t,\upsilon) &= P_2(1,\ t,\ 1) + P_2(1,\ t,\ i_1) + P_2(1,\ t,\ i_3) + P_2(1,\ t,\ i_7) \\ &+ P_2(i_1,\ t,\ 1) + P_2(i_1,\ t,\ i_1) + P_2(i_1,\ t,\ i_3) + P_2(i_1,\ t,\ i_7) \\ &+ P_2(i_2,\ t,\ 1) + P_2(i_2,\ t,\ i_1) + P_2(i_2,\ t,\ i_3) + P_2(i_2,\ t,\ i_7) \\ &+ P_2(i_4,\ t,\ 1) + P_2(i_4,\ t,\ i_1) + P_2(i_4,\ t,\ i_3) + P_2(i_4,\ t,\ i_7)\end{aligned} \tag{4.131}$$

By Corollary 4.108, we obtain

$$\begin{aligned}
16P_2(\sigma,t,\upsilon) &= \sum_{k=0}^{7}[\pi_k((1t)1)]^2Y_k + \sum_{k=0}^{7}[\pi_k((1t)i_1)]^2Y_k + \sum_{k=0}^{7}[\pi_k((1t)i_3)]^2Y_k \\
&+ \sum_{k=0}^{7}[\pi_k((1t)i_7)]^2Y_k + \sum_{k=0}^{7}[\pi_k((i_1t)1)]^2Y_k + \sum_{k=0}^{7}[\pi_k((i_1t)i_1)]^2Y_k \\
&+ \sum_{k=0}^{7}[\pi_k((i_1t)i_3)]^2Y_k + \sum_{k=0}^{7}[\pi_k((i_1t)i_7)]^2Y_k + \sum_{k=0}^{7}[\pi_k((i_2t)1)]^2Y_k \\
&+ \sum_{k=0}^{7}[\pi_k((i_2t)i_1)]^2Y_k + \sum_{k=0}^{7}[\pi_2((i_1t)i_3)]^2Y_k + \sum_{k=0}^{7}[\pi_2((i_1t)i_7)]^2Y_k \\
&+ \sum_{k=0}^{7}[\pi_k((i_4t)1)]^2Y_k + \sum_{k=0}^{7}[\pi_k((i_4t)i_1)]^2Y_k + \sum_{k=0}^{7}[\pi_k((i_4t)i_3)]^2Y_k \\
&+ \sum_{k=0}^{7}[\pi_k((i_4t)i_7)]^2Y_k
\end{aligned} \tag{4.132}$$

Hence

$$\begin{aligned}
16P_2(\sigma,t,\upsilon) &= \left(t_0^2Y_0 + t_1^2Y_1 + t_2^2Y_5 + t_3^2Y_6\right) + \left(t_1^2Y_0 + t_0^2Y_1 + t_3^2Y_5 + t_2^2Y_6\right) \\
&+ \left(t_0^2Y_3 + t_1^2Y_7 + t_2^2Y_2 + t_3^2Y_4\right) + \left(t_1^2Y_3 + t_0^2Y_7 + t_3^2Y_2 + t_2^2Y_4\right) \\
&+ \left(t_1^2Y_0 + t_0^2Y_1 + t_3^2Y_5 + t_2^2Y_6\right) + \left(t_0^2Y_0 + t_1^2Y_1 + t_2^2Y_5 + t_3^2Y_6\right) \\
&+ \left(t_1^2Y_3 + t_0^2Y_7 + t_3^2Y_2 + t_2^2Y_4\right) + \left(t_0^2Y_3 + t_1^2Y_7 + t_2^2Y_2 + t_3^2Y_4\right) \\
&+ \left(t_2^2Y_3 + t_3^2Y_7 + t_0^2Y_2 + t_1^2Y_4\right) + \left(t_3^2Y_3 + t_2^2Y_7 + t_1^2Y_2 + t_0^2Y_4\right) \\
&+ \left(t_2^2Y_0 + t_3^2Y_1 + t_0^2Y_5 + t_1^2Y_6\right) + \left(t_3^2Y_0 + t_2^2Y_1 + t_1^2Y_5 + t_0^2Y_6\right) \\
&+ \left(t_3^2Y_3 + t_2^2Y_7 + t_1^2Y_2 + t_0^2Y_4\right) + \left(t_2^2Y_3 + t_3^2Y_7 + t_0^2Y_2 + t_1^2Y_4\right) \\
&+ \left(t_3^2Y_0 + t_2^2Y_1 + t_1^2Y_5 + t_0^2Y_6\right) + \left(t_2^2Y_0 + t_3^2Y_1 + t_0^2Y_5 + t_1^2Y_6\right)
\end{aligned} \tag{4.133}$$

Simplifying gives,

$$\begin{aligned}16P_2(\sigma,t,\upsilon) &= 2Y_0\left(t_0^2+t_1^2+t_2^2+t_3^2\right)+2Y_1\left(t_0^2+t_1^2+t_2^2+t_3^2\right)\\&+2Y_2\left(t_0^2+t_1^2+t_2^2+t_3^2\right)+2Y_3\left(t_0^2+t_1^2+t_2^2+t_3^2\right)\\&+2Y_4\left(t_0^2+t_1^2+t_2^2+t_3^2\right)+2Y_5\left(t_0^2+t_1^2+t_2^2+t_3^2\right)\\&+2Y_6\left(t_0^2+t_1^2+t_2^2+t_3^2\right)+2Y_7\left(t_0^2+t_1^2+t_2^2+t_3^2\right)\end{aligned} \tag{4.134}$$

Therefore

$$P_2(\sigma,t,\upsilon)=\frac{Y_0+Y_1+Y_2+Y_3+Y_4+Y_5+Y_6+Y_7}{8} \tag{4.135}$$

Hence, player 2 is indifferent between all his pure quantum strategies.

Case 2: Player 2 employs the mixed quantum strategy μ which is a probability distribution over $\mathbb{S}^3$. Then the expected payoff to Player 2 is given by

$$\mathcal{E}_2(\sigma,\mu,\upsilon)=\int\limits_{\mathbb{S}^3\times\mathbb{S}^3\times\mathbb{S}^3} P_2(s,t,u)d(\sigma\times\mu\times\upsilon)(s,t,u) \tag{4.136}$$

Applying Fubini's Theorem (For more detail on probability measure and Fubini's theorem, see Appendix C), we obtain

$$\begin{aligned}
\mathcal{E}_2(\sigma,\mu,\upsilon) &= \int\limits_{\mathbb{S}^3\times\mathbb{S}^3\times\mathbb{S}^3} P_2(s,t,u)d(\sigma\times\mu\times\upsilon)(s,t,u) \\
&= \int\limits_{\mathbb{S}^3}\left[\int\limits_{\mathbb{S}^3\times\mathbb{S}^3} P_2(s,t,u)d(\sigma\times\upsilon)(s,u)\right]d\mu(t) \\
&= \int\limits_{\mathbb{S}^3}\left(\frac{1}{8}\sum_{k=0}^{7}Y_k\right)d\mu(t) \\
&= \frac{1}{8}\sum_{k=0}^{7}Y_k\mu\left(\mathbb{S}^3\right) = \frac{1}{8}\sum_{k=0}^{7}Y_k
\end{aligned} \tag{4.137}$$

Hence, Player II is indifferent between all his mixed quantum strategies. Hence, there is no incentive for player two to deviate from playing τ against the fixed pair (σ,υ).

In the same way, one verifies that if players two and three's strategies are given as in (4.118) and (4.119), respectively, then σ is an optimal response for player one.

In a similar manner, one checks that if players one and two's strategies are given as in (4.117) and (4.118), respectively, then υ is an optimal response for player three.

Therefore the special discrete distribution is a Nash equilibrium with expected payoff to the players given by

$$\frac{1}{8}\left(\sum_{k=0}^{7}X_k,\ \sum_{k=0}^{7}Y_k,\ \sum_{k=0}^{7}Z_k\right). \tag{4.138}$$

□

Note that Theorem 4.130 is a remarkable and amazing result in the sense that every three player, two strategy game quantized as above shares this common equilibrium, completely irrespective of the specific individual payoffs to the classical player.

4.6 Applications

We give two straightforward applications, one to a zero-sum game, the other to a dilemma game which is not a zero-sum game.

4.6.1 Nash-Shapley Poker Model

For our zero-sum game we consider the final stage of the Nash-Shapley Poker Model [42]. This final strategic form is a 3×2 zero-sum game with payoff function given in Figure 4.3, where we denote Player I's strategy space by $\{s_1,\ s_2\}$, Player II's strategy space by $\{t_1,\ t_2\}$, and Player III's strategy space by $\{u_1,\ u_2\}$.

I \ II	t_1	t_2
s_1	$(-2,-2,4)$	$(-2,6,-4)$
s_2	$(6,-2,-4)$	$(10,10,-20)$

Player III chooses u_1

I \ II	t_1	t_2
s_1	$(0,0,0)$	$(2,-4,2)$
s_2	$(-4,2,2)$	$(-3,-3,6)$

Player III chooses u_2

Figure 4.3: Nash-Shapley Poker Model

There are no classical pure strategy equilibria. But there is a unique classical mixed strategy equilibrium where Player I uses his/her second strategy with probability $p = \sqrt{\frac{7}{5}} - 1 \approx 0.18$, Player II uses his/her second strategy with probability $q = \sqrt{\frac{7}{5}} - 1 \approx 0.18$, and Player III uses his/her second strategy with probability $r = \frac{4p+8}{5p+12} \approx 0.68$ with expected payoff to the players approximately $(-0.40456,\ -0.40456,\ 0.80912)$. So quantizing this 3×2 game as above and applying Theorem 4.130 we have a mixed quantum strategy equilibrium given by each player utilizing their special discrete distribution with payoffs given by $(0.875,\ 0.875,\ -1.75)$.

We note that in the classical game, the third player has a distinct advantage but this advantage disappears in the quantized version. For more information on how this relates to playing poker over the coming quantum internet, see [9].

4.6.2 Three Player Dilemma Game

For our non-zero sum example, consider the three player dilemma game examined by Benjamin and Hayden [6]. In [6], the authors consider an EWL quantization of a classical three player dilemma game given in Figure 4.4 though using a different initial state than ours. The payoff function of this classical game is given by the below tables, where each player has the same strategy space denoted by $\{C,\ D\}$.

Player III chooses C

I \ II	C	D
C	$(0,0,0)$	$(-9,1,-9)$
D	$(1,-9,-9)$	$(9,9,1)$

Player III chooses D

I \ II	C	D
C	$(-9,-9,1)$	$(1,9,9)$
D	$(9,1,9)$	$(2,2,2)$

Figure 4.4: Three Player Dilemma Game

A classical analysis shows that there is a unique classical pure strategy equilibrium (D, D, D) with payoff to the players (2, 2, 2). So quantizing this 3×2 game as above and applying Theorem 4.130 we have a mixed quantum strategy equilibrium given by each player utilizing their special discrete distribution with payoffs given by (.5, .5, .5). We note that in the quantized game, the players get lower payoffs than the ones they get in the classical version when they all employ the special discrete distribution.

However, analyzing further our quantized Three Player Dilemma Game, we find that it admits the many following Nash equilibria.

4.6.2.1 A (pure, pure, pure) Nash Equilibrium

There is an equilibrium where each player employs a pure quantum strategy represented by a canonical octonionic basis element and this equilibrium is essentially the unique classical Nash equilibrium (D, D, D) in G which becomes an equilibrium in $G^{\mathcal{Q}_{\mathcal{I}}}$.

Proposition 4.139. *The Three Player Dilemma Game admits the following Nash equilibrium in pure quantum strategies:*

$$(i_4,\ i_6,\ i_7). \tag{4.140}$$

Proof. Take players two and three's strategies as given. Suppose player one plays the unit octonion $s = s_0 + s_1 i_1 + s_2 i_2 + s_3 i_4$. Then player one's payoff is

$$\begin{aligned} P_1(s, i_6, i_7) &= \sum_{j=0}^{7} [\pi_j((s i_6) i_7)]^2 X_j \\ &= \sum_{j=0}^{7} [\pi_j(-s_2 + s_3 i_1 + s_0 i_2 - s_1 i_4)]^2 X_j \\ &= s_2^2 X_0 + s_3^2 X_1 + s_0^2 X_2 + s_1^2 X_4 \\ &= s_2^2 \cdot 0 + s_3^2 \cdot 2 + s_0^2 \cdot 1 + s_1^2 \cdot 1 \\ &= s_0^2 + s_1^2 + 2 s_3^2 \end{aligned} \tag{4.141}$$

Player one needs to maximize (4.141) subject to the constraint that s must be a unit octonion, that is,

$$s_0^2 + s_1^2 + s_2^2 + s_3^2 = 1 \tag{4.142}$$

Then player one's best reply is to choose s such that $s_0 = s_1 = s_2 = 0$, that is, $s = i_4$.

Now, suppose player one plays a mixed quantum strategy μ. Then player one's expected payoff is given by

$$\begin{aligned} \mathcal{E}_1(\mu,\ i_6,\ i_7) &= \int_{\mathbb{S}^3} P_1(s,\ i_6,\ i_7) d\mu(s) \\ &= \int_{\mathbb{S}^3} (s_0^2 + s_1^2 + 2 s_3^2) d\mu(s); \quad \text{by (4.141)} \end{aligned} \tag{4.143}$$

Now, player one needs to maximize (4.143) subject to (4.142). Then player one's best reply is to choose a μ that assigns s_3 a probability of 1 and zero to everything else. Hence, player one's best reply is to employ $s = i_4$.

