Students' Reasoning about the Concept of Limit in the Context of reinventing the Formal Definition

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STUDENTS’ REASONING ABOUT THE CONCEPT OF LIMIT IN THE 
CONTEXT OF REINVENTING THE FORMAL DEFINITION

by

CRAIG ALAN SWINYARD

A dissertation submitted in partial fulfillment of the 
requirements for the degree of

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in
MATHEMATICS EDUCATION

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The abstract and dissertation of Craig Alan Swinyard for the Doctor of Philosophy in Mathematics Education were presented August 12, 2008, and accepted by the dissertation committee and the doctoral program.

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ABSTRACT


Title: Students' Reasoning about the Concept of Limit in the Context of Reinventing the Formal Definition

Many researchers (Artigue, 2000; Bezuidenhout, 2001; Cornu, 1991; Dorier, 1995) have noted the vital role limit plays as a foundational concept in analysis. The vast majority of topics encountered in calculus and undergraduate analysis are built upon understanding the concept of limit and being able to work flexibly with its formal definition (Bezuidenhout, 2001). The purpose of this study was to: 1) Develop insight into students' reasoning about limit in relation to their engagement in instruction designed to support their reinventing the formal definition of limit, and; 2) Inform the design of principled instruction that might support students' attempts to reinvent the formal definition of limit. The first objective was at the foreground of the study and was set against the broader background goal of contributing to an epistemological analysis (Thompson & Saldanha, 2000) of the concept of limit of a real-valued function and its formal definition. A central aim of epistemological analysis is to identify and understand key aspects of what might be
entailed in coming to understand a particular concept in relation to engagement with appropriate instruction.

In separate teaching experiments, two pairs of students successfully reinvented a definition of limit capturing the intended meaning of the conventional $\varepsilon$-$\delta$ definition. Analyses of the data generated in the teaching experiments revealed thematic elements of students' reasoning in the context of reinvention. For instance, the students' ability to shift from an *x-first perspective* (i.e., focusing first on $x$-values approaching the limiting value $a$ and then on corresponding $y$-values approaching a particular value $L$) to a *y-first perspective* (i.e., considering first a range of output values around a predetermined limit candidate $L$ and then establishing the existence of an interval of input values that would result in corresponding output values within the specified range) appeared paramount in their attempts to reinvent and reason coherently about the formal definition. This dissertation traces the evolution of the students' definitions over the course of two ten-week teaching experiments, and highlights thematic findings which point to what might be entailed in coming to reason flexibly and coherently about limit and its formal definition.
To my grandmothers, Wilma Harris and Betty Swinyard,
for emphasizing family, education, and a diligent work ethic.
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I would also like to thank the other members of my dissertation committee. I thank Dr. Karen Marrongelle, who patiently helped to steer me when I had no direction. Karen’s insights, questions, and encouragement speak volumes about her commitment to high quality research. I thank Dr. Sean Larsen, who generously devoted hours of his time and expertise to help construct tasks that would evoke the type of reasoning I sought. In addition, Sean’s editorial suggestions and guidance as I wrote preliminary articles relating to this dissertation study helped me refine my thinking about my research in important ways. I thank Dr. J. Michael Shaughnessy for modeling how to be an expert teacher and quality human being. Mike’s passion for mathematics education is infectious – thank you, Mike, for your advice and encouragement these past eight years. I thank Dr. Christine Cress for her early involvement in this study, and Dr. Stephen Isaacson for his willingness to
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Many years ago, when I was unsure of what I should do with my occupational life, my Uncle Lewis gave me an opportunity to stand in front of his classroom. In a unique and significant way, his encouragement gave me the confidence to pursue this path. Everyone should be fortunate enough to have an Uncle Lewis in their life.

If one word could describe my immediate family, it would be “supportive.” I thank my sister, Jill, for always looking out for me and for teaching me how to relentlessly pursue what you feel is rightfully yours. And I thank my parents, without whom I never would have had the chance to, nor would have learned the importance of, pursuing my passion. I am blessed.

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Chapter 1 – Introduction

Many researchers (e.g., Artigue, 2000; Bezuidenhout, 2001; Cornu, 1991; Dorier, 1995) have noted the vital role limit plays as a foundational concept in calculus and analysis. Cornu (ibid) notes that the limit "holds a central position which permeates the whole of mathematical analysis – as a foundation of the theory of approximation, of continuity, and of differential and integral calculus" (p.153). Indeed, limits arise in a variety of mathematical contexts, including differentiation and integration, the convergence and divergence of infinite sequences and series, applications related to determining measurable quantities of geometric figures (e.g., arc length, area, and volume), and describing the behavior of real-valued functions.

The formal definition of limit has the idea of real-valued functions as its object of focus. Robust understanding of the formal definition of limit is foundational as students proceed to more formal, rigorous mathematics – the vast majority of topics encountered in an undergraduate analysis course, where students study the theoretical underpinnings of calculus, are built upon formal limit understanding. “Well-constructed mental representations of the network of relationships among calculus concepts are essential for a thorough understanding of the conceptual underpinnings of the calculus, which includes the fundamental role of the limit concept” (Bezuidenhout, 2001, p.487). Continuity (both point-wise and uniform), derivatives, integrals, and Taylor series approximations are just a few of the topics studied in an analysis course for which limit serves as a foundational component.
Further, the formal definition of limit often serves as a starting point for developing facility with formal proof techniques, making sense of rigorous, formally-quantified mathematical statements, and transitioning to abstract thinking. Tall (1992) notes that the ability to think abstractly is a prerequisite for the transition to advanced mathematical thinking; Ervynck (1981) cites the limit as an opportunity for students to develop the ability to think abstractly. For all of the reasons discussed, the limit concept holds an important place in pedagogical considerations.

Research also indicates that understanding the concept of limit is problematic for students. Numerous studies (Bezuidenhout, 2001; Cornu, 1991; Cottrill et al., 1996; Davis & Vinner, 1986; Monaghan, 1991; Tall, 1992; Tall & Vinner, 1981; Williams, 1991) have enumerated the misconceptions students develop in their initial, informal explorations of the concept. A much smaller proportion of the research base has focused on the difficulty students have with formal treatments of the concept (Cornu, ibid; Dorier, 1995; Gass, 1992; Tall, ibid; Tall & Vinner, ibid; Williams, ibid). A few of these researchers have made conjectures about the source of students’ difficulty with the formal definition. The conventional $\varepsilon$-$\delta$ definition is rich with quantification and notation, and, according to Cornu (ibid), is cognitively sophisticated for first semester calculus students. Dorier (ibid) notes that the formal definition was “conceived for solving more sophisticated problems and for unifying all of them” (p.177), yet at the outset of calculus and introductory analysis, students likely have difficulty understanding the importance of a definition designed to unify problems they have yet to encounter. The general consensus seems clear –
calculus students have great difficulty reasoning coherently about the formal definition of limit. What remains unclear, however, is how students may come to understand the formal definition of limit. Indeed, this is an open question with few insights from research to inform it. The overarching purpose of this dissertation study was to generate such insights and to move toward the elaboration of a cognitive model of what might be entailed in coming to understand this formal definition. Specifically, the intent of this dissertation study was to: 1) Develop insight into students' reasoning in relation to their engagement in instruction designed to support their reinventing the formal definition of limit, and; 2) Inform the design of principled instruction that might support students' attempts to reinvent the formal definition of limit. To be clear, the first objective listed above was at the foreground of my study. Further, this first objective was set against the broader background goal of contributing to an epistemological analysis (in the sense of Thompson & Saldanha, 2000) of the concept of limit of a real-valued function and its formal definition. To contribute to an epistemological analysis is to gain insight into what is entailed in coming to understand a particular mathematical idea in relation to engagement in instruction designed to support the development of that understanding.

1.1 - Origin of Research Objectives and Research Questions

Extensive research (Bezuidenhout, 2001; Cornu, 1991; Davis & Vinner, 1986; Tall, 1992; Tall & Vinner, 1981; Williams, 1991) delineates the misconceptions
students commonly develop as they study the concept of limit informally. Significantly less is known, however, about how students reason about the formal definition of limit. Numerous studies (Cornu, ibid; Dorier, 1995; Fernandez, 2004; Gass, 1992; Larsen, 2001; Tall, ibid; Tall & Vinner, ibid; Williams, ibid) have indicated that the formal definition is decidedly complex for introductory calculus students. The formal definition of limit is a rigorous, formal statement that contains complex logical quantification and is rich with symbols and notation that introductory calculus students have not seen. "If a traditional $\varepsilon$-$\delta$ definition is attempted at the usual point early in the term, one runs the risk of confusing, discouraging, and alienating a sizable proportion of the class, thereby jeopardizing the prospects for a successful course" (Gass, ibid, p.9). Cornu (ibid) agrees - "[T]his unencapsulated pinnacle of difficulty occurs at the very beginning of a course on limits presented to a naïve student. No wonder they find it hard" (p.163). While the research has suggested that the formal definition of limit is difficult for students to understand, not much is known about how they might come to understand it. Indeed, the research base lacks empirically-driven insights about the conceptual entailments of this definition that could contribute to the development of a model of how students might come to understand it.

In their research, Cottrill et al. (1996) provide an initial genetic decomposition of the limit concept. This genetic decomposition is a seven step conjecture describing how students might come to understand the limit concept both informally and formally. Based on student data collected during the study, Cottrill
et al. refined the initial three steps of their genetic decomposition to better describe how students might come to reason about limits in an informal manner. Unfortunately, students in the study did not show sufficient evidence of reasoning at more sophisticated levels, and thus, the latter steps of the genetic decomposition were not refined or elaborated. Cottrill et al. conjectured that the latter steps of their genetic decomposition (i.e., reasoning coherently about the formal definition of limit) entailed formalizing one’s informal understandings of limit (i.e., formalizing one’s understanding of how to find limits). Larsen (2001) suggests, however, that developing an understanding of the formal definition of limit requires one to think in a very different manner – the thought process entailed in finding limits (which Cottrill et al. describe in the first three steps of their genetic decomposition) requires one to first consider x-values near the limiting value (x=a) and then consider corresponding function values (i.e., corresponding f(x)’s). To the contrary, the thought process entailed in validating limits (a role the formal definition plays) is fundamentally different, in that one must first consider a range of values along the y-axis, and only then consider a corresponding range of values along the x-axis. In other words, whereas informal discussions of limit primarily focus on the process of finding a viable candidate for the limit of a function, it is the formal definition that is used to validate such a candidate. Larsen’s study provides evidence that seeing the distinction between these thought processes is an important step in developing coherent understanding of the formal definition of limit. Other

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1 In this document, I will refer to the thought process associated with finding limits as an x-first perspective and the thought process associated with validating limits as a y-first perspective.
research (Bezuidenhout, 2001; Fernandez, 2004; Juter, 2006) indicates that students miss this important distinction.

Another important step in developing coherent understanding of the formal definition of limit appears to be building on one's informal understanding of the concept. Fernandez (2004) suggests that students may reason more coherently about the formal definition when allowed to build upon their spontaneous conceptions. In particular, she notes that when students in her study were not forced to use notation traditionally associated with the definition, they were more likely to reason coherently about the formal definition.

In the case of both Fernandez (2004) and Larsen (2001), students' reasoning about the formal definition of limit emerged in the context of interpreting the definition, in that students had been introduced to the formal definition in some manner prior to reasoning about it. Upon reflection, it occurred to me that interpreting a definition may very well result in a very different type of reasoning than if one were to attempt to reinvent the definition. Indeed, the formal definition of limit constructed by Cauchy, and subsequently Weierstrass, was motivated by a need to specify the local behavior of functions in a precise manner. Neither mathematicians' respective definitions were reformulations or interpretations of the traditional formal definition – to the contrary, these mathematicians constructed their definitions in response to an inherent need to classify functional behavior. It seemed, then, that I might stand to learn a great deal about how students reason about the formal definition of limit if I were to engage them in activities designed
to foster their reinvention of the formal definition of limit. Further, reinvention appeared to be a promising context for allowing students to build upon their spontaneous conceptions and informal understandings of limit, as Fernandez (ibid) suggests.

To reiterate, then, the intent of this dissertation study was to engage students in a sequence of instructional tasks with two objectives:

1. To develop insight into students' reasoning in relation to their engagement in instruction designed to support their reinventing the formal definition of limit, and;

2. To inform the design of principled instruction that might support students' attempts to reinvent the formal definition of limit

The first objective listed above was at the foreground of my study, and efforts to address it were guided by three central research questions:

i. Can students fitting the specified selection criteria, and in the context of guided reinvention, reinvent a definition of limit which captures the intended meaning of the conventional \( e-\delta \) definition?

ii. In the process of reinvention, what cognitive difficulties do students experience which hinder their progress, and how are such difficulties resolved?

iii. In their attempts to reinvent the definition of limit, do students reason from an \textit{x-first perspective} initially and, if so, what types of tasks help initiate a shift to a \textit{y-first perspective}?
The research objectives and related research questions presented here were designed with the aim of contributing to the research base on students' understanding of the formal definition of limit.

In this study, in the context of their engagement with instructional tasks designed to support coherent reasoning about the concept of limit, two pairs of above-average undergraduate students who had completed a three-term introductory Calculus sequence were each able to successfully reinvent a definition of limit capturing the intended meaning of the conventional \( \varepsilon-\delta \) definition. Evidence from the two teaching experiments which comprised this study: 1) corroborate Larsen's claim (2001) that reasoning about the formal definition of limit requires one to use a thought process distinct from the thought process typically employed when finding limit candidates; 2) suggest that defining closeness might support students in operationalizing the notion of infinite closeness, an idea which is fundamental to the limit concept, and; 3) underscore the importance of implementing the notion of arbitrary closeness as a way to resolve the cognitive difficulties that arise from reasoning from a potential infinity perspective. In the chapters that follow, I detail the literature relevant to this study, theoretical perspectives which framed the study, the methodological design and analytic approaches that were employed, and the nature of the two teaching experiments. I view this dissertation study as unique, in that inferences about how students reason about the formal definition of limit are made in the context of reinvention, as opposed to interpretation, of the formal definition.
1.2 - Chapter Abstracts

Beyond this introductory chapter, this document is comprised of six chapters, each devoted to articulating particular aspects of the study. In Chapter 2, I develop a critical examination of the body of research relevant for my study, drawing implications from it and situating my research within the current base of knowledge of the subject. The purpose of the second chapter is twofold: 1) to inform the reader of notable research previously conducted on the limit concept; and, 2) to inform the reader as to how this prior research informed this dissertation study.

In Chapter 3, I elaborate the theoretical perspectives that framed the study. The theoretical perspectives that framed my study are situated on two levels. Radical constructivism and the research paradigms of developmental research and teaching experiment methodology served as meta-level perspectives, and are described in Sections 3.1 and 3.2. The mathematical analysis of limit I provide in Section 3.3 was a domain-specific perspective that guided my methodology.

In Chapter 4, I explicate the research methodology and design, as well as the analytic procedures, which were employed in the study. In Section 4.1, I provide a general overview of the research design, which includes descriptions of the stages of research, the research cycle, and the data collection methods that were employed. In Section 4.2, I discuss participant selection and provide background information about the four students who participated in the teaching experiment phase of the study. In Section 4.3, I summarize the research instruments that were used in the two phases of the study, and I explain the overarching purpose of each
instrument. In Section 4.4, I discuss how data analysis was conducted, and include a detailed outline of the phases of the analysis. In Section 4.5, I address issues of validity and ethics related to the study.

In Chapters 5 and 6, I detail the results of the first and second teaching experiments, respectively, highlighting three products of each experiment which emerged in tandem: the phases of instruction, the evolution of the students’ characterization of limit, and the emergent themes which characterize student reasoning and point to subsequent pedagogical implications. These two chapters each consist of two main parts. In Part 1, I provide an overview of the instructional sequence, painting in broad strokes the unfolding of instruction across the teaching experiment and highlighting instructional goals and tasks. In Part 2, as I describe in greater detail the evolution of the students’ characterization of limit, I discuss the themes which emerged from my analysis of the data.

The seventh and concluding chapter consists of five parts. In Section 7.1, I summarize the central findings of the study, focusing on thematic elements which emerged during the two teaching experiments. In Section 7.2, I discuss the pedagogical implications of the central findings presented in Section 7.1. In Section 7.3, I describe how the study helps to address a gap in the research base on students’ understanding of limit. In Section 7.4, I discuss three limitations of the study germane to the specific research objectives that guided my work. Finally, in Section 7.5, I suggest possibilities for future research, based partially on the implications of the limitations discussed in Section 7.4.
Chapter 2 – Literature Review

The purpose of this chapter is twofold: 1) to inform the reader of notable research previously conducted on the limit concept; and 2) to inform the reader as to how this prior research informs this dissertation study. Though there are numerous ways the existing research on limit could be categorized, I choose to separate the literature into two broad categories: 1) informal limit research; and 2) formal limit research. I define informal limit research here as research that does not have, as its focus, the ways in which students reason about the formal definition of limit. Informal limit research comprises the vast majority of the research base on limit and will be the focus of Section 2.1 of this chapter. The informal limit research can subsequently be separated into four sub-categories. The first sub-category of informal limit research, described in Section 2.1.1, is comprised of research that addresses four important pre-calculus conceptions that affect one’s reasoning about the limit concept: algorithmic procedures, variable, covariational reasoning and domain and range, and the understanding of functions. The second sub-category, described in Section 2.1.2, demarcates the many misconceptions that students experience, at least initially. The themes I develop here center on substitution and continuity, informal language, and difficulties with infinity. The third sub-category, described in Section 2.1.3, is research that seeks to address the

Like any classification system, this one has both positive and negative aspects. It was ultimately chosen because it aligns with and helps illustrate what I believe are some of the deficiencies in the research base.
misconceptions discussed in Section 2.1.2 through the use of instructional innovations, such as counterexamples and other intended means of cognitive conflict, including the use of technology. The fourth sub-category of informal limit research, described in Section 2.1.4, is comprised of research that attempts to describe what students do understand about limits, as opposed to what misconceptions they possess.

In contrast to informal limit research, very few studies have explored how students reason about the formal definition of limit. While the volume of formal limit research is much less than that of informal limit research, it is nevertheless important because it provides useful models that can serve as starting points for studying students' conceptions of the formal definition of limit. In Section 2.2 of this chapter I discuss research that has suggested sources for students' struggle with the formal definition of limit, as well as research that describes how students reason about mathematical definitions. I also discuss research related to an important distinction that exists between the processes of finding limit candidates and validating limit candidates.

In Section 2.3, I conclude the chapter with a brief discussion of the implications for research that follow from the existing studies. All of the research described in this chapter is content-related (i.e., directly or indirectly related to the student's understanding of the limit concept). There are other studies that inform my research that are not content-related. These are discussed in Chapter 3.
2.1 - Informal Limit Research

2.1.0 Some Preliminary Comments

Students are likely to have experienced elements of the limit concept prior to their first calculus course. For example, in their study of functional behavior in pre-calculus, students may have worked with asymptotic behavior. As such, students likely enter the calculus classroom with some initial understanding of aspects of the limit concept. Typically however, students' first opportunity to engage knowingly with the limit concept occurs during differential calculus. It is in this first encounter with limit that researchers are likely to have the purest view of students' conceptions of limit, as in subsequent encounters the limit notion is embedded in other topics such as derivative and integral. It comes as no surprise, then, that the vast majority of research to date on students' conceptions of limit focuses on first quarter calculus students' understanding.

I assert that there are a host of concepts from pre-calculus that influence a student’s ability to understand limits. These include the use of algorithmic procedures, algebraic notions such as variable, absolute value, and inequalities, and concepts related to the theory of functions, such as covariational thinking, and domain and range. Before describing what research reveals about students' misconceptions of limit, I will first briefly discuss research related to important pre-calculus conceptions that impact students' understandings of limits. Following that,

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3 For the purpose of this document, I will discuss the research and my own findings within the structure of the quarter system and will use the phrases "first quarter calculus" and "Differential calculus" interchangeably.
I will describe the central findings of the misconception research, instructional approaches that address student misconceptions, and informal limit research that focuses on what students do understand about limits.

2.1.1 Important Pre-Calculus Conceptions

2.1.1.1 Algorithmic Procedures. Students' approaches to handling a new concept is shaped by their prior experience in that field, be it mathematics, physics, biology or otherwise. Like any field, students learn associated conventional skills and behaviors and then rely on those skills and behaviors as they encounter new concepts. It comes as no surprise, then, that many students initially attempt to impose algorithms or procedures on the limit concept. Prior to limits, the majority of mathematical problems a student has encountered have succumbed to some type of algorithmic or procedural approach. When students initially encounter limits, it is, for some, the first time in their mathematical career that the problem at hand cannot be solved with a straightforward algorithmic or procedural approach. According to Cornu (1991), this is likely troubling for students because it fails to match their mathematical behavior to date.

[T]he initial stages of calculus no longer rely purely on simple arithmetic and algebra....This obstacle makes the comprehension of the limit concept extremely difficult, particularly because a limit cannot be calculated directly using familiar methods of algebra and arithmetic (p.161).

Cornu's comment suggests that the nature of limit requires students to make cognitive accommodations (in the spirit of von Glasersfeld, 1995) for the concept from the outset.
2.1.1.2 Variable. The comment above by Cornu (1991) suggests that students, in reasoning about limits, can no longer rely solely on the same procedures and algorithms they have used previously in their mathematical careers. However, the very nature of the limit concept does require students to rely on other pre-calculus concepts, most notably their conceptual understanding of variable and covariational thinking, both of which presumably precede the study of calculus. Work by White and Mitchelmore (1996) indicates that most students have an abstract-apart understanding of variable — that is, they “treat variables as symbols to be manipulated rather than as quantities to be related” (p.91). White and Mitchelmore claim that, “abstract-apart ideas are formed without any true abstraction” (p.92) and posit that a prerequisite to success in understanding key calculus concepts, including limit, is having an abstract-general understanding of variable, wherein one can both successfully manipulate symbols and understand the intended conventional meaning that underlies those symbols. White and Mitchelmore are careful to point out that the abstract-apart and abstract-general notions are not meant to establish a dichotomy. Rather, a continuum exists between the two poles. Students characterized as having an abstract-apart conceptual understanding are capable of working with symbols proficiently with little or no understanding of the contextual meaning. The conventional definition of limit is laden with quantification and notation designed to capture significant ideas. If students have merely an abstract-apart understanding of variable, they likely will fail to develop a flexible, conceptual understanding of the formal definition of limit. According to
White and Mitchelmore, "[t]he inevitable conclusion is that a prerequisite to a successful study of calculus is an abstract-general concept of a variable" (p.93).

2.1.1.3 Covariational Reasoning and Domain & Range. Carlson, Larsen, and Jacobs (2001) believe that covariational reasoning, defined as "coordinating images of two varying quantities and attending to the ways in which they change in relation to each other" (p.145), plays an important role in limit understanding. In a study conducted by Carlson, Larsen, and Jacobs (ibid), 24 students were asked to determine the limit at \( x = 2 \) for two different functions. All but two of the students gave correct responses for the first problem and all but three of the students gave correct responses for the second problem. In follow-up interviews, each student gave explanations for their thinking that relied heavily on covariational reasoning, suggesting that the dynamic relationship that exists between the \( x \) and \( y \) variables is a pertinent matter to which students who can successfully compute limits attend. It seems that students must recognize and understand the existing dynamic between the independent variable \( x \) and the dependent variable \( y \) if they are to comprehend the most important aspects of the limit concept. Other research (Cottrill et al., 1996; Craighead & Fleck, 1997; Ferrini-Mundy & Graham, 1991) echoes the necessity for strong covariational reasoning prior to the study of calculus.

