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Equidistant Sets in Spaces of Bounded Curvature

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by

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Abstract

Given a metric space (X, d), and two nonempty subsets $A, B \subseteq X$, we study the properties of the set of points of equal distance to A and B, which we call the equidistant set $E(A, B)$. In general, the structure of the equidistant set is quite unpredictable, so we look for conditions on the ambient space, as well as the given subsets, which lead to some regularity of the properties of the equidistant set. At a minimum, we will always require that X is path connected (so that $E(A, B)$ is nonempty) and A and B are closed and disjoint (trivially, $\overline{A} \cap \overline{B} \subset E(A, B)$).

Historically, the equidistant set has primarily been studied with the assumptions: (i) X is Euclidean space and A, B are closed and disjoint; or (ii) X is a compact smooth surface and A, B are singleton sets. We combine and extend on these requirements by examining equidistant sets with the conditions that X is a compact Alexandrov surface (of curvature bounded below) and A, B are compact and disjoint. Significantly, we find $E(A, B)$ is always a finite simplicial 1-complex. The techniques developed are also applied to answer two open problems concerning equidistant sets in the Euclidian plane. In particular, we show that if A and B are disjoint closed subsets of $\mathbb{R}^2$, then $E(A, B)$ is a topological 1-manifold, and the Hausdorff dimension of $E(A, B)$ is 1.

Dedication

To Hannah, for her patience and support.

Acknowledgments

First and foremost, I would like to thank my advisor, J.J.P. Veerman, for introducing me to the subject of equidistant sets and guiding my research on the topic. I am also very grateful to Peter Oberly and Chris Aagaard for their many contributions during our research meetings. Their insights were particularly helpful in the development of Theorem 5.9 and Theorem 5.13, respectively. Finally, I would like to extend a special thanks to Joel Shapiro for not only agreeing to be on my committee, but also for providing a platform – namely, the weekly analysis seminar – for me to practice presenting my research over the past few years.

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1 Introduction

1.1 Overview

Given two nonempty sets A and B in a metric space X, we will denote by $E(A, B)$ the set of points whose distance to A is the same as their distance to B. That is, $E(A, B)$ is the set of points *equidistant* to A and B. Perhaps unsurprisingly, the structure of the set $E(A, B)$ depends not only on the qualities of the sets A and B, but also the properties of the ambient space X. The goal is to examine conditions under which $E(A, B)$ is 'well-behaved' in some sense.

Our particular interest in equidistant sets is motivated by the work of Veerman et al. in [VB06], [BV07], and [HPV17], where many results are given for equidistant sets determined be pairs of points in compact Riemannian manifolds. We generalize many of these results to equidistant sets determined by nonempty disjoint closed sets in compact Alexandrov spaces.

1.2 History and Motivation

The idea of an equidistant set has strong ties to classical geometry. For example, in the Euclidean plane, we can realize the conics as equidistant sets. If A is a line and B is a point (not on the line A), then $E(A, B)$ is a parabola. Similarly, if A is a circle and B is a point, then $E(A, B)$ is either an ellipse (when B is in the region bounded by A) or a hyperbola (when B is outside of the region bounded by A).

The modern study of equidistant sets seems to begin with the work of

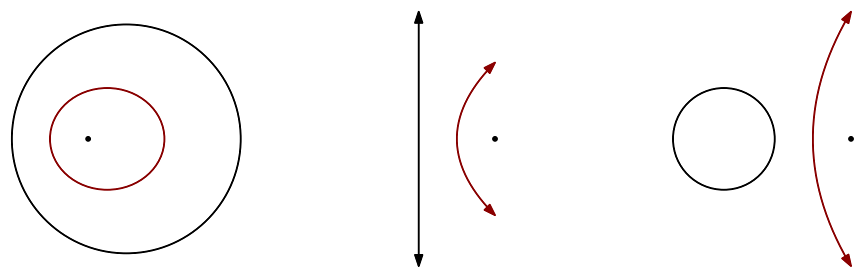

Figure 1: *From left to right: an ellipse, a parabola, and a hyperbola each realized as an equidistant set.*

Busemann [Bus55], Wilker [Wil75], and Loveland [Lov76]. However, the scope and goal of each author is as different as their chosen terminology (one finds *bisector*, *equidistant set*, and *midset*, respectively). Besides these references, one finds little if any mention of equidistant sets until the 1990s, where equidistant sets see a major role in the development of computer aided design and modeling applications. See for example, [EK98a], [EK98b], [Pet00], [SAR96], and [SPB96]. In these cases, the terms *bisector set* or *Voronoi surface* (a reference to Voronoi diagrams) are commonly used in place of equidistant set. While the direct applicability of these papers is significant, all of the work is done in Euclidean 2- or 3-space, which limits the overall scope.

Our interest stems from [VPRS00], where some qualities of equidistant sets are examined in metric spaces known as *Brillouin spaces*. These spaces arise from solid-state physics and the study of *Brillouin zones*, which can be used to describe the behavior of electrons in atomic crystal structures. In Brillouin spaces, equidistant sets determined by two points are interesting in that they not only separate the space into two components, but are minimal separating (i.e. they do not properly contain any other separating set). In [VB06] it is shown that a large class of metric spaces, including Riemannian manifolds, are Brillouin spaces. This motivates [BV07] and [HPV17], which classify many properties of equidistant sets (which they call *mediatrices*) determined by pairs of points in compact Riemannian manifolds. After observing that Alexandrov spaces with lower curvature bound are also Brillouin spaces, we generalize many of their results in Section 5.

1.3 Organization and Main Results

Section 2 reviews basic facts about metric spaces and length spaces. Sections 3 and 4 review Alexandrov spaces and Brillouin spaces, respectively. These sections provide any necessary preliminary results, as well as basic properties of equidistant sets in proper length spaces with no branching geodesics.

Section 5 combines techniques from [ST96], [VB06], and [BV07] to characterize equidistant sets in compact 2-dimensional Alexandrov spaces of curvature bounded below. In particular, we have the following theorem.

Theorem 5.7. *Let X be a compact 2-dimensional Alexandrov space of curvature bounded below (possibly with boundary). For any pair of disjoint nonempty closed subsets $A, B \subseteq X$, the equidistant set $E(A, B)$ is homeomorphic to a finite closed simplicial 1-complex.*

In fact, under the right circumstances, we can even bound the number of closed loops that appear in the equidistant set.

Theorem 5.13. *Let X be a compact 2-dimensional Alexandrov space of curvature bounded below (without boundary). If $A, B \subseteq X$ are nonempty disjoint* *and compact, then* $H_1(E(A,B)) = \mathbb{Z}^k$ *for some positive integer* k *satisfying*

$$1 \leq k \leq \dim H_1(X; \mathbb{Z}_2) + \dim H_0(A) + \dim H_0(B) - 1.$$

Finally, Section 6 is dedicated to the two open questions found in [PS14] concerning equidistant sets in the Euclidean plane. We achieve the following classification.

Theorem 6.3. *If* A *and* B *are nonempty disjoint closed subsets of the Euclidean plane, then the Hausdorff dimension of* $E(A,B)$ *is 1.*

2 Preliminaries

2.1 Metric Spaces

Throughout this manuscript, the primary ambient space will be a metric space. Recall that a *metric space* is the pair (X, d) where X is a set and $d : X \times X \to \mathbb{R}$ is the metric, which satisfies

(i) $d(x, y) \geq 0$ and $d(x, y) = 0 \iff x = y$ (positive definite)

(ii) $d(x, y) = d(y, x)$ (symmetric)

(iii) $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality)

for all $x, y, z \in X$. Following convention, when there is no need to specify the metric we will refer to a metric space as simply X.

Given any point x_0 in a metric space X and real number $\rho > 0$, we denote by $B_\rho(x_0)$ the open ball of radius ρ centered at x,

$$B_\rho(x_0) = \{x \in X : d(x, x_0) < \rho\}.$$

Similarly, we denote the closed ball of radius ρ and boundary of radius ρ by

$$\overline{B}_\rho(x_0) = \{x \in X : d(x, x_0) \leq \rho \quad \text{and} \quad \partial B_\rho(x_0) = \{x \in X : d(x, x_0) = \rho\}$$

respectively.

For our first examples, we will describe a few fundamental Reimannian manifolds; namely, Euclidean space, spherical space, and hyperbolic space.

However, our description of these spaces will avoid the Riemannian metric[1] and will be entirely from the metric space viewpoint of synthetic geometry (see also [BH99, Chapter I.2]).

Recall that n-dimensional Euclidean space – which we will simply denote $\mathbb{R}^n$ – comes with the inner product

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$$

which induces the norm $\|x\| = \sqrt{\langle x, x \rangle}$, and subsequently the metric $d(x, y) = \|x - y\|$.

For n-dimensional spherical space, we consider the set

$$\mathbb{S}^n = \Big\{ x \in \mathbb{R}^{n+1} : \|x\| = 1 \Big\}$$

together with the metric $d_{\mathbb{S}}(x, y) = \arccos\Big(\langle x, y \rangle\Big)$.

For n-dimensional Hyperbolic space, we use the Poincaré ball model,

$$\mathbb{H}^n = \Big\{ x \in \mathbb{R}^n : \|x\| < 1 \Big\}$$

with the metric

$$d_{\mathbb{H}}(x, y) = \operatorname{arcosh}\left(1 + \frac{2\|x - y\|^2}{(1 - \|x\|^2)(1 - \|y\|^2)}\right).$$

[1]A Riemannian metric is a smoothly-varying inner product on the tangent space. The metric we describe can be obtained by using the Riemannian metric to integrate along shortest paths connecting points.

We can now describe $\mathbb{M}_k^n$, the n-dimensional model spaces of constant sectional curvature k. Of particular importance are the spaces $\mathbb{M}_k^2$, which we will use to establish our bounded curvature conditions in Section 3.

There are three essential cases for $\mathbb{M}_k^n$. First, if $k = 0$, then $\mathbb{M}_k^n$ is precisely $\mathbb{R}^n$. Second, if $k > 0$, then $\mathbb{M}_k^n$ is the set $\mathbb{S}^n$, but with the scaled metric $d_k = d_{\mathbb{S}}/\sqrt{k}$. And third, if $k < 0$, then $\mathbb{M}_k^n$ is the set $\mathbb{H}^n$, but with the scaled metric $d_k = d_{\mathbb{H}}/\sqrt{-k}$. Note that $(\mathbb{M}_1^n, d_1) = (\mathbb{S}^n, d_{\mathbb{S}})$ and $(\mathbb{M}_{-1}^n, d_{-1}) = (\mathbb{H}^n, d_{\mathbb{H}})$.

One final important observation is that for $k > 0$, the distance between any two points in $\mathbb{M}_k^2$ is at most $\pi/\sqrt{k}$. This will influence many future hypotheses, so we define the *diameter* of a metric space by

$$\operatorname{diam} X = \sup\{d(x,y) : x, y \in X\}.$$

For the model spaces, we have

$$\operatorname{diam} \mathbb{M}_k^n = \begin{cases} \pi/\sqrt{k} & \text{if } k > 0 \\ \infty & \text{if } k \leq 0. \end{cases}$$

2.2 Distance Functions

Given a nonempty set $A \subseteq X$, the distance to the set

$$d(x, A) = \inf\{d(x,a) : a \in A\}$$

is fundamental to our study of equidistant sets.

Lemma 2.1. *If A is a nonempty subset of a metric space X, then $d(\cdot, A)$:*

$X \to \mathbb{R}$ is 1-Lipschitz continuous and $d(x, A) = d(x, \overline{A})$.

Proof. Let $x, y \in X$ be given. For any $a \in A$, the triangle inequality gives us

$$d(x, a) \le d(x, y) + d(y, a) \quad \text{and} \quad d(y, a) \le d(x, y) + d(x, a)$$

so taking the infimum over all $a \in A$ gives

$$d(x, A) \le d(x, y) + d(y, A) \quad \text{and} \quad d(y, A) \le d(x, y) + d(x, A).$$

Therefore, $|d(x, A) - d(y, A)| \le d(x, y)$, so $d(\cdot, A)$ is 1-Lipschitz continuous.

To verify that $d(x, A) = d(x, \overline{A})$, first note that the inclusion $A \subseteq \overline{A}$ implies $d(x, \overline{A}) \le d(x, A)$, so it suffices to show the reverse inequality. Let $x \in X$ and $\varepsilon > 0$ be given, and fix $\bar{a} \in \overline{A}$ such that $d(x, \bar{a}) < d(x, A) + \varepsilon/2$. If $B_{\varepsilon/2}(\bar{a})$ is the open ball of radius $\varepsilon/2$ centered at $\bar{a}$, then $B_{\varepsilon/2}(\bar{a}) \cap A$ is nonempty. For any $a \in B_{\varepsilon/2}(\bar{a}) \cap A$,

$$d(x, A) \le d(x, a) \le d(x, \bar{a}) + d(\bar{a}, a) < d(x, \overline{A}) + \varepsilon.$$

Given that the above holds for any $\varepsilon > 0$, we have $d(x, A) \le d(x, \overline{A})$. $\square$

Given $A \subseteq X$ and $x \in X$, we define $P_A(x)$ to be the *metric projection* of x onto A;

$$P_A(x) = \{a \in A : d(x, a) = d(x, A)\}.$$

In general, $P_A(x)$ may be empty. However, if we assume that A is compact (which we generally will) then $P_A(x)$ is nonempty for every $x \in X$.