In a similar manner, one verifies that if players one and three's strategies are as given in (4.140), then player two's optimal response is to choose $t = i_6$. In the same way, player three's optimal response is to play $u = i_7$.

Therefore (i_4, i_6, i_7) is a Nash equilibrium in $G^{Q_\mathcal{I}}$ with payoffs to the players $(2, 2, 2)$.

□

4.6.2.2 Equilibria of Type (pure, pure, mix of 2)

Proposition 4.144. *The Three Player Dilemma Game admits the following Nash equilibria*

$$\left(\frac{1+i_2}{2},\ 1,\ i_3\right) \tag{4.145}$$

$$\left(1,\ \frac{1+i_5}{2},\ i_3\right) \tag{4.146}$$

$$\left(1,\ i_5,\ \frac{1+i_3}{2}\right) \tag{4.147}$$

with expected payoffs to the players $(9, 5, 5)$, $(5, 9, 5)$, *and* $(5, 5, 9)$, *respectively.*

Proof. We show that (4.145) is a Nash equilibrium, the others follow symmetrically. Take players two and three's strategies as given and suppose that player one plays the pure quantum strategy represented by the unit octonion $s = s_0 + s_1 i_1 + s_2 i_2 + s_3 i_4$. Then player one's payoff is given by

$$\begin{aligned} P_1(s,1,i_3) &= \sum_{j=0}^{7}[\pi_j((s1)i_3)]^2 X_j \\ &= \sum_{j=0}^{7}[\pi_j(s_0 i_3 + s_1 i_7 + s_2 i_5 - s_3 i_6)]^2 X_j \\ &= s_0^2 X_3 + s_1^2 X_7 + s_2^2 X_5 + s_3^2 X_6 \\ &= 9s_0^2 - 9s_1^2 + 9s_2^2 - 9s_3^2. \end{aligned} \tag{4.148}$$

Player one needs to maximize (4.148) subject to the constraint (4.142). So player one's optimal response is to play s such that $s_1 = s_3 = 0$; for example $s = 1$ and $s = i_2$ are best replies for player one.

Now suppose player one employs a mixed quantum strategy, that is, a probability distribution μ over $\mathbb{S}^3$. Then player one's expected payoff is given by

$$\begin{aligned} \mathcal{E}_1(\mu,\ 1,\ i_3) &= \int_{\mathbb{S}^3} P_1(s,\ 1,\ i_3)d\mu(s) \\ &= \int_{\mathbb{S}^3} (9s_0^2 - 9s_1^2 + 9s_2^2 - 9s_3^2)d\mu(s); \quad \text{by (4.148)} \end{aligned} \tag{4.149}$$

Player one's goal is to maximize (4.149) subject to (4.142). Then player one's best reply is to choose a μ that assigns s_1 and s_3 zero probabilities and s_0 and s_2 nonzero probabilities; for example $s = (1 + i_2)/2$ is a best reply for player one.

In a similar manner, one verifies that if players one and three's strategies are as given in (4.145), then player two's optimal response is to choose $t = 1$. In the same way, player three's optimal response is to play $u = i_3$.

Therefore, (4.145) is a Nash equilibrium with expected payoff to the players

$$
\begin{aligned}
G^{mQ_\mathcal{I}}\left(\frac{1+i_2}{2}, 1, i_3\right) &= \frac{1}{2}\left[G^{Q_\mathcal{I}}(1,1,i_3) + G^{Q_\mathcal{I}}(i_2,1,i_3)\right] \\
&= \frac{1}{2}\left[\sum_{j=0}^{7} \pi_j^2((11)i_3)(X_j, Y_j, Z_j) + \sum_{j=0}^{7} \pi_j^2((i_2 1)i_3)(X_j, Y_j, Z_j)\right] \\
&= \frac{1}{2}(X_3 + X_5, Y_3 + Y_5, Z_3 + Z_5) \\
&= (9, 5, 5).
\end{aligned}
$$

□

4.6.2.3 Equilibria of Types (mix of 2, mix of 2, mix of 2)

The Three Player Dilemma Game admits the following Nash equilibria where each player mixes two canonical octonionic basis elements with equal probabilities.

$$
\begin{aligned}
&\text{Player 1 plays the octonions } 1 \text{ and } i_1, \text{ each with probability } 1/2. \\
&\text{Player 2 plays the octonions } i_5 \text{ and } i_6, \text{ each with probability } 1/2. \\
&\text{Player 3 plays the octonions } i_3 \text{ and } i_7, \text{ each with probability } 1/2.
\end{aligned}
\tag{4.150}
$$

$$
\begin{aligned}
&\text{Player 1 plays the octonions } i_2 \text{ and } i_4, \text{ each with probability } 1/2. \\
&\text{Player 2 plays the octonions } 1 \text{ and } i_1, \text{ each with probability } 1/2. \\
&\text{Player 3 plays the octonions } i_3 \text{ and } i_7, \text{ each with probability } 1/2.
\end{aligned}
\tag{4.151}
$$

$$\begin{aligned}
&\text{Player 1 plays the octonions } i_2 \text{ and } i_4, \text{ each with probability } 1/2.\\
&\text{Player 2 plays the octonions } i_5 \text{ and } i_6, \text{ each with probability } 1/2. \qquad (4.152)\\
&\text{Player 3 plays the octonions } 1 \text{ and } i_1, \text{ each with probability } 1/2.
\end{aligned}$$

$$\text{All three players play the octonions } 1 \text{ and } i_1 \text{ each with probability } 1/2. \qquad (4.153)$$

Proposition 4.154. *The strategic profiles given by (4.150), (4.151), (4.152), and (4.153) are Nash equilibria.*

Proof. We begin by proving that (4.150) is a Nash equilibrium. Take players two and three's strategies as given by (4.150) and let $\tau = \frac{1}{2}i_5 + \frac{1}{2}i_6$ and $\nu = \frac{1}{2}i_3 + \frac{1}{2}i_7$. Suppose player one plays the pure quantum strategy represented by the unit octonion $s = s_0 + s_1i_1 + s_2i_2 + s_3i_4$. Then player one's expected payoff is

$$\begin{aligned}
&\mathcal{E}_1(s,\tau,\nu) = \frac{1}{4}\left[P_1((si_5)i_3) + P_1((si_5)i_7) + P_1((si_6)i_3) + P_1((si_6)i_7)\right]\\
&= \frac{1}{4}\sum_{j=0}^{3}\left\{[\pi_j((si_5)i_3)]^2 + [\pi_j((si_5)i_7)]^2 + [\pi_j((si_6)i_3)]^2 + [\pi_j((si_6)i_7)]^2\right\}X_j\\
&= \frac{1}{4}\sum_{j=0}^{3}\{[\pi_j(s_2 - s_3i_1 - s_0i_2 + s_1i_4)]^2 + [\pi_j(-s_3 - s_2i_1 + s_1i_2 + s_0i_4)]^2\\
&+ [\pi_j(-s_3 - s_2i_1 + s_1i_2 + s_0i_4)]^2 + [\pi_j(-s_2 + s_3i_1 + s_0i_2 - s_1i_4)]^2\}X_j\\
&= \frac{1}{4}[(2s_2^2 + 2s_3^2)X_0 + (2s_2^2 + 2s_3^2)X_1 + (2s_0^2 + 2s_1^2)X_2 + (2s_0^2 + 2s_1^2)X_4]\\
&= \frac{1}{4}[(2s_2^2 + 2s_3^2)\cdot 0 + (2s_2^2 + 2s_3^2)\cdot 2 + (2s_0^2 + 2s_1^2)\cdot 1 + (2s_0^2 + 2s_1^2)\cdot 1]\\
&= \frac{1}{4}[4s_0^2 + 4s_1^2 + 4s_2^2 + 4s_3^2]\\
&= s_0^2 + s_1^2 + s_2^2 + s_3^2 = 1 \qquad (4.155)
\end{aligned}$$

Hence, player one is indifferent between all his pure quantum strategies.

Now suppose player one employs the mixed quantum strategy σ, that is, a probability distribution over the set of unit octonions which are real linear combinations of the elements $1,\ i_1,\ i_2,\ i_4$. Then player one's expected payoff is given by

$$\begin{aligned}
\mathcal{E}_1(\sigma,\tau,\nu) &= \int\limits_{\mathbb{S}^3\times\mathbb{S}^3\times\mathbb{S}^3} P_1(s,t,u)d(\sigma\times\tau\times\nu)(s,t,u) \\
&= \int\limits_{\mathbb{S}^3}\left[\int\limits_{\mathbb{S}^3\times\mathbb{S}^3} P_1(s,t,u)d(\tau\times\nu)(t,u)\right]d(\sigma)(s); \quad \text{By Fubini's Theorem} \\
&= \int\limits_{\mathbb{S}^3} 1d(\sigma); \quad \text{By (4.155)} \\
&= \sigma\left(\mathbb{S}^3\right) = 1
\end{aligned}$$

Therefore, player one is indifferent between all his mixed quantum strategies.

In the same way, if players one and three's strategies are given as in (4.150), then the mixed quantum strategy represented by the octonion $\frac{1}{2}i_5 + \frac{1}{2}i_6$ is an optimal response for player two.

In a similar manner, if players one and two's strategies are given as in (4.150), then the mixed quantum strategy represented by the octonion $\frac{1}{2}i_3 + \frac{1}{2}i_7$ is an optimal response for player three.

Therefore, (4.150) is a Nash equilibrium.

In a similar manner, one verifies that the quantum strategic profiles (4.151), (4.152), and (4.153) are all Nash equilibria. □

Proposition 4.156. *In the Nash equilibria given as in (4.150), (4.151), (4.152), and*

(4.153), the players' expected payoffs are $(1,0,0)$*,* $(0,1,0)$*,* $(0,0,1)$*, and* $(1,1,1)$*, respectively.*

Proof. We will show that the Nash equilibrium given as in (4.150) yields the expected payoffs $(1,0,0)$ to the players, respectively. By a similar calculation, one verifies that the Nash equilibria given as in (4.151), (4.152), and (4.153) yield the expected payoffs $(0,1,0)$, $(0,0,1)$, and $(1,1,1)$ to the players, respectively. Then

$$G^{mQ_{\mathcal{I}}}\left(\frac{1+i_1}{2}, \frac{i_5+i_6}{2}, \frac{i_3+i_7}{2}\right)$$
$$= \frac{1}{8}[G^{Q_{\mathcal{I}}}(1,\ i_5,\ i_3) + G^{Q_{\mathcal{I}}}(1,\ i_5,\ i_7) + G^{Q_{\mathcal{I}}}(1,\ i_6,\ i_3) + G^{Q_{\mathcal{I}}}(1,\ i_6,\ i_7)$$
$$+ G^{Q_{\mathcal{I}}}(i_1,\ i_5,\ i_3) + G^{Q_{\mathcal{I}}}(i_1,\ i_5,\ i_7) + G^{Q_{\mathcal{I}}}(i_1,\ i_6,\ i_3) + G^{Q_{\mathcal{I}}}(i_1,\ i_6,\ i_7)]$$

Applying Corollary 4.114, one obtains

$$\frac{1}{8}\sum_{j=0}^{7}\{[\pi_j((1i_5)i_3)]^2(X_j,Y_j,Z_j) + [\pi_j((1i_5)i_7)]^2(X_j,Y_j,Z_j)$$
$$+ [\pi_j((1i_6)i_3)]^2(X_j,Y_j,Z_j) + [\pi_j((1i_6)i_7)]^2(X_j,Y_j,Z_j)$$
$$+ [\pi_j((i_1i_5)i_3)]^2(X_j,Y_j,Z_j) + [\pi_j((i_1i_5)i_7)]^2(X_j,Y_j,Z_j)$$
$$+ [\pi_j((i_1i_6)i_3)]^2(X_j,Y_j,Z_j) + [\pi_j((i_1i_6)i_7)]^2(X_j,Y_j,Z_j)\}$$
$$= \frac{1}{8}[(X_2,Y_2,Z_2) + (X_4,Y_4,Z_4) + (X_4,Y_4,Z_4) + (X_2,Y_2,Z_2)$$
$$+ (X_4,Y_4,Z_4) + (X_2,Y_2,Z_2) + (X_2,Y_2,Z_2) + (X_4,Y_4,Z_4)]$$
$$= \frac{1}{2}(X_2+X_4, Y_2+Y_4, Z_2+Z_4)$$
$$= \frac{1}{2}(1+1, 9+(-9), 9+(-9)) = (1,0,0)$$

as expected. □

The above result is remarkable in the sense that if a player employs a mixed strategy evenly supported on the common octonionic basis elements 1 and i_1, then the corresponding Nash equilibrium yields that player a payoff of one while the other players get 0. If all three players play the unit octonions 1 and i_1, each with probability 1/2, then they all get a payoff of 1.