Related to the notion of covariational reasoning are the mathematical notions of domain and range, two concepts that research suggests students must be comfortable with before adequate understanding of limit can occur. Research by Cottrill et al. (1996) indicates that before students can be cognitively prepared for
more sophisticated levels of limit understanding, they must first be comfortable with the concepts of domain and range, as well as important connections between them. Specifically, Cottrill et al. posit that students must first construct a domain process for limit, then construct a range process for limit, and finally construct a coordinated schema between the two processes. Cottrill et al. suggest that students' failure to appreciate these connections between a function's domain and range makes them less likely to construct the necessary coordinated schema between the two processes.

2.1.1.4 Understanding of Functions. Students' understanding of the concept of functions also appears to impact the way in which they view the concept of limit. Ferrini-Mundy and Graham (1991) suggest that students who hold a restricted process conception of function and cannot view a function also as an object subsequently have a dynamic, process viewpoint of limit.

There is little evidence that the students see functions as objects of study in mathematics; rather, when a function is given, in equation form, usually one is expected to do something to it, such as substitute in a value. This part of studying functions...seems to be firmly established and becomes their way of working with other calculus concepts such as limit (p.630).

Ferrini-Mundy and Graham's research suggests that the ability to think about functions flexibly, as both a dynamic process and as a static object, is an important precursor to reasoning coherently about limits.

The studies referenced here speak little of student misconceptions. However, in a linear progression of learning, they provide insight into some pre-existing inhibitors that students may bring to the calculus classroom. In sum, students who
fail to have developed both an abstract-general view of variable and strong covariational reasoning are perhaps predisposed to some of the common misconceptions discussed below.

2.1.2 Student Misconceptions

A reasonable starting point for researching student cognition in any area is to assess whether student thinking fits with the type of thinking for which educators are hoping. Rather quickly it can become obvious that, at times, there are some significant gaps between the type of rich, flexible thinking desired by educators and the type of thinking initially prevalent among students. The goal then becomes one of describing and narrowing that gap. One way in which this gap can be described is by producing a laundry list of the deficiencies in student thinking. This approach, which I characterize as “misconception research,” is heavily characteristic of much of the research on limit. In the coming subsections of this part of the chapter, I describe what the research reveals to be common misconceptions of limit prevalent in student thinking, describing first confusion related to the use of substitution and the notion of continuity, then the numerous misconceptions that arise out of the use of informal language, and finally the difficulty students have with the notion of infinity.

2.1.2.1 Substitution and Continuity. The formal definition of limit, which is laden with notation and quantifiers, is cognitively sophisticated for first semester
calculus students to understand and interpret\(^4\). The cognitive sophistication entailed in understanding the formal definition has pedagogical implications. Pedagogically, it is often common practice to begin the exploration of a sophisticated mathematical concept by studying examples that are simplistic in nature. For example, when students study functions for the first time, often the examples students initially encounter are limited to linear, quadratic, and other "nicely behaved" functions. Research by Schwarz and Hershkowitz (1999) suggests that these *function prototypes* (i.e., examples of functions that receive extensive attention from teachers) might hinder students' conceptual development with functions. "[S]tudents who continue to hold only these images of functions may either become overly restrictive or not restrictive enough in deciding whether a particular expression represents a function" (Clement, 2001, p.747). When students begin their study of limits, they likely have not experienced the type of erratic behavior prevalent among functions studied in an undergraduate analysis course. In an effort to build upon students' prior knowledge, teachers may choose well-known continuous functions to introduce notions of limit (e.g., \(\lim_{x \to 3} x^2\)). Unfortunately though, in an effort to simplify the initial study of limits, students are too often presented with continuous functions whose limit can be computed by simply evaluating the function at the limiting value. Through no fault of their own, students quickly conclude that the limit of a function is simply the function value at

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\(^4\) A review of the literature pertaining to students' understanding of the formal definition of limit is provided in Section 2.2.
the point of interest (Bezuidenhout, 2001; Cottrill et al., 1996; Davis & Vinner, 1986; Tall, 1992; Williams, 1991). It comes as no surprise, then, that students later struggle to see the importance of continuity, for in their minds they see the notion of continuity as nothing new – this is merely their definition of limit. Bezuidenhout (ibid) reports that 37% of the students in her study thought that the existence of a limit implied continuity. She reports, “[t]his misconception may be mainly due to the use of a method of substitution to find limits algebraically, while there is a lack of understanding of the conceptual content underlying the procedure” (p.495).

Although the theory of limits originally developed in response to geometric problems, the theory evolved during the 17th and 18th centuries to address analytic problems as well. In fact, the limit concept became the basis of key ideas in calculus. For example, Boyer (1949) reports that Bolzano “gave a definition of continuous function which, for the first time, indicated clearly that the basis of the idea of continuity was to be found in the limit concept” (p.268). Hence, the limit concept was the basis for describing continuity. However, it is often discontinuous functions that receive the majority of attention when students are introduced to the limit concept. Indeed, if all functions were continuous, the need to consider the limits of functions might be lost since the height the function “intends” to reach at a particular x-value would be the height it actually reaches (i.e., its corresponding function value). To help students see the need for limits of functions, then, a

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5 Despite the fact that functions do not have human qualities, I ascribe the verb “intends” to functions because I have found that such a construct has pedagogical benefits when describing the limit concept to students.
teacher might provide a multitude of functions that have removable discontinuities or "holes." There is evidence (Williams, 1991) that some students who, rightfully, believe that the true benefit of limits is describing the behavior of functions around points of discontinuity, subsequently become confused and conclude that the limit concept is not applicable to continuous functions. As Williams reports, "considerable mental work is involved in getting the limit notion to work for continuous functions, because it involves imagining the point as not there" (p.226). A response from one student in Williams's study illustrates the confusion that can arise for students as they reason about the limit of a function at a point of continuity: "You never would really want to find a limit, you know, where there is a point, a continuous function. So it's, you know, kind of a moot point" (p.226). It is clear from the student's response that for this student, the concept of limit is restricted to points of discontinuity.

2.1.2.2 Informal Language. The preceding discussion suggests that prototypes can significantly impact a student's developing understanding of limit. Monaghan (1991) believes that "instructional paradigms intended to assist students' early understanding can create conceptual obstacles later" (p.24). Although some misconceptions in student thinking may arise out of teachers' overuse of prototypical examples, research shows that prototypes are not the only source of student misconceptions — confusion exists between the colloquial and mathematical meanings of key language that surrounds the limit concept. In an effort to develop students' intuitive understanding of limit and avoid premature use of formal limit
language involving $\varepsilon$’s and $\delta$’s, teachers often use informal language (Tall & Schwarzenberger, 1978). Examples include thinking of a limit as something that the function *tends to*, *converges to*, or *approaches*. These three verbs, as well as *limit*, evoke informal meanings for students that can cloud their mathematical understanding of limit. Reporting on a study that explored students’ interpretations of each of these four terms, Monaghan reports, “[f]rom the response to the open questions it is clear that the four [terms] generate everyday connotations that are at odds with the mathematical meanings” (p.23). Phrases such as “we can let $x$ get as close as we like to $a$” may evoke a myriad of colloquial meanings that research indicates are problematic for students’ development of static conceptions of limit. Davis and Vinner (1986) believe that words such as “limit” and phrases such as “$n$ goes to infinity” are such that they “unavoidably remind us of ideas that should not be part of our mental representation of the mathematical concept. The variable $n$ is not ‘going’ anywhere, and within many mathematical systems ‘infinity’ is not any place you could go to” (p.299). Research (Ferrini-Mundy & Graham, 1994; Monaghan, ibid) indicates that students’ concept images contain misleading ideas that are formed based on their prior experience with limit language that has been used colloquially. Below, I discuss common misconceptions that research suggests stems from students’ inaccurate application of colloquial language to the limit concept. These include the following notions: limits can be approached, but not

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6 *Concept image* refers to the non-verbal associations in a student’s mind related to the concept name (i.e., the collection of impressions or experiences related to the concept name) (Tall & Vinner, 1981).
reached; limits are bounds; limits carry with them an implied monotonicity; limits include the presence of motion; and finally, limits are an approximation.

One of the strongest misconceptions that students have with regard to limits is the notion that a limit is a value that is approached but never reached. An extensive list of studies provides clear evidence of this misconception (Cornu, 1991; Davis & Vinner, 1986; Ferrini-Mundy & Lauten, 1993; Lauten, Graham, & Ferrini-Mundy, 1994; Tall, 1980; Tall & Vinner, 1981; Williams, 1991). Students with this concept image of limit appear to be thinking of the limit in a way similar to that of a vertical asymptote. We see evidence of this in the following student responses regarding sequences:

A speed limit, like one on the highway, defines only a point beyond which you are not supposed to go. But the limit of a sequence is never reached by that sequence (Davis & Vinner, ibid, p.296).

The limit of a sequence is something that the limit approaches....Although the sequence approaches the value, it never actually gets there (Davis & Vinner, ibid, p.297).

This line of reasoning employed by the students above is problematic. For example, a student who believes that the limit of a sequence is never reached by the sequence may subsequently struggle to explain the behavior of constant sequences. Monaghan (1991) suggests that informal use of the phrases tends to and approaches is likely a source of this misconception. Monaghan also reports that students interpret tends to and approaches as synonymous, both representing movement towards the limit without getting there. This confusion is illustrated in the following response from a student in Monaghan's study: “It never actually gets
there, which is what tends to means to me. It means it approaches it or comes close to it but it won’t actually finally get there” (p.23).

A closely related misconception of limit that students often hold is that a limit is either an upper or lower bound (Cornu, 1991; Davis & Vinner, 1986; Frid, 1994). It is important to note that this misconception is slightly different than the one discussed previously, in that students here may allow for the limit to be reached, but not surpassed. In regards to sequences, we see evidence of this misconception in the following two student responses in Davis and Vinner’s study:

A limit is a boundary beyond which the sequence cannot go (p.296).
The limit is the point past which the sequence does not go...It will not reach whatever lies beyond this point (p.296).

Monaghan (1991) reports that the source of this student misconception is likely meanings derived from everyday use of words like limit. Monaghan suggests that students connect limits to speed limits, physical limits and mental limits – all things beyond which one should not or cannot go. “Whichever view students take, the everyday meaning of a limit as a boundary is clearly present” (p.23). Davis and Vinner also cite the influence of language as the source of this misconception:

Given that these students have had massive life experiences with boundaries, fences, speed limits, minimum wages, 10 o’clock curfews, and so on, it is not at all surprising that ideas from these fields continue to intrude into their attempt to represent mathematical concepts...such as “limit” [italics added] (p. 299).

Students who hold this belief regarding limits would struggle to explain the end-behavior of periodic functions, like those whose Cartesian graphs are a dampened sine wave.
Students with either of the two previous misconceptions (limit as unreachable or limit as a bound) are also likely to incorrectly assume that an implicit monotonicity accompanies the limit concept. Monaghan (1991) suggests that use of the phrases tends to or approaching in discussing limits leads students to believe that a sequence or function converges to its limit from a single direction. Students in the study conducted by Davis and Vinner (1986) reveal that this implied monotonicity influences the way in which they define limit. One student suggested that “the limit \( L \) is the smallest number such that \( a_n \leq L \) for all values \( n \)” (p.288). This definition illustrates the close relationship that exists between limit as a bound and implied monotonicity.

Student responses found in the preceding paragraphs suggest that elements of motion and movement are embedded in students’ concept images of limit. Tall (1980) reports that “many students tend to visualize a limit as a dynamic process rather than a numerical quantity” (p.173). Tall and Vinner (1981) suggest that the limit concept is most often discussed initially in a dynamic form – “\( f(x) \) approaches \( c \) as \( x \) approaches \( a \)” (p.155). However, I would point out that this dynamic form does not describe what a limit actually is (a numerical quantity) but rather foregrounds how one might imagine a function moving towards that numerical quantity. Understanding that a limit is a number would seem to entail a static notion of the concept as well. However, a dynamic notion of limit is admittedly the focus of most informal discussions of limit. Hence, while both a dynamic and static notion of limit are ultimately desirous, students rarely attain a static understanding,
due in large part to the strong influence of the concept image of limit (as a dynamic process) they build initially.

A final student misconception that stems from a colloquial interpretation of certain language terms is the notion that a limit is an approximation. The alarming frequency with which this misconception occurs is evident in a study by Williams (1976) – 85% of the 600 students in his study responded in the affirmative when asked if they believed a limit is an approximation. This misconception may be a result of instructors’ use of phrases such as “we can let $x$ get as close as we like to $a$,” in referring to the domain process that is embedded in the limit concept. Empirical evidence from Bezuidenhout’s study (2001) suggests that some students believe that one need only take functional values “close to” the limiting value $a$, and that “close to” is a matter of opinion. Ferrini-Mundy and Graham (1994) also provide evidence of this misconception, noting that Sandy, a student of interest in their study, felt that “in solving problems involving limits, she is not at all sure that the answers are ‘concrete’” (p.44). It is important to note that approximation may be seen as an appropriate steppingstone towards determining limits. In fact, research by Oehrtman (2004) suggests that approximation metaphors may assist students in developing correct intuitions about formal limit ideas. What is critical however is that students understand that approximations and limits are not equivalent. Davis and Vinner (1986) point out that viewing the two notions as equivalent is inappropriate in that while approximations can be used to “deduce
precise results," they can not be used to “compute precise results” (Davis & Vinner, ibid, p.295).

2.1.2.3 Difficulties with Infinity. Embedded in many of the misconceptions discussed above are the struggles that students experience with the notion of infinity. The limit itself is a numerical quantity $L$, which is verified as the limit via an infinite process of determining a delta neighborhood around the limiting point $a$ for each of an infinite number of epsilon neighborhoods around the limit $L$. Thus, understanding how students reason about limit must also include insight into how they reason about infinity. In regards to sequences, Tall and Vinner (1981) suggest that students may develop the notion that a limit is unreachable because they believe that for the limit to be reached, $n$ would have to actually reach infinity, something which they believe cannot happen. In response to this belief, Tall and Vinner report “Thus, ‘point nine recurring’ is not equal to one because the process of getting closer to one goes on forever without ever being completed” (p.159).

Reporting on a study done by Fischbein et al. (as cited in Tall, 1980), Tall notes that 84% of the 107 students in the study, in response to the following task, believed the process of continuously cutting a length in half would never end.

Given a segment $AB = 1$ meter. Let us add to $AB$ a segment $BC = \frac{1}{2}$ meter. Let us continue in the same way adding segments of $\frac{1}{4}$ meter, $1/8$ meter, etc. Will this process of adding segments come to an end? (p.173).

It appears that students become preoccupied with the limiting process itself, one that they believe cannot possibly terminate, instead of focusing on the numerical quantity to which the limit is equivalent. This might help explain why many
students hold only dynamic, and not also static, interpretations of the limit. Cornu (1991) echoes these sentiments:

[T]he initial teaching tends to emphasize the process of approaching a limit, rather than the concept of limit itself...Thus it is that students develop images of limits and infinity which relate to misconceptions concerning the process of 'getting close' or 'growing large' or 'going on forever' (p.156).

Work by Sierpinska (1987) descriptively classifies each of the 31 sixteen year-old pre-calculus students in her study by the way in which they view infinity. By doing so, Sierpinska offers insight into the connection between students' beliefs about infinity and their misconceptions of limit. For instance, she describes a "potentialist" model of limit as one in which "the limit of a sequence is what that sequence is infinitely approaching without ever reaching it; the impossibility of reaching the limit is implied by the impossibility of running through infinity in a finite time" (p.385). A "potentialist" then would likely hold the misconception that a limit is unreachable.

When one considers the potential cognitive conflict created by notions of infinity throughout the historical development of the concept of limit, it is not surprising that students experience the difficulties discussed above. Limits represent a significant mental shift for students, as it is the first infinite process they are likely to encounter in mathematics. The misinterpretation of informal language, combined with confusion regarding the notion of infinity, help create the misconceptions previously discussed.
2.1.3 Instructional Approaches that Address Misconceptions

The vast majority of limit research has focused on delineating the misconceptions held by students. This research has been discussed above. Some researchers however have attempted to move beyond identifying misconceptions by studying the effects that various instructional approaches have on students’ misconceptions of limits. The central research question of this type of methodology has been: Do student misconceptions persist when students experience X? Many studies (Cooley, 1997; Cottrill et al., 1996; Craighead & Fleck, 1997; Heid, 1988; Kidron & Zehavi; 2002; Lauten, Graham, & Ferrini-Mundy, 1994; Monaghan, Sun, & Tall, 1994) have explored the ways in which technology can be used to either evoke cognitive change in students’ informal limit conceptions or alleviate the misconceptions that are prevalent in initial student thought. Specifically, technological approaches have been implemented in hopes of broadening students’ views of limit beyond merely the procedural aspects that are used in determining a candidate for limit. A study conducted by Williams (1991) attempted to evoke cognitive change in students’ informal limit conceptions through the use of *discrepant events*. The work of Williams will be discussed here.

Williams (1991), referencing work by Tall (1980), believes that students have what Williams describes as a “prerigorous” understanding of limit that result in

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7 The extent to which technology supports students in resolving misconceptions and developing dynamic and static notions of limit is difficult to gauge, as the studies listed have reached conflicting results.

8 Williams (1991) defines a *discrepant event* as an event that “serves as an anomaly and produces cognitive conflict” with the aim of leading “students to a state of dissatisfaction with current conceptions” (p.220).
"informal models of limit" that "lead to more serious misunderstandings and interfere with future learning" (p. 219). Working from this belief, the goal of Williams’s research was to determine what prerigorous understanding of limits first time calculus students have and then subsequently determine what could be done to change the informal models of limits students possess. In Williams’s study, ten students were selected from an original subject population of 341 students to take part in five individual interviews over the course of seven weeks. In the first of these interviews, Williams attempted to identify clearly each student’s working definition of limit. In the second, third, and fourth interview, each student was given the working definitions of limit for two hypothetical students, Student 1 and Student 2, each of whom possessed an informal view of limit that varied from the other in some way. The students in the study were asked to choose between the two hypothetical students’ models of limit. The students were subsequently given various limit problems that were designed to be what Williams refers to as discrepant events for the students in the study. For example, had a student accepted a hypothetical student’s model of limit that included the idea that the limit of a function is never reached, a discrepant event might be a problem wherein a student is asked to compute the limit of a constant function. Williams’s hope was that students would experience cognitive conflict as a result of these discrepant events and thus would alter their working definitions of limit, eventually arriving at a formal definition of limit. With this in mind, Williams gave each of the ten students the opportunity at the end of sessions 2-4 to change their working definition of
limit. In the fifth and final session, students were asked to explain to the interviewer why their working definition of limit had or had not changed over the course of sessions 2-4 as a result of the discrepant events they had experienced.

While Williams's (1991) study revealed many of the informal student views of limit discussed in other studies, his research also suggests that a student's informal view of limit may be more complex than originally assumed. Williams found that students may simultaneously accept multiple informal views of limit, and thus, he concluded that it is difficult to classify students as having a specific informal view of limit. Williams also concluded that despite his efforts to evoke cognitive conflict in the students, they by and large were unwilling to alter their initial conceptions of limit. Indeed, his research indicates that students value a view of limit that is both expedient and simple. One student's struggle to adopt a formal view of limit is explained quite clearly: "I just learned enough to get the problems done, I'd say" (p.233). Another student seemed to view the formal understanding of limit as something unattainable and thus settled for a view that was "easier to understand" (p.233). It was not important to the students to change their informal views of limits but rather to have a view of limit with which they felt comfortable. Williams feels the current curriculum gives students little motivation to make much effort at adopting the formal understanding of limits that many instructors hope students will eventually obtain (p.235).
2.1.4 Research that Starts with what Students Understand

The research described above has approached student understanding largely from a deficit perspective, focusing on what students do not understand and the ways in which student understanding of limit fails to measure up to a formal, expert view of limit. While this substantial body of research has helped explicate both the types of difficulties students experience and possible sources of those difficulties, it has not described the ways in which student understanding may be valuable to build upon. In classifying the other research that exists on student understanding of limit, Williams (2001) comments:

Much of the previous work on limit, indeed, relied upon comparing students' notions to accepted, formal mathematical descriptions of limit. Often, the historical development of limit was compared to students' own personal development of the idea, with the $\varepsilon$-$\delta$ definition being the ultimate goal in both cases....In all cases, the big ideas of limit are external, and the focus is on the formal, complete, expert view of limit (p.344).

Ferrini-Mundy & Graham (1994), Oehrtman (2003, 2004), and Williams (2001) take a decidedly different approach in their research. Each views student thinking as a valuable tool to be used in the construction of flexible limit understanding. Below I summarize the work of each of these researchers, highlighting the significant value resulting from research that "starts with what students understand."

2.1.4.1 Ferrini-Mundy & Graham. As a departure from the misconceptions research discussed previously, Ferrini-Mundy and Graham (1994) conducted
interviews in hopes of describing college calculus students' understandings of function, limit, continuity, derivative, and definite integral. Using Piaget's constructivist viewpoint as their theoretical perspective, Ferrini-Mundy and Graham believe that mathematical knowledge is constructed through a process of reflective abstraction and that cognitive structures are under continual development.

The following assumptions are fundamental to their work:

> [S]tudents' constructions are rational and subject to explanation. We view the student's constructions not as errors or misconceptions to be eradicated and replaced with the 'correct' and publicly shared interpretations of major ideas, but rather as expected phenomena that are natural in the learning process (p.32).

Ferrini-Mundy and Graham report that students' understanding of key calculus concepts, such as limit, is "deeply intertwined" with much of the natural language discussed previously (e.g., approaches, as near as we like, gets closer to). Ferrini-Mundy and Graham's work is distinctive in the way in which they view the effects that this natural language has on student understanding of limit. While other research has focused on how such language is a source of students' misconceptions, Ferrini-Mundy and Graham provide evidence that students' interpretations of informal language that is traditionally used may serve as a tool to eventual understanding.

It appears that the traditional language used to explain the limit notion to students may be helping Sandy to shape a very fuzzy concept of what is meant by finding the limit of a function at a particular point...Somehow what Sandy perceives as the appropriateness of approximation as a central feature in the limit concept transfers to her views of answers to limit problems....She compares finding limits to sketching graphs, and argues that the answers "weren't that specific" (p.38).
Ferrini-Mundy and Graham report that the sense students make of problem situations is heavily influenced by their previous experiences and knowledge. While students’ understanding of limit may not perfectly resemble the formal, mathematical understanding held by experts, embedded in their informal viewpoints are valuable ideas and constructs upon which new meanings can be built during the evolution of their understanding of the concept.

Calculus students will actively formulate their own theories, build their own connections, and readily construct meaning for problem situations. Sandy was constantly forming her own connections, many of which were unexpected by the interviewer, and most of which were rational, based in experience, and adequate for certain problems (p.43).

The term *expected phenomena* (Ferrini-Mundy & Graham, ibid), then, seems to be an appropriate way to describe students’ rational thought processes. Indeed, students’ informal interpretations of new concepts should not be viewed as naïve conceptualizations to be eliminated, but rather as expected phenomena which may serve as a helpful tool for developing rich, flexible conceptual understanding.