2.3 Length Spaces

A *path* (or *curve*) in a metric space X is a homeomorphism $\gamma : [a, b] \to X$, where $[a, b]$ is an interval of $\mathbb{R}$ (in other words, by path or curve we mean a Jordan arc). The *length* of any path γ is denoted $L(\gamma)$ and is the supremum of the distance along finite partitions of the path:

$$L(\gamma) = \sup \left\{ \sum_{k=1}^{n-1} d\Big(\gamma(t_k), \gamma(t_{k+1})\Big) : a = t_1 < t_2 < \cdots < t_n = b \right\}.$$

When the length is finite, we say that the path is *rectifiable*.

A path γ is a *shortest path* if $d(\gamma(s), \gamma(t)) = |s - t|$ for all $s, t \in [a, b]$. Note that shortest paths are distance minimizing and parametrized by arc length. A path is a *geodesic* if it is locally a shortest path. Typically, we take the domain of a shortest path γ to be $[0, T]$ so that $t = d(\gamma(0), \gamma(t))$. However, it can be convenient to reparametrize a path to some other domain. Therefore, we say that a shortest path $\gamma : [c, d] \to X$ is *linearly parametrized* if $d(\gamma(s), \gamma(t)) = \lambda|s - t|$ for all $s, t \in [c, d]$, where λ is the constant

$$\lambda = \frac{d(\gamma(c), \gamma(d))}{d - c}.$$

A *length space* is a metric space such that the metric is equivalent to the infimum over paths connecting any two points. That is to say,

$$d(x, y) = \inf \Big\{ L(\gamma) : \gamma \text{ connects } x \text{ and } y \Big\}$$

for all $x, y \in X$. For our purposes, the distance between any two points in a metric space must be finite, so a length space must be path connected. If (Y, d) is a metric space such that every pair of points can be connected by a rectifiable path, then we can form a length space (Y, d') by using the above formula to define the metric d'.

We close this section with two theorems that will be used throughout. First, in a complete and locally compact length space, the Hopf-Rinow Theorem guarantees the existence of shortest paths.

Theorem 2.2 (Hopf–Rinow)**.** *If X is a complete and locally compact length space, then every closed and bounded subset of X is compact, and any two points in X can be connected by a shortest path.*

Finally, as a consequence of the Arzelá–Ascoli theorem, any sequence of shortest paths in a compact length space always contains a uniformly convergent subsequence.

Theorem 2.3 (Arzelà–Ascoli)**.** *If X is a separable metric space and Y is a compact metric space, then every uniformly equicontinuous sequence of functions $f_n : X \to Y$ has a subsequence that converges uniformly to a continuous function $f : X \to Y$.*

Corollary 2.4. *If X is a compact length space and $\{\gamma_n\}_n$ is a sequence of linearly reparametrized shortest paths, $\gamma_n : [0, 1] \to X$, then $\{\gamma_n\}_n$ contains a subsequence which converges uniformly to a linearly reparametrized shortest path.*

2.4 Hausdorff Measure and Dimension

We will give a brief, albeit very incomplete, overview of the Hausdorff measure. In particular, we are interested in the Hausdorff dimension, which we use to describe our ambient space as well as the equidistant sets we will examine. For a more detailed description see, for example, [BBI01, §1.7] or [Fal85, Chapter 1].

Definition 2.5. Let $d \geq 0$ and $\varepsilon > 0$ be given. For any set E in a metric space, we define

$$\mathscr{H}^d_\varepsilon(E) = \inf \left\{ \sum_{i \in I} (\operatorname{diam} A_i)^d : E \subseteq \bigcup_{i \in I} A_i, \operatorname{diam} A_i \leq \varepsilon \right\}$$

(with the conventions $\operatorname{diam} \varnothing = 0$ and $\inf \varnothing = \infty$). The *d-dimensional Hausdorff measure* is given by

$$\mathscr{H}^d(E) = \lim_{\varepsilon \to 0^+} \mathscr{H}^d_\varepsilon(E).$$

It is well-known that the Hausdorff measure is a Borel measure [Fal85, Theorem 1.5], so we can freely apply it to any open or closed subsets of a metric space.

In order to better illustrate the dimension associated with the Hausdorff measure, suppose that $s < t$. For any set A satisfying $\operatorname{diam} A \leq \varepsilon$ for some $\varepsilon > 0$, the fact that $s - t < 0$ yields the inequality

$$(\operatorname{diam} A)^s = (\operatorname{diam} A)^{s-t} (\operatorname{diam} A)^t \geq \varepsilon^{s-t} (\operatorname{diam} A)^t.$$

For any measurable set E, it follows from the above inequality that $\mathscr{H}^s_\varepsilon(E) \geq \varepsilon^{s-t}\mathscr{H}^t_\varepsilon(E)$. Therefore, if $\mathscr{H}^t(E)$ is positive (and finite), then $\mathscr{H}^s(E)$ must be infinite. Alternatively, if $\mathscr{H}^s(E)$ is positive (and finite), then $\mathscr{H}^t(E)$ must be zero. As such, there is at most one value $d \geq 0$ for which $\mathscr{H}^d(E)$ is not zero or infinity. This motivates the notion of *Hausdorff dimension* of a set.

Definition 2.6. Given a set E in a metric space, we define the *Hausdorff dimension* of E by

$$\dim_{\mathscr{H}} E = \inf\{d \geq 0 : \mathscr{H}^d(E) = 0\} = \sup\{d \geq 0 : \mathscr{H}^d(E) > 0\}.$$

For the purposes of this monograph, we only really need to measure rectifiable curves. Conveniently, the measure of such a curve is precisely its length.

Lemma 2.7. *If $\gamma : [0,1] \to X$ be a rectifiable curve, then $L(\gamma) = \mathscr{H}^1(\gamma([0,1]))$.*

Proof. Let $\varepsilon > 0$ be given and let $\{t_k\}_{k=1}^n$ be a partition of $[0,1]$ such that $\operatorname{diam}\gamma([t_k,t_{k+1}]) < \varepsilon$. One can check that any partition satisfying

$$\sum_{k=1}^{n-1} d(\gamma(t_k),\gamma(t_{k+1})) > L - \varepsilon/2 \quad \text{and} \quad d(\gamma(t_k),\gamma(t_{k+1})) < \varepsilon/2$$

is such a partition.

For each k, we can fix $s_{2k-1}, s_{2k} \in (t_k, t_{k+1})$, with $s_{2k-1} < s_{2k}$, such that

$$\operatorname{diam}\gamma([t_k,t_{k+1}]) < d(\gamma(s_{2k-1}),\gamma(s_{2k})) + \varepsilon/2n.$$

Notice that this partition satisfies

$$
\begin{aligned}
\mathscr{H}^1_\varepsilon(\gamma([0,1])) &\leq \sum_{k=1}^{n-1} \operatorname{diam}\gamma([t_k,t_{k+1}]) \\
&< \sum_{k=1}^{2n-1} \Big(d(\gamma(s_k),\gamma(s_{k+1})) + \varepsilon/2n \Big) \\
&\leq L(\gamma) + \varepsilon.
\end{aligned}
$$

Therefore, $\mathscr{H}^1(\gamma([0,1])) \leq L(\gamma)$.

On the other hand, the reverse inequality is a consequence of the following fact: For any connected set E, $\mathscr{H}^1(E) \geq \operatorname{diam} E$. This can be proven by observing that (i) on the real line, the 1-dimensional Hausdorff measure is equal to the Lebesgue measure; and (ii) if f is a 1-Lipschitz map, then $\mathscr{H}^d(A) \geq \mathscr{H}^d(f(A))$. Fixing $e, e' \in E$ such that $\operatorname{diam} E < d(e,e') + \varepsilon$, we apply the 1-Lipschitz map $d_e(x) = d(e,x)$ and find

$$\mathscr{H}^1(E) \geq \mathscr{H}^1(d_e(E)) \geq d(e,e') > \operatorname{diam} E - \varepsilon.$$

Now, letting $\{t_k\}_{k=1}^n$ be a partition of $[0,1]$ such that

$$L(\gamma) - \varepsilon < \sum_{k=1}^{n-1} d(\gamma(t_k),\gamma(t_{k+1})).$$

Since each $\gamma([t_k, t_{k+1}])$ overlaps on a set of measure zero, we find

$$\begin{aligned}
\mathscr{H}^1(\gamma([0,1])) &= \sum_{k=1}^{n-1} \mathscr{H}^1(\gamma([t_k, t_{k+1}])) \\
&\geq \sum_{k=1}^{n-1} \operatorname{diam} \gamma([t_k, t_{k+1}]) \\
&\geq \sum_{k=1}^{n-1} d(\gamma(t_k), \gamma(t_{k+1})) \\
&> L(\gamma) - \varepsilon. \qquad \square
\end{aligned}$$

3 Alexandrov Spaces

This section collects necessary definitions and results concerning the fundamentals of Alexandrov spaces. The vast majority of the content of this section can be found in the standard survey articles [BGP92], [Pla02], and [Shi93], as well as the text [BBI01].

3.1 Comparison Triangles and Angles

We denote by $\mathbb{M}^2_k$ the 2-dimensional complete simply-connected Riemannian manifold of constant sectional curvature k (see Section 2.1 for the metric space description of $\mathbb{M}^n_k$). If X is a length space and $x, y, z \in X$ satisfy

$$d(x, y) + d(y, z) + d(x, z) < \operatorname{diam} \mathbb{M}^2_k$$

then we can fix three points $\bar{x}, \bar{y}, \bar{z} \in \mathbb{M}^2_k$ such that

$$d(x, y) = d_k(\bar{x}, \bar{y}), \quad d(x, z) = d_k(\bar{x}, \bar{z}), \quad \text{and} \quad d(y, z) = d_k(\bar{y}, \bar{z}).$$

The points $\bar{x}$, $\bar{y}$, and $\bar{z}$, together with the shortest paths joining them, form a geodesic triangle in $\mathbb{M}^2_k$, which we call the *comparison triangle* and denote it by $\Delta^k(x, y, z)$. Note that such a comparison triangle is unique up to isometry. The interior angle in the geodesic triangle $\Delta^k(x, y, z)$ (in $\mathbb{M}^2_k$) with vertex $\bar{x}$ is denoted $\angle^k(y, z)$ and referred to as the *comparison angle* at x. If the identity of the vertex is unclear, we may also denote it $\angle^k_x(y, z)$.

Given two shortest paths in X, say $\gamma : [0, T] \to X$ and $\eta : [0, S] \to X$, with

$\gamma(0) = \eta(0)$, define the *upper angle* between γ and η as

$$\angle^+(\gamma, \eta) = \limsup_{s,t \to 0} \angle^k(\gamma(t), \eta(s))$$

or equivalently,

$$\angle^+(\gamma, \eta) = \lim_{\varepsilon \to 0} \sup \Big\{ \angle^k \left(\gamma(t), \eta(s)\right) : s, t \in (0, \varepsilon] \Big\}.$$

Note that the definition of upper angle can be applied to any paths which share a basepoint, but we will generally only use it for shortest paths. Furthermore, the definition of upper angle is independent of the space form we choose for the comparison angles (all Riemannian manifolds are infinitesimally Euclidean). However, the choice of k becomes significant when defining a bound on the curvature of our space.

In general, the upper angle does not retain many of the qualities we may expect for the angle. Importantly, however, it does satisfy a triangle inequality.

Lemma 3.1. *Let X be a length space and let γ, η, and σ be three shortest paths emanating from a common point p. Then*

$$\angle^+(\gamma, \eta) \leq \angle^+(\gamma, \sigma) + \angle^+(\sigma, \eta).$$

Proof. Note that $\angle^+(\gamma, \eta) \leq \pi$, so if $\angle^+(\gamma, \sigma) + \angle^+(\sigma, \eta) \geq \pi$, then the result is trivial. As such, we assume that $\angle^+(\gamma, \sigma) + \angle^+(\sigma, \eta) < \pi$. By way of contradiction, suppose that there is an $\varepsilon > 0$ such that

$$\angle^+(\gamma, \eta) > \angle^+(\gamma, \sigma) + \angle^+(\sigma, \eta) + \varepsilon. \tag{1}$$

For simplicity (and since the upper angle is independent of the curvature of the space form), we will use Euclidean space, $(\mathbb{M}_0^2, \|\cdot\|)$, for our comparison triangles. By the definition of limit superior, there is a $\delta > 0$ such that

$$\angle^0(\gamma(t), \eta(r)) > \angle^+(\gamma, \eta) - \varepsilon/3 \quad \text{for some } t, r < \delta \tag{2}$$

$$\angle^0(\gamma(t), \sigma(s)) < \angle^+(\gamma, \sigma) + \varepsilon/3 \quad \text{for all } t, s < \delta \tag{3}$$

$$\angle^0(\sigma(s), \eta(r)) < \angle^+(\sigma, \eta) + \varepsilon/3 \quad \text{for all } s, r < \delta. \tag{4}$$

Fix t and r satisfying (2) and let $\bar{p}, \bar{x}, \bar{y} \in \mathbb{M}_0^2$ be such that $t = \|\bar{x} - \bar{p}\|$, $r = \|\bar{y} - \bar{p}\|$, and

$$\angle^0(\gamma(t), \eta(r)) > \theta_{\bar{x},\bar{y}} > \angle^+(\gamma, \eta) - \varepsilon/3$$

where $\theta_{\bar{x},\bar{y}}$ is the angle between the segments $[\bar{p}, \bar{x}]$ and $[\bar{p}, \bar{y}]$. The left side of the above inequality tells us that $d(\gamma(t), \eta(r)) > \|\bar{x} - \bar{y}\|$. Combining the right side of the above inequality with (1), we have

$$\theta_{\bar{x},\bar{y}} > \angle^+(\gamma, \sigma) + \angle^+(\sigma, \eta) + 2\varepsilon/3.$$

Therefore, we can fix $\bar{z}$ along the segment $[\bar{x}, \bar{y}]$ such that

$$\theta_{\bar{x},\bar{z}} > \angle(\gamma, \sigma) + \varepsilon/3 \quad \text{and} \quad \theta_{\bar{z},\bar{y}} > \angle(\sigma, \eta) + \varepsilon/3.$$

Set $s = \|\bar{z} - \bar{p}\|$. Since $\|\bar{z} - \bar{p}\| \leq \max\{\|\bar{x} - \bar{p}\|, \|\bar{y} - \bar{p}\|\} < \delta$, by (3) and (4) we have

$$\theta_{\bar{x},\bar{z}} > \angle^0(\gamma(t), \sigma(s)) \quad \text{and} \quad \theta_{\bar{z},\bar{y}} > \angle^0(\sigma(s), \eta(r)).$$

It follows that $\|\bar{x} - \bar{z}\| > d(\gamma(t), \sigma(s))$ and $\|\bar{z} - \bar{y}\| > d(\sigma(s), \eta(r))$. Thus, we have

$$d(\gamma(t), \eta(r)) > \|\bar{x} - \bar{y}\| = \|\bar{x} - \bar{z}\| + \|\bar{z} - \bar{y}\| > d(\gamma(t), \sigma(s)) + d(\sigma(s), \eta(r))$$

which contradicts the triangle inequality in X. $\square$

3.2 Bounded Curvature Conditions

Let X be a length space. Generally speaking, bounded curvature in the sense of Alexandrov requires that locally, all triangles in X are at least as thick (for curvature bounded below) or at least as thin (for curvature bounded above) as the respective comparison triangle in $\mathbb{M}_k^2$ for some fixed k. This is formalized as follows.