Proposition 4.157. *The quantum strategic profile* $\left(\sigma = \frac{1+i_2}{2}, \tau = \frac{1+i_5}{2}, \upsilon = \frac{1+i_3}{2}\right)$ *is a Nash equilibrium in* $G^{m\mathbb{Q}_{\mathcal{I}}}$*, where* G *is the Three Player Dilemma Game, with expected payoff of* $\frac{19}{4}$ *to each player.*

Proof. Take players two and three's strategies as given and suppose player one responds with the pure quantum strategy represented by the unit octonion $s = s_0 + s_1 i_1 + s_2 i_2 + s_3 i_4$. Then player one's expected payoff is given by

$$\begin{aligned}
\mathcal{E}_1(s, \tau, \upsilon) &= \frac{1}{4}\left[P_1(s,1,1) + P_1(s,1,i_3) + P_1(s,i_5,1) + P_1(s,i_5,i_3)\right] \\
&= \frac{1}{4}\sum_{j=0}^{7}\left[\pi_j^2((s1)1) + \pi_j^2((s1)i_3) + \pi_j^2((si_5)1) + \pi_j^2((si_5)i_3)\right] X_j \\
&= \frac{1}{4}\sum_{j=0}^{7}[\pi_j^2(s_0 + s_1 i_1 + s_2 i_2 + s_3 i_3) + \pi_j^2(s_0 i_3 + s_2 i_5 - s_3 i_6 + s_1 i_7) \\
&+ \pi_j^2(s_0 i_5 + s_1 i_6 - s_2 i_3 + s_3 i_7) + \pi_j^2(-s_0 i_2 + s_1 i_4 + s_2 - s_3 i_1)]X_j \\
&= \frac{1}{4}[(s_0^2 + s_2^2)(X_0 + X_2 + X_3 + X_5) + (s_1^2 + s_3^2)(X_1 + X_4 + X_6 + X_7)] \\
&= \frac{1}{4}[(s_0^2 + s_2^2)(0 + 1 + 9 + 9) + (s_1^2 + s_3^2)(2 + 1 - 9 - 9)] \\
&= \frac{19}{4}s_0^2 - \frac{15}{4}s_1 + \frac{19}{4}s_2^2 - \frac{15}{4}s_3^2
\end{aligned} \tag{4.158}$$

Player one's goal is to maximize (4.158) subject to the constraint that s must be a unit octonion. Then player one's best response is to choose s such that $s_1 = s_3 = 0$; for example $s = 1$ and $s = i_2$ or any real convex linear combination of the unit octonions 1 and i_2, i.e. $p \cdot 1 + (1-p) \cdot i_2$ are best replies.

Take players one and three's strategies $\sigma' = p \cdot 1 + (1-p) \cdot i_2$ and v, respectively, as given and suppose player two responds with the pure quantum strategy represented by the unit octonion $t = t_0 + t_1 i_1 + t_2 i_5 + t_3 i_6$. Then player two's expected payoff is given by

$$\begin{aligned}
\mathcal{E}_2(\sigma', t, v) &= \frac{p}{2} P_2(1,t,1) + \frac{p}{2} P_2(1,t,i_3) + \frac{1-p}{2} P_2(i_2,t,1) + \frac{1-p}{2} P_2(i_2,t,i_3) \\
&= \frac{p}{2}\left\{\sum_{j=0}^{7} [\pi_j^2((1t)1) + \pi_j^2((1t)i_3)] Y_j\right\} \\
&+ \frac{1-p}{2}\left\{\sum_{j=0}^{7} [\pi_j^2((i_2t)1) + \pi_j^2((i_2t)i_3)] Y_j\right\} \\
&= \frac{p}{2}[(t_0^2 Y_0 + t_1^2 Y_1 + t_2^2 Y_5 + t_3^2 Y_6) + (t_0^2 Y_3 + t_1^2 Y_7 + t_2^2 Y_2 + t_3^2 Y_4)] \\
&+ \frac{1-p}{2}[(t_0^2 Y_2 + t_1^2 Y_4 + t_2^2 Y_3 + t_3^2 Y_7) + (t_0^2 Y_5 + t_1^2 Y_6 + t_2^2 Y_0 + t_3^2 Y_1)]
\end{aligned}$$

By setting $Y_0 = 0, Y_1 = 2, Y_2 = 9, Y_3 = 9, Y_4 = -9, Y_5 = 1, Y_6 = 1, Y_7 = -9$, and by combining the t_j's, one obtains

$$\mathcal{E}_2(\sigma', t, v) = (5 - .5p)t_0^2 + .5(1+p)t_1^2 + (4.5 + .5p)t_2^2 + (-3.5 - .5p)t_3^2 \quad (4.159)$$

Player two's goal is to maximize (4.159) subject to the constraints that t must be a unit octonion and $0 \leq p \leq 1$. Note that the coefficient of t_0^2, $5 - .5p$, is the largest when

$0 \leq p < .5$ and the coefficient of t_2^2, $4.5+.5p$, is the largest when $.5 < p \leq 1$. Moreover, $5 - .5p = 4.5 + .5p$ if $p = .5$. Then player two's best response is to choose $t = 1$ when $0 \leq p < .5$, $t = i_5$ when $.5 < p \leq 1$, and is indifferent between 1 and i_5 when $p = .5$. Hence, any real convex linear combination of the unit octonions 1 and i_5, i.e. $q \cdot 1 + (1-q) \cdot i_5$ is a best reply when $p = .5$.

Similarly, one verifies that if $0 \leq p < .5$, player three's best response to the fixed pair $(p \cdot 1 + (1-p) \cdot i_2,\ 1)$ of opponents' strategies is to choose $u = i_3$. If $p = .5$, player three's best reply to the fixed pair $(.5 \cdot 1 + .5 \cdot i_2, q \cdot 1 + (1-q) \cdot i_5)$ of opponents's strategies is to play $u = 1$ when $0 \leq q < .5$, $u = i_3$ when $.5 < q \leq 1$, and any real convex linear combination $r \cdot 1 + (1-r) \cdot i_3$ when $q = .5$ because of her indifference between the strategies 1 and i_3. If $.5 < p \leq 1$, player three's best response to the fixed pair $(p \cdot 1 + (1-p) \cdot i_2,\ i_5)$ is to play $u = 1$.

Therefore the quantum strategic profile (σ, τ, v) is a Nash equilibrium with expected payoffs to the players given by

$$
\begin{aligned}
G^{mQ_{\mathcal{I}}}(\sigma,\tau,v) &= \frac{1}{8}\sum_{j=0}^{7}[\pi_j^2((11)1) + \pi_j^2((11)i_3) + \pi_j^2((1i_5)1) + \pi_j^2((1i_5)i_3) \\
&+ \pi_j^2((i_21)1) + \pi_j^2((i_21)i_3) + \pi_j^2((i_2i_5)1) + \pi_j^2((i_2i_5)i_3)](X_j, Y_j, Z_j) \\
&= \frac{1}{8}[(2X_0 + 2X_2 + 2X_3 + 2X_5, 2Y_0 + 2Y_2 + 2Y_3 + 2Y_5, 2Z_0 + 2Z_2 + 2Z_3 + 2Z_5)] \\
&= \left(\frac{19}{4}, \frac{19}{4}, \frac{19}{4}\right).
\end{aligned}
$$

□

Chapter 5

SUMMARY AND FUTURE DIRECTIONS

To conclude, we present a summary of the results developed in this thesis and put forward a number of open problems.

5.1 A Brief Summary

5.1.1 Two Player, Two Strategy Games

Generic two player, two strategy games quantized according to the EWL protocol with a maximally entangled initial state with equal superpositions given in the Dirac notation by $\psi = (|00\rangle + |11\rangle)/\sqrt{2}$ were analyzed by Steven Landsburg . Via a quaternionic representation of the payoff function, Landsburg classified all the potential Nash equilibria of such games. However, there is an entire circle of maximally entangled states with equal superpositions of the form $\psi = (|00\rangle + e^{i\theta}|11\rangle)/\sqrt{2}$, where $\theta \in [0, 2\pi)$, which could be used in these quantizations. Landsburg's quaternionic construction was extended to games where the initial state is chosen arbitrarily from this circle and for these constructions, it was shown that Landsburg's classification of the potential Nash equilibria applies. In particular, the quantum strategic profile consisting of the special discrete distributions, i.e. $p = q = \frac{1}{4}(1 + i + j + k)$ is a Nash equilibrium for all $\theta \in [0, 2\pi)$ and regardless of the specific individual payoffs to the classical player.

5.1.2 Three Player, Two Strategy Games

Generic three player, two strategy games quantized according to the EWL protocol with a maximally entangled initial state given in the Dirac notation by $\psi = (|000\rangle + |111\rangle)/\sqrt{2}$ was considered next. An octonionic representation of the payoff function of such games was presented. This construction provided a fresh computational framework and gave the potential to classify all possible Nash equilibria for these games. While the full classification remains a goal of future research, our representation established the existence of certain Nash equilibria in these quantized games. For example, a remarkable and amazing fact about these games is that every such game shares a common equilibrium which consists of identical strategic choices completely irrespective of the specific individual payoffs to the classical player. This construction was applied to the Nash-Shapley Poker Model and a Three Player Dilemma Game where a number of interesting Nash equilibria were identified.

5.2 Open Problems

Here are a number of problems arising from the work performed thus far:

1. D. Robinson and D. Goforth put forward 144 equivalence classes [49] of two player, two strategy ordinal games. An open problem is to characterize those classes of games in which players "do better" in the maximally entangled EWL-quantized version. A longer term project is to complete a similar work for three

player, two strategy games.

2. From the work herein, a natural conjecture is that Landsburg's discretization theorem for two player, two strategy quaternionized quantum games can be extended to three player, two strategy octonionized quantum games, that is, "*Every mixed quantum strategy is equivalent to a mixed quantum strategy supported on at most four points and those four points can be taken to form an orthonormal basis of* $\mathbb{R}^4$ "

3. An immediate goal is to establish the complete classification of Nash equilibria in three player, two strategy octonionized quantum games. A best response analysis and the evidence obtained to date suggest a conjectural breakdown of the Nash equilibria in three player, two strategy octonionized quantum games:

   - **Equilibria of type "pure, pure, pure"**: Each player plays a pure quantum strategy represented by a canonical octonionic basis element, i.e. from the four point set $\{1, i_1, i_2, i_4\}$ for player one, $\{1, i_1, i_5, i_6\}$ for player two, and $\{1, i_1, i_3, i_7\}$ for player three.
   - **Equilibria of type "pure, pure, mix of two"** (up to permutations): Two players choose canonical octonionic basis elements, one player chooses a mixed strategy supported on two orthonormal points.
   - **Equilibria of type "mix of two, mix of two, mix of two"**: Each player plays a mixed quantum strategy supported on two orthonormal points (canonical octonionic basis elements), each played with probability 1/2.
   - **Equilibria of type "mix of three, mix of three, mix of three"**: Player

one's strategy is supported on three of the four octonions $\{1, i_1, i_2, i_4\}$, player two's strategy is supported on three of the four octonions $\{1, i_1, i_5, i_6\}$, and player three's strategy is supported on three of the four octonions $\{1, i_1, i_3, i_7\}$.

- **Equilibria of type "mix of four, mix of four, mix of four"**: Each player employs a mixed quantum strategy supported on four orthonormal unit octonions. For example, player one's strategy is $\frac{1}{4}(s_0 + s_1 + s_2 + s_3)$, where s_0, s_1, s_2, and s_3 are four orthonormal unit octonions and where each s_j is generated by the basis elements $\{1, i_1, i_2, i_4\}$. Recall that we established the case where the four orthonormal unit octonions are the canonical basis elements for each player.

4. Recall that The Ahmed-Bleiler-Khan octonionic representation focused on games with a specific maximally entangled state. However, as in the two player case, there is an entire circle of this type of maximally entangled states which could be used in these quantizations, an extension of this construction to games where the initial state is chosen arbitrarily from this circle should be relatively direct.

5. For four player, two strategy games, the sedonions, a 16 dimensional real algebra is available to co-ordinatize the payoff function. However, there are zero divisors in the sedonions and hence multiplicative issues with the norm. Is it possible to represent the quantum strategies away from the zero divisors and hence be able to co-ordinatize the payoff function with invertible sedonions?

6. For multi-player classical games collusion between the players, both explicit and implicit, forms a barrier to analysis. As collusion represents the co-ordinatization

of strategic choices, can collusion be effectively modeled by the quantum phenomenon of entanglement which coordinates the observation of states? A cohesive theory of such modeling could have a significant classical as well as quantum consequences.

7. Quantum games have been shown to play a significant role in quantum logic synthesis, see [44] [32] [31]. What is the role of equilibria here?

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Appendix A

REAL DIVISION ALGEBRAS

Much of the material below is taken from [4] and a number of papers and books that are cited as needed.

A.1 Introduction

The real numbers form a complete ordered field. The complex numbers are algebraically complete but not ordered. There are exactly four normed division algebras: the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$, and the octonions $\mathbb{O}$. Of these, the quaternions are non-commutative, and additionally the octonions are both non-commutative and non-associative.