Ferrini-Mundy and Graham’s work informs my dissertation study in a significant way – the instructional design for this study, which I describe in Chapter 4, reflects Ferrini-Mundy and Graham’s viewpoint that embedded in students’ informal interpretations of a concept are often valuable ideas and constructs upon which new meanings can be built during the evolution of their understanding of the concept.

2.1.4.2 Oehrtman. Similar to Ferrini-Mundy and Graham (1994), Oehrtman (2003, 2004) views student thinking as an informative starting point for how flexible conceptual understanding of limit might be developed. Oehrtman describes
students' intuitive understandings as the metaphors for limits they use to make sense of problematic situations. Student responses in written work and interviews were “analyzed for the following properties of instrumental use: support of implicative reasoning, commitment to the context of the metaphor, change in understanding of the problem, and change in meaning of the context” (2003, p.399). Eight metaphorical contexts emerged. Five were listed as “strong” metaphors because they displayed all four properties listed above, while three others were listed as “weak” metaphors, because they exhibited none of these instrumental properties. The five strong metaphors were collapse, approximation, closeness, infinity as a number, and physical limitation. The three weak metaphors identified were motion, zooming, and arbitrary smallness. Central to Oehrtman’s work is the contention that a difference exists between students’ structural knowledge of limits and their functional use of limit knowledge. Whereas structural knowledge of limits might describe the organizational structures students apply to the limit concept (i.e., the ways in which they cognitively organize the numerous ideas and notions related to the concept), it does not describe how students use or apply limit knowledge to problems involving limits (i.e., their functional use of limit knowledge). Oehrtman (2003) reports, “Students’ reported structural organization of mathematical concepts does not account for their actual use of those ideas” (p.404). His claim is that the majority of research has focused on the former, whereas research on the latter may provide valuable insight into the positive ways in which student’s intuitive ideas support their developing notions of limit.
“[R]esearch must look at richer data on their functional application of ideas in addition to their structure and logic” (2003, p.404). Oehrtman feels students’ metaphors for limits provide such insight into the functional use of limit knowledge.

Central to Oehrtman’s findings (2003, 2004) is the idea that students are able to use metaphors in productive ways to improve their conceptual and functional understanding of limits. While the metaphors used by students are not necessarily sound mathematically, Oehrtman cautions against dismissing such intuitive ideas.

Even though the collapse metaphor is mathematically incorrect, students like Karrie were able to use it to see valid connections between different types of limits..., between different contexts involving the same limits..., and between different representations of limits....Making such connections enabled these students to organize their thoughts for further inquiry and to make substantial progress conceptualizing the meaning of limits in difficult contexts (2003, p. 401).

Approximation metaphors emerged as productive tools for first year calculus students in this instrumentalist study of reasoning about limit concepts. Students’ intuitions about approximation mirrored several important aspects of $\varepsilon$-$\delta$ and $\varepsilon$-$N$ structures, were spontaneously employed to help tackle challenging problems about limits, guided exploration of the relevant formal structures of other important calculus concepts, and served as a basis for the abstraction of formal limit structures....[S]tudents’ spontaneous reasoning about approximation can serve as a productive foundation for limit concepts...(2004, p.95).

Oehrtman’s findings suggest that students’ use of metaphors is a natural and productive means through which sense can be made of formal ideas. Even though students may not initially be prepared to use formal language in their descriptions, the informal language and metaphorical descriptions used to describe their understanding may contain powerful, valuable elements of advanced
understanding. Oehrtman's research suggests the use of "strong" metaphors should be encouraged in the classroom, and that attempts to avoid and discourage such use disconnects students from an opportunity to build conceptual understanding based on their own experience and intuitive ideas.

2.1.4.3 Williams. Although his approach is slightly different from that of Oehrtman's (2003, 2004), Williams's (2001) research also provides evidence that students' intuitive notions of limit can serve as productive tools for building rich, flexible limit understanding. The goal of his research is to provide a new theory of student understanding of limit that "is sensitive to and begins with the informal notions brought to bear in complex domains" (p.342). Williams contrasts his research with traditional research that gives less attention to student voice. "Thus, instead of beginning with a mathematical analysis of limit, and the assumption of certain mental objects and processes..., this method begins with students' own reports of what is significant about limit" (p.365). Similar to Oehrtman, Williams feels that metaphors play an important role in student understanding, noting that students "establish and give meaning to informal notions of limit by way of metaphorical extensions from physical experience" (p.341).

Williams (2001) reports that while the use of metaphorical thinking allowed students to develop their conceptual understanding of limit in many areas, they were ultimately unwilling to let go of their initial guiding metaphors for more formal models. Reason for their unwillingness to adopt a more formal model appears to be related to the struggle students experience with the notion of actual
infinity. Williams notes that students' understanding of limit is contingent upon their ability to make the jump from finite to infinite. Williams points out that this jump might be assisted by a belief in actual infinity, yet formal mathematics discourages such a belief.

The notion of reaching a limit rests on the fundamental distinction between actual and potential infinity. Moreover, the ε-δ definition categorically rejects the concept of actual infinity (Tirosh, 1991). Thus with the concept of limit, students' informal conceptions that depend on coming to grips with actual infinity meet mathematical formalism head on. Given that the avoidance of actual infinity within modern mathematics is motivated by the desire to avoid rather subtle and mathematically complex paradoxes, it is not surprising that few students see the need to reject their informal model...Actual infinity may thus be the most important cognitive obstacle to learning the formal definition (p. 364).

It seems then, that in an effort to be mathematically consistent, calculus instructors may resist students' urges to make sense of actual infinity, a notion which might serve students in a productive way towards developing rich, flexible limit understanding. Williams’s research makes an important contribution to the literature by suggesting that mathematically invalid notions like actual infinity may be necessary for students to understand prior to an evolution to a more formal, mathematically sound model of limit.

The contributions of all of the studies discussed previously (Ferrini-Mundy & Graham, 1994; Oehrtman, 2003; Oehrtman, 2004; and Williams, 2001) are significant in that they lend credence to research that incorporates the voice of the

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9 Tirosh (1991) describes potential infinity and actual infinity, in relation to the history of mathematical development, as follows – "[T]he two competing ideas of infinity were potential infinity in which a mathematical process can be carried out for as long as required to approach a desired objective, and actual infinity in which one contemplates the totality of infinity, through, for example, conceiving the totality of all natural numbers at one time" (p.200).
student. It is apparent that students’ informal, intuitive understandings contain valuable elements that may assist them in developing limit understanding. The research community likely will benefit from conducting studies that approach understanding in a similar vain.

2.2 - Formal Limit Understanding

The majority of the research base on limit addresses students’ informal conceptual understanding of limit, focused largely on the misconceptions students possess, as described in Section 2.1 of this chapter. In contrast, relatively little research has sought to describe how students reason about limits in a more formal manner. I define formal limit reasoning as the reasoning that is associated with trying to make conceptual sense of the traditional $\epsilon-\delta$ definition of limit. What follows is a synthesis of the research that has been conducted on formal limit reasoning, including a discussion of the sources for students’ struggles with the formal definition, how students reason about mathematical definitions in general, and the distinction that exists between the processes of finding limit candidates and validating limit candidates using the formal definition.

2.2.0 Students and the Formal Definition of Limit: An Introduction

In short, there is a paucity of research that has looked directly at how students reason about or understand the formal definition of limit. While some have suggested pedagogical approaches for teaching students about the formal definition of limit (Gass, 1992; Steinmetz, 1977), very little research exists regarding how
students come to understand and reason about the formal definition. The research that does exist suggests that students, for a variety of reasons, struggle to understand and reason about the formal definition of limit (Cornu, 1991; Cottrill et al., 1996; Gass, 1992; Larsen, 2001; Tall, 1992; Tall & Vinner, 1981; Vinner, 1991; Williams, 1991), which is rich with quantification and notation, and, according to Cornu, cognitively sophisticated for first quarter calculus students. For instance, Vinner (ibid) reports that out of fifteen mathematically gifted calculus students who had spent significant time with the limit concept, only one was able to provide a formal definition for limit that might indicate "reasonably deep understanding of the concept" (p.78), and for this single student, the universal condition on epsilon was not explicit. Similarly, none of the students in the study conducted by Cottrill et al. (ibid) demonstrated the ability to progress to a point of reasoning about the limit concept in a formal manner. The general consensus seems clear – calculus students have great difficulty reasoning coherently about the formal definition of limit.

Some question whether the formal definition of limit is even appropriate for introductory calculus students. For instance, Tall (1992) reports that formal definitions are not appropriate as cognitive tools for developing conceptual understanding: "[F]ormal definitions of mathematics...are less appropriate as cognitive roots for curriculum development. Their subtlety and generality are too great for the growing mind to accommodate all at once..." (p.508). Dorier (1995) points out that "less formalized tools were used to solve most of the problems
[related to limits], while the 'ε-δ-definition' was conceived for solving more sophisticated problems and for unifying all of them" (p.177), yet at the outset of calculus, students struggle to see the significance of a unifying definition in that they have yet to encounter many problems. The current research, then, suggests that the formal definition of limit may be too cognitively complex to serve as a starting point from which to build conceptual understanding about limit. Regardless of when students should optimally encounter the formal definition of limit, the question remains what conceptual difficulties they will likely experience in coming to reason about limits formally. In the pages that follow, I will first discuss possible sources for students' struggles with the formal definition of limit, as cited in the literature. These sources include pedagogy that runs counter to the historical development of the limit concept, underdeveloped algebraic skills, and difficulties with quantification and notation. Next, I will discuss the role of mathematical definitions – both how students reason about mathematical definitions in general, and when and how students choose to use mathematical definitions. Finally, in conjunction with discussing the cognitive framework proposed by Cottrill et al. (1996) for how students reason about limits, I will carefully describe the distinction between the processes of finding limit candidates and validating the existence of limits using the formal definition.
2.2.1 Possible Sources of Students' Struggles with the Formal Definition of Limit

2.2.1.1 Historical Considerations and Algebraic Preparedness. There are a variety of explanations for why students struggle with the formal definition of limit. One compelling reason for the lack of evidence of formal limit understanding in the research is that the majority of studies have been conducted with first quarter calculus students, who likely lack the mathematical experience and maturity needed to reason coherently about the formal definition of limit at the outset of their calculus experience. Students who are presented with the formal definition of limit in the early stages of their experience with limits are being asked to develop their cognition in a manner that is counter to the historical development of the concept. Research by Juter (2006) suggests that students develop their conception of limit in a manner similar to the concept's historical development. Pedagogically it is worth considering that the great mathematicians of past centuries came to a collective formal understanding of the limit concept only after hundreds of years of struggling with informal notions. It is no surprise, then, that the research reveals first quarter calculus students struggle with the $\varepsilon$-$\delta$ definition. In cases where the formal definition is introduced at the outset, students likely struggle to see the need for a formal definition – indeed, they have far less motivation to understand the value of, and far fewer tools to comprehend, the formal definition than Weierstrass had when he developed the definition. Cornu (1991) agrees – “[T]his unencapsulated pinnacle
of difficulty occurs at the very beginning of a course on limits presented to a naïve student. No wonder they find it hard!” (p.163).

Another explanation for students' difficulties with the formal treatment of limits focuses on algebraic skills that precede calculus. The traditional formal definition of limit includes statements using both absolute value and compound inequalities. Realizing that the notion of distance is embedded in a conceptual understanding of absolute value and being proficient with compound inequality statements are necessary when making coherent sense of the traditional formal definition of limit. Researchers (Cornu, 1991; Cottrill et al., 1996; Ervynck, 1981) report that students' algebraic preparation in these areas often fails to prepare them sufficiently for studying the limit concept formally.

2.2.1.2 Quantification. Other research (Cottrill et al., 1996; Davis & Vinner, 1986; S. Larsen, personal communication, December 20, 2006; Tall & Vinner, 1981) suggests that students' difficulty with the formal definition of limit is partially attributable to the struggles they experience with quantification. Tall and Vinner (ibid) report that "students have great initial difficulties with the use of quantifiers 'all' and 'some' and the standard definitions of limits and continuity...can present problems to the student" (p. 160). Cottrill et al. (ibid) believe it is not the formality of the $\varepsilon$-$\delta$ definition that inhibits students from constructing a formal definition of limit, but rather it is "the need for a sophisticated use of existential and universal quantification...that makes the limit
concept so inaccessible to most students" (p.190). Dubinsky, Elterman, and Gong (1989) add:

The ability to work with existential and universal quantification of logical propositions is one of the most important and useful tools for accessing a vast array of mathematical ideas. Quantification is, on the other hand, one of the least often acquired and most rarely understood concepts at all levels (p.44).

According to research (Davis & Vinner, ibid; Dubinsky & Yiparaki, 2000; S. Larsen, personal communication, December 20, 2006), the struggle students experience with quantification appears to be related to the temporal order of quantifiers (i.e., distinguishing between “For all..., there exists...” (AE) statements and “There exists..., for all...” (EA) statements). Dubinsky and Yiparaki report that most of the students in their study “could not distinguish between AE and EA statements in mathematics and did not seem to be aware of the standard mathematical conventions for parsing statements” (p.239). In fact, their research shows that students were far more likely to interpret EA statements as AE statements than vice versa. “We continued to see a strong tendency to interpret statements as AE, even when the interviewer suggested that the statement at hand may have an EA form” (p.249). Dubinsky and Yiparaki provide a few reasons for why students may interpret EA statements as AE statements. First, they claim that natural language statements (i.e., non-mathematical statements made in everyday life) are more likely to be AE than EA and thus, there is a natural comfort and familiarity with the former. Second, given both the AE and EA version of a statement, the EA version implies the AE version. Hence, the AE version is more
likely to be true than the EA statement. "So a tendency to favor truth would result in a tendency to favor an AE interpretation" (Dubinsky & Yiparaki, ibid, p.253). Finally, in trying to verify the veracity of AE and EA statements, Dubinsky and Yiparaki claim that it is cognitively easier for students to determine whether AE statements are false than it is to determine whether EA statements are false. "AE statements, thanks to the order of quantifiers, provide a starting point, a strategy, for a student who wants to begin assessing the truth value of the statement" (p.254). Research by Zaslavsky and Shir (2005) support these findings.

Larsen, on the other hand, has found evidence that conflict with these findings (S. Larsen, personal communication, December 20, 2006). First, Larsen suggests that people who are labeled as interpreting EA statements as AE statements in these previously cited studies may, in fact, be interpreting all statements as AA (i.e., "for all e and for all δ..." - in other words, the quantified statement is true for every e-δ pair). Evidence from his ongoing research substantiates this possibility. Larsen also has found that while students may initially interpret EA statements as AE statements, they are likely to reverse their interpretations as their experience with quantified statements increase. In a study conducted with advanced calculus students (at the 300 level), Larsen found that some students, once they knew that the order of quantification has an affect on the meaning of quantified statements, were more likely to interpret both AE and EA statements as EA statements. Larsen posits that the reason for this interpretive shift is that students, once they have experienced proving quantified statements, more easily reason about EA statements.
than AE statements, and thus, prefer quantified statements to have an EA structure. The confusion students experience distinguishing between AE and EA statements is particularly relevant to this dissertation study, in that students, as part of the reinvention process, likely need to make sense of quantification order to develop conceptual understanding of the formal definition of limit.

2.2.1.3 Notation. Research by Fernandez (2004) suggests that students struggle with the abundant notation present in the conventional $\varepsilon$-$\delta$ definition of limit. In Fernandez's study, students were first asked to read about the formal $\varepsilon$-$\delta$ definition of limit and write down at least three things they did not understand about the formal definition. This was then followed by two separate, 100-minute lessons. The primary intent of the first lesson was for the teacher to understand how the students were reasoning about the formal definition. As a class, the students and teacher read through the formal definition material together, with the teacher stopping periodically to elicit students' questions and concerns about the formal definition. Students were then broken into smaller groups (3-5 students) and asked to read through, and write down questions about, worked examples related to the formal definition, wherein limits of functions were verified via the formal definition.

The second lesson in Fernandez's study (2004) was designed to respond to students' concerns and questions about the formal definition expressed during Lesson 1. Students' concerns included: the origin of $\varepsilon$ and $\delta$; the relationship between $\varepsilon$ and $\delta$, $L$ and $c$, and $x$ and $y$; the relationship between $\varepsilon$, $L$, and $y$; the relationship between $\delta$, $c$, and $x$; how to interpret inequalities; and why $|x-c|$ is
required to be positive while \(|f(x) - L|\) is not. Fernandez also found that the abundant notation in the formal definition was problematic for students. "[T]here was general agreement that the definition contained too much notation and that the need for this notation should be motivated" (p.45). Fernandez designed Lesson 2 so that students' concerns might be addressed. During the second lesson, linear examples (e.g., \(\lim_{x \to 2} 2x - 1\)) were explored both informally and formally. In regards to formal exploration, students were asked to suppose that a non-calculus student did not believe the function heights for \(f(x) = 2x - 1\) could be made arbitrarily close to 3 as \(x \to 2\). The students were asked to determine how close \(x\)-values would need to be to the value 2 in order to ensure that the function heights would fall within the pre-assigned distance of 3 provided by the non-calculus student. Students took turns playing the role of the non-calculus student so that multiple choices could be made for \(\varepsilon\), and eventually the students noted an emerging pattern relating pre-assigned values for \(\varepsilon\) and corresponding choices for \(\delta\). As they became aware of the need to generalize the results of their exploration, a need prompted by their experience with this role-playing game, the students studied two versions of the formal definition of limit. Version 1 carried with it less notation than the traditional formal definition (Version 2) that uses absolute value statements and symbolic quantifiers: "To ensure we can get \(f(x)\) within a distance \(\varepsilon\) of \(L\), we need to find a distance \(\delta\) around \(c\) so that if \(x\) lands within \(\delta\) of \(c\), this implies \(f(x)\) lies within \(\varepsilon\) of \(L\) OR if \(x \in (c - \delta, c + \delta)\), then \(f(x) \in (L - \varepsilon, L + \varepsilon)\)" (p.49).
A central finding in Fernandez's study (2004) was that students are able to reason coherently about the formal definition of limit when not obstructed by notation traditionally present in the formal definition. "Avoiding the notation in the book's version of the \( \varepsilon-\delta \) definition and using familiar interval notation proved a big step in raising students' comfort level for studying this concept" (p.50).

Following the second session, in an effort to further assess the students' reasoning about limits, the students were asked to use either Version 1 or 2 to verify a given limit. The less notation-intensive version (Version 1) was used by 40 of the 48 students, suggesting that a less notation-intensive version of the definition may be more meaningful and cognitively accessible for students.

Fernandez's study (2004) informed my dissertation study in two powerful ways. Methodologically, the design of her study suggests the value in first coming to understand how students are reasoning about limit in a formal manner, instead of imposing on students an expert's expectations for how they should reason formally about limit. During the first lesson, Fernandez's goal was not to instruct, but rather she desired to elicit students' misconceptions and perceptions of the formal definition. In her own words, "the lessons do suggest the value in a teacher's eliciting student ideas to convey a difficult mathematical topic....In thinking through options, it occurred to me that students' viewpoints could be a resource in lesson planning" (p.44). The second lesson was likely more pedagogically powerful because it addressed concerns that the students were invested in resolving. Second, Fernandez's study suggests that instead of attempting to lead students to understand
a formal definition that is meaningful to mathematics educators, it is likely more effective to first develop their understanding of a formal definition that is in some ways meaningful for them. The traditional definition of limit is laden with notation that, according to Fernandez, is troublesome for students. It seems, then, that an intermediate step to developing students' understanding of the traditional, notation-laden definition might be to first develop their understanding of equivalent, less notation-laden versions of the definition (i.e., Version 1 shown previously).

2.2.2 Mathematical Definitions

The research discussed above describes some of the possible sources of students' difficulties with the formal definition of limit, including poor or non-existent motivation for the purpose of the definition, a lack of algebraic preparedness, and inexperience dealing with quantifiers and notation. The last three of these sources each refer to *particular aspects* of the formal definition. On a more global level, it is worth considering how students reason about mathematical definitions in general. Below I discuss both students' conceptions of mathematical definitions, as well as how and when they choose to use them in mathematical contexts.

2.2.2.1 Students' Conceptions of Mathematical Definitions. Research by Zaslavsky and Shir (2005) indicates that students use differing strategies when defining mathematical concepts, depending upon the type of mathematical concept being defined. In a study conducted with four, high-level 12th-grade students,
Zaslavsky and Shir examined students’ conceptions of four mathematical definitions: square, isosceles triangle, increasing function and local maximum. Students demonstrated two different types of reasoning in defining the concepts. For geometric concepts (square and isosceles triangle), they used definition-based reasoning, meaning justifications for a particular definition were based on features or roles of mathematical definitions that the students deemed important (ibid, p.327). For instance, one student rejected a proposed statement as a definition of square because he felt the definition was too procedural (i.e., the definition was a set of instructions on how to construct a square), noting “a definition should not be given in the form of building instructions” (p.327). This same student accepted a different statement as a definition of square because the statement was minimal (i.e., it included no superfluous conditions). Hence, in Zaslavsky and Shir’s study, students’ acceptance or rejection of statements as definitions of geometric concepts was based on definition-based reasoning. In contrast, students used example-based reasoning for the analytic concepts (increasing function and local maximum), wherein justifications for a particular definition rely on examples or counterexamples to convince themselves or others regarding a statement about the concept (p.326). In fact, Zaslavsky and Shir report that students viewed the classification of examples and non-examples of a concept as one of the central purposes of mathematical definitions, commenting that “the students pointed to its power in ‘refuting functions’” (p. 334).
Apparently, dealing with the rather straightforward geometric concepts allowed the students to focus on the notion of a definition, whereas dealing with the more subtle analytic concepts led them to a process of monster-barring (Lakatos, 1976), wherein the students iteratively modified their definition to better reflect the concept image they held (p.328).

It appears, then, that the use of examples and counterexamples can be a powerful tool for students as they construct and reason about analytic definitions – examples and counterexamples served as a means by which Zaslavsky and Shir’s students’ understanding of analytic concepts evolved. The limit concept is similar to the concept of local maximum (one of the analytic concepts defined in the Zaslavsky and Shir study), in that the latter pertains to a numerical value along the $y$-axis and is usually defined formally with quantifier-intensive notation. This raises the possibility that students might gain similar traction from examples and counterexamples as they formally define limit.

The research by Zaslavsky and Shir (2005) also had methodological implications for my study. The intent of the study’s design was to place the students in control of the construction of meaning. While the researchers did create the questionnaire taken individually by students during the first session, the second session consisted of the students completing the same questionnaire as a group without any assistance or interference from the researchers. Thus, conclusions about the role of mathematical definitions came out of student discussion, with little input from the researchers.
In addition to the cognitive value of argumentation, these kinds of discussions move students away from treating the researcher (or teacher) as the sole authority on what is right or wrong in mathematics toward relying on their own sound reasoning (as advocated in NCTM, 2000) (p.342).

By challenging the students to support claims with the use of examples and counterexamples, the researchers engaged the students in activities that assisted students in developing example-based reasoning. A methodological approach that reflects the valuable aspects of Zaslavsky and Shir’s study described above proved fruitful as I studied how students reason about the formal definition of limit.

2.2.2.2 How and When Students Use Mathematical Definitions. Research by Zaslavsky and Shir (2005) provides evidence of how students reason about mathematical definitions. A related issue is how and when students choose to use mathematical definitions. Vinner (1991) suggests that students’ cognitive structures are split between two distinct cells – the students’ concept image (CI) and concept definition (CD)\(^\text{10}\). It is worth noting that there may not be overlap between these two cells – a student’s CD may be devoid of meaning for the student, and thus may not be included in the student’s CI.