Definition 3.2. We say that X is of *curvature bounded below* (or curvature $\geq k$) if the following holds: At every point of X there is a neighborhood U such that for every triangle $\Delta(a, b, c) \subseteq U$ and respective comparison triangle $\Delta^k(a, b, c) \subseteq \mathbb{M}_k^2$ (assuming the perimeter of $\Delta(a, b, c)$ is less than $2 \operatorname{diam} \mathbb{M}_k^2$), we have

$$d(u, v) \geq d_k(\bar{u}, \bar{v})$$

where u and v are points in $\Delta(a, b, c)$ and $\bar{u}$ and $\bar{v}$ are their respective comparison points in $\Delta^k(a, b, c)$. Conversely, we say that X is of *curvature bounded above* (or curvature $\leq k$) if the comparison configuration instead yields

$$d(u, v) \leq d_k(\bar{u}, \bar{v}).$$

See Figure 2 for a diagram of the aforementioned comparison triangles and comparison points. The neighborhood U in Definition 3.2 is referred to as a *region of bounded curvature.*

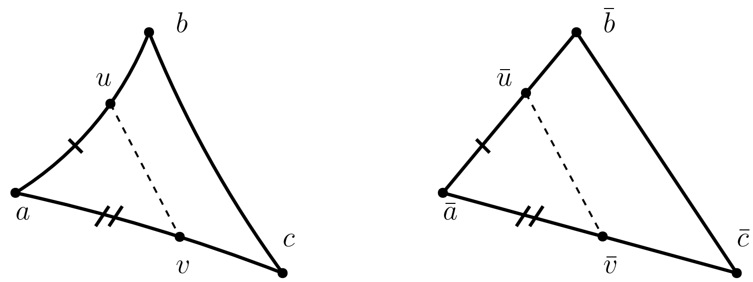

Figure 2: *The triangle $\Delta(a, b, c)$ (left) and its comparison triangle $\Delta^k(a, b, c)$ (right).*

It is worth noting at this point that the idea of bounding curvature by triangular configurations is fairly common. The bounded curvature conditions of Hadamard spaces, CAT(k) spaces, Busemann spaces, and Gromov hyperbolic spaces can all be expressed as a condition on triangles. On the other hand, there are more than a few equivalent ways to describe bounded curvature in the sense given above.

The following proposition gives one equivalent condition for bounded curvature using monotonicity of the comparison angle. Note that our primary concern for this exposition is curvature bounded below, so the statements of the proposition are formulated as such. However, the complimentary result for curvature bounded above can be found in [BBI01] or [BH99].

Proposition 3.3. *Let X be a length space such that there is a shortest path connecting every pair of points. Then the following are equivalent:*
(i) X is of curvature $\geq k$.
(ii) At every point, there is a neighborhood U such that for any shortest paths $\gamma : [0,T] \to U$ and $\eta : [0,S] \to U$ with $\gamma(0) = \eta(0)$, the map $t \mapsto \angle^k(\gamma(t), \eta(s))$ is nonincreasing for any fixed $s \in (0,S]$.

Proof. First, suppose X is of curvature $\geq k$ and let $s \in (0,S]$ and $t, t' \in (0,T]$ be given with $t' < t$. We will consider two comparison triangles, $\Delta^k(\gamma(t), \eta(s), \gamma(0))$ and $\Delta(\gamma(t'), \eta(s), \gamma(0))$. Let $\bar{u} \in \Delta^k(\gamma(t), \eta(s), \gamma(0))$ satisfy

$$t' = d_k(\bar{u}, \overline{\gamma(0)}) \quad \text{and} \quad t - t' = d_k(\overline{\gamma(t)}, \bar{u})$$

(see Figure 3). By the definition of curvature bounded below, we have

$$d_k(\overline{\gamma(t')}, \overline{\eta(s)}) = d(\gamma(t'), \eta(s)) \geq d_k(\bar{u}, \overline{\eta(s)}).$$

It follows that

$$\angle^k(\gamma(t'), \eta(s)) \geq \angle^k(\gamma(t), \eta(s)).$$

Thus, $t \mapsto \angle^k(\gamma(t), \eta(s))$ is nonincreasing.

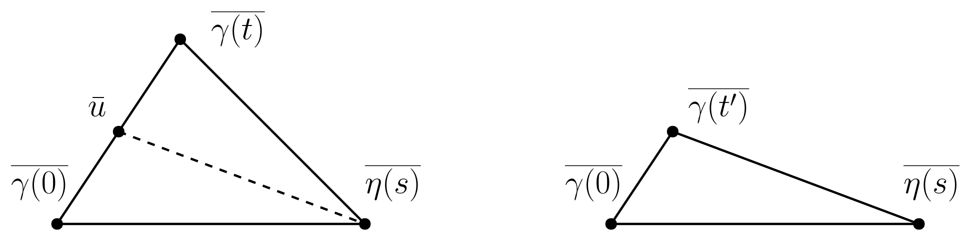

Figure 3: *The first pair of comparison triangles in Proposition 3.3.*

Conversely, suppose $t \mapsto \angle^k(\gamma(t), \eta(s))$ is nonincreasing. For a given $t \in (0, T]$ and $s \in (0, S]$, let $\bar{u}$ and $\bar{v}$ be points in the comparison triangle $\Delta^k(\gamma(t), \eta(s), \gamma(0))$. Without loss of generality, we can assume that $\bar{u}$ lies on the side of the comparison triangle containing $\overline{\gamma(0)}$ and $\overline{\gamma(t)}$, and $\bar{v}$ lies on the side of the comparison triangle containing $\overline{\gamma(0)}$ and $\overline{\eta(s)}$. Fixing t' and s' such that

$$t' = d_k(\bar{u}, \overline{\gamma(0)}) \quad \text{and} \quad s' = d_k(\bar{v}, \overline{\gamma(0)})$$

(notice that $t' \leq t$ and $s' \leq s$) we now also consider the comparison triangle $\Delta^k(\gamma(t'), \eta(s'), \gamma(0))$. By our nonincreasing hypothesis,

$$\angle^k(\gamma(t'), \eta(s')) \geq \angle^k(\gamma(t), \eta(s)).$$

Relabeling $u = \gamma(t')$ and $v = \eta(s')$, we find $d_k(u, v) \geq d_k(\bar{u}, \bar{v})$. $\square$

As the above proposition illustrates, the angle between paths plays a significant role in for our spaces of bounded curvature. In fact, we not only have upper angles, but the following lemma demonstrates that the angle between shortest paths is unambiguously defined by the limit along shortest paths.

Lemma 3.4. *Let X be a length space of curvature bounded above or below. If*

γ and η are shortest paths in X with $\gamma(0) = \eta(0)$, then

$$\limsup_{s,t\to 0} \angle^k(\gamma(t), \eta(s)) = \liminf_{s,t\to 0} \angle^k(\gamma(t), \eta(s)).$$

Proof. Assume that X is of curvature bounded below by k, and let γ and η be given. Without loss of generality, we can assume that γ and η are entirely contained in some region of bounded curvature. By Proposition 3.3 we have $t \mapsto \angle^k(\gamma(t), \eta(t))$ is nonincreasing (and bounded, since the upper angle is always between 0 and π), so $\lim_{t\to 0} \angle^k(\gamma(t), \eta(t))$ exists. Given any pair of sequences $\{t_n\}_n$ and $\{s_n\}_n$ such that $t_n, s_n > 0$ and $t_n, s_n \to 0$, setting $m_n = \min\{t_n, s_n\}$ and $M_n = \max\{t_n, s_n\}$ gives

$$\begin{aligned}
\limsup_{n\to\infty} \angle^k(\gamma(t_n), (s_n)) &\leq \limsup_{n\to\infty} \angle^k(\gamma(m_n), \eta(m_n)) \\
&= \liminf_{n\to\infty} \angle^k(\gamma(M_n), \eta(M_n)) \\
&\leq \liminf_{n\to\infty} \angle^k(\gamma(t_n), \eta(s_n)).
\end{aligned}$$

In the case X is of curvature bounded above, a symmetrical argument gives the result. □

Following Lemma 3.4, we define the *angle* between two paths as

$$\angle(\gamma, \eta) = \lim_{s,t\to 0^+} \angle^k\Big(\gamma(t), \eta(s)\Big)$$

assuming the limit exists; which it does for shortest paths in spaces of bounded curvature. We now give our formal definition of Alexandrov space.

Definition 3.5. An *Alexandrov space* is a complete and locally compact length space of curvature bounded below.

Although this definition is widely accepted, there is some variation in how authors define Alexandrov spaces. For example, some authors may not require local compactness. Others may also allow curvature bounded above. Much of our discussion will rely on the Hopf–Rinow theorem and the lower curvature bound, so we use the given definition to save time in the statements of our propositions.

3.3 No Branching Geodesics

Possibly the most significant consequence of the lower curvature bound (for our purposes), is that it removes the possibility of branching geodesics. Intuitively, two geodesics which emanate from the same point are said to *branch* if they initially overlap for some time interval, but then become disjoint at some point on the interior of both paths.

Definition 3.6. Let X be a length space and let $\gamma, \eta : [0,1] \to X$ be linearly parametrized shortest paths with $\gamma(0) = \eta(0)$. If there are values $t, s \in (0,1)$ such that

$$\gamma\Big([0,t]\Big) = \eta\Big([0,s]\Big) \quad \text{and} \quad \gamma\Big([t,t+\varepsilon]\Big) \cap \eta\Big([s,s+\varepsilon]\Big) = \gamma(t)$$

for some $\varepsilon > 0$, then γ and η are said to *branch*. The point $\gamma(t)$ is the *branch point* between γ and η.

Example 3.7. Consider a length space formed by two cones with their vertices identified. Any pair of shortest paths which begin at the same point on one cone and end at distinct equidistant points on the other cone will branch at the shared vertex.

Lemma 3.8. *If X is a length space of curvature $\geq k$, then there are no branching geodesics in X.*

Proof. Let γ and η be as in Definition 3.6, with branch point $\gamma(t)$. For simplicity, we assume that γ and η are parameterized so that $\gamma(t) = \eta(t)$. Fix $a, b \in (t, 1)$ such that $d(\gamma(a), \gamma(t)) = d(\eta(b), \eta(t))$ and $d(\gamma(a), \eta(b)) > 0$. Then the comparison triangle $\Delta^k(\gamma(0), \gamma(a), \eta(b))$ is a non-degenerate isosceles triangle.

Next, we fix $\bar{u}, \bar{v} \in$ on each side of the vertex $\overline{\gamma(0)}$ such that

$$d_k\Big(\bar{u}, \overline{\gamma(a)}\Big) = d\Big(\gamma(t), \gamma(a)\Big) \quad \text{and} \quad d_k\Big(\bar{v}, \overline{\eta(b)}\Big) = d\Big(\eta(t), \eta(b)\Big).$$

Since the comparison triangle is non-degenerate, $d_k(\bar{u}, \bar{v}) > 0 = d(\gamma(t), \eta(t))$ which contradicts that X is of curvature $\geq k$. $\square$

Note that the converse of Lemma 3.8 is not necessarily true, even for compact length spaces. This is illustrated by the following example.

Example 3.9. Let X be the surface of revolution obtained by rotation the graph of $z = \sqrt{x}$, for $0 \leq x \leq 1$, around the z-axis. In other words,

$$X = \Big\{(x, y, z) \in \mathbb{R}^3 : z = \sqrt[4]{x^2 + y^2}\ ,\ x^2 + y^2 \leq 1\Big\}.$$

Since X is the continuous image of a compact set, it is compact. First equipping X with the subspace metric induced by the Euclidean norm, we can then equip X with an intrinsic metric by defining $d(x, y)$ to be the infimum of lengths of paths from x to y. This space does not have branching geodesics; however, there is no lower curvature bound, which we can see by examining a triangle with one vertex at the singular point $(0, 0, 0)$.