As the story goes, in October 1843, Hamilton was out walking with his wife along the Royal Canal in Dublin when he discovered the quaternions. He later wrote, "That is to say, I then and there felt the galvanic circuit of thought close; and the sparks which fell from it were the fundamental equations between i, j, and k; exactly such I have used them ever since." Then, in a famous act of mathematical vandalism, he carved these equations into the stone of the Brougham Bridge:

$$i^2 = j^2 = k^2 = ijk = -1.$$

The next day, Hamilton wrote to his friend John T. Graves about his discovery. Two month later, in December 1843, Graves replied with a description of his "octaves"–the octonions. In July 1843, Hamilton wrote to Graves pointing out that the octonions were non-associative: "$A \cdot BC = AB \cdot C = ABC$, if A, B, C be quaternions, but not so, generally, with your octaves." By this statement, Hamilton first invented the term "associative", so the octonions may have played a role in clarifying the importance of this concept.

The classification of real division algebras began in 1878, when Georg Frobenius [22] showed that (up to isomorphism) there are exactly three such algebras which are associative: $\mathbb{R}, \mathbb{C}$, and $\mathbb{H}$. In 1898, Adolph Hurwitz [27] showed secondly that the octonions are the only non-associative algebra with a multiplicative norm. Then, in 1930, Max Zorn [58] generalized the results of Frobenius and Hurwitz, showing that $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ are the only alternative (See Definition **A.3** below) real division algebras.

In 1940, topologist Heinz Hopf [26] showed that, as vector spaces, division algebras over the real numbers necessarily have dimension 2^n for some integer $n \geq 0$. Of course, the four classic examples, $\mathbb{R}, \mathbb{C}, \mathbb{H}$, and $\mathbb{O}$, show the existence of real division algebras in dimensions 1, 2, 4, and 8, respectively. In 1958, Rauol Bott and John Milnor [10] and Michel Kervaire [30] independently proved the deep result that real division algebras in higher dimensions do not exist:

Theorem A.1. ((1,2,4,8)-Theorem) *Let A be a division algebra over the real numbers. Then (as a vector space) A has dimension either 1, 2, 4, or 8.*

To date, there has not been a purely algebraic proof of the (1,2,4,8)-Theorem. Indeed the Bott-Milnor-Kervaire proofs of this theorem are obtained as corollaries to a result on a topological property not discussed here, called the *parallelizability of the n-sphere*.

In this appendix, we describe the fundamental results on real division algebras used herein and outline some key constructions.

A.2 Preliminaries

We begin with some definitions.

Definition A.2. *Let* $\mathbf{F}$ *be a field. An* algebra A *over* $\mathbf{F}$ *is a pair* (A, m)*, where* A *is a finite-dimensional vector space over* $\mathbf{F}$ *and* multiplication $m : A \times A \to A$ *is an* $\mathbf{F}$*-bilinear map; that is, for all* $\lambda \in \mathbf{F}$*,* $x, y, z \in A$*,*

$$m(x, \lambda y + z) = \lambda m(x, y) + m(x, z),$$
$$m(\lambda x + y, z) = \lambda m(x, z) + m(y, z).$$

When clear from the context, we write $m(x, y) = xy$ for all $x, y \in A$.

Definition A.3. *Let A be an algebra over* $\mathbf{F}$*. Then A is said to be*

1. alternative *if* $x(xy) = (xx)y$ *and* $x(yy) = (xy)y$ *for all* $x, y \in A$*,*
2. associative *if* $x(yz) = (xy)z$ *for all* $x, y, z \in A$*,*

3. commutative *if* $xy = yx$ *for all* $x, y \in A$, *and*

4. unital *if there is a* $\mathbf{1} \in A$ *such that* $x\mathbf{1} = x = \mathbf{1}x$ *for all* $x \in A$.

If A is unital, then the *identity* $\mathbf{1}$ is uniquely determined.

Throughout, unless stated explicitly we do not assume that an algebra A is alternative, associative, commutative, or unital.

Definition A.4. *Let* A *be an algebra over* $\mathbf{F}$. *For* $x, y, z \in A$, *define the* associator $[x, y, z]$ *of* x, y, *and* z *by* $x(yz) - (xy)z$.

It is straightforward to check the following facts about the associator:

- The associator $[x, y, z] \mapsto x(yz) - (xy)z$ is a trilinear map $A^3 \to A$.
- If A is alternative, then the associator *alternates*, that is,

$$[x, y, z] = -[y, x, z] = -[x, z, y] = -[z, y, x] \tag{A.5}$$

  for all $x, y, z \in A$.

Lemma A.6. *Let* A *be an alternative algebra over* $\mathbf{F}$. *Then the following hold*

1. *the* flexible *law:* $x(yx) = (xy)x$ *for all* $x, y \in A$, *and*

2. *the* Moufang *identity:* $(zx)(yz) = z(xy)z$ *for all* $x, y, z \in A$.

3. *If we define* x^n *for* $n \in \mathbb{Z}^+$ *recursively by* $x^1 = x$ *and* $x^{n+1} = x^n x$, *then* A *is* power-associative, *that is,* $x^m x^n = x^{m+n}$ *for all* $x \in A$, *and* $m, n \in \mathbb{Z}^+$.

Proof. To prove (1), observe that $[x, y, x] = 0_A$ since A is alternative. Thus, $x(yx) - (xy)x = 0$, or equivalently, $x(yx) = (xy)x$ for all $x, y \in A$.

To prove (2), observe first that, when A is alternative, repeated use of the identities given by (A.5) yields

$$\begin{aligned}(zx)(yz) - ((zx)y)z &= [zx, y, z] = [y, z, zx] = y(z^2x) - (yz)(zx) \\ &= y(z^2x) - [yz, z, x] - (yz^2)x \\ &= [y, z^2, x] - [yz, z, x] = [y, z^2, x] - [x, yz, z] \\ &= [y, z^2, x] - x(yz^2) + (x(yz))z \\ &= [y, z^2, x] + [x, y, z]z - [x, y, z^2] = [x, y, z]z.\end{aligned}$$

Hence, if A is alternative, then

$$\begin{aligned}(zx)(yz) &= [x, y, z]z + ((zx)y)z \\ &= [x, y, z]z - [z, x, y]z + z(xy)z \\ &= z(xy)z\end{aligned}$$

for all $x, y, z \in A$.

To prove (3), we apply induction, the flexible law, and the Moufang identity. First, let us show that $x^{n+1} = xx^n$ for all $n \in \mathbb{Z}^+$. Indeed, the base case $xx^1 = x^2$ holds; and if $x^{n+1} = xx^n$ for some $n \geq 1$, then by the flexible law, $x^{n+2} = x^{n+1}x = (xx^n)x = x(x^nx) = xx^{n+1}$. Now, because $x = x^1$ we have shown that $x^{m+n} = x^mx^n$ in the base case $m = 1$. Assume by induction on m that $x^{m+1} = x^mx^n$ for some $m \geq 1$ and $n \geq 2$

(the case $n = 1$ is trivial). Then, by the Moufang identity, $x^{m+1}x^n = (xx^m)(x^{n-1}x) = xx^{m+n-1}x = x^{m+n+1}$ as required. □

Definition A.7. *An algebra A over* **F** *is said to be a* division algebra *if A is not trivial and $xy = 0_A \Rightarrow x = 0_A$ or $y = 0_A$ for all $x, y \in A$.*

Note that the term *division algebra* in Definition **A.7** comes from the following proposition, which shows that, in such an algebra left and right division can be unambiguously performed.

Proposition A.8. *Let A be an algebra over* **F**. *Then A is a division algebra if, and only if, A is not trivial and for all $a, b \in A$ with $b \neq 0_A$, the equations $bx = a$ and $yb = a$ have unique solutions $x, y \in A$.*

Proof. ($\Rightarrow$) Fix $b \in A$, say with $b \neq 0_A$, and let $\phi : A \to A$ be the linear transformation defined by $\phi(x) = bx$. If A is a division algebra, then $ker\phi = \{0_A\}$, thus ϕ is injective. But A is finite-dimensional as a vector space, so ϕ is actually bijective. Thus, the equation $bx = a$ has a unique solution. Similarly, one verifies that $yb = a$ has a unique solution by considering the linear transformation $y \mapsto yb$.

($\Leftarrow$) Suppose that $xy = 0_A$. If $x = 0_A$, then we're done. Otherwise, by assumption, if $x \neq 0_A$, there is a unique $y \in A$ such that $xy = 0_A$. But $x0_A = 0_A$, so $y = 0_A$. Therefore, A is a division algebra. □

Corollary A.9. *Let A be a division algebra over* F. *If A is alternative, then A is unital.*

Proof. Fix $b \in A$ with $b \neq 0_A$. Since A is a division algebra, by Proposition **A.8** the equation $yb = b$ has a unique solution $y = 1$. Furthermore, $1(1b) = 1b$. Since A is alternative, $1^2 b = 1b$ which implies $(1^2 - 1)b = 0_A$ and hence $1^2 = 1$. It follows that $1(1x - x) = 1(1x) - 1x = 1^2 x - 1x = 0_A$. But $1 \neq 0_A$ since $b \neq 0_A$. Therefore, $1x - x = 0_A$ and $1x = x$ for all $x \in A$. Similarly, $x1 = x$ for all $x \in A$, by considering the product $(x1 - x)1$. Thus A is unital. □

In the following we assume $\mathbf{F} = \mathbb{R}$ and consider classes of division algebras over $\mathbb{R}$ or *real division algebras* for short.

A.3 Quaternions and Octonions

We recall the algebras of quaternions $\mathbb{H}$ and octonions $\mathbb{O}$. Together with the real and complex numbers, these form the four classical division algebras over the real numbers. The quaternions and octonions are alternative division algebras that extend the real and complex numbers in a natural way. Under an appropriate identification,

$$\mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset \mathbb{O}. \tag{A.10}$$

A.3.1 Quaternions

We begin with a definition.

Definition A.11. *Let $\mathbb{H}$ be the 4-dimensional real algebra defined by*

$$\mathbb{H} = span_{\mathbb{R}}\{1,\ i,\ j,\ k\} \tag{A.12}$$

with identity 1 *whose multiplication is polynomial subject to Hamilton's relation* $i^2 = j^2 = k^2 = ijk = -1$. *We call* $\mathbb{H}$ *the algebra of quaternions.*

One identifies the real numbers as a subset of the quaternions via the natural inclusion map $\lambda \hookrightarrow \lambda 1$ for all $\lambda \in \mathbb{R}$. As an example, let us perform the product pq of the quaternions $p = 1 + i + k$ and $q = 2j - 3k$.

$$\begin{aligned}
pq &= (1+i+k)(2j-3k) \\
&= 1(2j-3k) + i(2j-3k) + k(2j-3k) && \text{(distributive law)} \\
&= 2(1j) - 3(1k) + 2(ij) - 3(ik) + 2(kj) - 3(k^2) && \text{(distributive law)} \\
&= 2j - 3k + 2k - 3(-j) + 2(-i) - 3(-1) \\
&= 3(1) - 2i + 5j - k && \text{(collect terms)}
\end{aligned}$$

Definition A.13. *Let* $x = a + bi + cj + dk \in \mathbb{H}$ *for some* $a, b, c, d \in \mathbb{R}$. *The* quaternionic conjugate *of* x, *denoted by* $\bar{x}$, *is defined by*

$$\bar{x} = a - bi - cj - dk \in \mathbb{H}. \tag{A.14}$$

The norm *of* x, *denoted by* $||x||$, *is defined by*

$$||x|| = \sqrt{\bar{x}x} \geq 0. \tag{A.15}$$

Note that if the quaternionic norm is well-defined and $\bar{x}x = x\bar{x}$ then $\mathbb{H}$ is a division algebra. Indeed, given $0 \neq x \in \mathbb{H}$, $||x|| > 0$ and $x^{-1} = ||x||^{-2}\bar{x}$ is the inverse of x as shown below.

$$xx^{-1} = \frac{x\bar{x}}{||x||^2} = \frac{||x||^2}{||x||^2} = 1 = \frac{\bar{x}x}{||x||^2} = x^{-1}x \tag{A.16}$$

Hence, if $xy = 0$, then $x = 0$ or $y = x^{-1}xy = x^{-1}0 = 0$ and, thus, $\mathbb{H}$ ia a division algebra. Note that this approach requires that $\mathbb{H}$ be associative. We will see later that the Cayley-Dickson process does not assume associativity in $\mathbb{H}$ when showing that $\mathbb{H}$ is a division algebra.

A.3.2 Octonions

An excellent survey on the octonions can be found in [5] and [13]. Let us begin with a definition.

Definition A.17. *Let* $\mathbb{O}$ *be the* 8*-dimensional real algebra defined by*

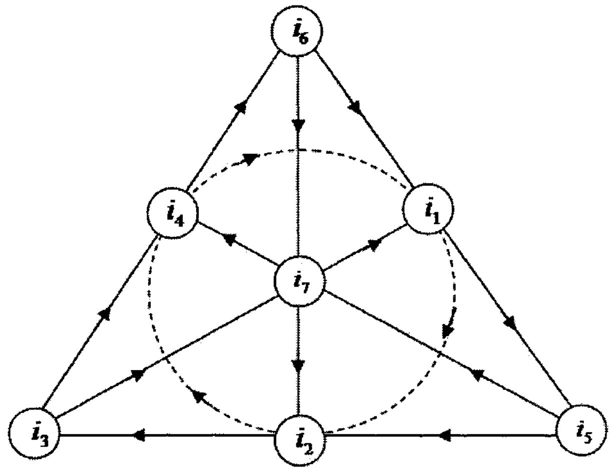
$$\mathbb{O} = span_{\mathbb{R}}\{1, i_1, i_2, i_3, i_4, i_5, i_6, i_7\}$$

with identity 1, $i_j^2 = -1$ *for* $j = 1, 2, \cdots, 7$*, and whose multiplication is polynomial subject to the relation* $i_j^2 = i_k^2 = i_l^2 = i_j i_k i_l = -1$ *if the* i_j, i_k, i_l*'s are cyclically ordered as in the edge oriented Fano plane below.*

One verifies that octonionic multiplication is non-associative by considering, for example, the associated products $i_1(i_2 i_3)$ and $(i_1 i_2)i_3$.