[T]o acquire a concept means to form a concept image for it. To know by heart a concept definition does not guarantee understanding of the concept. To understand...means to have a concept image. Certain meaning should be associated with the words (p.69).

Vinner claims that in non-technical contexts, it is unnatural for students to consult definitions because conversations in such contexts are dominated by one’s CI. Vinner believes there is an inherent expectation at the collegiate level however, that

\(^{10}\) CI refers to the non-verbal associations in a student’s mind related to the concept name, whereas CD refers to whatever verbal definition is given by the student when asked.
as students engage in technical contexts "the concept image will be formed by means of the concept definition and will be completely controlled by it" (p.71). Vinner reasons that even in technical contexts, students' concept formation is dominated by their CI: "It is hard to train a cognitive system to act against its nature and to force it to consult definitions" (p.72). This model for concept formation certainly seems to apply in the case of limits, where research (Tall & Vinner, 1981; Williams, 1991) suggests students' concept formation is dominated by informal, dynamic notions of limit that reflect their experiences with the concept to date. One reason for students' reluctance to consult their CD as they reason about the limit concept may be that their CI is sufficient for the majority of tasks they initially experience while engaging with the limit concept. "Needless to say, that in most of the cases, the reference to the concept image cell will be quite successful. This fact does not encourage people to refer to the concept definition cell" (Vinner, p.73). It seems, then, that for a student's CD to be an integral/usable part of his or her understanding (i.e., part of his or her CI), he or she must be presented with a reason for deriving meaning from the CD.

2.2.3 Finding Limits vs. Validating Limits

2.2.3.1 General Comments. Research suggests that as students engage informally with the limit concept, they may not understand the unique role that the definition plays in regards to limit. As Fernandez (2004) points out, "the informal approach focuses on techniques for finding limits, while the formal definition is
concerned with why limits exist and why they are what they are” (p.43). In other words, whereas informal discussions of limit primarily focus on the process of finding a viable candidate for limit, it is the formal definition that is used to validate such a candidate. Research (Bezuidenhout, 2001; Fernandez, ibid; Juter, 2006) suggests that students miss that important distinction. Fernandez reports that students in her study, as they looked through written examples showing how the formal definition of limit can be used to prove the limit of a function, “questioned what the authors were ‘trying to prove’ since the ‘limit is already known in the examples’” (p.46). In response to this student concern, Fernandez began the second part of her study by “ensuring students’ understanding the distinction between the informal approach (which focuses on techniques for finding limits) and the formal one (which is used to rigorously verify the limits found)” (pp.46-47). Students in Bezuidenhout’s study displayed a similar lack of understanding of the unique role of the formal definition. Bezuidenhout conjectures that students’ over-reliance on the method of substitution when finding limits not only can have a negative influence on their conceptual understanding of limit, it also makes it difficult for them to “appreciate the relevance of the (ε-δ)-definition” (p.495). It is not surprising that a student who believes a limit is merely a single function value would take issue with the intricacies of formality of the definition. Juter believes examples that motivate the definition are critical to students developing their formal understanding of limits. “It is important that students get examples that they
cannot solve without the sharpness of a strict definition; otherwise, they see no reason to learn a strict definition” (p.427).

2.2.3.2 Lessons from Cottrill et al. Cottrill et al. (1996) provide what they call a genetic decomposition (GD) of how students might reason about the limit concept. This GD is a clearly stated description of the process that a student experiences as he or she constructs a formal understanding of limit. Cottrill et al. set out to propose an alternative to the conceptual dichotomy of students either having: a) a dynamic/process conception of limit; or, b) a static/formal conception of limit. Cottrill and his colleagues suggest instead that the concept of limit might eventually be thought of as a schema that is the collection of actions, processes and objects. I believe the cognitive model suggested by Cottrill et al. serves as a particularly useful starting point for studying students’ formal conceptions of limit, for several reasons that I elaborate in this section.

The work of Cottrill et al. (1996) stands out for its unique approach – rather than listing the components of limit that students do not understand, or offering pedagogical remedies for such misconceptions, Cottrill et al. have instead provided a framework that is suggestive of the cognitive process that students might experience in coming to understand the concept of limit. They constructed the GD in two stages. First, Cottrill et al. created an initial decomposition for how the concept of limit might be learned. Based on that initial decomposition, instructional

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11 This theoretical perspective is commonly referred to as APOS theory (action-process-object-schema). For a detailed description, see Dubinsky (1992).

12 This initial version of their genetic decomposition could be thought of as a mathematical analysis of the concept. I provide my own mathematical analysis of limit in Chapter 3.
strategies were designed and implemented in hopes of both providing evidence that might validate the initial decomposition as well as gathering evidence that might shed light on ways the initial decomposition might be modified and improved. After one iterative cycle, Cottrill et al. offered a refined version of the GD. This hypothesized framework for how students may come to understand the limit concept is as follows:

1. The action of evaluating \( f \) at a single point \( x \) that is considered to be close to, or even equal to, \( a \).
2. The action of evaluating the function \( f \) at a few points, each successive point closer to \( a \) than was the previous point.
3. Construction of a coordinated schema as follows.
   (a) Interiorization of the action of Step 2 to construct a domain process in which \( x \) approaches \( a \).
   (b) Construction of a range process in which \( y \) approaches \( L \).
   (c) Coordination of (a), (b) via \( f \). That is, the function \( f \) is applied to the process of \( x \) approaching \( a \) to obtain the process of \( f(x) \) approaching \( L \).
4. Perform actions on the limit concept by talking about, for example, limits of combinations of functions. In this way, the schema of Step 3 is encapsulated to become an object.
5. Reconstruct the processes of Step 3(c) in terms of intervals and inequalities. This is done by introducing numerical estimates of the closeness of approach, in symbols, \( 0 < |x - a| < \delta \) and \( |f(x) - L| < \varepsilon \).
6. Apply a quantification schema to connect the reconstructed process of the previous step to obtain the formal definition of limit.
7. A completed \( e-\delta \) conception applied to a specific situation. (Cottrill et al., ibid).

The most significant difference between the initial decomposition and the revised decomposition seen above are the substeps listed under the third step. Evidence collected in student interviews led Cottrill et al. to believe that the construction of a coordinated schema happens in a three-part process. The majority of their analysis focused on this third step, as well as the two preceding steps.
Cottrill et al. (1996) hypothesize that students move through a hierarchy of understandings when working with limits. The researchers report that most students initially view the limit of a function as the action of evaluating the function $f$ at a single point $x$ that is considered to be close to, or even equal to, $a$, or as the action of evaluating the function $f$ at a few points, each successive point closer to $a$ than was the previous point. In this initial perspective of limits, students do not see the limit as a dynamic process but rather as an action performed on a discrete number of input values. Cottrill et al. suggest that students then progress to viewing the limit as the action of evaluating the function $f(x)$ at a finite number of values, each value nearing the limiting value. At this point students still do not view the limit as an infinite process, but rather as a finite number of computations. Cottrill et al. report that students whose understanding evolves beyond viewing the limit as an action experience a shift in their understanding; they come to view the limit as a process in three separate stages. First, students view the action of computing numerous function values as a domain process wherein $x$ approaches the limiting value $a$. Second, students recognize a corresponding range process, wherein function values $y$ approach the limit $L$. However, despite constructing both a domain process and a range process, students often fail to make a connection between the two and see them as unrelated entities. Finally, some students progress to observing that the domain process of $x$ values approaching the limiting value $a$ results, via the function $f$, in a range process of $y$ values that approach the limit $L$. 
At this stage, Cottrill et al. postulate that students will have constructed a coordinated image between domain and range.

The fourth step listed in the genetic decomposition offered by Cottrill et al. (1996) describes a transition to thinking of the limit as an object, which they define as being "constructed through the encapsulation of a process...achieved when the individual becomes aware of the totality of the process, realizes that transformations can act on it, and is able to construct such transformations" (pp.171-172). An object conception of limit is seen by the authors as an advance from that of a process conception of limit. Unfortunately, there was little compelling evidence in the study that students think of the limit as an object. Cottrill believes that the limit is not an idea that lends itself well to being thought of as an object (J. Cottrill, personal communication, February 24, 2006). In their 1996 paper, Cottrill et al. suggested that a student's application of a limit law (ex: \( \lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) \)) might indicate that the student is thinking of the limit as an object on which actions can be performed. However, it seems far more likely that when students apply such limit laws, although they are technically performing the action of addition (or subtraction) on two or more limits, they are conceiving of the action as taking place on two or more numbers. Thus, any object conception being discussed is really that of number, not limit. According to Cottrill et al., in order to have an object conception of something, one must be able to de-encapsulate that something so as to think of it as an action or process. For instance, the concept of function can be easily thought of in this way. However, it is difficult
to imagine deencapsulating a limit if one thinks of a limit as merely a number. Thus, it is hard to conceive of how a number might be deencapsulated into its action or process components. This difficulty may help explain the struggle Cottrill et al. faced in finding evidence of students at the fourth step above.

There was no evidence that students in the study conducted by Cottrill et al. (1996) were able to move to the level of actually thinking of the limit as a schema—that is, a “coherent collection of actions, processes, objects and other schemas that are linked in some way and brought to bear upon a problem situation” (p.172). Further, none of the students’ thinking evolved to the point of having a formal conceptual understanding of limit. This is not surprising in that, according to the authors, the ability to understand fully the ε-δ definition of limit requires one to be able to encapsulate the limit schema.

The GD suggested by Cottrill et al. (1996) implies that in order to have a formal understanding of limit, one must merely formalize one’s informal notions of limit. In the decomposition above, that equates to formalizing the first three steps by reconstructing the coordinated schema described in step 3c in terms of intervals and inequalities. I argue however, that the formalization process is not so straightforward. Indeed, a formal understanding requires one to think in terms of intervals and inequalities, but I contend that this transition to formal thinking is not merely a reconstruction of what is described in the first three stages of the genetic
decomposition offered by Cottrill et al. Research by Larsen (2001) substantiates this perspective. Most students in Larsen's study did not make connections between their respective formal understanding and the rest of their concept image, which was comprised mostly of informal conceptions described in the first three steps of the GD. Larsen suggests that "the formal definition is structurally different from the dynamic conception as described by the first four steps of the genetic decomposition", thus making it "unlikely that a student could successfully interpret the syntax in terms of their dynamic conception" (p.29).

I recommend, then, that a clearer distinction be made between informal and formal understanding of limit. One can think of informal understanding as that which is employed when the goal is to find a candidate for the limit. For example, a student asked to find $\lim_{x \to \pi} \sin(x)$ might apply a variety of strategies, the most likely of which would be direct substitution. However, regardless of the strategy that is applied, a student response of "0" most certainly should not suggest a formal understanding of limit, for the formal definition does not address how one might find a candidate. Instead, as Larsen (2001) suggests, the formal definition addresses how one might validate the choice of a candidate. Selection and validation are two different processes. In calculus courses, students are taught a variety of strategies for selection – direct substitution, algebraic manipulation, and tabular and graphical

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13 Fernandez (2004) and Juter (2006) have also suggested that validating limits involves a process distinct from the process used to find limits. Their perspectives, in addition to Larsen's perspective discussed here, have assisted me in articulating my own thinking on the distinction between these two processes.
inspection. However, none of these satisfy the formal definition’s requirement of validation. Research (most notably Cottrill et al., 1996) provides evidence that when students select a candidate for the limit of a function, they do so utilizing an \textit{x-first perspective}. By \textit{x-first perspective} I mean that students focus their attention first on the inputs (x-values) and then on the corresponding outputs (y-values). The selection of a candidate is made based on what numeric value the y-values are getting closed to as x-values get closer to \( a \). In contrast, the validation of a candidate for a limit requires that one begin with a given candidate. Hence, the formal definition is dependent upon a candidate having already been selected. Validating a candidate, however, relies on one’s ability to reverse his or her thinking. Instead of going from x-values to y-values, a student must first consider what is taking place along the y-axis.

In order to understand the definition of a limit, a student must coordinate an entire interval of output values, imagine reversing the function process and determine the corresponding region of input values. The action of a function on these values must be considered simultaneously since another process (one of reducing the size of the neighborhood in the range) must be applied while coordinating the results (Carlson, Oehrtman, & Thompson, 2007, p.160).

Thus, the process of validating a candidate requires a student to utilize a \textit{y-first perspective}, considering first a range of output values around the candidate, projecting back to the x-axis, and subsequently determining an interval around the limit value that will produce outputs within the pre-selected y-interval.

Evidence from Larsen’s study (2001) suggests that the ritual of utilizing what I call an \textit{x-first perspective} is indeed strong for students. The following selection
includes a portion of a student interview¹⁴ in Larsen’s study, followed by Larsen’s interpretation of the student’s reasoning.

**Erik:** So what did you say? You said for every epsilon there is a delta?...How bout the other way around, can you pick delta first and then, then epsilon?

**I:** So, why do you ask that?

**Erik:** I don’t know just curious.

**I:** Is there some reason why it seems like that would be better?

**Erik:** Cause I’m just used to putting things into the input instead of changing the output, just changing the input and see what happens. Because of my preconceived math notions, prior educations....

It appears that Erik is trying to connect the formal definition with the rest of his concept image....[H]e expresses the desire for the formal definition to work the way his informal understanding works, by “changing the input and see what happens” (p.23).

The intricacies involved in utilizing a y-first perspective are arguably far more complex than merely formalizing an x-first perspective, as Cottrill et al. (1996) conjectured. The very complex nature of the formal definition makes it highly unlikely that a student with a strong x-first viewpoint of functions would be able to conceive of a new concept in such a y-first manner, particularly when the focus during a first term calculus course is on finding limits, not validating them.

**2.2.3.3 Conclusion.** In summary, I view the GD offered by Cottrill et al. (1996) as a framework which provided a helpful starting point for my own research. Instead of working from a deficit perspective, Cottrill et al. provided a positive description of student thought. Due to the complexity of the limit concept, their work only was able to provide evidence of how students reason about limits informally in the context of finding limit candidates. What is needed is a clearer

¹⁴ In the interview here, “Erik” is the student, and “I” stands for the interviewer.
understanding of how students reason about limits formally in the context of validating limit candidates\textsuperscript{15}. It is reasonable to work with students who have had sufficient time to develop their informal understanding. Because the limit concept is traditionally seen informally throughout Differential calculus, it is sensible to explore students’ formal understanding after students have completed this initial calculus course. Hence, students who have studied the limit informally are viable candidates for such research.

2.3 - Implications for Research

This review of the literature suggests that the mathematics education community knows a great deal about students’ informal reasoning about limits. Specifically, many studies have revealed what students do \textit{not} know about limit. The “misconceptions” research discussed in Section 2.1 is extensive and has more than adequately provided evidence of the difficulties students experience during their initial interactions with the limit concept. Unfortunately, however, very little research has been conducted that focuses on what students \textit{do} know about limit, both informally and formally. In broad terms, the research of Ferrini-Mundy and Graham (1994), Oehrtman (2003, 2004), and Williams (2001), described in Section 2.1, as well as the work of Fernandez (2004), described in Section 2.2, are helpful models of how research might better focus on what students do know about the limit concept and, in Fernandez’s case, for instructional approaches that might

\textsuperscript{15} The distinction between the process of finding limit candidates and validating limit candidates, which I have articulated in Section 2.2.3.2, is a central theme in this document, one that I develop in the next chapter.
support students developing an understanding of the formal definition of limit. Specifically, these models inform my study in the following ways. Ferrini-Mundy and Graham report that students make sense of tasks based on their own experiences and that their initial conceptions are not to be viewed as "errors or misconceptions to be eradicated and replaced with the 'correct' and publicly shared interpretations of major ideas, but rather as expected phenomena that are natural in the learning process" (p.32). These findings suggest that students might develop understanding about the formal definition of limit from their own informal conceptions. Similarly, Oehrtman’s work, as well as Williams’s, suggests students’ metaphors for limit may serve as productive seeds for cognitive growth, more so, perhaps, than mathematicians’ perspectives on the concept. “Thus, instead of beginning with a mathematical analysis of limit, and the assumption of certain mental objects and processes (as in, e.g., genetic decomposition), this method begins with students’ own reports of what is significant about limit” (Williams, 2001, p.365). The methodology for this dissertation study, discussed in Chapter 4, reflects these perspectives.

Further, more research is needed that might describe the cognitive processes through which students progress as their conceptual and procedural understanding is formed. To date, the GD proposed by Cottrill et al. (1996) is the most descriptive and informative attempt to articulate the cognitive progression students experience in regards to the limit concept. While their work has provided evidence of how students reason about limit informally, there is a dearth of data describing how
students reason formally about limit. Thus, it seems that more research is needed that might elucidate the latter stages of their GD. While other studies (e.g., Larsen, 2001; Fernandez, 2004) have sought to describe how students interpret the formal definition, my research seeks to address this need by focusing on how students reason about the formal definition of limit in the context of reinvention.
Chapter 3 – Theoretical Perspectives

The theoretical perspectives that frame my study are situated on two levels. *Radical constructivism* and the research paradigms of *developmental research* and *teaching experiment methodology* serve as meta-level perspectives, and are described in Sections 3.1 and 3.2. The *mathematical analysis of limit* I provide in Section 3.3 is a domain-specific perspective that guided my methodology.

3.1 - Radical Constructivism

von Glasersfeld (1995), drawing on Piaget’s (1971, 1977) genetic epistemology, developed a psychological theory of knowing which is known as *radical constructivism* (RC). Two central tenets of RC are:

1. Knowledge is not passively received either through the senses or by way of communication, but is actively built up by the cognizing subject.
2. The function of cognition is adaptive, in the biological sense of the term, tending towards fit or viability and serves the subject’s organization of the experiential world, not the discovery of an objective ontological reality (von Glasersfeld, 1995, p.51).

Thus, for the radical constructivist, to speak of objective truth is meaningless, in that there is no way to conceive of what is outside of one’s own experience. “[Radical constructivism] replaces the notion of ‘truth’ (as true representation of an independent reality) with the notion of ‘viability’ within the subjects’ experiential world” (von Glasersfeld, ibid, p.22). The construction of knowledge, then, is necessarily dependent upon one’s prior experiences and one’s current conceptual structures. Indeed, knowledge is characterized as being built up by way of two
mechanisms: assimilation and accommodation. Assimilation is the process whereby an experience is relatively seamlessly integrated into one’s existing conceptual structures. von Glasersfeld characterizes assimilation as follows:

The cognitive organism perceives (assimilates) only what it can fit into the structures it already has. This, of course, is a description from the observer’s point of view. It has actually the important implication that when an organism assimilates, it remains unaware of, or disregards, whatever does not fit into the conceptual structures it possesses (p.63).

In von Glasersfeld’s theory, when the individual is unable to assimilate an experience or information encountered in a situation, he/she will likely experience a perturbation. Such perturbations are apt to impel the individual to reassess the nature of the cognitive structure with which he/she tried to assimilate the current experience. von Glasersfeld describes what results from this reassessment.

This review may reveal characteristics that were disregarded by assimilation. If the unexpected outcome of the activity was disappointing, one or more of the newly noticed characteristics may effect a change in the recognition pattern and thus in the conditions that will trigger the activity in the future. Alternatively, if the unexpected outcome was pleasant or interesting, a new recognition pattern may be formed to include the new characteristic, and this will constitute a new scheme. In both cases there would be an act of learning and we would speak of an ‘accommodation’ (pp.65-66).

The tacit assumption in von Glasersfeld’s theory is that organisms are coherence-seeking beings who are always aiming to make sense of their experiential world. He describes this goal-directed process as an action scheme, which is comprised of three parts: 1) the recognition of a certain situation; 2) a specific activity associated with that situation; and, 3) the expectation that the activity produces a certain
previously experienced result. von Glasersfeld elaborates his tripartite scheme as follows:

The ‘recognition’ in part 1 is always the result of assimilation. An experiential situation is recognized as the starting-point of a scheme if it satisfies the conditions that have characterized it in the past. From an observer’s point of view, it may manifest all sorts of differences relative to past situations that functioned as trigger, but the assimilating organism (e.g., the child) does not take these differences into account. If the experiential situation satisfies certain conditions, it triggers the associated activity. The activity, part 2, then produces a result which the organism will attempt to assimilate to its expectation part 3. If the organism is unable to do this, there will be a perturbation (Piaget, 1974a, p. 264). The perturbation, which may be either disappointment or surprise, may lead to all sorts of random reactions, but one among them seems particularly likely: if the initial situation 1 is still retrievable, it may now be reviewed, not as a compound triggering situation, but as a collection of sensory elements” (pp.65-66).

Hence, for the radical constructivist, learning is characterized as the act of making accommodations to cognitive structures so as to reestablish cognitive equilibrium. In short, then, as a cognizing subject interacts with the world, he/she seeks to assimilate new experiences. If new experiences cannot be assimilated, the cognizing subject experiences a perturbation, the result of which may be a cognitive accommodation.

In this study, I drew on radical constructivism in a couple of important ways. First, radical constructivism functioned as a guiding framework methodologically, both in regards to the dynamic I aimed to create between participants in each teaching experiment, and in regards to how I selected participants for the teaching experiment phase. The sequence of instructional tasks I implemented in the two teaching experiments was designed to create a dynamic in which students might
experience frequent perturbations, and thus, have the opportunity to make cognitive accommodations. Hence, the study’s methodology was in line with the two tenets of radical constructivism: instructional activities were designed to motivate the cognizing subject to organize his or her experiential world and thus, actively build up knowledge. Also, participant selection included a criterion that participants be active seekers of viability and fit between their mathematical understandings. An important distinction is worth making, however. While I agree with the tacit assumption in von Glasersfeld’s (1995) theory that organisms are coherence-seeking beings, I also believe that in educative settings, some students are more coherence-seeking than others. This belief is reflected in the selection criteria I used for the study. Students selected to participate in this study had demonstrated a greater effort and desire, relative to other students, to consistently make sense of their experiential world as it relates to complex mathematical ideas.

Second, radical constructivism served as a lens through which I analyzed the data generated in the two teaching experiments. How one interprets the tasks he/she is presented is necessarily dependent upon one’s prior experiences. As the students engaged with the instructional activities, their observable actions and behaviors provided evidence of how they might be interpreting said tasks. In a manner consistent with Steffe and Thompson’s (2000) description of modeling students’ interpretations, I compared my models of the students’ interpretations with those targeted in instruction, so that I could make subsequent revisions for future iterations of the research cycle, and so that research findings could be cast as
inferences about student reasoning given particular interpretations of instructional tasks. Also, given the impossibility of discovering ontological reality, the intention of data analysis was not to generate statements of fact about how students reason about or understand limits, but rather to generate viable interpretations of students' reasoning and understanding – i.e., interpretations that fit with their observed actions/behavior, in the sense that were they to reason in the ways I theorize, those ways might well express themselves in the observed behaviors\textsuperscript{16}.

3.2 - Instructional Design Heuristics Drawn From Developmental Research, and the Teaching Experiment Methodology

In this section I describe aspects drawn from the perspectives of developmental research and the teaching experiment methodology that guided my construction of instructional tasks designed to support students in reinventing the formal definition of limit.