3.4 Finite Dimensional Alexandrov Spaces

For the most part, particularly in Section 5, we will work in 2-dimensional Alexandrov spaces, which we will call an *Alexandrov surface.*

When referring to the dimension of an Alexandrov space, we generally mean the Hausdorff dimension. However, this is known to be equivalent to topological dimension [Pla02, Theorem 156]. See [BBI01, Chapter 10] (the source for each point of the following proposition) for a more complete discussion. For our purposes, the most important fact is that every Alexandrov surface is a topological 2-manifold.

Proposition 3.10 ([BBI01])**.** *Let X be an Alexandrov space.*

(i) The Hausdorff dimension of X is an integer or infinity.

(ii) All open subsets of X have the same Hausdorff dimension.

(iii) If X is 2-dimensional, then it is a topological 2-manifold, possibly with boundary.

The proof of the above proposition is quite technical, so we will avoid leave it to the references. The basic idea is to use *strainers*, which generalize an orthogonal frame in $\mathbb{R}^n$, in order to define the dimension locally. What is known as the *strainer number* is then shown to be equivalent to the Hausdorff dimension.

3.5 Space of Directions and Tangent Cone

Let X be an Alexandrov space and fix some $x \in X$. Let Γ_x be the set of all nonconstant shortest paths emanating from x. We define an equivalence relation on Γ_x by

$$\gamma \sim \eta \iff \angle(\gamma, \eta) = 0$$

and let Σ_x be the set of equivalence classes of $\Gamma_x/\sim$. The angle is a metric on Σ_x (it is easy to check that $\angle$ is symmetric and positive definite, and Lemma 3.1 proves the triangle inequality). We define the *space of directions*, S_xX, as the completion of $(\Sigma_x, \angle)$. For any shortest path γ, we denote by $[\gamma]$ (or simply γ when there is no ambiguity) its equivalence class in S_xX.

Lemma 3.11. *Let X be an Alexandrov space of curvature $\geq k$. For any shortest paths $\gamma : [0, T] \to X$ and $\eta : [0, S] \to X$, with $\gamma(0) = \eta(0)$, $\angle(\gamma, \eta) = 0$ if and only if one path is a subpath of the other.*

Proof. Let γ and η be given, and suppose that $T \leq S$. First, assume that γ is a subpath of η. Then for all sufficiently small t, we have $\gamma(t) = \eta(t)$, so

$$\angle(\gamma, \eta) = \lim_{t \to 0^+} \angle^k(\gamma(t), \eta(t)) = 0.$$

Now suppose that γ is not a subpath of η. By Lemma 3.8, there is no $\varepsilon > 0$ such that $\gamma|_{[0,\varepsilon]}$ is a subpath of η. Therefore, we can find a sequence $\{t_n\}_n$

such that $t_n \to 0$ and $d(\gamma(t_n), \eta(t_n)) > 0$ for all n. By the angle comparison condition for curvature $\geq k$, for any sufficiently large N,

$$\angle(\gamma, \eta) \geq \angle^k(\gamma(t_N), \eta(t_N)) > 0.$$

Thus, if $\angle(\gamma, \eta) = 0$, then γ is a subpath of η. $\square$

Theorem 3.12 ([BBI01, Theorem 10.8.6])**.** *If X is an n-dimensional Alexandrov space of curvature $\geq k$ with $n \geq 2$, then for every $x \in X$, the space of directions S_xX is a compact $(n-1)$ dimensional Alexandrov space of curvature ≥ 1.*

For 2-dimensional Alexandrov spaces, Theorem 3.12 tells us that the space of directions is either a line segment or a circle. Furthermore, by the radius sphere theorem [GP93], $\operatorname{diam} S_xX \leq \pi$.

Example 3.13. Let C be the surface of a 3-dimensional unit cube. Notice that C is flat everywhere except at its corners where we can construct arbitrarily small equilateral triangles with $90°$ angles. Therefore C is a 2-dimensional Alexandrov space of curvature ≥ 0. If the point p is one of the corners of C, then $\operatorname{diam} S_pX = 3\pi/4$. All other $x \in C$ have $\operatorname{diam} S_xX = \pi$.

The *tangent cone* (or tangent space) at x, which we denote T_xX, is the Euclidean cone over S_xX:

$$T_xX = \Big(S_xX \times [0, \infty)\Big) / \Big(S_xX \times \{0\}\Big).$$

We equip the tangent cone with the metric induced by the law of cosines,

$$d\Big((\gamma,t),(\eta,s)\Big) = \sqrt{t^2 + s^2 - 2st\cos\Big(\angle(\gamma,\eta)\Big)}.$$

When X is an Alexandrov space of curvature $\geq k$, T_xX is an Alexandrov space of curvature ≥ 0 for all $x \in X$.

Theorem 3.14 ([BBI01, Corollary 10.9.5]). *If X is an n-dimensional Alexandrov space of curvature $\geq k$, then for every $x \in X$, the tangent cone T_xX is an n-dimensional Alexandrov space of curvature ≥ 0.*

Although the tangent cone has nice metric properties, mapping between an Alexandrov space and its tangent cone at a point is not nearly as well behaved as in the case of Riemannian manifolds and their tangent spaces. For example, a given direction in S_xX may not be realized by any shortest path emanating from the point x (see Example 5.10). However, we can still define an exponential and logarithmic maps, but the domain (particularly of the exponential map) is limited.

Definition 3.15. Let X be an Alexandrov space. Given a point $p \in X$, we define the *exponential map* $\exp_p : T_xX \to X$ by $\exp_p(\gamma,t) = x$ if there is a shortest path from p to x with direction γ and $d(p,x) = t$. Of course, $\exp_p(\gamma,0) = p$ for any direction γ, and $\exp(\gamma,t)$ is undefined whenever there is no shortest path with direction γ and length t.

We also define the *logarithmic map*, $\log_p$, as the preimage of the exponential map: for any $x \in X$, $\log_p(x)$ is the set of all pairs (γ,t) such that there is a shortest path from p to x with direction γ and $t = d(p,x)$.

3.6 Doubling Theorem

Let X be an n-dimensional Alexandrov space with nonempty boundary ∂X, and let $\phi : X \to Y$ be an isometry. We define the doubling of X, which we denote $\widetilde{X}$, as the gluing of X with itself (i.e. with Y) along the boundary,

$$\widetilde{X} = X \cup_{\phi|_{\partial X}} Y.$$

After identifying ∂X with $\phi(\partial X)$, we can equip the doubled space with the metric

$$\tilde{d}(x,y) = \begin{cases} d_X(x,y) & \text{if } x, y \in X \\ d_Y(x,y) & \text{if } x, y \in Y \\ \inf\{d(x,z) + d(y,z) : z \in \partial X\} & \text{if } x \in X \text{ and } y \in Y. \end{cases}$$

It is straightforward to see that $\widetilde{X}$ is a length space, but it is in fact also an n-dimensional Alexandrov space of curvature $\geq k$.

Theorem 3.16 (Doubling Theorem [Per91])**.** *Given an n-dimensional Alexandrov space X of curvature $\geq k$ with nonempty boundary, the doubled space $\widetilde{X}$ is an n-dimensional Alexandrov space of curvature $\geq k$ with empty boundary.*

Since Perelman's doubling theorem was never formally published,[2] we will also quote Petrunin's gluing theorem, which generalizes the doubling theorem to any two spaces with isometric boundaries.

[2]Anton Petrunin kindly keeps a copy of the preprint available on his website: `https://anton-petrunin.github.io/papers/`

Theorem 3.17 (Gluing Theorem [Pet97]). *Let X and Y be Alexandrov spaces with nonempty boundary and curvature $\geq k$. Let there be an isometry $\phi : \partial X \to \partial Y$, where ∂X and ∂Y are considered as length spaces with the induced metric from X and Y. Then the glued space $Z = X \cup_{\phi} Y$ is an Alexandrov space with curvature $\geq k$.*

The doubling theorem follows naturally from the gluing theorem, as the boundary of X is certainly isometric to itself. Moving forward, we can now essentially assume that all Alexandrov spaces are without boundary, since any Alexandrov space with boundary is simply a subspace of the doubling.

4 Equidistant Sets

Definition 4.1. Let (X, d) be a metric space. Given two nonempty sets $A, B \subseteq X$, the *equidistant set* (also *mediatrix* or *midset*) determined by A and B is the set of points of equal distance to A and B;

$$E(A, B) = \{x \in X : d(x, A) = d(x, B)\}.$$

In the case of singleton sets, say $\{a\}$ and $\{b\}$, we use $E(a, b) = E(\{a\}, \{b\})$ for simplicity of notation.

It should be observed that one can easily construct a metric space for which the equidistant set determined by two points is actually empty. However, this is easily avoided by assuming the space is path connected (which, in particular, length spaces always are).

Lemma 4.2. *Let X be a path connected metric space. If A and B are nonempty subsets of X, then $E(A, B)$ is nonempty.*

Proof. Define the function $f_{AB} : X \to \mathbb{R}$ by

$$f_{AB}(x) = d(x, A) - d(x, B)$$

and notice that $E(A, B) = f_{AB}^{-1}(\{0\})$. Fixing some $a \in A$ and $b \in B$ and letting $\sigma : [0, 1] \to X$ be a path from a to b, the intermediate value theorem tells us that $f_{AB}(\sigma(t)) = 0$ for some $t \in [0, 1]$. $\square$

Example 4.3. Let D be the metric space obtained by mapping each page of this dissertation to the integers by page number. Pages before the first page are mapped to the nonpositive integers, with the list of figures being mapped to 0 and the title page being mapped to -6. We then equip the dissertation with the subspace metric induced by the absolute value. One could also think of this as the metric naturally induced by the pdf reader on your computer, or the metric induced by printing the pages and stapling them together in the appropriate order.

In the metric space we have described, this page is the equidistant set determined by the title page and the last page (p. 70). In other words, you could consider this to be the true middle of the dissertation. From here, any other page you choose to read is either closer to the end or the beginning, as there are no other pages in this equidistant set.

On the other hand, let us equip the pages of this dissertation with the discrete metric. We could think of this as the metric induced by printing all of the pages, shuffling them, and then throwing them on the floor. Having completely removed any order to the pages – and perhaps lost the will to read them – any pair of pages are essentially equally far apart. Approaching the pile of paper and picking any number of pages with each hand, we will let L and R be the pages we hold in our left and right hand, respectively. Regardless of which pages we hold (as long as there is at least one in each hand), $E(L, R)$ is precisely the set of pages left on the floor.

A significant aspect of equidistant sets determined by points is that they separate a space into two components. In particular, we are interested in the case that the equidistant set is *minimal separating*.

Definition 4.4. For any subset E of a connected space X, if $X \setminus E$ consists of more than one component, then E is said to be *separating* in X. If no proper subset of E separates X, then E is *minimal separating*.

Example 4.5. Let p and q be two distinct points in Euclidean space $\mathbb{R}^n$. Then we have $E(p,q) = \{v : \langle p-q, v\rangle = 0\}$. In this case, $E(p,q)$ is minimal separating.

Example 4.6. Consider the normed space $(\mathbb{R}^2, \|\ \|_1)$, where

$$\|(x_1, x_2)\|_1 = |x_1| + |x_2|.$$

Letting $p = (0,0)$ and $q = (1,1)$, we find $E(p,q)$ is not minimal separating. This is illustrated in Figure 4.

Example 4.7. Consider the real line with the following metrics

$$d_1(x,y) = \frac{|x-y|}{1+|x-y|} \quad \text{and} \quad d_2(x,y) = \begin{cases} |x-y|, & \text{if } |x-y| < 1 \\ 1, & \text{otherwise.} \end{cases}$$

The metrics d_1 and d_2 are strongly equivalent with $d_1(x,y) \leq d_2(x,y) \leq 2d_1(x,y)$. It is easy to verify that for any $p, q \in (\mathbb{R}, d_1)$, we have $E(p,q) =$

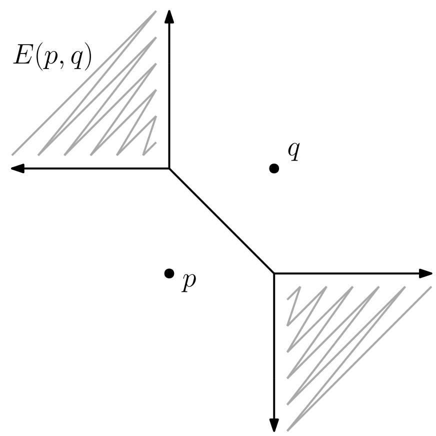

Figure 4: *The equidistant set described in Example 4.6.*

$\{\frac{p+q}{2}\}$. However, in $(\mathbb{R}, d_2)$, if $p = 2$ and $q = -2$,

$$E(p,q) = (-\infty, -3] \cup [-1, 1] \cup [3, \infty).$$

4.1 Brillouin Spaces

The study of Brillouin spaces – or more precisely, Brillouin zones – has significant application in solid state physics. In [VPRS00], Brillouin spaces were introduced as a class of metric spaces for which Brillouin zones could be studied. In particular, for any pair of distinct points, say p and q, in a Brillouin space, the equidistant set $E(p,q)$ is minimal separating. We begin with the following definitions from [VPRS00], as well as the main classification theorem from [VB06].