On one hand

$$i_1(i_2 i_3) = i_1 i_5 = i_6.$$



On the other hand

$$(i_1 i_2) i_3 = i_4 i_3 = -i_6.$$

In general, associativity of basic octonionic elements fails up to a sign.

As with the quaternions, we identify the real numbers as a subset of the octonions using the natural inclusion map $\lambda \hookrightarrow \lambda 1$ for all $\lambda \in \mathbb{R}$.

As an example, consider the associated octonionic products $s(tu)$ and $(st)u$, where $s = 1 + i_1$, $t = 2i_2 + 3i_3$, and $u = 4i_4 - 5i_5$.

On one hand

$$\begin{aligned}
s(tu) &= (1 + i_1)[(2i_2 + 3i_3)(4i_4 - 5i_5)] \\
&= (1 + i_1)[8(i_2 i_4) - 10(i_2 i_5) + 12(i_3 i_4) - 15(i_3 i_5)] \\
&= (1 + i_1)(8i_1 + 10i_3 + 12i_6 - 15i_2) \\
&= 8i_1 + 10i_3 + 12i_6 - 15i_2 + 8i_1^2 + 10(i_1 i_3) + 12(i_1 i_6) - 15(i_1 i_2) \\
&= -8 + 8i_1 - 15i_2 + 10i_3 - 15i_4 - 12i_5 + 12i_6 + 10i_7.
\end{aligned}$$

On the other hand

$$
\begin{aligned}
(st)u &= [(1+i_1)(2i_2+3i_3)](4i_4-5i_5) \\
&= [2i_2+3i_3+2(i_1i_2)+3(i_1i_3)](4i_4-5i_5) \\
&= [2i_2+3i_3+2i_4+3i_7](4i_4-5i_5) \\
&= 8(i_2i_4)-10(i_2i_5)+12(i_3i_4)-15(i_3i_5)-8-10(i_4i_5)+12(i_7i_4)-15(i_7i_5) \\
&= 8i_1+10i_3+12i_6-15i_2-8-10i_7+12i_5+15i_4 \\
&= -8+8i_1-15i_2+10i_3+15i_4+12i_5+12i_6-10i_7.
\end{aligned}
$$

For $x = a_0+\sum_{j=1}^{7} a_j i_j \in \mathbb{O}$, the *octonionic conjugate* of x, denoted by $\bar{x}$, is defined as

$$\bar{x} = a_0 - \sum_{j=1}^{7} a_j i_j \in \mathbb{O},$$

and the *norm* of x, denoted by $||x||$, is defined as $||x|| = \sqrt{\bar{x}x} \geq 0$.

Note that if the octonionic norm is well-defined and $\bar{x}x = x\bar{x}$ then $\mathbb{O}$ is a division algebra. Indeed, given $0 \neq x \in \mathbb{O}$, $||x|| > 0$ and $x^{-1} = ||x||^{-2}\bar{x}$ is the inverse of x as shown below.

$$xx^{-1} = \frac{x\bar{x}}{||x||^2} = \frac{||x||^2}{||x||^2} = 1 = \frac{\bar{x}x}{||x||^2} = x^{-1}x \tag{A.18}$$

Hence, if $xy = 0$, then $x = 0$ or $y = x^{-1}xy = x^{-1}0 = 0$ and, thus, $\mathbb{O}$ ia a division algebra. Note that this approach requires that $\mathbb{O}$ be alternative. We will see later that the Cayley-Dickson process does not assume alternativity in $\mathbb{O}$ when showing that $\mathbb{O}$ is a division algebra.

A.4 Cayley-Dickson Process

The Cayley-Dickson process for constructing families of algebras with conjugation explains why the complex numbers are commutative, but not real; the quaternions are associative, but not commutative; and the octonions are alternative, but not associative. It also explains why $\mathbb{R}, \mathbb{C}, \mathbb{H}$, and $\mathbb{O}$ are division algebras, yet no division algebras extend the octonions. The Cayley-Dickson process mimics the construction of complex numbers as pairs of real numbers.

Definition A.19. *Let A be an algebra over $\mathbb{R}$. Then A is said to be a* $\star$-algebra *if there exists a linear map called* conjugation $\star : A \longrightarrow A$ *(acting exponentially) such that*

$$x^{\star\star} = x, \quad (xy)^\star = y^\star x^\star, \quad \textit{for all } \ x, y \in A$$

We call a $\star$-algebra A real *if $x^\star = x$ for all $x \in A$, and* nicely-normed *if A is unital, $x + x^\star$ is real, and $x^\star x = xx^\star \geq 0$ for all nonzero $x \in A$.*

Note that both the real numbers $\mathbb{R}$ and the complex numbers $\mathbb{C}$ are $\star$-algebras under the usual complex conjugation $x^\star \mapsto \bar{x}$ for all $x \in \mathbb{C}$. Both $\star$-algebras are nicely-normed since both have an identity and

$$x + x^\star = x + \bar{x} = 2Re(x) \in \mathbb{R} \quad \text{and} \quad x^\star x = xx^\star = x\bar{x} = |x|^2 \geq 0$$

for all $x \in \mathbb{C}$. Moreover, $\mathbb{R}$ is real since $x^\star = \bar{x} = x$ for all $x \in \mathbb{R}$; yet $\mathbb{C}$ is not real since $i^\star = \bar{i} = -i \neq i$ and $i \in \mathbb{C}$.

Definition A.20. *Let A be a nicely-normed $\star$-algebra, and let $x \in A$. The* norm *of x, denoted by $||x||$, is defined by $||x|| = \sqrt{x^\star x} \geq 0$.*
If $x \neq 0$, the inverse *of x, denoted by x^{-1}, is defined by $x^{-1} = ||x||^{-2}x^\star$.*

From the above definitions, one verifies the following proposition.

Proposition A.21. *The norm and inverse as defined in Definition* **A.20** *are well-defined.*

Proof. Since the $\star$-algebra A is nicely-normed, $x^\star x \geq 0$ for all $x \in A$, with equality if and only if $x = 0$. Thus, $\sqrt{x^\star x} \geq 0$ exists and the norm is well-defined for all $x \in A$. If $x \neq 0$, it follows that

$$xx^{-1} = \frac{x^\star x}{||x||^2} = \frac{||x||^2}{||x||^2} = 1 = \frac{||x||^2}{||x||^2} = \frac{xx^\star}{||x||^2} = x^{-1}x,$$

where x and $x^\star$ commute since A is nicely-normed. Therefore x^{-1} is the unique (two-sided) inverse of x and well-defined for all non-zero $x \in A$. □

Corollary A.22. *Let A be a nicely-normed $\star$-algebra. If A is alternative, then A is a division algebra.*

Proof. Let $xy = 0$ for some $x, y \in A$, and suppose that $x \neq 0$. To prove that A is a division algebra, we must show that $y = 0$. Because A is alternative, by the Moufang identity of Lemma **A.6**,

$$yx^{-1} = 1(yx^{-1}) = (x^{-1}x)(yx^{-1}) = x^{-1}(xy)x^{-1} = x^{-1}0x^{-1} = 0,$$

where $x^{-1} = ||x||^{-2}x^\star$ is well-defined by Proposition **A.21**. Hence $yx^{-1} = y||x||^{-2}x^\star =$

0 implies $yx^\star = 0$, which in turn implies $xy^\star = (yx^\star)^\star = 0^\star = 0$. Thus,

$$x(y + y^\star) = xy + xy^\star = 0 + 0 = 0. \tag{A.23}$$

Since A is nicely-normed, $y + y^\star \in \mathbb{R}$; but $x \neq 0$, so $y + y^\star = 0$ by Equation (A.23), or equivalently, $y = y^\star$. Therefore, again since A is alternative,

$$-||y||^2 x = x(-yy^\star) = x(yy) = (xy)y = 0y = 0. \tag{A.24}$$

We conclude that $||y|| = 0$ which occurs if and only if $y = 0$. □

Definition A.25. *Let A be a $\star$-algebra. The* Cayley-Dickson extension *of A, denoted by A', is the $\star$-algebra $A \times A$ satisfying*

- *addition:* $(a, b) + (c, d) = (a + c, b + d)$
- *scalar product:* $\lambda(a, b) = (\lambda a, \lambda b)$
- *multiplication:* $(a, b)(c, d) = (ac - db^\star, a^\star d + cb)$
- *conjugation:* $(a, b)^\star = (a^\star, -b)$

for all $a, b, c, d \in A$ and $\lambda \in \mathbb{R}$.

Note that, up to isomorphism, $\mathbb{R}' = \mathbb{C}$, $\mathbb{C}' = \mathbb{H}$, and $\mathbb{H}' = \mathbb{O}$.

Clearly $\mathbb{R}' = \mathbb{C}$ by setting $(0, 1) = i$. Since $\mathbb{R}$ is real, $x^\star = x$ for all $x \in \mathbb{R}$; hence, the relations for multiplication and conjugation in $\mathbb{R}'$ satisfy

$$(a, b)(c, d) = (ac - db, ad + cb) \quad \text{and} \quad (a, b)^\star = (a, -b)$$

for all $a, b, c, d \in \mathbb{R}$ where in $\mathbb{C}$ these satisfy

$$(a + bi)(c + di) = (ac - db) + (ad + cb)i \quad \text{and} \quad \overline{a + bi} = a - bi.$$

Similarly, one verifies that, up to isomorphism, $\mathbb{C}' = \mathbb{H}$ and $\mathbb{H}' = \mathbb{O}$ by considering the identifications

$$i = (i, 0), \quad j = (0, 1), \quad \text{and} \quad k = (0, -i)$$

for the quaternions, and

$$i_1 = (i, 0), \quad i_2 = (j, 0), \quad i_3 = (0, 1), \quad i_4 = (k, 0),$$

$$i_5 = (0, -j), \quad i_6 = (0, k), \quad \text{and} \quad i_7 = (0, -i)$$

for the octonions.

Theorem A.26. *(Properties of Extensions). Let A be a $\star$-algebra. Then*

(1) A' is never real (unless trivially $A = 0$).

(2) A is real (and thus commutative) $\Leftrightarrow$ A' is commutative.

(3) A is commutative and associative $\Leftrightarrow$ A' is associative.

(4) A is associative and nicely-normed $\Leftrightarrow$ A' is alternative and nicely-normed.

(5) A is nicely-normed $\Leftrightarrow$ A' is nicely-normed.

Proof. For (1), choose $b \in A$ such that $b \neq 0$. Then $(0, b) \in A'$. But $(0, b)^\star = (0, -b) = -(0, b) \neq (0, b)$. Thus, A' is not real.

For (2), suppose first that A is real. Then A is also commutative. Hence, A' is commutative, since for any $(a, b), (c, d) \in A$

$$\begin{aligned}(a,b)(c,d) &= (ac - db^\star, a^\star d + cb)\\ &= (ca - bd^\star, d^\star a + bc)\\ &= (c,d)(a,b).\end{aligned}$$

Conversely, suppose that A' is commutative and let $a \in A$. Then $(a^\star, 0) = (0, a)(0, -1) = (0, -1)(0, a) = (a, 0)$. Hence, $a^\star = a$ for all $a \in A$ and A is real.