3.2.1 Developmental Research and the Principle of Guided Reinvention

Gravemeijer (1998) describes the goal of developmental research as follows: “to design instructional activities that (a) link up with the informal situated knowledge of the students, and (b) enable them to develop more sophisticated, abstract, formal knowledge, while (c) complying with the basic principle of intellectual autonomy” (p.279). I view the goal of developmental research as being in line with the epistemological stance of radical constructivism described earlier in

\textsuperscript{16} In Chapter 4, I detail the analytic methods employed in this study.
this chapter, in that developmental research views knowledge as being constructed by individuals based on informal knowledge that is situated in their own experiences. Developmental research in education typically unfolds in cycles that are driven by two reflexively related phases – a developmental phase and a research phase. The former is characterized by the development of instructional activities and tasks designed to assist students in developing previously identified understandings related to a particular mathematical topic or idea. The instructional activities and tasks are developed based on a local instructional theory. The instructional theory includes a stated rationale for why the chosen instructional activities might best support students in developing the previously identified understandings. The instructional theory is local in the sense that it is focused on a particular mathematical topic or idea. Neither the instructional theory nor the instructional activities are static entities. Rather, an evolving dialectic exists between the theory and the activities. The latter research phase is characterized by analysis of student activity and reasoning as they engage in the instructional activities and tasks. This analysis, in turn, then serves as a guide in further developing the local instructional theory, and in refining the instructional activities and tasks to be implemented in subsequent research cycles.

A heuristic commonly associated with developmental research is guided reinvention. Guided reinvention is a well-established heuristic that has been employed in various content areas of post-secondary mathematics education (see Larsen, 2004; Marrongelle & Rasmussen, 2006; Weber & Larsen, 2005). Guided
reinvention is described by Gravemeijer et al. (2000) as “a process by which students formalize their informal understandings and intuitions” (p.237). An important aspect of this process is the identification of plausible instructional starting points from which students might naturally formalize their informal understandings and intuitions. Traditionally, there have been two approaches for determining appropriate starting points for instruction – 1) Analyses of the historical evolution of the mathematical topic with an eye toward identifying motivating problems or contexts for conceptual development; and, 2) Examination of students’ informal strategies and interpretations of contextual problems that are directly related to the mathematical concept. Gravemeijer et al. (2000) describe the history of mathematics and students’ informal interpretations as “sources of inspiration” for the researcher, who “tries to formulate a tentative, potentially revisable learning trajectory along which collective reinvention...might be supported” (Gravemeijer et al., 2000, p. 239). The first approach is intended to assist the researcher in formulating a learning trajectory in response to historical cognitive barriers and subsequent discoveries. While I did not analyze the historical evolution of limit with the aim of formulating a specific learning trajectory that would mirror the evolution of the concept, I did attempt to create an environment intended to mimic important aspects of the mathematical setting that Cauchy and Weierstrass experienced. That is, students selected for my study had no prior experience with the conventional $\varepsilon$-$\delta$ definition, and were posed with the challenge of characterizing local functional behavior using precise mathematical language. In
this way, the historical evolution of the definition informed my selection of a starting point for instruction. The second approach aims to inform the researcher as to how he/she might provide students with authentic opportunities to experience perturbations and to make subsequent accommodations. My analysis of students' responses to an Informal Limit Reasoning Survey\textsuperscript{17}, as well as my examination of existing research on students' informal reasoning about limit guided the design of initial instructional tasks for the teaching experiment phase of the study.

On a meta-level, the theoretical perspectives of radical constructivism and developmental research interacted in the following manner in my study. As each teaching experiment progressed, the intention of data analysis was not to generate statements of fact about how students reason about or understand limits, but rather to generate viable interpretations of students' reasoning and understanding. Thus, radical constructivism served as an epistemological foundation for analysis of student reasoning. This analysis, in turn, informed the construction of instructional tasks designed to leverage students' informal reasoning with the aim of supporting their development of more sophisticated, abstract, formal knowledge. Hence, the design of instruction in my study reflected the goals of developmental research.

\textbf{3.2.2 Teaching Experiment Methodology and Cognitive Modeling}

The purpose of my research was two-fold: 1) to develop an empirically-grounded instructional sequence designed to support student reinvention of the

\textsuperscript{17} The nature of this survey is described in Chapter 4.
formal definition of limit; and, 2) to develop insight into students’ reasoning in relation to their engagement in said instruction. Addressing the first purpose was important, as doing so allowed me to address the second purpose. However, the second purpose was at the foreground of my research, taking priority over the other. Hence, while the end product included an instructional element, the focus of my research was on modeling student thinking, along the lines articulated by proponents of the teaching experiment methodology (Steffe & Thompson, 2000). A teaching experiment is comprised of a sequence of teaching episodes, each of which involve a teacher-researcher, one or more students, a witness of the teaching episode, and a way to record the data generated during the teaching episode. Subsequent teaching episodes are designed based on analysis of records of previous teaching episodes. An important aspect of the teaching experiment methodology is the distinction that is made between students’ mathematics and mathematics of students, which Steffe and Thompson explicate below.

[W]e have to accept the student’s mathematical reality as being distinct from ours. We call those mathematical realities “students’ mathematics,” whatever they might be. Students’ mathematics is indicated by what they say and do as they engage in mathematical activity, and a basic goal of the researchers in a teaching experiment is to construct models of students’ mathematics. “Mathematics of students” refers to these models, and it includes the modifications students make in their ways of operating (p.269).

Hence, the researcher’s central purpose in a teaching experiment is to construct a model of student thinking or reasoning in relation to a particular concept or idea. Indeed, while a researcher’s understanding and knowledge of the particular concept or idea may help frame the teaching experiment, the researcher should not view his
or her own thinking as that which the student should come to learn (Steffe & Thompson, 2000). “In a teaching episode...the students' reasoning is the focus of attention” (p.282). In this way, the teaching experiment methodology was an appropriate framework to help me address the second purpose of my research. Further, this framework is consistent with the theoretical perspectives of radical constructivism and developmental research – focusing the attention of analysis on generating viable interpretations of students' reasoning and understanding honors the epistemological stance of radical constructivism and acknowledges the individual as having intellectual autonomy.

The meta-level theoretical perspectives described in the first two sections of this chapter were useful for addressing my central research objective. The aim of my research was to model students' reasoning in relation to their engagement in instruction designed to guide them in reinventing the formal definition of limit. The teaching experiment methodology served as an orienting and guiding framework for this central research objective, as Steffe and Thompson (2000) suggest: “In particular, our goal is to bring forth the schemes that students have constructed through spontaneous development and to use them in the formulation of the major research hypotheses of the teaching experiment” (p.290). My research objective was also in line with the goal of developmental research – to design instructional activities that allow students to autonomously build upon their informal knowledge as they develop more sophisticated, abstract, formal knowledge. Finally, the guided reinvention principle oriented my selection of starting points for instruction –
efforts were made to place participants in mathematical settings similar to those experienced historically by Cauchy and Weierstrass, and students' informal interpretations of the concept guided the development of instruction.

3.3 - A Mathematical Analysis of Limit

3.3.0 Introduction

The central research objective of this study was to develop insight into students' reasoning in relation to their engagement in instruction designed to support their reinventing the formal definition of limit. In previous chapters, I conjectured that the way one might reason about the concept of limit while reinventing the formal definition of limit is distinct from how one might reason about the concept while interpreting the formal definition's meaning for the first time. Based on this conjecture, the study was designed to assess students' reasoning about limit in the context of reinvention, rather than interpretation, of the formal definition.

The genetic decomposition proposed by Cottrill et al. (1996) is a conjectured model of how students might come to understand the concept of limit and its formal definition. The formulation of this genetic decomposition undoubtedly framed both the methodological design and analysis of their study. Likewise, my research included an a priori mathematical analysis of the concept of limit and its formal definition. This analysis consisted of two parts – the mathematical-conceptual analysis and the mathematical-symbolic analysis – which collectively
address understandings of limit which might arise in the context of reinvention and
the context of interpretation. The former explicates conceptual entailments to which
I conjectured a student would need to attend in the process of successfully
*reinventing* a coherent formal definition of limit. The latter deconstructs the
conventional \( \varepsilon-\delta \) definition atomically, resulting in a list of understandings
someone *interpreting* the definition would presumably possess had he or she
previously reinvented a formal definition of limit. Collectively, these two analyses
served as a domain-specific framework for instructional design and data analysis.
In Section 3.3.1, I present the mathematical-conceptual analysis. This is followed
by my presentation of the mathematical-symbolic analysis in Section 3.3.2.
Following the presentation of each mathematical analysis, I specify the ways in
which that analysis influenced methodological decisions in my study.

### 3.3.1 The Mathematical-Conceptual Analysis of Limit

The mathematical-conceptual analysis of limit I present here consists of two
parts. First, I describe what I feel is the central idea captured by the concept of
limit\(^\text{18}\). The purpose of this description is to explicitly articulate the fundamental
notion I expected students to be characterizing in their attempts to reinvent the
formal definition of limit. Second, I summarize conceptual entailments to which I
conjectured a student would need to attend in order to successfully reinvent a
cohercent formal definition of limit.

\(^{18}\) The limit concept extends to numerous contexts. However, the focus of this study is the limits of
functions. Hence, references made to "the notion of limit" are done so with this focus in mind.

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3.3.1.1 The Central Idea Captured by the Concept of Limit. The purpose of this first section is to develop a careful articulation of the central idea captured by the concept of limit. First and foremost, the concept of limit addressed here has the idea of a real-valued function as its object of focus, by which I mean a functional relation between two real-valued quantities that systematically governs the manner in which the quantities co-vary. Thus, a function, $f$, determines the corresponding values of the dependent quantity $f(x)$ that arise by choosing values of the independent quantity $x$ from the domain of the function. It is this correspondence and coupled variation between the quantities $x$ and $f(x)$ that is commonly referred to as the behavior of the function $f$. More specifically, the concept of limit was developed as a way to describe this functional behavior on a local level — that is, to describe how the quantity $f(x)$ varies or "behaves" given small variations in the quantity $x$ around a particular fixed value $x=a$. Given a real-valued function $f$, an informal characterization of the concept of limit might be as follows:

**Definition of Limit (Informally):** The function $f$ has a limit of $L$ as the independent variable $x$ approaches $a$ if $f(x)$ (i.e., the function values or vertical heights on the graph of the function) becomes arbitrarily close to $L$ as $x$ gets sufficiently close to $a$.

The power and significance of this concept becomes most apparent when one considers its instantiation in the case of discontinuous functions. Indeed, one might argue that the concept is really devised as a means of adequately describing the

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19 I acknowledge that one could interpret a function’s “local behavior” as including higher-order behavior such as differentiability and concavity. For the purposes of this document, however, the phrase “local behavior” will always be in reference to how the quantity $f(x)$ varies given small variations in the quantity $x$ around a particular fixed value $x=a$. 

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local behavior of a function around values where the function is discontinuous, including values where the function is not even defined. This development can be seen as an effort to unify the idea of "local behavior" across the two classes of continuous and discontinuous functions. An argument and an example help drive this point home.

If all functions were continuous everywhere, then their local behaviors would not be difficult to describe: it would suffice to note that a function simply attains the value \( f(a) \) whenever \( x=a \). There would be no need to consider the values of the function at points near \( x=a \). This latter is essentially how mathematical analysis conventionally treats functions at points of discontinuity, with respect to describing how they behave in proximity to such points. For instance, the function 
\[
f(x) = \frac{\sin(x)}{x}
\]
has a discontinuity at \( x = 0 \). Thus, its local behavior around this value cannot be described by simply specifying what value it attains there. Instead, one attempts to specify how \( f \) behaves within small ranges of the unattainable value in relation to corresponding small ranges around \( x=0 \). This type of specification is unnecessary for functions that are everywhere continuous, precisely because there is no issue of the function having any "holes". Indeed, continuity just says that the limit of a function at \( x=a \) is equal to the function value, \( f(a) \):

A function \( f \) is said to be **continuous** at \( x = a \) iff \( \lim_{x \to a} f(x) = f(a) \).

Thus, in a world with only continuous functions, the concept of limit would become rather innocuous, tending to lose its power and significance.
The aforementioned specification of the local behavior of a function at a point is conventionally expressed in mathematical analysis by the \(\varepsilon-\delta\) formulation of the concept, which is often attributed to Weierstrass (Boyer, 1949). The intent of my study was to gain insight into students' reasoning about limit as they reinvented a formal definition capturing the intended meaning of the formalization provided by Weierstrass. In the next section, I summarize the a priori conjectures I made about difficulties students might experience as they reinvent the formal definition of limit.

3.3.1.2 Conjectured Difficulties Students Might Experience During Reinvention. In the paragraphs that follow, I describe three cognitive hurdles I anticipated would arise for students in the context of reinventing the formal definition of limit.

Finding vs. Validating

In Chapter 2, I noted an important distinction between the process of finding candidates for a limit at a point and subsequently validating that such candidates are indeed the limit. This distinction was made in reference to Larsen's (2001) conjectures about the cognitive implausibility of the genetic decomposition proposed by Cottrill et al. (1996). I elaborate this distinction here as an issue to which students reinventing the definition of limit would likely need to attend.

There are a variety of approaches used for finding candidates for limits (i.e., algebraic techniques that allow for direct substitution, as well as graphical and tabular techniques that help make potential candidates evident), but the formal
definition is not one of them. On the contrary, the formal definition assumes the
preexistence of a candidate for $L$ — the candidate $L$ is only truly the limit if it
satisfies the quantified conditions stated in the formal definition. Thus, the formal
definition only gives a mechanism for validating that a given candidate $L$ is a limit;
it gives no mechanism for finding candidates for $L$. Hence, finding a candidate for
$L$ necessarily precedes using the formal definition to validate that a given candidate
is, in fact, the limit. Students’ initial experience with the limit concept is most often
dominated by learning, and subsequently employing, algebraic techniques designed
to find limit candidates. With such a strong initial focus on finding limits, it seems
unlikely that students would appreciate that the purpose of the formal definition is
not to find $L$, but rather to validate it as the limit. It would appear, then, that for
students to reinvent the formal definition of limit, they must be able to note the
distinction between finding and validating, so that they might become motivated to
construct a formal definition. I anticipate that the failure to make this distinction
will hinder students’ efforts to reinvent the formal definition, as they would
presumably lack the necessary motivation to precisely define limit.

**x-First Perspective vs. y-First Perspective**

The distinction made in the previous paragraphs between finding limit
candidates and subsequently validating them relates to a second cognitive difficulty
students may experience in reinventing the formal definition of limit. As Larsen
(2001) suggests, the thought process required by the formal definition is reverse
from that utilized when finding limit candidates. To be clear, finding a limit
candidate presumably utilizes what I call an *x-first perspective*, wherein one imagines *x*-values approaching the limiting value, *a*, and subsequently inspects what *y*-value the corresponding function outputs are approaching. Validating a candidate, however, relies on one’s ability to reverse his or her thinking. Carlson, Oehrtman, and Thompson (2007) agree: “In order to understand the definition of a limit, a student must coordinate an entire interval of output values, imagine reversing the function process and determine the corresponding region of input values” (p.160). Thus, the process of validating a candidate requires a student to utilize what I call a *y-first perspective*, considering first a range of output values around the candidate, projecting back to the *x*-axis, and subsequently determining an interval around the limit value that will produce outputs within the pre-selected *y*-interval. Prior to calculus, students’ interactions with functions are almost entirely from an *x*-first perspective. As a result, I conjecture that students may find reasoning from a *y*-first perspective counterintuitive and thus, will state their initial characterizations of limit from an *x*-first perspective.

*Algebraic vs. Graphical Representations*

An assumption underlying this mathematical-conceptual analysis of limit is that reinventing a formal definition capturing the intended meaning of the conventional *ε*-*δ* definition of limit relies on one’s ability to characterize precisely (either with words or symbols) imagistic features of a function’s *graphical representation*. Indeed, I conjecture that reinventing the definition of limit requires one to place an
emphasis on describing the visual aspects of limits as related to the conventional ε-δ illustration seen in Figure 3.1 (Stewart, 2001).

![Figure 3.1 - ε-δ Illustration of Limit](image)

**Figure 3.1 - ε-δ Illustration of Limit**

It is worth noting, however, that students' prior experiences with limits are comprised largely of learning, and subsequently employing, algebraic techniques to find limit candidates. Thus, I anticipate that students may be more likely to focus their attention on functions' *algebraic representations*. Further, because students study algebraic expressions and equations in mathematics prior to their introduction to graphs, they may be reluctant to accept that functions can exist in graphical form only. As such, they may feel as though a function's algebraic representation is always accessible and, in turn, may believe that they can simply apply direct substitution via one of their learned algebraic techniques to determine a limit with certainty. As a result, students may be reluctant to analyze function's graphical representations, which subsequently may hinder their progress in reinventing a formal definition.
3.3.1.3 How the Mathematical-Conceptual Analysis Influenced Methodology. The mathematical-conceptual analysis presented here influenced my study's methodology in a couple of important ways. First, the instructional tasks implemented in each teaching experiment session were informed by ongoing analysis of student reasoning in previous sessions. This mathematical-conceptual analysis served as one lens for analyzing student reasoning. Hence, I analyzed student reasoning with an eye towards evidence of the cognitive difficulties I explicated in the preceding pages. As I identified these cognitive difficulties arising for the students, I then constructed instructional tasks designed to support the students in resolving these difficulties. Thus, the mathematical-conceptual analysis influenced my ongoing analysis of student reasoning, and subsequently, the design of instruction for future teaching experiment sessions.

The mathematical-conceptual analysis also influenced the post analysis I conducted after each individual teaching experiment, and my retrospective analysis (Cobb, 2000) of the entire data corpus. Specifically, in these analyses I looked for evidence of student reasoning indicative of the cognitive difficulties I had conjectured.

3.3.2 The Mathematical-Symbolic Analysis of Limit

3.3.2.1 Introduction. The conventional ε-δ definition is a symbolic- and notation-rich formal mathematical statement. A student's understanding of the mathematical symbols, notation, and logical structure of the conventional definition
should not be taken to indicate that he or she understands the meaning of the concept that the conventional definition is designed to capture. However, if a student were to reinvent and understand a definition of limit capturing the intended meaning of the conventional \( \epsilon-\delta \) definition, one might expect that student to possess an understanding of the concept that would allow him or her to subsequently carry out an analytic deconstruction of the meaning of the definition. My approach in analyzing the formal definition of limit is to decompose the formal statement, piece by piece, into some of it its key semantic constituents and components. This method of analysis deconstructs the meaning of the definition atomically, following its formal statements piece by piece as if one were trying to interpret its meaning. Given this method of analysis, two related caveats are in order: First, there is no insinuation on my part that such a method is how a first-time reader might best come to understand this definition. To the contrary, reinvention was employed in this study based on the conjecture that such a context would serve as a means of better supporting student understanding of the formal definition. Thus, this analysis might be thought of as my attempt to identify components of the conventional \( \epsilon-\delta \) definition that someone who had previously reinvented the formal definition of limit would likely be able to understand and/or interpret coherently. Second, the folding back from form (i.e., syntactic aspects) to the intended semantics of the definition entails having imagistic representations of these ideas in mind. Thus, there should be no misunderstanding of my motives in this analysis: I would not
necessarily impute conceptual understanding of the idea of limit to "symbol pushing" behavior with respect to this formal definition.

3.3.2.2 The Mathematical-Symbolic Analysis. The following is the conventionally accepted formal definition of limit:

**Definition:** The function $f$ is said to have a limit $L$ as $x$ approaches $a$, denoted $\lim_{x \to a} f(x) = L$, provided that: for every $\varepsilon > 0$, there exists a $\delta > 0$, such that $0 < |x-a| < \delta \rightarrow |f(x) - L| < \varepsilon$

My mathematical-symbolic analysis deconstructs the stated conditions in the definition.

1) "For every $\varepsilon > 0$, there exists a $\delta > 0$ such that"

The above phrase raises two issues about: 1) how $\varepsilon$ and $\delta$ are individually quantified; and, 2) the logical implication of the order in which the quantifiers appear.

The two words "such that" at the end of the phrase "For every $\varepsilon > 0$, there exists a $\delta > 0$ such that" are intended to indicate that the relationship between the epsilons and corresponding deltas needs to be such that a particular condition or set of conditions not yet stated are satisfied. In fact, the conditions that follow must be satisfied not for just a single number called $\varepsilon$, but in fact, for every choice of $\varepsilon$. Further, for every choice of $\varepsilon$, there must merely exist a single $\delta$ for which the conditions yet to be stated are satisfied. Hence, the two statements, when taken together as a coordinated whole, collectively indicate that one must consider each and every possible choice of $\varepsilon$, yet need only establish the existence of a single $\delta$ for each possible choice of $\varepsilon$. Someone having a coherent understanding of the
formal definition would be able to distinguish the difference between the types of quantification on \( \varepsilon \) and \( \delta \) — \( \varepsilon \) is universally quantified, whereas \( \delta \) is existentially quantified.

The phrase "For every \( \varepsilon > 0 \), there exists a \( \delta > 0 \)" indicates a particular relationship between \( \varepsilon \) and \( \delta \). For each and every positive \( \varepsilon \), there is a corresponding positive \( \delta \). The first two phrases stated together might suggest an ordered relationship to a first time reader — epsilons first, deltas second. Hence, the choice of \( \delta \) depends on the choice of \( \varepsilon \), not vice versa. The definition does not say "There exists a \( \delta > 0 \) such that for every \( \varepsilon > 0 \)..." Such a definition would indicate the existence of a single \( \delta > 0 \) that satisfies stated conditions for every choice of \( \varepsilon > 0 \). Someone who had reinvented the definition of limit and understood its underlying meanings would presumably be able to understand the nature of this relationship between \( \varepsilon \) and \( \delta \).

2) "\( 0 < |x-a| < \delta \), \( |f(x)-L| < \varepsilon \)"

The informal definition of limit describes the limit \( L \) as a number that the function values \( f(x) \) become arbitrarily close to as \( x \) gets sufficiently close to \( a \). The phrase "\( f(x) \) becomes arbitrarily close to \( L \) as \( x \) gets sufficiently close to \( a \)" is rather ambiguous in describing closeness to \( L \) and \( a \). The formal definition is designed to describe this closeness to both \( L \) and \( a \) explicitly. Absolute value in mathematics represents distance on the real number line. Hence, the first phrase, "\( 0 < |x-a| < \delta \)," states that the distance between \( x \) and \( a \) is positive, yet smaller than \( \delta \), which was previously chosen based on \( \varepsilon \). Similarly, the second phrase, "\( |f(x)-L| < \varepsilon \)," specifies
that the distance between the function value $f(x)$ and $L$ is less than some positive number $\epsilon$. Figure 3.2 (Stewart, 2001) shows the pictorial representation of what the two absolute value statements describe. Closeness to $a$ can be thought of in terms of an interval around $a$, the width ($\delta$) of which can be made as small as necessary. Likewise, closeness to $L$ can be thought of as an interval around $L$, the width ($\epsilon$) of which can also be made as small as necessary.

![Figure 3.2 - Conventional $\epsilon-\delta$ Illustration](image)

The reader should note here that distance is non-negative, but unlike the statement in the definition that describes the distance between $x$ and $a$, there is nothing restricting the distance between $f(x)$ and $L$ from being 0 in this part of the specification.