Definition 4.8. A metric space (X,d) is said to be *metrically consistent* if given any $x \in X$, for all $R > r > 0$ with r sufficiently small, and for each $a \in \partial B_R(x)$, there is a $z \in \partial B_r(x)$ satisfying $B_{d(a,z)}(z) \subseteq B_R(x)$ and $\partial B_{d(z,a)}(z) \cap$

$\partial B_R(x) = \{a\}$.

Definition 4.9. A *Brillouin space* is a path-connected proper metric space X which is metrically consistent and for every distinct pair of points $p, q \in X$, the equidistant set $E(p, q)$ is minimal separating.

Theorem 4.10 ([VB06, Theorem 2.1])**.** *Let X be a proper, path-connected metric space such that*
(i) any two points in X can be connected by a shortest path; and
(ii) for any three distinct points $p, x, y \in X$, if two shortest paths, one from p to x and the other from p to y, have a common segment, then one of the paths is a subset of the other.
Then X is Brillouin.

With some slight revision of language, and taking into consideration the Hopf-Rinow Theorem (Theorem 2.2), the above theorem can be rephrased as follows.

Theorem 4.11 ([VB06])**.** *Let X be a complete and locally-compact length space. If X has no branching geodesics, then X is Brillouin.*

This immediately yields the following proposition.

Proposition 4.12. *Every Alexandrov space of curvature bounded below is Brillouin.*

Proof. This follows from our definition of an Alexandrov space (Definition 3.5), the Hopf-Rinow Theorem, Lemma 3.8, and Theorem 4.10. $\square$

4.2 Basic Properties of Equidistant Sets

We will now generalize some of the preliminary results of [VB06], but for equidistant sets determined by disjoint closed subsets. Our ambient metric space will always a proper length space with no branching geodesics, which we now know includes Alexandrov spaces.

Definition 4.13. Given a point x and a compact set A, the *metric projection* $P_A(x)$ is the set of points of A which realize the distance $d(x, A)$,

$$P_A(x) = \{a \in A : d(x, a) = d(x, A)\}.$$

Whenever we say that $\gamma : [0, T] \to X$ is a shortest path from a point x to a set A, we mean that $\gamma(0) = x$ and $\gamma(T) \in P_A(x)$.

The following lemma, and its corollaries, are adapted from [VB06, §2].

Lemma 4.14. *Let X be a proper length space with no branching geodesics and let $A, B \subseteq X$ be nonempty disjoint and closed. If $x \in E(A, B)$ and $\gamma : [0, T] \to X$ is a shortest path with with $\gamma(0) = x$ and $\gamma(T) \in P_A(x)$, then*

$$d(\gamma(t), A) < d(\gamma(t), B)$$

for all $t > 0$.

Proof. Let $x \in E(A, B)$ and γ with $\gamma(T) = a \in P_A(x)$ be given. For the sake of contradiction, suppose that there is a $t_0 \in (0, T)$ such that $d(\gamma(t_0), B) \leq d(\gamma(t_0), A)$. Fix $b_0 \in P_B(\gamma(t_0))$. Since $d(x, a) \leq d(x, b_0)$ and $d(\gamma(t_0), b_0) \leq$

$d(\gamma(t_0), a)$,

$$\begin{aligned} d(x, \gamma(t_0)) + d(\gamma(t_0), b_0) &= d(x, a) - d(\gamma(t_0), a) + d(\gamma(t_0), b_0) \\ &\leq d(x, b_0) - d(\gamma(t_0), a) + d(\gamma(t_0), a) \\ &= d(x, b_0) \end{aligned}$$

so $d(x, b_0) = d(x, \gamma(t_0)) + d(\gamma(t_0), b_0)$. Given that $a \neq b_0$, this implies that γ is branching, which contradicts our hypothesis. Thus, $d(\gamma(t), A) < d(\gamma(t), B)$ for all $t > 0$. □

Corollary 4.15. *If X is a proper length space with no branching geodesics, then for any nonempty disjoint closed sets $A, B \subseteq X$, the equidistant set $E(A, B)$ has empty interior.*

Proof. Let $x \in E(A, B)$ be given. By Lemma 4.14, an open ball of any radius centered at x cannot be contained in $E(A, B)$ since any shortest path from x to A (or B) immediately leaves $E(A, B)$. □

Our next corollary is a generalization of the idea of minimal separating introduced in the study of Brillouin spaces. In our case, $E(A, B)$ still separates the space, but it may not separate the space into exactly two components. However, it is still minimal separating in the sense that if we remove any point from the equidistant set, we can now define a path which begins in the set A and ends in the set B.

Corollary 4.16. *Let X is a proper length space with no branching geodesics and $A, B \subseteq X$ be nonempty disjoint closed sets. Given any $x \in E(A, B)$, if we* *remove x from $E(A,B)$, then there is a path connecting some $a \in A$ to some $b \in B$ which does not intersect $E(A,B) \setminus \{x\}$.*

Proof. Given any $x \in E(A,B)$, if γ if a shortest path from $a \in P_A(x)$ to x, and η is a shortest path from x to $b \in P_B(x)$, then By Lemma 4.14, the concatenation of γ and η connects a to b without intersecting $E(A,B)\setminus\{x\}$. $\square$

If we additionally assume that A and B are connected, then $E(A,B)$ is minimal separating in the sense of Definition 4.4.

Corollary 4.17. *Let X be a proper length space with no branching geodesics. If $A, B \subseteq X$ are nonempty disjoint closed and connected, then $E(A,B)$ is minimal separating.*

Proof. Define the sets

$$X_A = \{x \in X : d(x,A) < d(x,B)\} \quad \text{and} \quad X_B = \{x \in X : d(x,B) < d(x,A)\}$$

so that X is the union of the disjoint sets X_A, X_B, and $E(A,B)$.

First, we will show that X_A is in fact a single component of $X \setminus E(A,B)$. Certainly $A \subseteq X_A$, and given that A is connected, A must lie in one component of X_A. Letting x_0 be any element of X_A and $\gamma : [0,T] \to X$ be a shortest path from x_0 to A, the same reasoning as Lemma 4.14 shows that the image of γ is contained in X_A. In particular, x_0 is in the same component as A. Therefore, X_A (and subsequently, X_B) is a single component.

Applying Corollary 4.16, we see that removing any point from $E(A,B)$ allows for a path from X_A to X_B. Thus, $E(A,B)$ is minimal separating. $\square$

4.3 Directions to the Equidistant Set

Given an Alexandrov space X, a compact set $A \subseteq X$, and a point $x \in X \setminus A$, we define Θ_A to be the set of directions of shortest paths from x to A:

$$\Theta_A = \{[\gamma] \in S_x X : \gamma \text{ is a shortest path from } x \text{ to } A\}.$$

Lemma 4.18. *Let X be an Alexandrov space of curvature $\geq k$, and let A and B be disjoint compact subsets of X. For any $x \in E(A, B)$, the sets Θ_A and Θ_B are disjoint compact subsets of $S_x X$.*

Proof. First, we verify that Θ_A and Θ_B are disjoint. By way of contradiction, suppose that there are paths γ_a and γ_b from x to $a \in P_A(x)$ and $b \in P_B(x)$, respectively, such that $\angle(\gamma_a, \gamma_b) = 0$. By Lemma 3.11, one of these paths is a subpath of the other. But since they have the same length, we must have $\gamma_a = \gamma_b$, which is a contradiction since A and B are disjoint.

To show that Θ_A (and subsequently, Θ_B) is compact, let $\{v_n\}_n$ be a sequence in Θ_A. For each n let $\gamma_n : [0, d(x, A)] \to X$ be a shortest path in the equivalence class v_n. Since $L(\gamma_n) = d(x, A)$ for each n, by Corollary 2.4 and the compactness of A, $\{\gamma_n\}_n$ contains a subsequence which converges uniformly to a shortest path $\gamma : [0, d(x, A)] \to X$ which connects x to A. Therefore, $\{v_n\}_n$ contains a convergent subsequence which converges to $[\gamma] \in \Theta_A$. Thus, Θ_A is compact. $\square$

4.4 The Equidistant set and the Cut Locus

We end this section with a quick detour to establish some connections between equidistant sets and the cut locus. See [ST96, §§1,2] for further discussion on the content of this section.

Definition 4.19. Let X be an Alexandrov space of curvature bounded below, and $K \subseteq X$ be compact. A point $x \in X$ is a *cut point* to K if there is a shortest path γ from x to K such that γ is not properly contained in any other shortest path to K. The *cut locus* to K, denoted $C(K)$, is the set of all cut points to K.

Since Alexandrov spaces do not have branching geodesics, and each $x \in E(A, B)$ admits at least two shortest paths to $A \cup B$, we have the following lemma. A similar observation was made by [Zam04, p. 378].

Lemma 4.20. *If X is an Alexandrov space, and $A, B \subseteq X$ are nonempty disjoint compact sets, then $E(A, B) \subseteq C(A \cup B)$.*

While the equidistant set is always a subset of the cut locus, it is not necessarily a proper subset. For example, if a and b are distinct points on the 2-sphere, then $E(a, b) = C(\{a, b\})$. However, cases like this are exceptional.

The significance of Lemma 4.20 is that it allows us to use the techniques developed in [ST96] when discussing equidistant sets in 2-dimensional Alexandrov spaces; the main content of the next section.

5 Equidistant Sets on Alexandrov Surfaces

We will now exclusively assume that X is a compact 2-dimensional Alexandrov space (i.e. compact Alexandrov surface) and A and B are nonempty closed subsets of X. We additionally assume that X is without boundary, but in the case of Theorem 5.7, the doubling theorem allows for spaces with boundary.

This section generalizes many of the main theorems of [VB06], [BV07], and [HPV17]. The referenced papers examine equidistant sets determined by two points on a compact smooth surface. Our contribution is proving analogous results for equidistant sets determined by disjoint compact sets on a compact Alexandrov surface.

5.1 The Equidistant Set is a Simplicial 1-Complex

The discussion here is essentially an adaptation of [BV07, §1]. However, we examine Alexandrov spaces instead of Riemannian manifolds, and disjoint nonempty compact sets instead of distinct points.

The following definition gives a construction of 'wedge-shaped' neighborhoods which we will use to describe the local behavior of $E(A, B)$. These are essentially the same as the so-called *sectors* of [ST96]. We assume the radius ρ always satisfies $\rho < \inf_{a \in A} d(a, B)$, as well as being sufficiently small so that the ball $B_\rho(x)$ is homeomorphic to a disk and the boundary

$$\partial B_\rho(x) = \{y \in X : d(x, y) = \rho\}$$

is homeomorphic to a circle. The fact that X is a 2-manifold (Proposition

3.10) guarantees such a ρ always exists.

Definition 5.1. Fix directions $\theta_A \in \Theta_A$ and $\theta_B \in \Theta_B$ such that there is a path (not necessarily a shortest path) in S_xX beginning at θ_A and ending at θ_B which does not pass through any other points of Θ_A or Θ_B. Then fix shortest paths γ_A and γ_B to A and B, respectively, such that $[\gamma_A] = \theta_A$ and $[\gamma_B] = \theta_B$. A *wedge* at x of radius ρ, which we will denote $W_\rho(x)$, is the open set bounded by γ_A, γ_B, and $\partial B_\rho(x)$ such that no other paths from x to A or B intersect $W_\rho(x)$. The *wedge boundary*, $\partial W_\rho(x)$, is the segment of $\partial B_\rho(x)$ contained in the closure of $W_\rho(x)$.

See Figure 5 for clarification of wedge and wedge boundary. Note that every point x admits at least two wedges (and always an even number), but only finitely many.

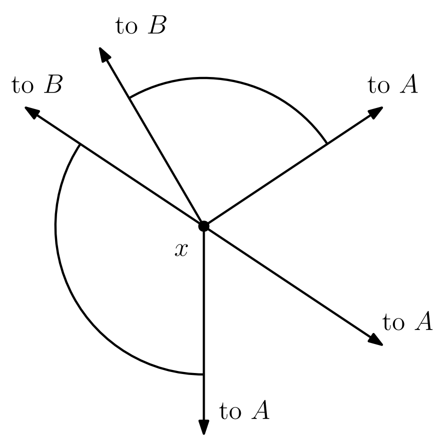

Figure 5: *A point x which admits two wedges.*

Lemma 5.2. *Let X be a compact 2-dimensional Alexandrov space of curvature bounded below and let $A, B \subseteq X$ be disjoint closed sets. For any $x \in E(A, B)$, there are only finitely many wedges at x.*

Proof. By way of contradiction, suppose there are infinitely many wedges at $x \in E(A,B)$. Then there must be infinitely many distinct directions $\{\theta_{A,n}\}_{n=1}^{\infty} \subseteq \Theta_A$ and $\{\theta_{B,n}\}_{n=1}^{\infty} \subseteq \Theta_B$ which form the wedges. Furthermore, we can assume that these sequences are alternating in the sense that $\theta_{B,n}$ is always between $\theta_{A,n}$ and $\theta_{A,n+1}$ (i.e., if we start from $\theta_{A,1}$ we can move along a geodesic in S_xX which passes through $\theta_{B,1}$, then $\theta_{A,2}$, then $\theta_{B,2}$, and so on). Since Θ_A is compact, there is a convergent subsequence

$$\theta_{A,n_k} \to \bar{\theta} \in \Theta_A.$$

The subsequence $\{\theta_{B,n_k}\}_k$ of directions between θ_{A,n_k} and θ_{A,n_k+1} also converges to $\bar{\theta}$. Thus, $\bar{\theta}$ is the direction of a shortest path to both A and B, which contradicts Lemma 4.18. □

The following two lemmas are adapted from [ST96]. The original statement of these results concerns the cut locus, but the proofs apply to the equidistant set (see Lemma 4.20). To illustrate, we will prove the first lemma. Except for small notational adjustments, the proof is taken directly from [ST96, Basic Lemma].