For (3), suppose that A is commutative and associative and let $(a, b), (c, d), (e, f) \in A'$. Then

$$\begin{aligned}&(a,b)[(c,d)(e,f)]\\ &\quad= (a,b)(ce - fd^\star,\ \ c^\star f + ed)\\ &\quad= (a[ce - fd^\star] - [c^\star f + ed]b^\star,\ \ a^\star[c^\star f + ed] + [ce - fd^\star]b)\\ &\quad= (ace - afd^\star - c^\star fb^\star - edb^\star,\ \ a^\star c^\star f + a^\star ed + ceb - fd^\star b)\\ &\quad= (ace - db^\star e - fd^\star a - fb^\star c^\star,\ \ c^\star a^\star f - bd^\star f + ea^\star d + ecb)\\ &\quad= ([ac - db^\star]e - f[a^\star d + cb]^\star,\ \ [ac - db^\star]^\star f + e[a^\star d + cb])\\ &\quad= (ac - db^\star,\ \ a^\star d + cb)(e,f)\\ &\quad= [(a,b)(c,d)](e,f)\end{aligned}$$

which shows that A' is associative. Conversely, suppose that A' is associative and let

$a, b, c \in A$. Then A is commutative since

$$(0, ab) = (a^\star, 0)(0, b) = [(0, a)(0, -1)](0, b)$$
$$= (0, a)[(0, -1)(0, b)] = (0, a)(b, 0) = (0, ba).$$

Also A is associative since

$$(a(bc), \quad 0) = (a, 0)(bc, 0) = (a, 0)[(b, 0)(c, 0)]$$
$$= [(a, 0)(b, 0)](c, 0) = (ab, 0)(c, 0) = ((ab)c, 0).$$

For (5), suppose that first A is nicely-normed. Let $(a, b) \in A'$. Then $(a, b) + (a, b)^\star = (a, b) + (a^\star, -b) = (a + a^\star, 0) \in \mathbb{R}$. Also, if $a \neq 0$ or $b \neq 0$, then

$$\begin{aligned}
(a, b)(a, b)^\star &= (a, b)(a^\star, -b) \\
&= (aa^\star + bb^\star, \quad -a^\star b + a^\star b) \\
&= (aa^\star, 0) = (aa^\star, 0) + (bb^\star, 0) > 0 \\
&= (a^\star a + b^\star b, \quad ab - ab) \\
&= (a^\star, -b)(a, b) = (a, b)^\star (a, b).
\end{aligned}$$

Hence, A' is nicely-normed. Conversely, assume A' is nicely-normed and let $a \in A$. Then $a + a^\star \equiv (a, 0) + (a^\star, 0) = (a, 0) + (a, 0)^\star \in \mathbb{R}$. Similarly, if also $a \neq 0$, then $aa^\star \equiv (a, 0)(a^\star, 0) = (a, 0)(a, 0)^\star > 0$ and $a^\star a \equiv (a^\star, 0)(a, 0) = (a, 0)^\star (a, 0)$. Thus, A is nicely-normed.

For (4), suppose first that A is associative and nicely-normed. Then A' is nicely-normed by (5). It is also straightforward to check that A' is alternative. Conversely, now suppose that A' is alternative and nicely-normed. Then A is nicely-normed by (5). It remains to show that A is associative. For this, let $a, b, c \in A$, then one verifies that $a(bc) = (ab)c$. □

It follows from Theorem **A.26** that:

$$\begin{gathered}\mathbb{R} \text{ is a real commutative associative nicely-normed } \star\text{-algebra} \Rightarrow \\ \mathbb{C} \text{ is a commutative associative nicely-normed } \star\text{-algebra} \Rightarrow \\ \mathbb{H} \text{ is an associative nicely-normed } \star\text{-algebra} \Rightarrow \\ \mathbb{O} \text{ is an alternative nicely-normed } \star\text{-algebra}\end{gathered}$$

and therefore $\mathbb{R}, \mathbb{C}, \mathbb{H}$, and $\mathbb{O}$ are division algebras. It also follows that the octonions are neither real, nor commutative, nor associative.

A.5 Remarks

Given any nonzero $\star$-algebra A, the Cayley-Dickson extension A' is clearly a $\star$-algebra with twice the dimension of A. Hence, with the initial input $A = \mathbb{R}$, the Cayley-Dickson extensions inductively yields a nested sequence of real division algebras with conjugation beginning with the 2-dimensional complex numbers, the 4-dimensional quaternions, and the 8-dimensional octonions. Yet, as illustrated by Theorem **A.26**, each extension loses a property of its predecessor: the complex numbers are not real, the quaternions are not commutative, the octonions are not associative, and the *sedonions* $\mathbb{S}$ are not alternative. The sedonions $\mathbb{S}$ are not a division algebra because they have zero-divisors. However, sedonions are power-associative and non-zero sedonions which are not zero-divisors have inverses. Therefore, only the first four algebras in the sequence $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}, \mathbb{S}, \cdots$ are division algebras. This fact is a special case of Theorem **A.1**. See [28] and [35] for further information and references on the sedonions.

Appendix B

PROOF OF COROLLARIES 4.108 AND 4.114

In this appendix, we present the proof of Corollary 4.108. The proof of Corollary 4.114 follows directly from the proof of Corollary 4.108.

B.1 Proof of Corollary 4.108

Define the octonions a_+, a_-, b_+, and b_- as

$$a_+ = \frac{1}{2}\left[(s_{10} + s_{01})t_{10}\right]u_{01}; \quad a_- = \frac{1}{2}\left[(s_{10} - s_{01})t_{10}\right]u_{01};$$

$$b_+ = \frac{1}{2}\left[(s_{10} + s_{01})t_{00}\right]u_{00}; \quad b_- = \frac{1}{2}\left[(s_{10} - s_{01})t_{00}\right]u_{00}.$$

Recall that for an octonion x, $\pi_j(x)$ denotes the projection of the octonion x onto the subspace of $\mathbb{O}$ spanned by the vector basis element i_j, where we set $i_0 = 1$. Now let

$$\begin{aligned} o = & \sqrt{\pi_0^2(a_+) + \pi_0^2(a_-)} \cdot 1 + \sqrt{\pi_1^2(a_+) + \pi_1^2(a_-)} \cdot i_1 \\ & + \sqrt{\pi_2^2(b_+) + \pi_2^2(b_-)} \cdot i_2 + \sqrt{\pi_3^2(a_+) + \pi_3^2(a_-)} \cdot i_3 \\ & + \sqrt{\pi_4^2(b_+) + \pi_4^2(b_-)} \cdot i_4 + \sqrt{\pi_5^2(b_+) + \pi_5^2(b_-)} \cdot i_5 \\ & + \sqrt{\pi_6^2(b_+) + \pi_6^2(b_-)} \cdot i_6 + \sqrt{\pi_7^2(a_+) + \pi_7^2(a_-)} \cdot i_7, \end{aligned} \tag{B.1}$$

or in compact form

$$o = \sum_{j=0,1,3,7} \sqrt{\pi_j^2(a_+) + \pi_j^2(a_-)} \; i_j + \sum_{k=2,4,5,6} \sqrt{\pi_k^2(b_+) + \pi_k^2(b_-)} \; i_k \tag{B.2}$$

Then Theorem 4.32 says that if players one, two, and three employ the pure quantum strategies represented by the unit octonions s, t, and u, respectively, then the payoff to the players is given by

$$G^{\mathcal{Q}_{\mathcal{I}}}(s, t, u) = \sum_{j=0}^{7} [\pi_j(o)]^2 (X_j, Y_j, Z_j), \tag{B.3}$$

where o is the octonion given in (B.1) and X_j, Yj, Z_j are taken from the tables of Figure 4.1. Now Corollary 4.108 says that if the strategic profile (s, t, u) is of the form $(s_0 + s_1 i_1 + s_2 i_2 + s_3 i_4,\ i_l,\ i_m)$, $(i_k,\ t_0 + t_1 i_1 + t_2 i_5 + t_3 i_6,\ i_m)$, or $(i_k,\ i_l,\ u_0 + u_1 i_1 + u_2 i_3 + u_3 i_7)$, where $k = 0, 1, 2, 4; l = 0, 1, 4, 5; m = 0, 1, 3, 7$, then the payoff to the players is given by

$$G^{\mathcal{Q}_{\mathcal{I}}}(s,t,u) = \sum_{j=0}^{7}[\pi_j((st)u)]^2(X_j, Y_j, Z_j), \tag{B.4}$$

or equivalently

$$[\pi_j(o)]^2 = [\pi_j((st)u)]^2 \tag{B.5}$$

for all $j = 0, 1, \cdots, 7$. Therefore, proving Corollary 4.108 amounts to establishing the equality (B.5) for all $j = 0, 1, \cdots, 7$. For this, we need to consider three cases and for each case, 16 sub-cases. We go over Case 1, the others follow symmetrically.

<u>**Case 1**</u>: Suppose player one employs the pure quantum strategy represented by the unit octonion $s = s_{00} = s_0 + s_1 i_1 + s_2 i_2 + s_3 i_4$ and players two and three employ pure quantum strategies represented by unit octonions of the form $t = t_{00} = i_l,\ l = 0, 1, 5, 6$ and $u = u_{00} = i_m,\ m = 0, 1, 3, 7$, respectively. Then $s_{10} = -s_0 + s_1 i_1 + s_2 i_2 + s_3 i_4$ and $s_{01} = s_0 - s_1 i_1 + s_2 i_2 + s_3 i_4$ which imply in turn that $a_+ = [(s_2 i_2 + s_3 i_4) i_l] i_m$ and $a_- = [(-s_0 + s_1 i_1) i_l] i_m$.

Sub-case 1 : $i_l = 1$ and $i_m = 1$. Then $t_{10} = -1$ and $u_{01} = 1$. Hence

$$a_+ = [(s_2 i_2 + s_3 i_4)(-1)]1 = -s_2 i_2 - s_3 i_4$$

$$a_- = [(-s_0 + s_1 i_1)(-1)]1 = s_0 - s_1 i_1$$

$$b_+ = [(s_2 i_2 + s_3 i_4)1]1 = s_2 i_2 + s - 3 i_4$$

$$b_- = [(-s_0 + s_1 i_1)1]1 = -s_0 + s_1 i_1$$

So

$$o = \sqrt{s_0^2}\cdot 1 + \sqrt{s_1^2}\cdot i_1 + \sqrt{s_2^2}\cdot i_2 + \sqrt{s_3^2}\cdot i_4$$

and

$$(st)u = s_0 \cdot 1 + s_1 \cdot i_1 + s_2 \cdot i_2 + s_3 \cdot i_4.$$

Therefore, $[\pi_j(o)]^2 = [\pi_j((st)u)]^2$ for all $j = 0, 1, \cdots, 7$ as was to be shown.

Sub-case 2 : $i_l = 1$ and $i_m = i_1$. Then $t_{10} = -1$ and $u_{01} = -i_1$. Hence

$$a_+ = [(s_2 i_2 + s_3 i_4)(-1)](-i_1) = -s_2 i_4 - s_3 i_2$$

$$a_- = [(-s_0 + s_1 i_1)(-1)](-i_1) = -s_0 i_1 - s_1$$

$$b_+ = [(s_2 i_2 + s_3 i_4)1]i_1 = -s_2 i_4 + s_3 i_2$$

$$b_- = [(-s_0 + s_1 i_1)1]i_1 = -s_0 i_1 - s_1$$

So

$$o = \sqrt{s_1^2} \cdot 1 + \sqrt{s_0^2} \cdot i_1 + \sqrt{s_3^2} \cdot i_2 + \sqrt{s_2^2} \cdot i_4$$

and

$$(st)u = (s1)i_1 = -s_1 \cdot 1 + s_0 \cdot i_1 + s_3 \cdot i_2 - s_2 \cdot i_4.$$

Therefore, $[\pi_j(o)]^2 = [\pi_j((st)u)]^2$ for all $j = 0, 1, \cdots, 7$ as was to be shown.

Sub-case 3 : $i_l = 1$ and $i_m = i_3$. Then $t_{10} = -1$ and $u_{01} = i_3$. Hence

$$a_+ = [(s_2 i_2 + s_3 i_4)(-1)]i_3 = -s_2 i_5 + s_3 i_6$$

$$a_- = [(-s_0 + s_1 i_1)(-1)]i_3 = s_0 i_3 - s_1 i_7$$

$$b_+ = [(s_2 i_2 + s_3 i_4)1]i_3 = s_2 i_5 - s_3 i_6$$

$$b_- = [(-s_0 + s_1 i_1)1]i_3 = -s_0 i_3 + s_1 i_7$$

So

$$o = \sqrt{s_0^2} \cdot i_3 + \sqrt{s_2^2} \cdot i_5 + \sqrt{s_3^2} \cdot i_6 + \sqrt{s_1^2} \cdot i_7$$

and

$$(st)u = (s1)i_3 = s_0 \cdot i_3 + s_2 \cdot i_1 - s_3 \cdot i_2 - s_1 \cdot i_7.$$

Therefore, $[\pi_j(o)]^2 = [\pi_j((st)u)]^2$ for all $j = 0, 1, \cdots, 7$ as was to be shown.

Sub-case 4 : $i_l = 1$ and $i_m = i_7$. Then $t_{10} = -1$ and $u_{01} = i_7$. Hence

$$\begin{aligned} a_+ &= [(s_2 i_2 + s_3 i_4)(-1)]i_7 = s_2 i_6 + s_3 i_5 \\ a_- &= [(-s_0 + s_1 i_1)(-1)]i_7 = s_0 i_7 + s_1 i_3 \\ b_+ &= [(s_2 i_2 + s_3 i_4)1]i_7 = -s_2 i_6 - s_3 i_5 \\ b_- &= [(-s_0 + s_1 i_1)1]i_7 = -s_0 i_7 - s_1 i_3 \end{aligned}$$

So

$$o = \sqrt{s_1^2} \cdot i_3 + \sqrt{s_3^2} \cdot i_5 + \sqrt{s_2^2} \cdot i_6 + \sqrt{s_0^2} \cdot i_7$$

and

$$(st)u = (s1)i_7 = -s_1 \cdot i_3 - s_3 \cdot i_5 - s_2 \cdot i_6 - s_0 \cdot i_7.$$

Therefore, $[\pi_j(o)]^2 = [\pi_j((st)u)]^2$ for all $j = 0, 1, \cdots, 7$ as was to be shown.