In sum, an unambiguous characterization of infinite closeness along the $x$- and $y$-axes is a fundamental component of any formal definition of limit. Someone possessing a coherent understanding of the formal definition could presumably interpret the inequality statements used in the conventional $\epsilon-\delta$ definition as notation used to describe getting close to both $L$ and $a$. 
3) “$0 < |x-a| < \delta \rightarrow |f(x) - L| < \varepsilon$”

Putting the two phrases discussed above together via an implication, one sees that restricting the distance between $x$ and $a$ necessarily restricts the distance between $f(x)$ and $L$. Hence, closeness to $a$ and $L$ are related. Indeed, the definition says that sufficient closeness to $a$ (as described in #2 above) implies sufficient closeness to $L$ (as described in #2 above). Someone with a formal understanding of limit would seemingly be able to understand the role of an implication in the definition.

Further, the implication also introduces a universal quantifier on $x$ in the definition. Logically, the statement $p \rightarrow q$ denotes a necessity relationship – that statement $p$ implies statement $q$, meaning that whenever statement $p$ is true, statement $q$ will necessarily also be true. Here, $p$ is a statement about the distance between $x$ and $a$ being less than delta, yet positive. The arrow then indicates that if this condition $p$ is met, the conclusion $q$ will subsequently be true. Hence, the implication says that for every $x$-value within a positive distance $\delta$ of $a$ (but $\neq a$), the corresponding function value $f(x)$ will be within $\varepsilon$ of $L$. Pictorially, Figure 3.3 (Stewart, 2001) shows that $x$-values within the chosen $\delta$-neighborhood of $a$ have corresponding function values within the previously selected $\varepsilon$-neighborhood of $L$. 
Hence, $x$-values in the $\delta$-band (except possibly $x = a$) lead to $y$-values in the $\varepsilon$-band. Most importantly, this implication holds not just for a single $x$-value in a $\delta$-neighborhood of $a$, but for every $x$-value in that $\delta$-neighborhood (except possibly $x = a$). It is reasonable to assume that someone who had reinvented a definition of limit capturing the intended meaning of the conventional $\varepsilon$-$\delta$ definition would subsequently understand that the quantifier on $x$ is universal.

3.3.2.3 How the Mathematical-Symbolic Analysis Influenced My Methodology. The mathematical-symbolic analysis influenced my study in the following ways. First, in the midst of each teaching experiment session, and in ongoing analysis, I looked for evidence of the students using notation reminiscent of that used in the conventional $\varepsilon$-$\delta$ definition. In cases when such notation was suggested by a student, I subsequently posed tasks which might support the continued use of the notation by the students. Second, the mathematical-symbolic definition identified specific understandings that I conjectured a student would have had he or she previously reinvented a definition of limit capturing the
intended meaning of the conventional $\epsilon$-$\delta$ definition. To test this conjecture, I designed a specific task for the final paired session which assessed the extent to which the students possessed these identified understandings$^{20}$. In sum, then, the mathematical-symbolic analysis influenced both my ongoing analysis of student reasoning, and the design of particular instructional tasks.

3.3.3 The Interaction between Theoretical Frameworks

Some brief summarizing comments are in order. The theoretical frameworks I described in Sections 3.1 and 3.2 served as meta-level perspectives for my study. Radical constructivism guided the design and execution of my study epistemologically. For instance, careful attention was paid in data analysis to generating viable interpretations of students’ reasoning as opposed to assessing whether students possessed a pre-determined “correct” form of understanding. The perspectives of developmental research and the related heuristic of guided reinvention influenced the selection of starting points for instruction and the subsequent construction of instructional tasks informed by ongoing analysis of student reasoning. Finally, the goals of the teaching experiment methodology aligned with the central objective of my research – to model student reasoning. Each of these theoretical perspectives served as overarching frameworks that complimented the two-part domain-specific mathematical-analysis of limit described in Section 3.3. This domain-specific framework had methodological

\footnote{The details of this task can be found in subsequent chapters of this document.}
implications as well. For instance, it guided the design of instruction and served as one analytic lens. However, it was unique from the meta-level perspectives as it pertained specifically to the concept of limit. Thus, in a way that the meta-level perspectives could not, the mathematical-analysis of limit helped me differentiate between the types of understandings of limit I would expect someone to attend to in the process of reinventing a formal definition and the understandings particular to the conventional ε-δ definition that I would expect someone to possess had that person previously reinvented a formal definition of limit.
Chapter 4 – Research Methodology, Design and Analysis

In this chapter I explicate the research methodology and design, as well as the analytic procedures, which were employed in the study. The chapter is presented in five sections. In Section 4.1, I provide a general overview of the research design, which includes descriptions of the stages of research, the research cycle, and the data collection methods that were employed. In Section 4.2, I discuss participant selection and provide background information about the four students who participated in the teaching experiment phase of the study. In Section 4.3, I summarize the research instruments that were used in the two phases of the study, and I explain the overarching purpose of each instrument. In Section 4.4, I discuss how data analysis was conducted, and include a detailed outline of the phases of the analysis. In Section 4.5, I address issues of validity and ethics in this research project.

4.1 – Overview of Research Design

4.1.1 Description of Research Stages

The purpose of this section is to provide the reader with a general overview of the multiple research stages which comprised the study. Details of these stages are found in subsequent sections of the chapter.

The study was comprised of seven research stages, highlighted by two central research phases – Phase I: the preliminary survey, and Phase II: the teaching
experiments. Table 4.1 displays the chronology of the distinct research stages and identifies the phases comprised by them.

<table>
<thead>
<tr>
<th>Phase</th>
<th>Stage</th>
<th>Research Activity</th>
<th>Date</th>
</tr>
</thead>
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<tr>
<td>I</td>
<td>1</td>
<td>Participant Selection for Informal Limit Reasoning Survey</td>
<td>March 2007</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>Informal Limit Reasoning Survey</td>
<td>April 2007</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>Analysis of Surveys and Participant Selection for Teaching Experiments (TE)</td>
<td>May 2007</td>
</tr>
<tr>
<td>II</td>
<td>4</td>
<td>TE #1</td>
<td>May – July 2007</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>Post Analysis of TE #1 and Refinement of Instructional Sequence</td>
<td>Aug – Sept 2007</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>TE #2</td>
<td>Sept – Dec 2007</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>Post Analysis of TE #2 and Retrospective Analysis of Data Corpus</td>
<td>Jan – May 2008</td>
</tr>
</tbody>
</table>

Table 4.1 – Chronological outline of the various levels of the research study

Participant selection and the collection and analysis of data, then, transpired over the course of seven research stages. The first stage consisted of the selection of twelve student participants from the researcher’s Calculus III course\(^\text{21}\) to complete an Informal Limit Reasoning Survey. Students were selected on the basis of criteria described in Section 4.2. The second stage consisted of said students completing the Informal Limit Reasoning Survey. This survey provided the researcher data necessary for meeting the participant selection criteria (described in Section 4.2) for the Teaching Experiment phase of the study. The third stage consisted of the

\(^{21}\) This course was the third term of a three-term introductory Calculus sequence. The content focus of this particular course was infinite sequences and series.
researcher analyzing the Informal Limit Reasoning Survey and selecting four participants for the Teaching Experiment phase of the study, which was comprised of two separate teaching experiments. Each teaching experiment was conducted with a pair of students\(^{22}\). The first teaching experiment marked the fourth stage of the study, transpiring over the course of approximately ten weeks. The fifth stage was comprised of the researcher conducting a post analysis of the first teaching experiment, and subsequently refining the sequence of instructional tasks in anticipation of the second teaching experiment. Analysis of the first teaching experiment and preparation for the second teaching experiment took approximately two months. The second teaching experiment, also transpiring over the course of approximately ten weeks, marked the sixth stage of the study. The seventh, and final, stage followed the second teaching experiment, and was comprised of a post analysis of the second teaching experiment, as well as a retrospective analysis of the entire data corpus. These analyses were conducted over a five month period.

4.1.2 Description of Research Cycle

The sequences of instructional tasks designed for this study were created with a central purpose in mind: to develop insight into students' reasoning in relation to their engagement in instruction designed to support their reinventing the formal definition of limit, presumably with no prior experience with the definition. This was set against the broader background goal of contributing to an epistemological

\(^{22}\) The rationale for conducting the teaching experiments with pairs of students (as opposed to individual students) is explicated in Section 4.1.4.
analysis (Thompson & Saldanha, 2000) of the concept of limit of a real-valued function and its formal definition. The research was developmental in nature (Gravemeijer, 1998) in that an epistemological analysis of the formal definition of limit, and thus the sequence of instructional activities, evolved in tandem with analyses of student responses to tasks posed in instruction. I see contributing to an epistemological analysis as a process driven by the reiteration of a research cycle entailing three central stages: *design* of an instructional sequence, *engagement of students* in the instructional sequence, and *analysis* of students’ reasoning. Figure 4.1 illustrates this cyclic process.

![Figure 4.1 - Overview of Research Design](image)

The instructional design of the teaching experiments was shaped by two key factors: 1) A mathematical analysis of the concept of limit, with a particular focus
on the formal definition of limit; and, 2) Analyses and modeling of students’ reasoning as they engaged with instructional tasks. These factors were in a dialectical relation that was traceable to the pilot study that motivated the dissertation study. The pilot study was motivated and informed by an a priori mathematical analysis of the concept of limit. Analyses of the data generated in the pilot study, in turn, fed back into a refinement of the mathematical analysis of the concept. This refined mathematical analysis consisted of two parts – a mathematical-conceptual analysis and a mathematical-symbolic analysis. Collectively, this two-part analysis shaped the dissertation study, in that it pointed to a number of seemingly fundamental understandings that students would likely develop in the process of reinventing the formal definition. The first teaching experiment, then, was informed by both the mathematical analysis of limit and an analysis of student reasoning during the pilot study. The pilot study I conducted can be considered as a 0th iteration of the cyclic research process.

An epistemological analysis is not a static product. Rather, it is an ever-evolving product informed by the analysis of students’ understandings in specific settings (Thompson & Saldanha, 2000). This dissertation study entailed two iterations of the research cycle, employing a staggered design wherein both pairs of participants were engaged with a sequence of instructional tasks one pair at a time. The study’s design included two months time in between each teaching experiment.

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23 This mathematical analysis of limit can be found in Chapter 3.
24 For an account of the pilot study, see Swinyard & Lockwood, 2007.
to maximize the opportunity for analysis of student reasoning and refinement of the instructional sequence.

4.1.3 Data Collection Methods

The study was comprised of two central phases – the survey phase and the teaching experiment phase. With the aim of gathering information about how the initial participant pool reasoned about limits informally, I conducted a task-based Informal Limit Reasoning Survey with twelve undergraduate students from a large, urban university in the Pacific Northwest region of the United States. Each of the survey participants were students in two or more of the courses forming a three-quarter introductory calculus sequence I taught during the 2006-2007 academic year. The survey was designed to help guide the selection\(^{25}\) of four students to participate in the teaching experiment phase of the study. The students completed the surveys in my presence and were given as much time as needed to complete the survey. Generally, the students were able to complete the survey in 45-60 minutes. The surveys were not analyzed with the intent of determining varying levels of understanding based on some score. Rather, the surveys were analyzed with an eye toward detecting notable weaknesses or inconsistencies in students’ reasoning about limits. Also, it is worth acknowledging that students’ reasoning about complex mathematical ideas such as limit likely fluctuates in its coherence. As such, weaknesses or inconsistencies in students’ reasoning on this survey did not

\(^{25}\) I describe the criteria on which students were selected to participate in the second phase of the study in Section 4.2.
serve as a sole means by which to disqualify an individual as a potential participant in the teaching experiment phase. Four students were ultimately selected to form two pairs for the teaching experiment phase of the study based on a holistic evaluation of the four main criteria outlined in Section 4.2, combined with additional information that was gathered from this survey. Thus, the Informal Limit Reasoning Survey served as but one element of that holistic evaluation. The contents of the survey are described in greater detail in Section 4.3 and can be found in Appendix A.

Two teaching experiments, each involving a pair of students, formed the second phase of the study. Both teaching experiments consisted of ten, 60-100 minute paired sessions, and one 30-60 minute individual exit interview. The paired sessions were conducted in a classroom, with the students responding to instructional tasks on the blackboard in the front of the room. Only the participating students, researcher, and research assistant were present for each session. Each session was generally separated by a span of 6-10 days, allowing time for ongoing analysis between sessions and the subsequent construction of appropriate instructional activities based on the ongoing analysis. All sessions, including the individual exit interviews, were videotaped by a research assistant. These videotapes were the primary source of data for informing a contribution to an epistemological analysis of limit.

Ongoing analysis was the first of three types of data analysis employed in this study. The other two types of analysis were post and retrospective. The nature of each of the three types of analysis is detailed in Section 4.4.
4.1.4 Rationale for Conducting the Teaching Experiments with Pairs of Students

There were a couple of advantages in conducting the teaching experiments with pairs of students instead of with individual students. First, doing so encouraged more communication between the students than between the students and me, thus increasing the potential for creating a dynamic of authentic, self-motivated reasoning and shared ideas. I feared that conducting the teaching experiment with individual students would result in the emergence of a "guess and check" dynamic, wherein the student would try to guess what I wanted him or her to say and would continue to do so until I seemed satisfied. My concern was that such a dynamic would result in less authentic and self-motivated reasoning. Conducting the teaching experiments with pairs of students enabled the emergence of the type of reasoning I wished to explore. Second, working in pairs ostensibly allowed students time for reflection – while one student was reasoning aloud about the situation at hand, the other student presumably had the time to sit and think, feeling less pressure to speak. This structure appeared to provide students more time for processing the information at hand, which then seemingly led to more substantive reasoning. A central purpose of the teaching experiments was to provide a context for generating empirical evidence of student reasoning with respect to the formal definition of limit. Thus, having the students work in pairs created a dynamic in which reasoning of interest to me was more likely to emerge than would otherwise
have been the case if the students were responding directly to me, as opposed to one another.

4.2 – Selection of Teaching Experiment Participants

Participants for the teaching experiment phase were selected on the basis of the following criteria: 1) robust informal understanding of limit; 2) no prior experience with the formal definition of limit, be it in high school or other calculus courses taken at the university level; 3) demonstrated ability to communicate their reasoning freely and without hesitation and make sense of complex mathematical ideas; and, 4) demonstrated responsibility, maturity, and reliability. I discuss each of these criteria in greater detail below and then provide background information about the four teaching experiment participants.

4.2.1 Criterion 1: Robust Informal Understanding of Limit

Given the central purpose of the sequence of tasks employed in this study, participants for the survey phase, and subsequently, the teaching experiment phase were selected partially based on the extent to which they demonstrated strong informal understanding of limit. I hypothesized that reinventing the formal definition of limit would be improbable for students who possessed a weak informal understanding of limit to begin with. As an example, research (Cornu, 1991; Davis & Vinner, 1986; Ferrini-Mundy & Lauten, 1993; Lauten, Graham, &

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27 The central purpose of the dissertation study was to develop insight into students’ reasoning in relation to their engagement in instruction designed to support their reinventing the formal definition of limit, presumably with no prior experience with the definition.
Ferrini-Mundy, 1994; Tall, 1980; Tall & Vinner, 1981; Williams, 1991) indicates students often initially believe that the limit is a value that is approached but never reached. Students in my dissertation study holding such a belief would then have been reasoning about an idea distinct from the idea described by the formal definition of limit and thus, the subsequent data I collected would have been insufficient to contribute to an epistemological analysis (Thompson & Saldanha, 2000) of limit. To contribute to an epistemological analysis is to gain insight into what is entailed in coming to understand a particular mathematical idea in relation to engagement in instruction designed to support the development of that understanding. This study was designed so that I might generate and collect data that would allow me to contribute to such an analysis. As such, it was necessary to establish that students selected for the two teaching experiments could reason coherently about the limit concept on an informal level. Specifically, this meant that students were able to:

1) Discuss when a limit does exist and why
2) Discuss when a limit does not exist and why
3) Determine limits for both finite and infinite situations
4) Sketch graphs satisfying given conditions related to both finite and infinite limits
5) Provide an informal definition of limit that demonstrates viable conceptual understanding.

Students demonstrating these abilities were seen as viable candidates for producing data that could be used to contribute to an epistemological analysis of limit. Students were selected for the survey phase of the study based partially upon the degree to which they had developed and demonstrated informal understanding of the concept throughout the introductory calculus sequence. Specifically, a variety of assessment tools, including homework, classroom activities, and exams provided sufficient evidence from which to draw conclusions as to the students' informal understanding. The Informal Limit Reasoning Survey (see Appendix A) provided supplemental evidence of the students' informal understanding. Collectively, these assessment tools informed the selection of participants for the teaching experiment phase.

4.2.2 Criterion 2: No Prior Experience with the Formal Definition of Limit

The second criterion for selection was that participants have no previous experience with the formal definition of limit. Evidence (Swinyard & Lockwood, 2007) suggests that prior experience with the formal definition – regardless of how recent, substantial, or worthwhile that experience – results in students often attempting to remember as opposed to reason. I recognize that one cannot always control for students' prior mathematical experiences. However, I conjectured that

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28 By viable conceptual understanding, I mean the student possesses stable ways of thinking about limit that entail imagining, connecting, inferring, and thinking about situations involving limits in particular ways.
my ability to learn the ways in which students reason about the formal definition of limit were to be optimized by eliminating the potential for muddled attempts to regurgitate formalisms that were not meaningful to them. Evidence cited above illustrates the potential for remembering to cloud students’ reasoning. Selecting participants who had no prior experience with the formal definition of limit provided me a “cleaner slate” from which to begin engaging them in substantive reasoning.

During the 2006-2007 academic year, I taught a three-term introductory calculus sequence. Students enrolled in two or more of these courses served as a pool from which to draw twelve participants for the survey phase of the study. By selecting students from my own calculus course who had no prior calculus experience, I was able to control better for how they had interacted with limits. By this I mean I was more certain of the ways in which the participants for my study had engaged with the limit concept than if I had selected participants who had been taught calculus exclusively by other instructors. Information about students’ prior calculus experience was gathered via introductory student emails required at the outset of the course, in which students provided me with background information in response to specific questions, such as “Have you taken Calculus previously? If so, when and where did you take Calculus?” Students studied the idea of limit only informally during the three-term introductory sequence – thus, I had a much better

29 “Cleaner slate” is not meant here in the “tabula rasa” sense – that is, my epistemological perspective is not that students’ minds should be viewed as “clean slates” on which ideas are imprinted.
understanding of the ways in which they had encountered the concept than if I had chosen students from another participant pool. Selecting students from my own course allowed me to have greater control over the selection of appropriate candidates for my study, as well as make clearer conclusions following data analysis.

Of the twelve students selected for the survey phase of the study, five had taken calculus at other (secondary or post-secondary) institutions prior to their enrollment in the three-term introductory sequence I taught during the 2006-2007 academic year. Four of those five students were not chosen for the teaching experiment phase of the study, based largely on my inability to establish that they had not recently been introduced to the formal definition of limit. The fifth student, who was chosen for the teaching experiment phase, had previously taken one term of calculus at a different post-secondary institution, but had done so over ten years prior, and demonstrated no evidence of having been introduced to the formal definition.

As an additional precaution, as each teaching experiment progressed, teaching experiment participants were specifically asked not to consult mathematics textbooks or other resources, including other participants, for ideas related to the concept of limit, so that the integrity of the study could be maintained. Again, my intention was that students might reinvent, as opposed to interpret, the formal definition of limit.
4.2.3 Criterion 3: Demonstrated Ability to Communicate Reasoning and Make Sense of Complex Mathematical Ideas

The guiding purpose of the proposed dissertation study was to understand better how students reason about the formal definition of limit. To gain such understanding, I needed students who felt comfortable communicating their mathematical thoughts and ideas to one another. Those students who satisfied Criteria 1 and 2 and who showed comfort in communicating their ideas clearly, both verbally and in writing, served as strong candidates for the dissertation study. As their calculus instructor, I had sufficient opportunity to observe and assess these abilities. My assessment of students’ ability to communicate was primarily subjective, based on ongoing observation.

Further, to gain understanding of how students reason about limit in the context of reinvention, it was important for me to select participants who had proven to be active seekers of viability and fit between their mathematical understandings. Students selected to participate in this study had demonstrated a greater effort and desire, relative to other students, to consistently make sense of their experiential world as it relates to complex mathematical ideas.

4.2.4 Criterion 4: Demonstrated Responsibility, Maturity, and Reliability

Finally, data generation relied on students’ willingness and ability to demonstrate responsibility and reliability by following through with their commitment to make themselves available for scheduled instructional sessions.
Specifically, I engaged participants in the teaching experiment in pairs, requiring me to coordinate multiple people’s schedules. As such, it was imperative that only those students who had demonstrated responsibility and reliability were chosen to participate, so as not to compromise the data generation process. I was able to observe students’ dispositions with regard to this issue throughout the three-term introductory calculus sequence.

4.2.5 Teaching Experiment Participants

Four of the twelve students who completed the Informal Limit Reasoning Survey were subsequently selected for the teaching experiment phase of the study. These students were chosen on the basis of the criteria described in the preceding sections, as well as the researcher’s estimation of their ability to work effectively in tandem to reinvent the definition of limit. In order to preserve the confidentiality of the teaching experiment participants, I use pseudonyms: Amy, Mike, Chris, and Jason. Demographic background for the teaching experiment participants is as follows: one female and three males, with an age range from 19 to 28 years of age. Additional background information is provided in Table 4.2.

<table>
<thead>
<tr>
<th>Name</th>
<th>Academic Major</th>
<th>Calculus 1 Grade</th>
<th>Calculus 2 Grade</th>
<th>Calculus 3 Grade</th>
</tr>
</thead>
<tbody>
<tr>
<td>Amy</td>
<td>Linguistics</td>
<td>A</td>
<td>A</td>
<td>A</td>
</tr>
<tr>
<td>Mike</td>
<td>Mathematics</td>
<td>A</td>
<td>A</td>
<td>A</td>
</tr>
<tr>
<td>Chris</td>
<td>Computer Science</td>
<td>A</td>
<td>A-</td>
<td>B+</td>
</tr>
<tr>
<td>Jason</td>
<td>Philosophy</td>
<td>A</td>
<td>A-</td>
<td>P</td>
</tr>
</tbody>
</table>

Table 4.2 – Background Information of Teaching Experiment Participants
Two points of clarification in regards to the participants' grades in the three-term calculus sequence are worth mentioning. First, due to scheduling conflicts, Amy took and received her grade for Calculus 2 from a different instructor. This instructor assured me that none of the students in his course was introduced to the formal definition of limit. Hence, Amy's inclusion in the teaching experiment phase did not compromise the integrity of the study. Second, due to other academic commitments, and because he did not need a letter grade for his academic major, Jason chose to be graded in Calculus 3 on a pass/no pass basis. In accordance with his performance in the first two calculus courses, Jason demonstrated above-average ability in Calculus 3.

4.3 – Survey and Teaching Experiment Instruments

4.3.1 Informal Limit Reasoning Survey

The Informal Limit Reasoning Survey consisted of ten tasks/questions designed to elicit evidence of students' ways of reasoning about and understanding the concept of limit. Here, I describe the nature of, and a brief rationale for, each task. A copy of the survey can be found in Appendix A. The first two tasks required the students to agree or disagree with statements made about limits by hypothetical students. These tasks also required the students to provide justification for their responses. The first two tasks were designed to assess the extent to which students believe the existence of a limit can be justified on the basis of the presence of an algebraic or tabular functional representation. These tasks also provided me an
opportunity to assess whether the participating students' concept images contained any of the misconceptions detailed in Chapter 2. The next five tasks assessed the extent to which the students could generate prototypical examples and counterexamples of limits, as well as what justification they might provide for why a limit exists or fails to exist. The eighth task required the students to construct a graph satisfying multiple limit-related conditions, and was designed to assess students' understanding of one-sided limits and their ability to distinguish between finite and infinite limits. The ninth task required the students to rate their approval of five different informal definitions of limit\textsuperscript{30}, some of which were embedded with errors reflecting prevalent misconceptions that I described in Chapter 2. The final task provided the students an opportunity to state their personal definition of limit. This task helped me to identify any students who may have previously seen the formal definition of limit, as usage of conventionally used symbols like $\varepsilon$ and $\delta$ were a strong indication of the students' past introduction to the definition.