Lemma 5.3 ([ST96, Basic Lemma])**.** *Let $W_\rho(x)$ be a wedge at $x \in E(A,B)$. Then there exists a point $x' \in E(A,B) \cap W_\rho(x)$ and a Jordan arc $\sigma : [0,1] \to E(A,B) \cap \overline{W_\rho(x)}$ such that $\sigma(0) = x$ and $\sigma(1) = x'$.*

Proof. Let $x \in E(A,B)$ and $W_\rho(x)$ be given. We will call the shortest paths from x to A and B which bound the wedge γ_A and γ_B respectively. Let I_A and I_B be disjoint closed subarcs of $\partial W_\rho(x)$ such that $\gamma_A(\rho) \in I_A$ and

$\gamma_B(\rho) \in I_B$. Notice that if we take any sequence $\{x_n\}_n \subseteq W_\rho(x)$ such that $x_n \to x$, then the shortest paths from x_n to A (resp. x_n to B) must converge to γ_A (resp. γ_B). Therefore, there exists a sufficiently small $\varepsilon > 0$ such that for any $y \in \overline{W_\rho(x)} \cap \overline{B}_\varepsilon(x)$, any shortest path from y to A (resp. y to B) intersects I_A (resp. I_B).

We define the sets

$$\begin{aligned} &W_A = \{y \in \overline{W_\rho(x)} \cap \overline{B}_\varepsilon(x) : d(y, A) \leq d(y, B)\} \\ \text{and} \quad &W_B = \{y \in \overline{W_\rho(x)} \cap \overline{B}_\varepsilon(x) : d(y, B) \leq d(y, A)\}. \end{aligned}$$

Clearly $W_A \cap W_B = E(A, B) \cap \overline{W_\rho(x)} \cap \overline{B}_\varepsilon(x)$, so $W_A \cap W_B$ is a nonempty compact set. Notice further that for each $z \in W_A \cap W_B$, there is a wedge containing x which is bounded by shortest paths to A and B intersecting I_A and I_B respectively. We will call this wedge $W_{2\rho}(z; x)$ (the radius 2ρ ensures $I_A \cup I_B \subseteq \overline{W_{2\rho}(z; x)} \cap \overline{W_\rho(x)}$).

The point $x^* \in W_A \cap W_B$ is chosen to satisfy for all $z \in W_A \cap W_B$,

$$W_\rho(x) \cap W_{2\rho}(z; x) \subseteq W_\rho(x) \cap W_{2\rho}(x^*; x).$$

In this sense, x^* is maximal in $W_A \cap W_B$. We now assume that I_A and I_B were chosen so that

$$I_A \cup I_B = \partial W_\rho(x) \cap \overline{W_{2\rho}(x^*; x)}.$$

Furthermore, we redefine W_A (and similarly W_B) to be the set

$$W_A = \{y \in \overline{W_\rho(x)} \cap \overline{W_{2\rho}(x^*; x)} : d(y, A) \leq d(y, B)\}.$$

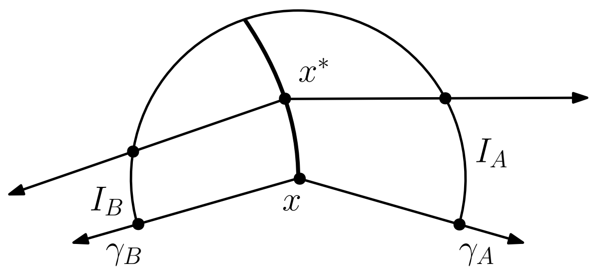

Figure 6: *The setup for the Jordan arc construction of Lemma 5.3.*

We will now show that $W_A \cap W_B$ is a Jordan arc by first constructing a continuous surjection $\phi : I_A \to W_A \cap W_B$. Since I_A is a closed subinterval of $\partial B_\rho(x)$, we know that it is homeomorphic to a closed interval, so we will identify it with the unit interval $[0, 1]$ (with $\phi(0) = x$). After constructing our continuous surjection ϕ, it will be clear that we can reparameterize I_A by some stepped piecewise linear function $\alpha : [0, 1] \to [0, 1]$ to get our homeomorphism $\sigma = \phi \circ \alpha$.

For each $t \in I_A$, if there is a shortest path from $z \in W_A \cap W_B$ to A which passes through t, then we define $\phi(t) = z$. Note that clearly $\phi(0) = x$ and $\phi(1) = x^*$. Given that shortest paths do not cross and every shortest path from $z \in W_A \cap W_B$ crosses some point in I_A, the map ϕ is well-defined and surjective.

In general, it is possible that $\phi(t_0) = \phi(t_1) = z_0$ for some $t_0 < t_1$, but this is not a problem as we will define $\phi(t) = z_0$ for all $t \in (t_0, t_1)$ and then use the reparameterization α to 'step' over this interval. In fact, any such $t \in I_A$ for which $\phi(t)$ is not defined as in the previous paragraph is between some t_0 and t_1 such that $\phi(t_0) = \phi(t_1) = z_0$, so this completes the construction of ϕ. Continuity of ϕ follows from the shortest paths passing through I_A converging uniformly to the shortest path γ_A. To clarify the existence of the reparameterization α, see also [Fal85, Lemma 3.1]. □

The second lemma that we will take from [ST96] is actually a consequence of Theorem 5.12. However, the details are quite technical, so we will omit the proof.

Lemma 5.4 ([ST96, Lemma 2.3])**.** *Every Jordan arc $\sigma : [0,1] \to E(A,B)$ constructed as in Lemma 5.3 is rectifiable.*

In fact, for sufficiently small radius, a wedge will only contain a single Jordan arc segment of the equidistant set.

Lemma 5.5. *Let $x \in E(A,B)$ be given and let $W_\rho(x)$ be a wedge at x. There is a sufficiently small δ such that the intersection of $E(A,B)$ and $\overline{W_\varepsilon(x)}$ is a Jordan arc for all $0 < \varepsilon < \delta$.*

Proof. By Lemma 5.3, there is a Jordan arc $\sigma : [0,1] \to E(A,B) \cap \overline{W_\rho(x)}$ with $\sigma(0) = x$. We assume that ρ is taken sufficiently small so that $\sigma(1) \in \partial W_\rho(x)$. By way of contradiction, suppose that $W_{1/n}(x)$ contains a point $x_n \in E(A,B) \setminus \sigma([0,1])$ for all n. Passing to a subsequence if necessary, we can assume that $\{x_n\}_{n \geq 1/\rho}$ is contained in one of the two components of $W_\rho(x) \setminus \sigma([0,1])$, so we will assume it is in the component whose boundary contains part of γ_A, a shortest path from x to A. For each n, let η_n be a shortest path from x_n to B. Since $x_n \to x$, we must have a subsequence of $\{\eta_n\}$ which converges uniformly to a shortest path from x to B. However, this is impossible since η_n cannot cross γ_A or σ (Lemma 4.14), and $W_{1/n}(x)$ does not intersect any shortest path from x to B. $\square$

Corollary 5.6. *There are only finitely many $x \in E(A, B)$ which admit more than two wedges.*

Proof. If there was an infinite sequence of distinct points $\{x_n\}_{n=1}^{\infty} \subseteq E(A, B)$ which admit more than two wedges, then by the compactness of $E(A, B)$, there is a subsequence which converges to some $\bar{x} \in E(A, B)$. By Lemma 5.2 there are only finitely many wedges at $\bar{x}$, so there is a wedge $W_\rho(\bar{x})$ containing infinitely many of these points for any $\rho > 0$, which contradicts Lemma 5.5. $\square$

Theorem 5.7. *Let X be a compact 2-dimensional Alexandrov space of curvature bounded below (possibly with boundary). For any pair of disjoint nonempty compact subsets $A, B \subseteq X$, the equidistant set $E(A, B)$ is homeomorphic to a finite closed simplicial 1-complex.*

Proof. First, suppose that X has no boundary. At each $x \in E(A, B)$ which admits more than two wedges (of which there are finitely many), we place a vertex at x. By compactness of $E(A, B)$ and Lemma 5.5, there is a natural concatenation of finitely many Jordan arcs, which forms an edge between appropriate vertices. In the case that a point x belongs to a component of $E(A, B)$ that is a simple closed curve (so there are no points with more than two wedges) we simply place a vertex at x and then create our edge of Jordan arcs as before. Thus, $E(A, B)$ is homeomorphic to a finite simplicial 1-complex.

In the case that X has nonempty boundary, we apply the doubling theorem to get a space without boundary. By our earlier discussion, $E(A, B)$ is a finite simplicial 1-complex in this space. By compactness, the boundary ∂X can only intersect $E(A, B)$ finitely many times. To see this, first note that $\partial X \cap E(A, B)$ cannot contain a Jordan arc. For the boundary to contain a Jordan arc segment of $E(A, B)$, there must be a wedge which is bisected by the boundary (Theorem 5.12), but then the directions that determine the wedge, γ_A and γ_B, must terminate at the same point, which is a contradiction. Therefore, the intersection happens at Jordan arcs in $E(A, B)$ passing through the boundary. If we assume there are infinitely many such intersections, then there must be a cluster point, but there can be no wedge at such a cluster point. Thus, removing the part of $E(A, B)$ contained in the 'doubled' part of the space and placing vertices at the finitely many points in ∂X leaves us with finite simplicial 1-complex. □

We can conclude from [ST96, Theorem A] that Hausdorff dimension of the equidistant sets (with the conditions of the above theorem) is always 1, since any 'endpoints' of the cut locus do not appear in the equidistant set (see [ST96, Example 4] for a fractal cut locus). However, it is possible to produce a non-fractal cut locus with infinite measure on a compact surface [Ito96, Theorem B]. Equidistant sets on compact surfaces, on the other hand, will always have finite measure.

Corollary 5.8. *Let X be a compact Alexandrov surface. If $A, B \subseteq X$ are nonempty disjoint and compact, then the* 1*-dimensional Hausdorff measure of $E(A, B)$ is nonzero and finite.*

Proof. Let $\{W_i\}_{i\in I}$ be a cover of $E(A,B)$ by open wedges which admit a unique rectifiable Jordan arc in their interior (see Lemma 5.5). Compactness of $E(A,B)$ implies we can find a finite subcover $\{W_k\}_{k=1}^n$. Let J_k be the unique Jordan arc contained in the closure of each wedge W_k of this finite subcover. Then we have

$$0 < \mathscr{H}^1(J_1) \leq \mathscr{H}^1(E(A,B)) \leq \sum_{k=1}^{n} \mathscr{H}^1(J_k) < \infty. \qquad \square$$

5.2 The Direction of the Equidistant Set

Although the equidistant set is not generally a smooth curve, given any point in the equidistant set, we can at least predict the directions from the starting point which stay in the equidistant set. These directions are precisely the bisector of any wedge at that point (Theorem 5.12). To see why this is true, first recall our function f_{AB} defined by

$$f_{AB}(x) = d(x,A) - d(x,B).$$

Since f_{AB} is a linear combination of distance functions (to compact sets) it always admits a one-sided directional derivative.

Theorem 5.9 ([FOV21])**.** *Let X be an Alexandrov space, $\gamma : [0,T] \to X$ a shortest path, and K a compact set not containing $\gamma(0)$. Then*

$$\lim_{t\to 0} \frac{d(\gamma(t),K) - d(\gamma(0),K)}{t} = -\cos(\angle_{\min})$$

where $\angle_{\min}$ is the infimum of angles between γ and any shortest path connecting

$\gamma(0)$ *to* K.

Letting x be a point of $E(A,B)$, and $W_\rho(x)$ be a wedge at x, if γ is a shortest path emanating from x, Theorem 5.9 gives us

$$\lim_{t\to 0^+} \frac{f_{AB}(\gamma(t)) - f_{AB}(\gamma(0))}{t} = -\cos\Big(\angle(\gamma,\Theta_A)\Big) + \cos\Big(\angle(\gamma,\Theta_B)\Big).$$

Since $E(A,B)$ is the level set $f_{A,B}=0$, the directional derivative in the equidistant set should be 0. Therefore, we expect the equidistant set to be locally well approximated by shortest paths satisfying

$$\cos\Big(\angle(\gamma,\Theta_A)\Big) = \cos\Big(\angle(\gamma,\Theta_B)\Big).$$

Let us now restrict ourselves again to 2-dimensional Alexandrov spaces. Fix $x \in E(A,B)$, and let $W_\rho(x)$ be a wedge bounded by the paths γ_A and γ_B, to A and B respectively. Given that there are no other shortest paths from x to A or B in the wedge,

$$\angle(\gamma,\Theta_A) = \angle(\gamma,\Theta_B) \iff \angle(\gamma,\gamma_A) = \angle(\gamma,\gamma_B).$$

Thus, we expect the direction of the equidistant set to be exactly halfway between the directions which determine the wedge. However, in an Alexandrov space there may be no shortest path which the desired direction, as the following example shows.

Example 5.10. Let C be the compact 2-dimensional Alexandrov space obtained by gluing two two closed unit disks along their boundaries. If a is the point at the center of one disk and b is the point at the center of the other disk, then $E(a,b)$ is precisely the identified boundary of the disks. At any point of $E(a,b)$, the space of directions is a unit circle, but there are no shortest paths which realize the two directions along the identified boundary.