Sub-case 5 : $i_l = i_1$ and $i_m = 1$. Then $t_{10} = i_1$ and $u_{01} = 1$. Hence

$$a_+ = [(s_2 i_2 + s_3 i_4) i_1] 1 = -s_2 i_4 + s_3 i_2$$
$$a_- = [(-s_0 + s_1 i_1) i_1] 1 = -s_0 i_1 - s_1$$
$$b_+ = [(s_2 i_2 + s_3 i_4) i_1] 1 = -s_2 i_4 + s_3 i_2$$
$$b_- = [(-s_0 + s_1 i_1) i_1] 1 = -s_0 i_1 - s_1$$

So

$$o = \sqrt{s_1^2} \cdot 1 + \sqrt{s_0^2} \cdot i_1 + \sqrt{s_3^2} \cdot i_2 + \sqrt{s_2^2} \cdot i_4$$

and

$$(st)u = (s i_1) 1 = -s_1 \cdot 1 + s_0 \cdot i_1 + s_3 \cdot i_2 - s_2 \cdot i_7.$$

Therefore, $[\pi_j(o)]^2 = [\pi_j((st)u)]^2$ for all $j = 0, 1, \cdots, 7$ as was to be shown.

Sub-case 6 : $i_l = i_1$ and $i_m = i_1$. Then $t_{10} = i_1$ and $u_{01} = -i_1$. Hence

$$a_+ = [(s_2 i_2 + s_3 i_4) i_1](-i_1) = [-s_2 i_4 + s_3 i_2](-i_1) = s_2 i_2 + s_3 i_4$$
$$a_- = [(-s_0 + s_1 i_1) i_1](-i_1) = [-s_0 i_1 - s_1](-i_1) = -s_0 + s_1 i_1$$
$$b_+ = [(s_2 i_2 + s_3 i_4) i_1] i_1 = [-s_2 i_4 + s_3 i_2] i_1 = -s_2 i_2 - s_3 i_4$$
$$b_- = [(-s_0 + s_1 i_1) i_1] i_1 = [-s_0 i_1 - s_1] i_1 = s_0 - s_1 i_1$$

So

$$o = \sqrt{s_0^2} \cdot 1 + \sqrt{s_1^2} \cdot i_1 + \sqrt{s_2^2} \cdot i_2 + \sqrt{s_3^2} \cdot i_4$$

and

$$(st)u = (si_1)i_1 = [-s_1 + s_0 i_1 + s_3 i_2 - s_2 i_4]i_1 = -s_0 \cdot 1 - s_1 \cdot i_1 - s_2 \cdot i_2 - s_3 \cdot i_4$$

Therefore, $[\pi_j(o)]^2 = [\pi_j((st)u)]^2$ for all $j = 0, 1, \cdots, 7$ as was to be shown.

Sub-case 7: $i_l = i_1$ and $i_m = i_3$. Then $t_{10} = i_1$ and $u_{01} = i_3$. Hence

$$a_+ = [(s_2 i_2 + s_3 i_4)i_1](-i_1) = [-s_2 i_4 + s_3 i_2]i_3 = s_3 i_5 + s_2 i_6$$

$$a_- = [(-s_0 + s_1 i_1)i_1](-i_1) = [-s_0 i_1 - s_1]i_3 = -s_1 i_3 - s_0 i_7$$

$$b_+ = [(s_2 i_2 + s_3 i_4)i_1]i_1 = [-s_2 i_4 + s_3 i_2]i_3 = s_3 i_5 + s_2 i_6$$

$$b_- = [(-s_0 + s_1 i_1)i_1]i_1 = [-s_0 i_1 - s_1]i_3 = -s_1 i_3 - s_0 i_7$$

So

$$o = \sqrt{s_1^2} \cdot i_3 + \sqrt{s_3^2} \cdot i_5 + \sqrt{s_2^2} \cdot i_6 + \sqrt{s_0^2} \cdot i_7$$

and

$$(st)u = (si_1)i_3 = [-s_1 + s_0 i_1 + s_3 i_2 - s_2 i_4]i_3 = -s_1 \cdot i_3 + s_3 \cdot i_5 + s_2 \cdot i_5 + s_0 \cdot i_7$$

Therefore, $[\pi_j(o)]^2 = [\pi_j((st)u)]^2$ for all $j = 0, 1, \cdots, 7$ as was to be shown.

Sub-case 8: $i_l = i_1$ and $i_m = i_7$. Then $t_{10} = i_1$ and $u_{01} = i_7$. Hence

$$a_+ = [(s_2 i_2 + s_3 i_4) i_1] i_7 = [-s_2 i_4 + s_3 i_2] i_7 = s_2 i_5 - s_3 i_6$$

$$a_- = [(-s_0 + s_1 i_1) i_1] i_7 = [-s_0 i_1 - s_1] i_7 = +s_0 i_3 - s_1 i_7$$

$$b_+ = [(s_2 i_2 + s_3 i_4) i_1] i_7 = [-s_2 i_4 + s_3 i_2] i_7 = s_2 i_5 - s_3 i_6$$

$$b_- = [(-s_0 + s_1 i_1) i_1] i_7 = [-s_0 i_1 - s_1] i_7 = s_0 i_3 - s_1 i_7$$

So

$$o = \sqrt{s_0^2} \cdot i_3 + \sqrt{s_2^2} \cdot i_5 + \sqrt{s_3^2} \cdot i_6 + \sqrt{s_1^2} \cdot i_7$$

and

$$(st)u = (si_1)i_7 = [-s_1 + s_0 i_1 + s_3 i_2 - s_2 i_4] i_7 = -s_0 \cdot i_3 + s_2 \cdot i_5 - s_3 \cdot i_6 - s_1 \cdot i_7$$

Therefore, $[\pi_j(o)]^2 = [\pi_j((st)u)]^2$ for all $j = 0, 1, \cdots, 7$ as was to be shown.

Sub-case 9: $i_l = i_5$ and $i_m = 1$. Then $t_{10} = i_5$ and $u_{01} = 1$. Hence

$$a_+ = [(s_2 i_2 + s_3 i_4) i_5] 1 = -s_2 i_3 + s_3 i_7$$

$$a_- = [(-s_0 + s_1 i_1) i_5] 1 = -s_0 i_5 + s_1 i_6$$

$$b_+ = [(s_2 i_2 + s_3 i_4) i_5] 1 = -s_2 i_3 + s_3 i_7$$

$$b_- = [(-s_0 + s_1 i_1) i_5] 1 = -s_0 i_5 + s_1 i_6$$

So

$$o = \sqrt{s_2^2} \cdot i_3 + \sqrt{s_0^2} \cdot i_5 + \sqrt{s_1^2} \cdot i_6 + \sqrt{s_3^2} \cdot i_7$$

and

$$(st)u = (si_5)1 = -s_2 \cdot i_3 - s_0 \cdot i_5 + s_1 \cdot i_6 + s_3 \cdot i_7$$

Therefore, $[\pi_j(o)]^2 = [\pi_j((st)u)]^2$ for all $j = 0, 1, \cdots, 7$ as was to be shown.

Sub-case 10: $i_l = i_5$ and $i_m = i_1$. Then $t_{10} = i_5$ and $u_{01} = -i_1$. Hence

$$a_+ = [(s_2 i_2 + s_3 i_4) i_5](-i_1) = [-s_2 i_3 + s_3 i_7](-i_1) = -s_3 i_3 - s_2 i_7$$

$$a_- = [(-s_0 + s_1 i_1) i_5](-i_1) = [-s_0 i_5 + s_1 i_6](-i_1) = -s_1 i_5 - s_0 i_6$$

$$b_+ = [(s_2 i_2 + s_3 i_4) i_5] i_1 = [-s_2 i_3 + s_3 i_7] i_1 = s_3 i_3 + s_2 i_7$$

$$b_- = [(-s_0 + s_1 i_1) i_5] i_1 = [-s_0 i_5 + s_1 i_6] i_1 = s_1 i_5 + s_0 i_6$$

So

$$o = \sqrt{s_3^2} \cdot i_3 + \sqrt{s_1^2} \cdot i_5 + \sqrt{s_0^2} \cdot i_6 + \sqrt{s_2^2} \cdot i_7$$

and

$$(st)u = (si_5)i_1 = [-s_2 i_3 + s_0 i_5 + s_1 i_6 + s_3 i_7] i_1 = s_3 \cdot i_3 + s_1 \cdot i_5 - s_0 \cdot i_6 + s_2 \cdot i_7$$

Therefore, $[\pi_j(o)]^2 = [\pi_j((st)u)]^2$ for all $j = 0, 1, \cdots, 7$ as was to be shown.

Sub-case 11: $i_l = i_5$ and $i_m = i_3$. Then $t_{10} = i_5$ and $u_{01} = i_3$. Hence

$$a_+ = [(s_2 i_2 + s_3 i_4) i_5] i_3 = [-s_2 i_3 + s_3 i_7] i_3 = s_2 - s_3 i_1$$

$$a_- = [(-s_0 + s_1 i_1) i_5] i_3 = [-s_0 i_5 + s_1 i_6] i_3 = s_1 i_4 + s_0 i_2$$

$$b_+ = [(s_2 i_2 + s_3 i_4) i_5] i_3 = [-s_2 i_3 + s_3 i_7] i_3 = -s_2 - s_3 i_1$$

$$b_- = [(-s_0 + s_1 i_1) i_5] i_3 = [-s_0 i_5 + s_1 i_6] i_3 = s_1 i_4 + s_0 i_2$$

So

$$o = \sqrt{s_2^2} \cdot 1 + \sqrt{s_3^2} \cdot i_1 + \sqrt{s_0^2} \cdot i_2 + \sqrt{s_1^2} \cdot i_4$$

and

$$(st)u = (s i_5) i_3 = [-s_2 i_3 + s_0 i_5 + s_1 i_6 + s_3 i_7] i_3 = -s_2 \cdot 1 - s_3 \cdot i_1 - s_0 \cdot i_2 + s_1 \cdot i_4$$

Therefore, $[\pi_j(o)]^2 = [\pi_j((st)u)]^2$ for all $j = 0, 1, \cdots, 7$ as was to be shown.

Sub-case 12: $i_l = i_5$ and $i_m = i_7$. Then $t_{10} = i_5$ and $u_{01} = i_7$. Hence

$$a_+ = [(s_2 i_2 + s_3 i_4) i_5] i_7 = [-s_2 i_3 + s_3 i_7] i_7 = -s_3 - s_2 i_1$$

$$a_- = [(-s_0 + s_1 i_1) i_5] i_7 = [-s_0 i_5 + s_1 i_6] i_7 = s_1 i_2 - s_0 i_4$$

$$b_+ = [(s_2 i_2 + s_3 i_4) i_5] i_7 = [-s_2 i_3 + s_3 i_7] i_7 = -s_3 - s_2 i_1$$

$$b_- = [(-s_0 + s_1 i_1) i_5] i_7 = [-s_0 i_5 + s_1 i_6] i_7 = s_1 i_2 - s_0 i_4$$

So

$$o = \sqrt{s_3^2} \cdot 1 + \sqrt{s_2^2} \cdot i_1 + \sqrt{s_1^2} \cdot i_2 + \sqrt{s_0^2} \cdot i_4$$

and

$$(st)u = (si_5)i_7 = [-s_2i_3 + s_0i_5 + s_1i_6 + s_3i_7]i_7 = -s_3 \cdot 1 - s_2 \cdot i_1 + s_1 \cdot i_2 + s_0 \cdot i_4$$

Therefore, $[\pi_j(o)]^2 = [\pi_j((st)u)]^2$ for all $j = 0, 1, \cdots, 7$ as was to be shown.

Sub-case 13: $i_l = i_6$ and $i_m = 1$. Then $t_{10} = i_6$ and $u_{01} = 1$. Hence

$$a_+ = [(s_2i_2 + s_3i_4)i_6]1 = s_3i_3 + s_2i_7$$

$$a_- = [(-s_0 + s_1i_1)i_6]1 = -s_1i_5 - s_0i_6$$

$$b_+ = [(s_2i_2 + s_3i_4)i_6]1 = s_3i_3 + s_2i_7$$

$$b_- = [(-s_0 + s_1i_1)i_6]1 = -s_1i_5 - s_0i_6$$

So

$$o = \sqrt{s_3^2} \cdot i_3 + \sqrt{s_1^2} \cdot i_5 + \sqrt{s_0^2} \cdot i_6 + \sqrt{s_2^2} \cdot i_7$$

and

$$(st)u = (si_6)1 = [s_0 + s_1i_1 + s_2i_2 + s_3i_4]i_6 = s_3 \cdot i_3 - s_1 \cdot i_5 + s_0 \cdot i_6 + s_2 \cdot i_7$$

Therefore, $[\pi_j(o)]^2 = [\pi_j((st)u)]^2$ for all $j = 0, 1, \cdots, 7$ as was to be shown.