Collectively, the ten tasks were designed to provide me with insight into students' reasoning about limit, so that I might subsequently select four qualified participants for the teaching experiment phase. Also, data from this survey informed the subsequent formation of instructional tasks for the first teaching experiment. Indeed, the methodology employed in this study reflects Ferrini-Mundy and Graham's (1994) viewpoint that embedded in students' informal

\textsuperscript{30} The informal definitions of limit used for this task were borrowed from Williams (1991).
interpretations of a concept are often valuable ideas and constructs upon which new meanings can be built during the evolution of their understanding of the concept.

4.3.2 Teaching Experiment Tasks

In line with the goals of developmental research (Gravemeijer, 1998) and the epistemological stance of radical constructivism (von Glasersfeld, 1995), each teaching experiment was framed by a sequence of instructional tasks which aimed to assist the students in developing previously identified understandings related to the concept of limit. The intent was to establish a learning environment in which the students would frequently be motivated by authentic perturbations to make cognitive accommodations to their informal understandings of limit. The ultimate instructional aim of each teaching experiment was for the students to reinvent a definition of limit that captured the intended meaning of the conventional $\varepsilon-\delta$ definition. The reinvention process was *guided* in the sense that instructional tasks and activities were carefully designed so that students might naturally formalize their informal understandings and intuitions.

The implementation of a developmental research methodology (Gravemeijer, 1998) resulted in the generation of an instructional task sequence for each of the two teaching experiments. The instructional activities and tasks were developed based on a *local instructional theory*, which rationalized why the chosen instructional activities might best support students in developing the previously

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31 These targeted understandings are summarized in the mathematical-conceptual analysis of limit presented in Chapter 3.
identified understandings. Neither the instructional theory nor the instructional activities were static entities. Rather, an evolving dialectic existed between the theory and the activities. Analysis of student activity and reasoning as they engaged in the instructional activities and tasks, in turn, served as a guide in further developing the local instructional theory, and in refining the instructional activities and tasks to be implemented in subsequent teaching experiment sessions.

Both instructional sequences were relatively similar in structure. During initial sessions, tasks were designed to further assess students' informal understandings of limit and motivate them to see the need for increasingly precise and rigorous formulations of the idea. Next, students were asked to generate prototypical examples and counterexamples of limit, which they subsequently used as tools for guiding the formulation of their definition. The remainder of each teaching experiment was characterized by the pair of students conjecturing and refuting formulations of the definition in an iterative manner. In this sense, the students' interactions were reminiscent of mathematical conversations described by Lakatos (1976). The reader will develop a fuller understanding of the specific tasks and activities which comprised the sequence of instructional tasks by reading the subsequent chapters and appendices of this document. A complete list of the instructional tasks employed in the first and second teaching experiments, as well as a rationale for those tasks, is provided in Appendices B and C, respectively.
4.4 – Description of Data Analysis

In this section I describe the methods I employed to analyze the data. The analytic approach I utilized is consistent with grounded theory methods (Glaser & Strauss, 1967), in which data analysis is a cyclic process in which hypotheses about students’ reasoning are generated, reflected upon, and subsequently refined until increasingly stable and viable hypotheses emerge. The analysis of data for this study occurred at a variety of levels. As each teaching experiment was unfolding, I conducted ongoing analysis, which informed my decisions about subsequent sessions within the same teaching experiment. Following the completion of each teaching experiment, I conducted a post analysis of the data generated by each pair of students. This provided me an opportunity to analyze each set of data more deeply, so as to begin to develop themes present throughout the data set. Finally, following the completion of both teaching experiments, I conducted a retrospective analysis (Cobb, 2000), in which I was able to analyze the entire corpus of data at a deeper level than the preceding analyses. What follows is a description of the ways in which I analyzed data both during and after the two teaching experiments.

4.4.1 Ongoing Analysis

Immediately following each session, I wrote down my initial reactions to what had just transpired. These thoughts included conjectures as to why the students had

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32 The data corpus for analysis consisted of twenty two videotaped sessions each lasting 60-100 minutes.
been either successful or unsuccessful in making progress towards a refined
definition of limit within the context of certain tasks. These thoughts also included
initial ideas about what types of tasks in subsequent sessions might facilitate
progress for the students. Following this initial reflection, I transcribed each session
in its entirety. This process provided me an opportunity to pay close attention to
their choice of words as they engaged with and responded to the tasks I presented
them. Following the completion of each transcription, I next re-read the
transcription, underlining the portions of the transcription that I felt were important
for any variety of reasons, including articulated thoughts that seemed to provide the
students leverage, the voicing of concerns or perceived hurdles that needed to be
overcome, or signs of progress and revision. Underlining these important excerpts
allowed me to more easily locate important interactions later upon deeper analysis.

Next, I viewed the session a second time and constructed a content log—a
record, described in broad strokes, of what transpired in each session. Included in
each content log was an account of what issues were being addressed by each
student, as well as the issues I was raising for them. An excerpt from one such
content log is seen below.

**Example from a Content Log**

24:00 I ask Mike if he agrees with Amy’s comments; Mike draws a jump
discontinuity (for a piecewise function) and points out the limit
wouldn’t exist, so plugging in wouldn’t work; it’s clear that Mike and
Amy both have a wealth of functions in their “function catalog” from
which they can work.

26:30 I recap for them what they’ve said about certainty in regards to different
representations of functions and ask them if a graph can exist in and of
itself; Amy says, "How do you know it's even a graph of a function? How do you know it's not just a pretty picture?" (I should have let Mike answer here)

28:00 Before Mike can answer I give them an example of a radar-created graph that has no corresponding algebraic representation and ask Amy to talk about why the limit is 0 by just talking about the graph

The reader will note that the content log is partially underlined. Following the completion of each content log, I read the content log in its entirety, once again underlining the portions that I deemed important, as I had for the session transcriptions discussed above.

The initial reflection following each session, as well as the construction of the session transcription and content log provided me an opportunity to form ideas about how best to proceed in the following session. Prior to each session, I composed an 8-10 page document which included the following information: email correspondence between myself and my committee members pertaining to the previous session, central purposes/objectives of the upcoming session, anticipated tasks that would be employed, as well as a rationale for each of those tasks, and finally, a copy of my reflections on the previous session, comprised partially of my initial reactions, as well as deeper reflections following the construction of the session transcription and content log. I entitled these 8-10 page documents “Ideas, Goals, Tasks, and Reflections.”

4.4.2 Post Analysis

Following the first teaching experiment, I reviewed all of the transcriptions for the first pair of students in chronological order. As I did so, I copied the underlined
portions that I deemed important in some manner into a new document. Importance was certainly a matter of my opinion – another person analyzing the data, perhaps looking through a different analytical lens, may have found a different subset of transcription excerpts important. Nevertheless, I considered excerpts to be important if I felt they shed light on the ways in which the students were reasoning about limit or if they illustrated significant cognitive obstacles or marked progress.

As I extracted these excerpts from the individual session transcriptions, I jotted down notes prior to each session excerpt that described what was transpiring in the excerpt and why I felt it was important. As I was doing this, I also noted which session the excerpt was from and when during the session the excerpt took place, so as to more easily locate important excerpts during later analyses. An example of this is seen below.

Mike’s response below seems to suggest a categorization he’s formed for when and how one can plug into a function (i.e., use direct substitution) (#2, pre-1:10:00)

Craig: Mike? What do you think Mike? What is it for you? What does it mean to plug in?
Mike: I’d say for the first case it was to replace x with 4, and then solve. In the second case, the same thing after we factor and stuff. In the second case, or this last case (pointing at the piecewise function) though, I don’t, I wouldn’t plug in 0 because, I don’t know, I guess I’m looking at inequalities as different then...

This process led to the construction of a 130 page document which traced the development of the first pair of students’ reasoning, as well as the evolution of their formal definition of limit. I called this document the “Learning Trajectory for Amy and Mike.”
The construction of the aforementioned learning trajectory was an important step in analysis, because it provided me an opportunity to view the data as a more collective whole. During the ongoing analysis, I was only analyzing each session one at a time. While some thought was given to prior sessions in the midst of the ongoing analysis, I was not considering the collective data set generated by the sessions with Amy and Mike as a whole. Indeed, doing so was impossible until all ten paired sessions had been completed. Thus, the learning trajectory helped me better see the development of Amy and Mike’s ideas over the course of the ten paired sessions, as well as the ways in which some of their ideas and beliefs about limits were consistent throughout the teaching experiment. Following the second teaching experiment, I constructed a similar learning trajectory for the second pair of students (Chris and Jason).

Finally, following the construction of the learning trajectory for the first pair of students that I described above, and prior to the second teaching experiment, I conducted what I call a task analysis of the tasks employed during the first teaching experiment. The purpose of this task analysis was to determine which tasks I should employ with the second pair of students (and the corresponding rationale) and which tasks I felt were not worth using with the second pair (and why). These decisions were made on the basis of my analysis of how effective the tasks had been during the first teaching experiment. An example taken from this task analysis document is shown below in Figure 4.2. The reader should note that the general task description, written prompt, and purpose were written prior to the
session in which the task was employed; the task analysis portion was the only part written at the time the task analysis document was created.

**Interview 4, Task 1:** Have students generate graphical examples of the different ways a function could have a (finite) limit of 2 at \( x=5 \).

**Written Prompt:** Please generate as many distinct examples of how a function could have a limit of 2 at \( x=5 \). In other words, what are the different scenarios in which a function could have a limit of 2 at \( x=5 \)?

**Purpose:** The thought here is it will be important for them to have these various scenarios in mind when they attempt to pin down what it means for a function to have a limit \( L \) as \( x\to a \). Part of defining limit (or anything for that matter) is being able to articulate conditions which would address ALL the different instantiations of such an object. Thus, I’ll start here by having them create their own examples. Another reason for having them do this (as opposed to having me provide these different ways for them) is so that they will have some ownership of these graphs – they’ve been wary of graphs that “come from someplace unknown,” so it seems to make sense to have them generate these themselves. I’m hoping this will increase the buy-in for them and that they’ll be more likely to engage in conversations about limits of these graphs, which is the first of two central goals for this session.

**Task Analysis:** This task ended up being very important – in fact, it was important enough that we revisited it in Session 9 so that we could have visible the different ways in which a function can have a finite limit \( L \) at \( x=a \) as Amy & Mike worked on a definition. A central focus of Amy & Mike’s as they constructed their definition of limit was having their articulation be such that it addressed all three scenarios simultaneously – hence, I think this task is very worthwhile. In fact, I think it may be helpful to move to this sooner than later – i.e., in the first or second session, not the 4th.

**Figure 4.2 – Task Analysis Example**
As part of the *post* analysis, the task analysis served as a means by which I was able to refine the instructional sequence to better address my research objectives and support the second pair of students in their efforts to reinvent the formal definition of limit.

### 4.4.3 Retrospective Analysis

Following the completion of both teaching experiments, and both *post* analyses, I next reviewed both learning trajectories in their entirety. The learning trajectories were lengthy and detailed in their analysis. Hence, the review of these two documents was yet another form of analysis, the aim of which was to situate events which transpired during each teaching experiment “in a broader theoretical context, thereby framing them as paradigmatic cases of more encompassing phenomena” (Cobb, 2000). As I reviewed these two documents, I generated a list of what I initially referred to as *themes*. This list of themes was constructed in free form – that is, I did not attempt to group, organize, or categorize the themes in any manner in the process of creating the list. Instead, I merely jotted down what I felt were consistent thought processes or reasoning patterns among some or all of the four students, as well as conceptual difficulties the students appeared to experience. The initial list contained over fifty themes. Next, seeking validation of my proposed themes (in the sense of Strauss & Corbin, 1998), I triangulated various data sources, including students’ surveys, videotapes of the teaching experiments, and written responses to specific tasks within each teaching experiment. Also, I
discussed this initial list of themes (or perhaps more aptly put, consistent ways of thinking and reasoning) with multiple colleagues. The act of triangulating data sources and describing these themes to others was yet another form of analysis—both actions led me to realize that some themes found in the original list could more appropriately be described as interesting things that came up during the teaching experiments. Following the triangulation of data sources and each presentation of my themes to colleagues, I was able to better organize the themes in terms of both importance and nature. The vast majority of the original themes became secondary in my estimation, either due to infrequency or lack of relative importance to other themes or the students' ultimate success in reinvention. The remaining themes I viewed as primary, and subsequently formed the backbone of the Results section of the dissertation (see Chapter 5 and Chapter 6). The resulting narrative in these two chapters emerged out of my effort to tell a coherent story about the development of both pairs of students' evolving definitions in a manner that pieced together and highlighted the importance of the distilled set of themes.

4.4.4 Addressing the Central Research Objective

The central objective of this dissertation study was to develop insight into students' reasoning about limit in the context of reinventing the formal definition. The analytic methods described in the preceding sections supported me in successfully addressing this objective. Specifically, the cyclic nature of the analytic approach I employed permitted me to generate, reflect upon, and subsequently
refine, hypotheses about students' reasoning until increasingly stable and viable hypotheses emerged. The depth and interaction of the analyses, combined with the ample data generated over the course of each teaching experiment, allowed for the statement of thematic findings which were consistently supported by the data and grounded in the students' reasoning.

4.5 – Issues of Validity and Ethics

4.5.1 Issues of Validity

To ensure the credibility and significance of my results, issues of validity were a guiding element of the design, execution, and analysis of this research project. The development of the sequence of activities employed in the first teaching experiment drew upon my analysis of data collected during the pilot study (Swinyard & Lockwood, 2007). Subsequently, the development of the sequence of activities employed in the second teaching experiment drew upon the task analysis I performed following the first teaching experiment. The first of these analyses was conducted in collaboration with another knowledgeable researcher who served as a witness to both the pilot study and the first teaching experiment. A second colleague served as a witness and co-developer for the second teaching experiment. Both colleagues assisted me by videotaping the instructional sessions described in Appendices B and C, as well as by participating in the ongoing, post, and retrospective analyses described in Section 4.4. Collectively, these two colleagues provided me an opportunity to validate research findings, as Steffe & Thompson
(2000) suggest. Further, retrospectively analyzing the data corpus and articulating proposed thematic findings to colleagues were two ways in which I was able to either corroborate or disconfirm initial conjectures I made about student reasoning. The disconfirmation of initial conjectures motivated the refinement of thematic findings to fit more viably with the raw data collected during the study. This approach was consistent with validation methods described by Strauss and Corbin (1998).

4.5.2 Issues Related to Ethics

Studies involving human subjects by nature require attention to ethical issues. While there were no real risks associated with participating in this study, it was possible that some students could have experienced feelings of uneasiness due to the following reasons. First, students were asked to engage with a concept that is both conceptually rich and complicated. As such, potential existed for students to become frustrated and anxious while working on tasks during the instructional sequence. It is worth noting however that the focus of this research was on student reasoning as opposed to evaluating student understanding and/or achievement. It was made clear to the participants that their reasoning was of utmost importance. Repeatedly making the focus of my research transparent to the participants appeared to alleviate some of the anxiety they might have felt had they thought my primary goal was to evaluate their understanding. Second, the potential existed for participants to feel self-conscious about being videotaped and about having their
written work analyzed. To help alleviate these concerns, participants were given pseudonyms which have been, and will continue to be, used during any discussion of student work with outsiders, as well as during any presentation or publication of results. Further, only those students who consented to having their written work and video images of their work used were eligible for selection for the teaching experiment phase of the study. This restriction did not ultimately have any bearing on participant selection for the teaching experiment phase of the study, as all twelve survey participants consented to having their written work and video images of their work used.

It is also worth noting that participants in the teaching experiment had the opportunity to benefit in at least two significant ways. First, in the course of engaging with the instructional tasks in the teaching experiment, each of the four participants had the opportunity to think deeply about, and develop understanding of, a complicated mathematical idea. Second, these four participants were financially compensated for their time, each receiving an honorarium of fifteen dollars per session.
Chapter 5 - The First Teaching Experiment

In the pages that follow, I detail the results of the first teaching experiment, highlighting three products of the experiment which emerged in tandem: the phases of instruction, the evolution of the students' characterization of limit, and eight emergent themes which characterize student reasoning and point to subsequent pedagogical implications. This chapter consists of two main parts. In Part 1, I provide an overview of the instructional sequence, painting in broad strokes the unfolding of instruction across the first teaching experiment and highlighting instructional goals and tasks. In Part 2, as I describe in greater detail the evolution of the students' characterization of limit, I discuss eight themes which emerged from my analysis of the data.33

5.1 – Part 1: Overview of Instructional Sequence

5.1.0 Introduction

The first teaching experiment consisted of ten paired sessions34 and a final individualized session with each student. The experiment unfolded in four distinct instructional phases. These four phases collectively suggest the following emergent instructional trajectory:

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33 In Part 2, I do occasionally include the description of particular tasks that marked significant moments in the teaching experiment. I do not, however, provide a complete listing of the instructional tasks employed in the second teaching experiment within this fifth chapter. The reader can find a complete description of the sequence of instructional tasks, as well as the rationale for those tasks, in Appendix B.

34 Each session was separated by a span of approximately seven days with two notable exceptions. Twelve days separated both Sessions 5 and 6 and Sessions 9 and 10.
Phase 1: Assessment and Attempts to Motivate Necessity (Sessions 1-3)

Phase 2: Initial Attempts to Define Limit at a Point via Graphical Conversations (Sessions 4-6)

Phase 3: Characterizing Limit at Infinity (Sessions 7-8)

Phase 4: Revisiting Limit at a Point (Sessions 9-10)

The transitions between phases were marked by instructional “sign posts” – shifts in instruction that were based on pedagogical decisions to alter the instructional trajectory in response to the students’ activity and reasoning. In Part 1 of this chapter, I will delineate the four phases of the first teaching experiment, painting in broad strokes the unfolding of instruction across the ten paired sessions and highlighting instructional goals and tasks.

5.1.1 Phase 1: Assessment and Attempts to Motivate Necessity (Sessions 1-3)

Two central pedagogical goals drove the instructional decisions that defined and demarcated the first phase of the teaching experiment. The first goal was to assess Amy and Mike’s informal understandings of limit, and so I engaged them in tasks designed to assess and leverage their informal notions of limit. Assessment tasks included having Amy and Mike determine limits using the usual algebraic techniques and discussing the extent to which different representations of functions (algebraic, tabular, and graphical) provide conclusive evidence regarding the existence of a limit.
The second central pedagogical goal of this phase was to help motivate within Amy and Mike a sense for the necessity of a rigorous justification process for limit. Prior to the second phase, their experience with limits had been primarily focused on finding limit candidates by using algebraic techniques to employ direct substitution, or by inspecting tables and/or graphs. They had not, however, been asked to justify or validate those limit candidates. Thus, my second pedagogical goal was to generate cognitive conflict within Amy and Mike that might in turn motivate them to begin questioning whether the validation of proposed limits requires a process inherently distinct from that used to find limit candidates. To motivate such necessity, I provided Amy and Mike tasks for which it was possible to propose limit candidates, but not possible to subsequently justify those candidates with certainty.

5.1.2 Phase 2: Initial Attempts to Define Limit at a Point via Graphical Conversations (Sessions 4-6)

My analysis of Amy and Mike’s reasoning over the course of the first three sessions of the experiment prompted me to make a pedagogical shift prior to the fourth session. Throughout the first three sessions, both students were reluctant to engage in graphical conversations about limits. Their reliance on algebraic representations and discomfort with graphical conversations was problematic, because I intended for them to reason about limits in a more formal manner, with
an emphasis on describing the visual aspects of limits as related to the conventional ε-δ illustration seen in Figure 5.1 (Stewart, 2001).

With these thoughts in mind, I made a pedagogical shift at the outset of the fourth session. Tasks and activities during Sessions 4-6 were primarily focused on discussing limits in a graphical setting, in hopes that the absence of analytic expressions might support the enrichment of the visual aspects of Amy and Mike's respective concept-images. Tasks included generating prototypical examples of limit. The prototypical examples Amy and Mike generated subsequently served as sources of motivation as they made initial attempts at precisely characterizing what it means for a function to have a limit. Phase 2 of the teaching experiment, then, constituted a period of iterative refinement for Amy and Mike; as they attempted to characterize limit precisely, the examples and counterexamples of limit that they encountered created cognitive conflict for them, which they sought to relieve by refining their characterization. The central focus of this phase of the teaching experiment was on having Amy and Mike incorporate explicit language in their
characterization of limit as they mulled over and wrestled with the essential characteristics and subtleties associated with the concept.

Amy and Mike’s initial characterizations of limit were from an *x-first perspective*, a perspective coherent with the act of finding limits. By *x-first perspective*, I mean that Amy and Mike’s characterizations focused first on inputs (x-values) progressively closer to \( a \), and only then on corresponding outputs (y-values) approaching \( L \). However, an *x-first* perspective does not align with the act of justifying limits, as the formal definition of limit places emphasis first on choosing an interval symmetric about a value along the y-axis. Thus, one of my central instructional goals during the second phase of the teaching experiment was to elicit a shift in Amy and Mike’s reasoning towards a *y-first perspective*.

5.1.3 Phase 3: Characterizing Limit at Infinity (Sessions 7-8)

Amy and Mike’s reasoning over the course of Sessions 4-6 prompted me to initiate a second pedagogical shift prior to the seventh session. By the end of the sixth session, Amy and Mike’s efforts to reinvent the definition of *limit at a point* was stalled by their struggles to explicitly characterize *infinite closeness* and their disinclination to assume a *y-first* perspective. In response, my instructional agenda shifted to focus their attention to defining *limit at infinity*, anticipating that their efforts to characterize and formalize *limit at infinity* might provide necessary support for defining *limit at a point*, as the two definitions are structurally similar. Further, because the definition of the former only requires characterizing *infinite*
closeness on one axis, it appeared to be a context more conducive to eliciting a precise description of infinite closeness and initiating a shift to a y-first perspective. For these reasons, characterizing limit at infinity was the central focus of Phase 3. Instructionally, this phase was similar to Phase 2; Amy and Mike first generated prototypical examples and counterexamples of limit at infinity, and then they used these as sources of motivation to iteratively refine their characterization of limit at infinity.

During Phase 3, a serendipitous observation by Amy led the pair to attempt to refine their definition so as to appropriately eliminate all y-values other than \( L \) from consideration as the limit. This refinement re-oriented Amy and Mike's attention to precisely characterizing infinite closeness. To facilitate their progress, I shifted Amy and Mike's attention to first defining closeness,\(^{35}\) In response they first characterized closeness to \( L \), and soon thereafter employed their conception of closeness to \( L \) to operationalize infinite closeness to \( L \). These efforts induced a noticeable shift among Amy and Mike to a y-first perspective and culminated in a joint characterization of limit at infinity that is conceptually synonymous to the conventional definition.