Although there may not be a shortest path which admits the desired direction, we can always achieve the desired direction as a limit of shortest paths. Note that for any sequence $\{x_n\}_n \subseteq E(A,B) \cap W_\rho(x)$ such that $x_n \to x$, if γ_n is the shortest path from x to x_n, we expect find

$$\lim_{n\to\infty}\left(\lim_{t\to 0^+}\frac{f_{AB}\big(\gamma_n(t)\big) - f_{AB}(x)}{t}\right) = 0$$

since $f_{AB}(x_n) = f_{AB}(x) = 0$. This idea is formalized by the Theorem 5.12, but first we establish a preliminary lemma.

Lemma 5.11. *Let $x \in E(A,B)$ be a wedge at x, and let $W_\rho(x)$ be a wedge at x bounded by shortest paths γ_A and γ_B to A and B respectively. If there exists a shortest path $\eta : [0,T] \to \overline{W_\rho(x)}$ with $\eta(0) = x$ such that the image of η intersects $E(A,B)$ in an infinite sequence of distinct points converging to x, then*

$$\angle(\eta,\gamma_A) = \angle(\eta,\gamma_B) = \begin{cases} \frac{1}{2}\angle(\gamma_A,\gamma_B) & \text{if } \angle(\eta,\gamma_A) \le \frac{1}{2}\operatorname{diam} S_xX \\ \operatorname{diam} S_xX - \frac{1}{2}\angle(\gamma_A,\gamma_B) & \text{if } \angle(\eta,\gamma_A) > \frac{1}{2}\operatorname{diam} S_xX. \end{cases}$$

Remark. When $\operatorname{diam} S_xX = \pi$, the angle formula matches our Euclidean intuition.

Proof. Let $\{t_n\}_{n=1}^{\infty}$ be the sequence such that $\eta(t_n) \in E(A, B)$ for all n and $\eta(t_n) \to x$ (clearly $t_n \to 0$). By Theorem 5.9,

$$\begin{aligned}
0 &= \lim_{n\to\infty} \frac{f_{AB}(\eta(t_n)) - f_{AB}(x)}{t_n} \\
&= \lim_{n\to\infty} \frac{d(\eta(t_n), A) - d(x, A)}{t_n} - \lim_{n\to\infty} \frac{d(\eta(t_n), B) - d(x, B)}{t_n} \\
&= \lim_{t\to 0^+} \frac{d(\eta(t), A) - d(\eta(0), A)}{t} - \lim_{t\to 0^+} \frac{d(\eta(t), B) - d(\eta(0), B)}{t} \\
&= -\cos\Big(\angle(\eta, \gamma_A)\Big) + \cos\Big(\angle(\eta, \gamma_B)\Big).
\end{aligned}$$

Thus, $\angle(\eta, \gamma_A) = \angle(\eta, \gamma_B)$, so $[\eta]$ is one of (up to) two directions in S_xX equidistant to $[\gamma_A]$ and $[\gamma_B]$. If the wedge is acute, then $\angle(\eta, \gamma_A) = \frac{1}{2}\angle(\gamma_A, \gamma_B)$. Conversely, if the wedge is obtuse, then $\angle(\eta, \gamma_A) = \operatorname{diam} S_xX - \frac{1}{2}\angle(\gamma_A, \gamma_B)$. $\square$

The following theorem shows that the the equidistant set has a (one-sided) tangent direction and it is the bisector of the directions which determine the wedge. Compare to [HPV17, Theorem A] or [ST96, Lemma 2.1].

Theorem 5.12. *Let X be a compact 2-dimensional Alexandrov space of curvature bounded below, $A, B \subseteq X$ be disjoint nonempty compact sets, and $x \in E(A, B)$. If $W_\rho(x)$ is a wedge at x, bounded by shortest paths γ_A and γ_B, to A and B respectively, then there is at least one arc length parameterized curve*

$$\sigma : [0, \tau] \to E(A, B) \cap \overline{W_\rho(x)}$$

with $\sigma(0) = x$*, and any such curve satisfies*

$$\angle(\sigma, \gamma_A) = \angle(\sigma, \gamma_B) = \begin{cases} \frac{1}{2}\angle(\gamma_A, \gamma_B) & \textit{if } \angle(\sigma, \gamma_A) \leq \frac{1}{2} \operatorname{diam} S_x X \\ \\ \operatorname{diam} S_x X - \frac{1}{2}\angle(\gamma_A, \gamma_B) & \textit{if } \angle(\sigma, \gamma_A) > \frac{1}{2} \operatorname{diam} S_x X. \end{cases}$$

Proof. By Lemmas 5.3 and 5.4, we know $E(A, B) \cap \overline{W_\rho(x)}$ contains a path of finite length beginning at x. Given that every path of finite length admits an arc lenght parameterization [BH99, Proposition I.1.20], the existence of an arc length parameterized path $\sigma : [0, \tau] \to E(A, B) \cap \overline{W_\rho(x)}$ is guaranteed, we only need to verify the formula for the angle. In fact, by Lemma 5.5, we can assume that σ is unique as long as τ is sufficiently small.

Let $\theta^* \in S_x X$ be the direction which is the desired bisector of $[\gamma_A]$ and $[\gamma_B]$, the directions determining our wedge. Let $\varepsilon > 0$ be given and let $B_\varepsilon(\theta^*)$ be the open interval of radius ε around $\theta^* \in S_x X$. In order to prove the theorem, we will show that for all sufficiently small $s > 0$,

$$\sigma(s) \in C_{\pm\varepsilon} := \exp_x \Big(B_\varepsilon(\theta^*) \times (0, \delta) \Big) \tag{5}$$

where $\delta > 0$ is appropriately small.

If (5) is not true, then there are two possibilities:

(i) there is a constant $c > 0$ such that $\sigma(s)$ is outside of $C_{\pm\varepsilon}$ whenever $0 < s < c$; or

(ii) there is a sequence $\{s_n\}_{n=1}^\infty$ with $s_n \to 0$ such that infinitely many $\sigma(s_n)$ are in $C_{\pm\varepsilon}$ and infinitely many $C_{\pm\varepsilon}$ are outside of $C_{\pm\varepsilon}$.

If case (i), then without loss of generality, we can assume that $d(z, A) >$

$d(z, B)$ for all z in $C_{\pm\varepsilon}$. Letting γ be a shortest path with $[\gamma] \in B_\varepsilon(\theta^*)$ and $\angle(\gamma, \gamma_A) < \angle(\gamma, \gamma_B)$, we have

$$\begin{aligned} 0 &= \lim_{t \to 0^+} \frac{f_{AB}(\sigma(t)) - f_{AB}(x)}{d(\sigma(t), x)} \\ &\leq \lim_{t \to 0^+} \frac{f_{AB}(\gamma(t)) - f_{AB}(x)}{t} \\ &= -\cos\Big(\angle(\gamma, \gamma_A)\Big) + \cos\Big(\angle(\gamma, \gamma_B)\Big) \\ &< 0 \end{aligned}$$

which is a contradiction.

If case (ii), then we can find a shortest path γ with $0 < \angle(\gamma, \theta^*) \leq \varepsilon$ such that σ intersects γ infinitely many times in a sequence which converges to x. However, this contradicts Lemma 5.11.

We conclude that $\sigma(s) \in C_{\pm\varepsilon}$ for all s sufficiently small. Since ε can be taken arbitrarily close to 0, we find $[\sigma]$ exists and $[\sigma] = \theta^*$. □

5.3 A Bound on the Homology of the Equidistant Set

Now that we know the equidistant set is a 1-complex, it is appropriate to ask if we can provide any further details on the structure. While the specifics of the equidistant set will depend greatly on the structure and placement of the sets A and B, we can at least provide an upper bound on the dimension of the first homology group (Theorem 5.13). This bound depends on both the genus of the surface as well as the number of disconnected pieces of A and B.

The following theorem is a generalization of [VB06, Theorem 4.2] for $n = 2$.

Theorem 5.13. *Let X be a compact Alexandrov surface (without boundary). If $A, B \subseteq X$ are nonempty disjoint and compact, then $H_1(E(A,B)) = \mathbb{Z}^k$ for some positive integer k satisfying*

$$1 \leq k \leq \dim H_1(X;\mathbb{Z}_2) + \dim H_0(A) + \dim H_0(B) - 1.$$

Proof. For simplicity of notation, let E be the equidistant set $E(A,B)$. Since the equidistant set is a simplicial 1-complex (Theorem 5.7), $\dim H_1(E) = \dim H_1(E;\mathbb{Z}_2)$, so we will use $\mathbb{Z}_2$ coefficients to avoid issues of orientability. Consider the following portion of the long exact sequence for the pair (X,E):

$$\cdots \xrightarrow{\partial_3} H_2(E;\mathbb{Z}_2) \xrightarrow{i_2} H_2(X;\mathbb{Z}_2) \xrightarrow{p_2} H_2(X,E;\mathbb{Z}_2)$$

$$\xrightarrow{\partial_2} H_1(E;\mathbb{Z}_2) \xrightarrow{i_1} H_1(X;\mathbb{Z}_2) \xrightarrow{\quad p_1 \quad} \cdots.$$

First, we observe that E has empty interior (Corollary 4.15), so $H_2(E;\mathbb{Z}_2) = 0$. Furthermore, X is a closed manifold (Proposition 3.10(iii)), so $H_2(X;\mathbb{Z}_2) = \mathbb{Z}_2$. We claim that $H_2(X,E;\mathbb{Z}_2) = (\mathbb{Z}_2)^\ell$ where ℓ is a positive integer satisfying

$$2 \leq \ell \leq \dim H_0(A;\mathbb{Z}_2) + \dim H_0(B;\mathbb{Z}_2). \tag{6}$$

Before proving the claim, we show how it proves the theorem. Filling in the known homology groups, our portion of the long exact sequence becomes

$$\cdots \xrightarrow{\partial_3} 0 \xrightarrow{i_2} \mathbb{Z}_2 \xrightarrow{p_2} (\mathbb{Z}_2)^\ell \xrightarrow{\partial_2} H_1(E;\mathbb{Z}_2) \xrightarrow{i_1} H_1(X;\mathbb{Z}_2) \xrightarrow{p_1} \cdots.$$

Exactness of this sequence tells us that $\ker(\partial_2) = \mathrm{im}(p_2) = \mathbb{Z}_2$, so we must have $\mathrm{im}(\partial_2) = (\mathbb{Z}_2)^{\ell-1}$. Given that $\ell \geq 2$, this gives us the lower bound for $\dim H_1(E; \mathbb{Z}_2)$. To get the upper bound, notice that $\ker(i_1) = \mathrm{im}(\partial_2) = (\mathbb{Z}_2)^{\ell-1}$, and $\mathrm{im}(i_1)$ is contained in $H_1(X; \mathbb{Z}_2)$, so we have $\dim H_1(E; \mathbb{Z}_2) \leq \dim H_1(X; \mathbb{Z}_2) + \ell - 1$.

To prove $H_2(X, E; \mathbb{Z}_2) = (\mathbb{Z}_2)^\ell$ with ℓ as in (6), we define the open sets

$$X_A = \{x \in X : d(x, A) < d(x, B)\} \quad \text{and} \quad X_B = \{x \in X : d(x, B) < d(x, A)\}.$$

Let ℓ_A and ℓ_B be the number of components of X_A and X_B respectively. First, we will show that ℓ_A (and subsequently, ℓ_B) is finite. Let

$$\varepsilon = \inf\{d(a, b) : a \in A, b \in B\}.$$

Since A and B are compact and disjoint, $\varepsilon > 0$. By compactness of A, we can find finitely many $a_i \in A$ such that $A \subseteq \bigcup_{i=1}^{N} B_{\varepsilon/3}(a_i)$. Furthermore, by the definition of ε,

$$\bigcup_{i=1}^{N} B_{\varepsilon/3}(a_i) \subseteq X_A.$$

By Corollary 4.16, X_A can have no more than N components (one for each a_i). Thus ℓ_A is finite. To be more precise, one can use the same reasoning as Corollary 4.17 to see that if $B_{\varepsilon/3}(a_i) \cap B_{\varepsilon/3}(a_j)$ is nonempty, then these balls lie in the same component of X_A. It follows that $1 \leq \ell_A \leq \dim H_0(A)$. Applying the same reasoning for ℓ_B, we see that $X \setminus E$ has $\ell_A + \ell_B$ components. We conclude, via Lefschetz duality, that

$$\dim H_2(X, E; \mathbb{Z}_2) = \dim H^0(X \setminus E; \mathbb{Z}_2) = \ell_A + \ell_B$$

where $2 \leq \ell_A + \ell_B \leq \dim H_0(A) + \dim H_0(B)$. $\square$

While the upper bound on $\dim H_1(E(A, B))$ given in Theorem 5.13 can be quite large (or even infinite), it is easy to show that there is no better upper bound.