Sub-case 14: $i_l = i_6$ and $i_m = i_1$. Then $t_{10} = i_6$ and $u_{01} = -i_1$. Hence

$$a_+ = [(s_2 i_2 + s_3 i_4) i_6](-i_1) = [s_3 i_3 + s_2 i_7](-i_1) = -s_2 i_3 + s_3 i_7$$

$$a_- = [(-s_0 + s_1 i_1) i_6](-i_1) = [-s_1 i_5 - s_0 i_6](-i_1) = s_0 i_5 - s_1 i_6$$

$$b_+ = [(s_2 i_2 + s_3 i_4) i_6] i_1 = [s_3 i_3 + s_2 i_7] i_1 = s_2 i_3 - s_3 i_7$$

$$b_- = [(-s_0 + s_1 i_1) i_6] i_1 = [-s_1 i_5 - s_0 i_6] i_1 = -s_0 i_5 + s_1 i_6$$

So

$$o = \sqrt{s_2^2} \cdot i_3 + \sqrt{s_0^2} \cdot i_5 + \sqrt{s_1^2} \cdot i_6 + \sqrt{s_3^2} \cdot i_7$$

and

$$(st)u = (s i_6) i_1 = [(s_0 + s_1 i_1 + s_2 i_2 + s_3 i_4) i_6] i_1 = s_2 \cdot i_3 + s_0 \cdot i_5 + s_1 \cdot i_6 - s_3 \cdot i_7$$

Therefore, $[\pi_j(o)]^2 = [\pi_j((st)u)]^2$ for all $j = 0, 1, \cdots, 7$ as was to be shown.

Sub-case 15: $i_l = i_6$ and $i_m = i_3$. Then $t_{10} = i_6$ and $u_{01} = i_3$. Hence

$$a_+ = [(s_2 i_2 + s_3 i_4) i_6] i_3 = [s_3 i_3 + s_2 i_7] i_3 = -s_3 - s_2 i_1$$

$$a_- = [(-s_0 + s_1 i_1) i_6] i_3 = [-s_1 i_5 - s_0 i_6] i_3 = s_1 i_2 - s_0 i_4$$

$$b_+ = [(s_2 i_2 + s_3 i_4) i_6] i_3 = [s_3 i_3 + s_2 i_7] i_3 = -s_3 - s_2 i_1$$

$$b_- = [(-s_0 + s_1 i_1) i_6] i_3 = [-s_1 i_5 - s_0 i_6] i_3 = s_1 i_2 - s_0 i_4$$

So

$$o = \sqrt{s_3^2} \cdot 1 + \sqrt{s_2^2} \cdot i_1 + \sqrt{s_1^2} \cdot i_2 + \sqrt{s_0^2} \cdot i_4$$

and

$$(st)u = (si_6)i_3 = [(s_0 + s_1 i_1 + s_2 i_2 + s_3 i_4)i_6]i_3 = -s_3 \cdot 1 - s_2 \cdot i_1 + s_1 \cdot i_2 + s_0 \cdot i_4$$

Therefore, $[\pi_j(o)]^2 = [\pi_j((st)u)]^2$ for all $j = 0, 1, \cdots, 7$ as was to be shown.

Sub-case 16: $i_l = i_6$ and $i_m = i_7$. Then $t_{10} = i_6$ and $u_{01} = i_7$. Hence

$$a_+ = [(s_2 i_2 + s_3 i_4)i_6]i_7 = [s_3 i_3 + s_2 i_7]i_7 = -s_2 + s_3 i_1$$
$$a_- = [(-s_0 + s_1 i_1)i_6]i_7 = [-s_1 i_5 - s_0 i_6]i_7 = -s_0 i_2 - s_1 i_4$$
$$b_+ = [(s_2 i_2 + s_3 i_4)i_6]i_7 = [s_3 i_3 + s_2 i_7]i_7 = -s_2 + s_3 i_1$$
$$b_- = [(-s_0 + s_1 i_1)i_6]i_7 = [-s_1 i_5 - s_0 i_6]i_7 = -s_0 i_2 - s_1 i_4$$

So

$$o = \sqrt{s_2^2} \cdot 1 + \sqrt{s_3^2} \cdot i_1 + \sqrt{s_0^2} \cdot i_2 + \sqrt{s_1^2} \cdot i_4$$

and

$$(st)u = (si_6)i_7 = [(s_0 + s_1 i_1 + s_2 i_2 + s_3 i_4)i_6]i_7 = -s_2 \cdot 1 + s_3 \cdot i_1 + s_0 \cdot i_2 - s_1 \cdot i_4$$

Therefore, $[\pi_j(o)]^2 = [\pi_j((st)u)]^2$ for all $j = 0, 1, \cdots, 7$ as was to be shown.

Case 2: Suppose player one employs a pure quantum strategy represented by $s = s_{00} = i_k,\ k = 0, 1, 2, 4$, player two employs a pure quantum strategy represented by the unit octonion $t = t_{00} = t_0 + t_1 i_1 + t_2 i_5 + t_3 i_6$, and player three employs a unit octonion represented by $u = u_{00} = i_m,\ m = 0, 1, 3, 7$. Then one verifies that $[\pi_j(o)]^2 =$

$[\pi_j((st)u)]^2$ for all $j = 0, 1, \cdots, 7$.

Case 3: Similarly, suppose player one employs a pure quantum strategy represented by $s = s_{00} = i_k,\ k = 0, 1, 2, 4$, player two employs a pure quantum strategy represented by $t = t_{00} = i_l,\ l = 0, 1, i_5, i_6$, and player three employs a unit octonion represented by $u = u_{00} = u_0 + u_1 i_1 + u_2 i_3 + u_3 i_7$. Then one verifies that $[\pi_j(o)]^2 = [\pi_j((st)u)]^2$ for all $j = 0, 1, \cdots, 7$.

This concludes the proof of Corollary 4.108.

B.2 Proof of Corollary 4.114

The result obtained in Case 1 holds for all unit octonion $s = s_0 + s_1 i_1 + s_2 i_2 + s_3 i_4$ and in particular when $s = i_k,\ k = 0, 1, 2, 4$. Similarly, the result of Case 2 holds for all unit octonions $t = t_0 + t_1 i_1 + t_2 i_4 + t_3 i_4$ and in particular for $t = i_l,\ l = 0, 1, 4, 5$. Also, the result of Case 3 holds for all unit octonions $u = u_0 + u_1 i_1 + u_2 i_3 + u_3 i_7$ and in particular for $u = i_m,\ m = 0, 1, 3, 7$. This concludes the proof of Corollary 4.114.

Appendix C

PROBABILITY MEASURE AND FUBINI'S THEOREM

Much of the material below is taken from [37].

C.1 Algebras

Union, intersection, and complementation are the three basic operations in set theory. A nonempty collection of sets closed under these operations is called an **algebra of sets**. More formally

Definition C.1. *Let Ω be a set. A nonempty collection $\mathcal{A}_0$ of subsets of Ω is called an* ***algebra*** *if the following two conditions are satisfied:*
(a) $A \in \mathcal{A}_0$ implies $A^c \in \mathcal{A}_0$.
(b) $A, B \in \mathcal{A}_0$ implies $A \cup B \in \mathcal{A}_0$.

It is straightforward to check the following facts:

- It follows from Definition **C.1** that an algebra is necessarily closed under intersection, that is, if $\mathcal{A}_0$ is an algebra and $A, B \in \mathcal{A}_0$, then $A \cap B \in \mathcal{A}_0$.
- An algebra is closed under finite unions and intersections, that is, if $\mathcal{A}_0$ is an algebra and $A_k \in \mathcal{A}_0$ for $k = 1, 2, \cdots, n$, then $\bigcup_{k=1}^n A_k \in \mathcal{A}_0$ and $\bigcap_{k=1}^n A_k \in$

$\mathcal{A}_0$.

- A nonempty collection of subsets of Ω is an algebra if it is closed under complementation and intersection.

Let Ω be a nonempty set. Then one verifies that each of the following is an algebra of subsets of Ω:

(1) the power set, $\mathcal{P}(\Omega)$, that is, the set of all subsets of Ω;

(2) the trivial algebra, $\{\emptyset, \Omega\}$; and

(3) $\{\emptyset, A, A^c, \Omega\}$, where A is a nonempty proper subset of Ω.

It is useful to know that given a collection of subsets, there is a smallest algebra containing the collection. The smallest algebra containing a collection $\mathcal{C}$ of subsets of Ω is called the **algebra generated** by $\mathcal{C}$ and is denoted $\mathcal{A}_0(\mathcal{C})$.

C.2 σ-Algebras

As we have seen, an algebra of sets is closed under finite unions (and intersections). It is useful to consider a stronger condition, namely, closure under countably-infinite unions (and intersections).

Definition C.2. *Let Ω be a set. A nonempty collection $\mathcal{A}$ of subsets of Ω is called a **σ-algebra** if the following two conditions are satisfied:*

(a) $A \in \mathcal{A}$ implies $A^c \in \mathcal{A}$

(b) $\{A_n\}_n \subset \mathcal{A}$ implies $\bigcup_n A_n \in \mathcal{A}$

Directly from Definition **C.2**, we note that any σ-algebra is an algebra. However, the converse is not true.

C.3 Probability Measure

We begin by considering the general concept of measure.

Definition C.3 *Let Ω be a set and $\mathcal{A}$ a σ-algebra of subsets of Ω. A* ***measure****, μ, on $\mathcal{A}$ is an extended real-valued function satisfying the following conditions:*

(a) $\mu(A) \geq 0$ for all $A \in \mathcal{A}$.

(b) $\mu(\emptyset) = 0$.

(c) If $A_1, A_2, \cdots$ are in $\mathcal{A}$, with $A_i \cap A_j = \emptyset$ for $i \neq j$, then

$$\mu\left(\bigcup_n A_n\right) = \sum_n \mu(A_n).$$

The pair $(\Omega, \mathcal{A})$ is called a ***measurable space*** *and the triple $(\Omega, \mathcal{A}, \mu)$ is called a* ***measure space****.*

The members of a σ-algebra $\mathcal{A}$ are often referred to as $\mathcal{A}$ **-measurable sets**.

A measure space, $(\Omega, \mathcal{A}, \mu)$, is called a *$\sigma$-finite measure space* if there is a sequence, $\{A_n\}_n$, of $\mathcal{A}$-measurable sets such that $\cup_n A_n = \Omega$ and $\mu(A_n) < \infty$ for each n.

Suppose that $(\Omega, \mathcal{A}, \mu)$ is a measure space. If $\mu(\Omega) = 1$, then $(\Omega, \mathcal{A}, \mu)$ is called a **probability space** and μ a **probability measure**.

As an example, consider the experiment of tossing a coin twice. The set of possible outcomes for that experiment is $\Omega = \{HH, HT, TH, TT\}$, where, for instance, HT denotes the outcome of a head on the first toss and a tail on the second toss. Set $\mathcal{A} = \mathcal{P}(\Omega)$ and, for $E \in \mathcal{A}$, define $\mu(E) = |E|/4$ where $|E|$ denotes the number of elements of E. Then $(\Omega, \mathcal{A}, \mu)$ is a probability space. Note that this is the appropriate measure space to use when the coin is equally likely to come up heads or tails. To illustrate, the probability of getting at least one head in two tosses of a balanced coin is $\mu(\{HH, HT, TH\}) = \frac{3}{4}$.

We recall the concept of measurability for real-valued functions on an abstract space.

Definition C.4. *Let $(\Omega, \mathcal{A})$ be a measurable space. A real-valued function f on Ω is said to be an $\mathcal{A}$* ***-measurable function*** *if $f^{-1}(O) \in \mathcal{A}$ for all open sets $O \subset \mathbb{R}$.*

C.4 Fubini's Theorem

We consider the iteration of integrals for complex-valued measurable functions. To ensure the existence of the integrals involved, an integrability condition is imposed.

Theorem C.5. (Fubini's Theorem)

Suppose that $(\Omega_1, \mathcal{A}_1, \mu_1)$ and $(\Omega_2, \mathcal{A}_2, \mu_2)$ are σ-finite measure spaces. Let f be a complex-valued $\mathcal{A}_1 \times \mathcal{A}_2$-measurable function on $\Omega_1 \times \Omega_2$ such that $f \in \mathcal{L}^1(\mu_1 \times \mu_2)$, i.e. at least one of the quantities,

$$(i) \int_{\Omega_1 \times \Omega_2} |f(x,y)| d(\mu_1 \times \mu_2)(x,y),$$
$$(ii) \int_{\Omega_1} \left[\int_{\Omega_2} |f(x,y)| d\mu_2(y) \right] d\mu_1(x)$$
$$(iii) \int_{\Omega_2} \left[\int_{\Omega_1} |f(x,y)| d\mu_1(x) \right] d\mu_2(y)$$

is finite. Then

$$\int_{\Omega_1\times\Omega_2} f(x,y)d(\mu_1\times\mu_2)(x,y) = \int_{\Omega_1}\left[\int_{\Omega_2} f(x,y)d\mu_2(y)\right]d\mu_1(x)$$
$$= \int_{\Omega_2}\left[\int_{\Omega_1} f(x,y)d\mu_1(x)\right]d\mu_2(y).$$

Note that Fubini's Theorem generalizes to n-dimensional product spaces. For example, if $n = 3$ and $f \in \mathcal{L}^1(\mu_1\times\mu_2\times\mu_3)$, then

$$\int_{\Omega_1\times\Omega_2\times\Omega_3} f(x,y,z)d(\mu_1\times\mu_2\times\mu_3)(x,y,z) = \int_{\Omega_{i_1}}\left[\int_{\Omega_{i_2}\times\Omega_{i_3}} f(x,y,z)d(\mu_{i_2}\times\mu_{i_3})\right]d\mu_{i_1}.$$

for each permutation, i_1, i_2, i_3, of 1, 2, 3.

For further information on measures, the reader is referred to [50].