Example 5.14. Let S^2 be the unit sphere embedded in $\mathbb{R}^3$, and consider the sets

$$A = \{(1, 0, 0), (-1, 0, 0)\} \quad \text{and} \quad B = \{(0, 1, 0), (0, -1, 0)\}.$$

Then $E(A, B)$ consists of two great circles which intersect at the north and south poles of the sphere (see Figure 7), so we have

$$\dim H_1(E(A, B)) = 3 = 0+2+2-1 = \dim H_1(S^2)+\dim H_0(A)+\dim H_0(B)-1.$$

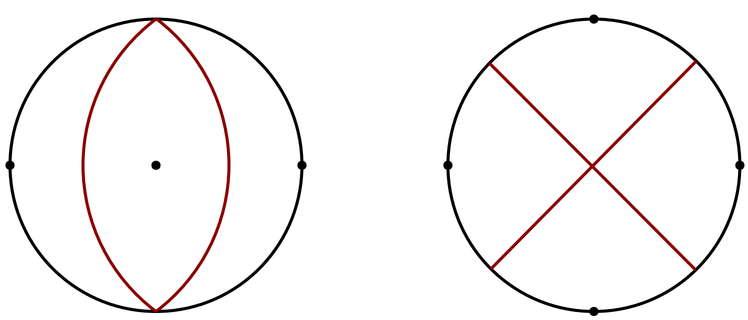

Figure 7: *Side view (left) and top view (right) of the equidistant set described in Example 5.14.*

The construction given in Example 5.14 is easily adjusted to produce an equidistant set with $\dim H_1(E(A, B)) = k$ for any odd positive integer, so there really is no upper bound on the homology. On the other hand, if we additionally assume that A and B are connected, then Corollary 4.17 gives the following corollary to Theorem 5.13. This bound for the homology is precisely that found in [VB06, Theorem 4.2] (for $n = 2$) when A and B are singleton sets.

Corollary 5.15. *Let X be a compact Alexandrov surface (without boundary). If $A, B \subseteq X$ are nonempty disjoint compact and connected, then $H_1(E(A, B)) = \mathbb{Z}^k$ for some positive integer k satisfying*

$$1 \leq k \leq \dim H_1(X; \mathbb{Z}_2) + 1.$$

6 Equidistant Sets in the Plane

We now use the techniques developed in previous sections to answer two prompts for equidistant sets in the plane given in [PS14].

6.1 Equidistant Sets Determined by Connected Sets

First, we address the following problem:

> [PS14, p. 32] *Problem*: Characterize all closed sets of $\mathbb{R}^2$ that can be realized as the equidistant set of two connected disjoint closed sets.

Interestingly, the answer to this question is already known: In [Bel75, Theorem 1] it is shown that the equidistant sets determined by two nonempty disjoint closed connected subsets of the Euclidean plane is a topological 1-manifold (i.e. the equidistant set is homeomorphic to a circle or the real line). However, the result is misquoted in [Lov76] (and subsequently in [PS14]) as being true only for *compact* subsets.[3] In order to illustrate the strength of some of the techniques developed in Section 5, we give a quick alternative proof of Bell's theorem.

Theorem 6.1. *If A and B are nonempty disjoint closed connected subsets of the Euclidean plane, then $E(A, B)$ is a topological 1-manifold.*

Proof. Let $x \in E(A, B)$ be given. In order to prove the theorem, we will show that for some $\varepsilon > 0$, $E(A, B) \cap B_\varepsilon(x)$ is the interior of a Jordan arc, and

[3] This is likely due to Bell's own abstract, which claims the result for "mutually disjoint plane continua" (a continuum being generally understood to be a connected compact set), but the statement of the theorem, and its proof, is for connected closed sets.

therefore every point of $E(A, B)$ is contained in a neighborhood homeomorphic to an open interval of the real line.

Fix a real number $r > d(x, A)$, and note that $\overline{B}_{3r}(x)$ is a compact Alexandrov surface (with boundary) of curvature ≥ 0. We define the compact sets

$$A_x = A \cap \overline{B}_{3r}(x) \quad \text{and} \quad B_x = B \cap \overline{B}_{3r}(x)$$

and consider $E(A_x, B_x)$ in the space $\overline{B}_{3r}(x)$. Since every $y \in \overline{B}_r(x) \cap E(A, B)$ satisfies

$$d(y, A) \leq d(y, x) + d(x, A) < 2r$$

we find $P_A(y) \subseteq \overline{B}_{3r}(x)$ (and the same is true for $P_B(y)$), so

$$E(A_x, B_x) \cap \overline{B}_r(x) = E(A, B) \cap \overline{B}_r(x).$$

Given that A and B are each connected, every $x \in E(A, B)$ admits exactly two wedges (if there were more than two wedges at a point, we could use the set A and two distinct paths from x to A to separate B from itself), so let $W_\rho^1(x)$ and $W_\rho^2(x)$ be two such wedges at x (for some sufficiently small $\rho > 0$). Applying Lemma 5.5 shows that we can find ε such that $0 < \varepsilon < \rho$ for which

$$E(A, B) \cap \overline{B}_\varepsilon(x) = \left(E(A, B) \cap \overline{W_\varepsilon^1(x)}\right) \cup \left(E(A, B) \cap \overline{W_\varepsilon^2(x)}\right)$$

is a Jordan arc. $\square$

6.2 Hausdorff Dimension of the Equidistant Set

We now answer the following question (the term *focal sets* refers to the sets A and B which determine the equidistant set):

> [PS14, p. 32] *Question*: Does there exist an equidistant set in the plane with connected disjoint focal sets, having Hausdorff dimension greater than 1? What about other notions of dimension? How does the dimension of an equidistant set depend on the dimension of its focal sets?

Note that the question does not require the *closure* of the sets be disjoint. This allows for interesting constructions, such as the example in [Wil75], which shows an equidistant set (with disjoint focal sets) can have Hausdorff dimension 2. This is achieved by using two 'interlocking combs' as the sets A and B. In fact, as the following example illustrates, we can exploit overlapping closures to produce equidistant sets in the plane of any Hausdorff dimension between 1 and 2.

Example 6.2. Let K be the (boundary of the) Koch snowflake. Define A as the bounded subset of $\mathbb{R}^2 \setminus K$ and B as the unbounded subset of $\mathbb{R}^2 \setminus K$. Then $A \cap B = \varnothing$, but $E(A, B)$ is precisely K, which has Hausdorff dimension $\log_3(4) \approx 1.26186$.

On the other hand, using two sets with overlapping closure is 'cheating' in some sense when working with equidistant sets, as we can have $d(x, A) = d(x, B) = 0$ even when x is outside of A and B. For this reason, it is more appropriate to ask the question for sets with disjoint closure. In this case, we find the equidistant set is always 1-dimensional, even if A and B are not connected.

Theorem 6.3. *If A and B are nonempty disjoint closed subsets of the Euclidean plane, then the Hausdorff dimension of $E(A,B)$ is* 1.

Proof. By the countable stability of the Hausdorff dimension [Fal90, p. 29],

$$\dim_{\mathscr{H}} \bigcup_{n=1}^{\infty} F_n = \sup_{1\leq n<\infty} \dim_{\mathscr{H}} F_n$$

so it is sufficient to show that the equidistant set is the union of countably many 1-dimensional sets. Letting $x \in E(A,B)$ and $n \geq 1$ be given, we will show that $\overline{B}_n(x) \cap E(A,B)$ is 1-dimensional, which implies that

$$\begin{aligned}\dim_{\mathscr{H}} E(A,B) &= \dim_{\mathscr{H}} \bigcup_{n=1}^{\infty} \left(\overline{B}_n(x) \cap E(A,B)\right) \\ &= \sup_{1\leq n<\infty} \dim_{\mathscr{H}} \overline{B}_n(x) \cap E(A,B) \\ &= 1.\end{aligned}$$

First, following the same reasoning as in Theorem 6.1, we will consider the compact subspace $\overline{B}_R(x)$, where $R > 2n + d(x,A)$, so that the equidistant set determined by the disjoint compact sets

$$A_x = A \cap \overline{B}_R(x) \quad \text{and} \quad B_x = B \cap \overline{B}_R(x)$$

satisfies $E(A,B) \cap \overline{B}_n(x) = E(A_x, B_x) \cap \overline{B}_n(x)$. It follows from Corollary 5.8

that $E(A_x, B_x)$ has Hausdorff dimension 1 and finite measure. Given that $E(A, B) \cap \overline{B}_n(x)$ contains some rectifiable curve,

$$0 < \mathscr{H}^1\Big(E(A, B) \cap \overline{B}_n(x)\Big) \leq \mathscr{H}^1\Big(E(A_x, B_x)\Big)$$

so $E(A, B) \cap \overline{B}_n(x)$ has Hausdorff dimension 1. $\square$

7 Conclusion

The central goal of this dissertation was to extend the study of equidistant sets on (compact) surfaces, which began with the pair of papers [VB06, BV07]. We were able to extend the theory in two key areas:

(i) Equidistant sets determined by disjoint compact sets behave in essentially the same manner as equidistant sets determined by distinct points.

(ii) Equidistant sets on compact Alexandrov surfaces retain the same properties as those on compact smooth surfaces.

We say 'essentially the same' in point (i) as there is of course one clear difficulty of passing from distinct points to disjoint compact sets. We can still guarantee that the equidistant set is a finite simplicial 1-complex (Theorem 5.7), but the equidistant set no longer separates the space into precisely two components. In fact, there is no upper bound on the number of components of $X \setminus E(A, B)$, even though the number is always finite (Theorem 5.13). On the other hand, as we have seen throughout, adding the assumption that A and B are connected returns this desirable property.

In this section, we will discuss further work and open questions in the study of equidistant sets.

7.1 Clarifying Rectifiability

We begin with a perhaps less significant question, which is motivated by Lemma 5.4. As mentioned before, this lemma was originally written for the cut locus to a compact set, but clearly applies to the equidistant set. However, it presents a challenge in that the proof given in [ST96] is very hard to follow.

It seems that this lemma would be a relatively straightforward consequence of Theorem 5.12, but limitations on time have prevented a more complete examination of this result, so we leave it as a question.

Question. Is it possible to improve the proof of Lemma 5.4, as given in [ST96, Lemma 2.3]? Moreover, can the proof be simplified by assuming we are in the equidistant set, rather than the cut locus?

7.2 Higher Dimensional Spaces

Theorem 5.7 gives a nice connection between the dimension of the ambient space and the dimension of equidistant sets: Loosely speaking, 2-dimensional spaces give rise to 1-dimensional equidistant sets. It is natural to ask what happens if the ambient space is 3-dimensional or even n-dimensional.

Simple examples show that the dimension of the equidistant set is always 1 less than that of the ambient space. However, it is difficult to prove this is true in general, as the techniques developed thus for (particularly, that of wedge neighborhoods) are specific to 2-dimensional spaces, so we pose the following question.

Question. If X is a compact n-dimensional Alexandrov space and $A, B \subseteq X$ are nonempty disjoint compact subsets, then is $E(A, B)$ homeomorphic to a finite simplicial $(n-1)$-complex?

One difficulty that arises for Alexandrov spaces of dimension $n > 2$ is that they are not generally n-dimensional topological manifolds (but they do contain an open dense set of points locally homeomorphic to an open set in

$\mathbb{R}^n$ [BBI01, Theorem 10.8.3]). As such, it may be more appropriate to assume that X is a Riemannian manifold. In fact, for Riemannian manifolds, one can easily generalize Theorem 5.13. See also [VB06, Corollary 4.6].

Theorem 7.1. *Let X be a compact n-dimensional Riemannian manifold (without boundary). If $A, B \subseteq X$ are nonempty disjoint and compact, then*

$$1 \leq \dim H_{n-1}(E(A, B); \mathbb{Z}_2) \leq \dim H_n(X; \mathbb{Z}_2) + \dim H_0(A) + \dim H_0(B).$$

Circling back to the question, suppose that X is a compact 3-dimensional Alexandrov space, and let $A, B \subseteq X$ be nonempty disjoint and compact. Fixing any point $x \in X$, we know that S_xX is an Alexandrov surface (of curvature ≥ 1) and $\Theta_A, \Theta_B \subseteq S_xX$ are nonempty disjoint and compact. Applying Theorem 5.7, we see that the equidistant set $E(\Theta_A, \Theta_B)$ is a simplicial 1-complex. One might hope that $E(A, B)$ would be locally well-approximated by applying the exponential map to $E(\Theta_A, \Theta_B)$. However, it it possible to construct an example for which this is not the case [BS09, Example 3.1]. Assuming there is a way to overcome this difficulty, it may then be possible to answer the question by induction on dimension.

7.3 Counting Minimal Separating Sets

Let M_g be a compact orientable smooth genus g surface without boundary. In [BV07], equidistant sets determined by distinct points on M_g are classified (up to homeomorphism) for $g = 0$ and $g = 1$. The classification for genus $g \geq 2$ is still an open question, and it is conjectured that the number of equidistant sets (determined by pairs of distinct points) is the same as the number finite simplicial 1-complexes that can be embedded in M_g as a minimal separating set [BV07, Conjecture 3.1].

Recall from Corollary 4.17, that if $A, B \subseteq M_g$ are nonempty disjoint compact and connected, then $E(A, B)$ is minimal separating in M_g. Applying Corollary 5.15, we quickly see that the same arguments used in [BV07] give us the exact same classification for equidistant sets when A and B are connected as when A and B are points. In other words, if $g = 0$, then $E(A, B)$ is homeomorphic to a circle; and if $g = 0$, then $E(A, B)$ is homeomorphic to one of five graphs given in [BV07, Theorem 2.4]. Given this information, we update the previous questions of classifying equidistant sets and minimal separating sets.

Question. Let M_g be a compact orientable smooth genus g surface without boundary. For each $g \geq 2$, how many distinct (up to homeomorphism) equidistant sets $E(A, B)$ can be produced from nonempty disjoint compact and connected subsets $A, B \subseteq M_g$?

Question. Let M_g be a compact orientable smooth genus g surface without boundary. For any fixed g, is the number of equidistant sets determined by nonempty disjoint compact and connected subsets the same as the number finite simplicial 1-complexes that can be embedded in M_g as a minimal separating set?

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