\(^{35}\) Infinite closeness or infinite proximity was the notion Amy and Mike tried to first explicitly characterize during Phase 2. However, in reference to this notion, the specific phrase they attempted to articulate was infinitely close, not infinite closeness. In an effort to remain consistent with the language they had previously used, I asked them during Phase 3 to define close as opposed to closeness. However, for readability, I substitute infinite closeness and closeness for infinitely close and close, respectively, throughout both this chapter and Chapter 6.
5.1.4 Phase 4: Revisiting Limit at a Point (Sessions 9-10)

The fourth and final phase of the teaching experiment marked a transition for Amy and Mike back to refining their definition of \textit{limit at a point}. Prior to Phase 4, their definition was neither precise nor mathematically valid. However, the work Amy and Mike had done during Phase 3 had seemingly provided necessary support for them to successfully refine their definition of \textit{limit at a point}. At the outset of Phase 4, I noted Amy and Mike's success in defining \textit{limit at infinity}, drawing particular attention to the increased precision each of their refinements had contributed to their definition. Having presented Amy and Mike with their evolving articulations of \textit{limit at infinity}, I next directed their attention to their most recent definition of \textit{limit at a point}, constructed during Session 6. Noting the difference in specificity in their definitions motivated them to make further revisions to their definition of \textit{limit at a point}. The fourth phase unfolded with Amy and Mike recalling their prototypical examples of \textit{limit at a point} and subsequently using them, as well as their definition of \textit{limit at infinity}, to ultimately characterize \textit{limit at a point} in a manner synonymous to that of the conventional $\varepsilon$-$\delta$ definition. The teaching experiment concluded with Amy and Mike interpreting the conventional $\varepsilon$-$\delta$ definition of limit in light of their own reinvention and verbalizing their opinions regarding the mathematical role of the definition of limit.
5.2 – Part 2: Emergent Themes

5.2.0 Introduction

The overview I provide in Part 1 of this chapter describes in broad strokes the evolution of Amy and Mike's definition of limit at a point. The purpose of the second part of this chapter is to describe in greater detail their reinvention of this complex definition. Figure 5.2 displays the key formulations developed by Amy and Mike in the reinvention process. The session number during which each definition was developed is shown in parentheses next to each definition so as to provide a chronological sense of progression.

<table>
<thead>
<tr>
<th>Amy and Mike's Evolving Definition of Limit</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Definition #1:</strong> ( f \text{ has a limit } L \text{ at } x=a \text{ provided as } x\text{-values get closer to } a, \text{ y-values get closer to } L. ) <em>(Session 4)</em></td>
</tr>
<tr>
<td><strong>Definition #2:</strong> If you could zoom forever and always get closer to ( a ) and ( L ), then you have a limit. <em>(End of Session 4)</em></td>
</tr>
<tr>
<td><strong>Definition #3:</strong> A function has a limit ( L ) at ( a ) when zooming in FOREVER both horizontally and vertically yields no gaps that have length ( &gt;0 ) AND that it looks like it approaches a finite number ( L ). <em>(Session 5)</em></td>
</tr>
<tr>
<td><strong>Definition #4:</strong> The limit ( L ) of a function at ( x=a ) exists if every time we look at the function more closely as we get infinitely close to ( x=a ), it bears out the same pattern of behavior, i.e., looks to be approaching some ( y ) value ( L ) w/no vertical gaps in the graph. <em>(Session 6)</em></td>
</tr>
<tr>
<td><strong>Definition #5:</strong> As ( x ) gets arbitrarily close to ( a ), (</td>
</tr>
<tr>
<td><strong>Definition #6:</strong> As ( x ) gets arbitrarily close to ( a ), (</td>
</tr>
<tr>
<td><strong>Definition #7:</strong> For any arbitrarily small ( \lambda ) we can find a value of ( x ) arbitrarily close to ( a ) such that (</td>
</tr>
<tr>
<td><strong>Definition #8:</strong> For any arbitrarily small ( \lambda ) we can find a value of ( x ) arbitrarily close to ( a ), i.e. (</td>
</tr>
</tbody>
</table>
Definition #9:  \[ \lim_{{x \to a}} f(x) = L \] provided that: given any arbitrarily small \( \lambda \), we can find an \((\delta > 0)\) such that \( |L - f(x)| \leq \lambda\) for all \( x \) in that interval except possibly \( x = a \). (Session 9 – Final Definition)

Figure 5.2 - Amy and Mike's Evolving Definition of Limit

The reader will recall that the intent of this dissertation study was to engage students in an instructional sequence with two objectives:

1. To develop insight into students' reasoning in relation to their engagement in instruction designed to support their reinventing the formal definition of limit, and;

2. To inform the design of principled instruction in relation to limit.

The first research objective is focused on student cognition, while the second is focused on instruction. It is worth noting that the nature of the teaching experiment was such that student cognition and pedagogical decisions were intimately intertwined – student reasoning informed decisions I made about instruction, which, in turn, influenced student reasoning. The interconnectedness of student reasoning and instruction is also seen in the themes that emerged during the first teaching experiment. My description of the evolution of Amy and Mike's definition in Part 2 of this chapter is situated among eight themes that emerged during the first teaching experiment. Each of these themes has both cognitive and pedagogical elements – the former characterize Amy and Mike's reasoning about limit in the context of reinvention, while the latter address instructional findings related to student cognition. Figure 5.3 gives a listing of the eight themes, including the sessions in which each theme emerged (or re-emerged).
Emergent Themes

Theme 1: Reliance on Algebraic Representations and Distrust of Graphs (Sessions 3-6)

Theme 2: Predominance of an $x$-First Perspective and the Counterintuitiveness of a $y$-First Perspective (Sessions 3-10)

Theme 3: Potential Infinity as a Hindrance to Characterizing Infinite Closeness (Sessions 3-6)

Theme 4: Limit at Infinity as a Context Conducive for Initiating Necessary Cognitive Shifts (Session 7)

Theme 5: Reinventing Limit at Infinity as Support for Reinventing Limit at a Point (Sessions 9-10)

Theme 6: Reinvention of the Formal Definition of Limit: An Existence Proof (Sessions 4-10)

Theme 7: Reinvention as Support for Coherently Interpreting Conventional Formulations of the Definition (Session 10)

Theme 8: Reinvention as Motivation for Need for Formal Definition (Session 10)

Figure 5.3 – Emergent Themes

Although evidence of particular themes was present throughout the teaching experiment, in general my description of themes in this chapter is provided chronologically. The first three themes describe cognitive difficulties Amy and Mike experienced that hindered their progress. Themes 4 and 5 address a beneficial pedagogical decision that helped to alleviate Amy and Mike’s cognitive difficulties and stimulate progress in their characterization of limit at a point. The final three themes point to insights I gained only after the entire reinvention process concluded. In the pages that follow, I will situate the unfolding of Amy and Mike’s
characterization of limit seen in Figure 5.2 among the eight themes presented in Figure 5.3.\(^{36}\)

5.2.1 Theme 1: Reliance on Algebraic Representations and Distrust of Graphs

The central instructional goal of the teaching experiment was for Amy and Mike to reinvent the formal definition of limit. Engaging Amy and Mike in thought and discussion about limits in a graphical setting seemed necessary, given the visual imagery that such a graphical setting stands to provide as a basis for the formal definition of limit. Indeed, reinventing and/or understanding the formal definition requires one to describe the visual aspects of limits as related to the conventional \(\varepsilon-\delta\) illustration seen in Figure 5.4 (Stewart, 2001).

![Figure 5.4 - \(\varepsilon-\delta\) Illustration of Limit](image)

Having Amy and Mike reason about limits from a graphical perspective was one of my main pedagogical aims during the initial phases of the teaching experiment. These initial sessions, however, shed light on Amy and Mike’s substantial reliance

\(^{36}\) The first of the eight themes addresses perspectives Amy and Mike held which precluded them initially from engaging in conversations designed to elicit a precise characterization of limit. Thus, the detailing of their evolving characterization does not begin until my discussion of the second theme.
on algebraic representations and general reluctance to engage in graphical conversations\textsuperscript{37}, unless they had been assured that such graphs had originated from an algebraic representation. Evidence of this emergent theme was abundant in the first two phases of the experiment. I provide such evidence in the pages that follow.

\textit{Issues of Certainty and Representational Accuracy}

At the outset of Phase 1, I engaged Amy and Mike in tasks designed to assess the extent to which different representations of functions (algebraic, tabular, and graphical) provide conclusive evidence regarding the existence of a limit. Their responses indicated that they not only preferred working with algebraic representations, but that they also felt that algebraic representations provide them 100\% certainty\textsuperscript{38} about a limit's value.

Craig: Okay, so...what would you want, if you were looking for a limit, and we've got these different representations - algebraic formula, graph, table - which would you prefer to have if you wanted to determine the limit with certainty?

Mike: I'll take a formula. Umm, an equation.

Craig: An equation? Umm, and if you had an equation, what degree of certainty would you say you could determine the limit?

Mike: 100\%.

Craig: How 'bout you?

Amy: 100\%.

Further evidence of Amy's confidence in algebraic representations arose later during the second session, when, in response to a task involving a piecewise

\textsuperscript{37} I use \textit{graphical conversations} here, and throughout this document, to mean conversations focused on limits in the context of graphical representations of functions.

\textsuperscript{38} "100\% certainty" is a phrase both Amy and Mike used throughout the teaching experiment to indicate complete confidence that a particular functional representation could ensure them that a proposed candidate for the limit of a function was indeed the limit.
function, Amy declared that the presence of an algebraic formula allowed her to justify a limit candidate. Amy compared her trust of algebraic representations to tabular representations.

Craig: [H]ow would you justify that that limit is 0?
Amy: Umm, because I have...a definition of the relationship between the input and the output of the function...for all cases. I mean it's like, it’s like having like a...corpus, a text in a foreign language or something. And, or, you have an unfamiliar word in it and you try and you see it used in various contexts and try to extrapolate, uh, what the word might mean versus just having a dictionary in front of you, and, and having, having the word defined for you. I mean, there it is....

Craig: So the formula for, it’s like the dictionary, it gives you the definition.
Amy: Exactly, whereas a table would be like a text.

Amy’s comments above suggest that she viewed tabular representations as being open to interpretation, whereas she believed algebraic representations provide objective certainty in all cases.

In regards to graphical representations, it was evident that Amy felt that graphs originate from one of the two other types of representations (tables or algebraic formulas), and the extent to which she derived certainty from graphs was dependent upon the representation from which the graph originated. When asked if she only had a graph but no corresponding algebraic formula, Amy responded as follows:

Amy: It would depend on whether I had any information to go along with the graph about how the graph was generated. If it was based off of a table of points, then I would say it was garbage just like the table, and if someone were to tell me that it was based off of the equation and show what the equation is doing at all points...or were a completely accurate graph, then that’s just as good as an equation in my book pretty much. But, if we don’t know the level of magnification and we don’t know the degree to which the graph
does in fact represent what the function is doing at all points, then it’s just as useless as a table.

Amy’s comments indicate that she felt strongly that tabular or algebraic representations pre-exist graphical representations. This viewpoint is significant in that it helps explain why Amy was reluctant to engage in graphical conversations without first having access to a particular algebraic representation. Amy’s perspective concerned me because I had anticipated that the success she and Mike would have in reinventing the formal definition of limit would be contingent upon their ability and willingness to think about and explore graphical representations of functions.

In an effort to shift Amy and Mike’s attention away from algebraic representations, I asked them to consider a scenario in which no algebraic representation for a function was provided, but rather a graphical representation existed in and of itself. Amy’s comments below corroborate that she did not separate graphical representations of functions from the algebraic equations that may define them.

Craig: [C]an a graph of a function exist in and of itself, separate from either coming from a table or coming from an equation or formula?
Amy: I mean, how do you know it’s a graph of a function? How do you know it’s a graph at all? It’s just like, you know, like a pretty picture?

Amy’s reliance on algebraic representations and general distrust for graphical representations support findings in the research literature (Knuth, 2000). Following Amy’s comments, I attempted to convince both she and Mike that functions could
exist without an algebraic representation if, for instance, such graphs were formed
by collecting empirical data. It is evident that Amy opposed such a possibility.

Amy: I don’t know Craig. I’m really having a hard time with this kind of
like abstract, umm, immaculately-conceived graph, because
somewhere along the line you have some device that is detecting a
certain given set of values.

The perspective that Amy shared above, and with which Mike concurred during the
third session, proved to be quite paralyzing, as I would later find, in that both
students were extremely hesitant to engage in graphical conversations about limits;
for them, certainty was based on the presence of algebraic formulas. Further, there
was an underlying concern that graphical representations carry with them the
possibility of misrepresenting the function. It is worth noting that the experiment
was designed to elicit responses from Amy and Mike about a concept that is very
visual in nature. Yet for Amy, it seemed imperative that we establish that the
visual elements were actually depicting accurately that which they were designed to
depict. When I drew a function on the board that had a removable discontinuity at
the point (5, 7) (see Figure 5.5) and asked Amy what she thought the limit of that
function was, and how she would justify her response, her concern about a graph
accurately depicting a function became even more evident.

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39 I remind the reader that an assumption underlying the mathematical-conceptual analysis of limit
provided in Chapter 3 is that reinventing a definition of limit capturing the intended meaning of the
conventional $\epsilon-\delta$ definition relies on one’s ability to attend to imagistic features of a function’s
graphical representation.
Craig: What is the limit of that function as $x$ approaches 5? So you have someone who comes and says we're learning limits. I have this graph. What's the limit and why is that limit what you say that it is?

Amy: You know, once again, I'm, I would say that, that a really good guess would be that, that the visual relationship that I perceive to hold around 5 would indicate that the limit would be 7 at 5, but

Craig: Assuming, assuming this function acts in good faith, say?

Amy: ....Umm, well I'm gonna go out on a limb here and like, I'm just gonna, you know, run with this metaphor for a second and say that all functions act in good faith. Its representations...based off of measurements that fail. And if I'm assuming that, that, that measurements and the representations based off of them are accurate, then I can feel really good about, and certain that's the value of the limit.

Craig: Okay, so how 'bout this? I think you said the limit presumably is 7 in this case. What would have to be true for that, to you, for that limit to be 7? What would have to be true?

Amy: Umm, that that graph is, umm, a perfect representation of the function at all points.

Hence, for Amy it seemed that the certitude with which she argued for a proposed limit from a graphical perspective relied on that graph being accepted as a perfect representation of the function. This perspective suggests that Amy viewed graphs

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40 In this chapter, Figures 5.5, 5.6, 5.7, 5.9, 5.10, and 5.17 are exact recreations of the graphs that were constructed during the actual teaching experiment. These recreations are provided solely for the purpose of improving readability, as the original images of these graphs were not adequately captured by video. All other graphs presented in this chapter are video images of the originally constructed graphs.
not as \textit{functions}, but rather as \textit{depictions of functions} that may or may not be accurate.

Amy: [W]e’re making a whole lot of assumptions about the graph, i.e., one of them being that like the fact that it looks like it goes to 7, or that, that we can perceive a relationship that’s consistent, umm, is not just an artifact of our big old clumsy human eyeballs, you know?

By the end of Phase 1, it appeared evident that it was going to be difficult to engage Amy in graphical conversations about limits unless she first accepted graphs as accurate representations of functions.

Mike initially appeared to be slightly less wary of graphical representations. In response to tasks posed during the first phase of the experiment, Mike was quick to graph the function of interest, either on his calculator, or on the board if the function were one he was familiar with. Despite his comfort with using graphs, though, Mike demonstrated more certainty in his response to limit tasks when he also had access to a corresponding algebraic representation. This suggests that certainty for Mike, too, was contingent upon access to a function’s algebraic formula.

Craig: What did the graph of the function offer you...?

Mike: Hmmm, I'd say nothing...I guess you could confirm your answer with the graph by actually plotting the points, but in the end I'd trust my algebra....

In general, Mike's response to limit tasks indicated that although he found the graph useful for providing visual confirmation of his conjectures regarding limits, he ultimately relied on access to the algebraic representation for certainty. Towards the end of the third session, I asked Amy and Mike to determine the limit of a
particular function for which I presented the algebraic formula. In response to this task, Mike immediately graphed the function on a calculator and proposed a value for the limit based on a graphical inspection. It is worth noting, however, that Mike made it evident that even though he had increased confidence in his graphical inspection because the graph originated from an algebraic formula, he would not prove that the limit of a function is $L$ using the graph.

Craig: So, if you were working with someone and you say, actually I've graphed this thing and it looks like it's heading towards a height of

Mike: Well, 1, and that wouldn't be my way to prove it to them.

Craig: Okay, how would you prove it to them?

Mike: I would prove it to them by solving this limit algebraically, but I don't know how right now.

Mike's response suggests that he believed that algebraic techniques are the only way one could justify a limit with certainty, and that the ability to justify a limit algebraically rests on the amount of knowledge one has of such techniques. The words “right now” in Mike's response suggest that proving the limit algebraically is merely a matter of knowing all of the appropriate algebraic techniques, which presumably comes with time as one becomes more mathematically experienced.

By the end of Phase 1, Amy and Mike had explicitly stated that while all three functional representations could provide them with an idea for what the limit might be, only algebraic representations would allow them to determine a limit with complete certainty. Their strategy for responding to limit tasks could be summarized as follows: For one to determine a limit with certainty, he/she must employ algebraic techniques (e.g. rationalizing the numerator, factoring, applying
L'Hospital's Rule) to the algebraic representation so as to eventually use direct substitution. Further, it was evident that they believed that a function could not exist without a defining algebraic representation, and that direct substitution would always be possible with algebraic representations. Thus, Amy's and Mike's progress towards precisely characterizing limit was initially hindered by their strong reliance on algebraic representations of functions. Their reliance on algebraic representations is not entirely surprising, given that much of their experience with limits in introductory Calculus had focused on learning algebraic techniques that allow one to find limits via direct substitution.

**Persistent Reliance on Algebraic Representations and Distrust of Graphs**

As the second phase of the experiment began, I pointed out to Amy and Mike that we had spent ample time during the first three sessions discussing the different representations of a function. I next told them that I wanted to shift the focus of our discussions to talking about limits as related to graphical representations of functions. Out of concern that they would be hesitant to engage in such discussions, I told them that they could assume that the graphs we were going to discuss originated from some unknown formula. Amy, in particular, appeared relieved by this concession. However, despite efforts to shift Amy and Mike's attention to graphical representations during Phase 2, it was evident that Amy still relied on, and showed a preference for, algebraic representations of the function. The following excerpt provides evidence of Amy's persisting belief that determining a
function’s limit requires consultation of the function’s analytic expression. When asked what her advice would be for a student seeking a function’s limit, Amy responded as follows:

Amy: I would tell them to go back to the function, because the function itself is the definition of what is happening and, and if...we’re interested in defining what is happening everywhere except this point, then, umm, the function hands you a definition on a silver platter.

Her reference to “the function” in the preceding excerpt is a reference to the algebraic representation of the function. As the conversation continued, I later pressed Amy on whether a student could determine a function’s limit from just a graphical representation. In response, Amy clearly articulated her distrust of graphs.

Craig: And, then the student says okay, so how do I get a sense for that pattern of behavior? So you’re saying I look at the pattern of behavior of everything else, and then from that I’m able to say, so therefore the limit is...2, let’s say.

Amy: I would say umm, you know, Dear student, umm, I highly recommend not actually trying to, to umm, make predictions about functions based off of graphical representations. Sincerely, Amy. P.S. Good luck. (laughs)

As the experiment progressed, it became increasingly apparent that any conclusions or claims Amy made about limits from a graphical perspective were subject to questioning on the basis that graphical representations fail to provide the same objective certainty provided by algebraic representations. The following excerpt illustrates Amy’s concern.

Craig: [W]hat would we need to do for a graph to be able to...help tell us what the limit is?
Amy: Well, I feel like we would have to agree...that the graph is a sort of like, like a theoretical perfect thing that actually...is not a material manifestation of something but actually is sort of like a perfect visual representation of...our abstraction...I'm having a hard time reconciling the idea of like a graph being completely accurate....

Indeed, throughout the second phase of the experiment, Amy exhibited ongoing reliance on algebraic representations and reluctance to graphical representations. The following excerpt illustrates that even as late as the sixth session, Amy still possessed an inherent distrust of graphical representations of functions, despite her increased willingness to discuss limits within such a context.

Amy: [I]t is always true that you don’t know conclusively based off of a graphical representation of something how accurate it is. And whether if you just zoomed in like, you know, a zillion more times,...you would discover something new about it that you couldn’t see just by looking at it from the wrong perspective, or from a different perspective....[T]he only way you could conclusively know is if you had an algebraic representation of the, of the little machine that generated that graph....[I]t seems like we keep dancing around some kind of concept that we have to talk about in a series of analogies or hypothetical situations, you know? Like if we had a graph that we knew was a perfect representation, you know? This like...hypothetical, like, graph....Because you can’t trust a graph, really.

While Mike was less demonstrative than Amy, he did continue to share her reluctance to trust graphs, noting that the best a graph could do is provide the student with an “educated guess.” To help alleviate their distrust of graphical representations, I told them towards the end of Phase 2 that the mathematical community unilaterally accepts that graphs are intended to depict functional behavior in good faith and that they could assume, for the sake of our
conversations, that all graphs we considered in our discussions were accurate
depictions of functions. This seemed to quell their concerns.

Summary

In summary, during the first two phases of the experiment, Amy and Mike both
displayed a substantial reliance on algebraic representations for justifying limit
candidates, noting that the only way to justify, or prove, the existence of a limit is
via direct substitution. They repeatedly expressed the opinion that all functions
originate from algebraic representations, and furthermore, that direct substitution is
always possible through algebraic manipulation. Amy, in particular, was reluctant
to engage in graphical conversations, as she felt there was no way of knowing
whether graphs accurately depict the algebraic representations from which they
originate. Evidence of this theme was abundant in the first two phases of the
experiment. It is likely that Amy and Mike’s reliance on algebraic representations
stems from their experience in Calculus 1, when a great deal of their initial
experience with limits focused on finding limits via the application of algebraic
techniques. Further, students’ experience with algebraic equations precedes
graphical explorations in the mathematics curriculum. As such, it is not surprising
that students might believe that graphical representations have algebraic origins. As
the experiment proceeded, Amy and Mike’s outward reliance on algebraic
representations and distrust of graphical representations waned. Their willingness
to engage in graphical conversations appears to have resulted from my concession
at the end of Phase 2 that if it helped them to do so, they could imagine that the graphs we were discussing had algebraic origins and accurately depicted those algebraic origins.

5.2.2 Theme 2: Predominance of an x-First Perspective and Counterintuitiveness of a y-First Perspective

In my review of the literature in Chapter 2, I spoke in detail about the cognitive distinction between finding limit candidates and validating limit candidates. In calculus courses, students are taught a variety of strategies for finding candidates for limits – direct substitution, algebraic manipulation, and tabular and graphical inspection. However, none of these satisfy the formal definition’s requirement of validation. Indeed, validating a limit candidate requires one to establish that the proposed candidate, \( L \), satisfies a universally quantified implication. Cottrill et al. (1996) provide evidence that when students find limit candidates, they use what I call an x-first perspective. By x-first, I mean that students focus their attention first on inputs (x-values) progressively closer to \( a \), and only then on the numeric y-value being approached by corresponding outputs (y-values). The selection of a candidate, then, is conventionally made on the basis of what numeric value the y-values are getting close to as x-values get closer to \( a \). It is worth noting, then, that the validation of a limit requires that one begin with a given candidate. The key to validating a candidate, however, entails reversing one’s thinking (i.e., using a y-first perspective). Instead of going from x-values to y-values, a student must first
consider what is taking place along the y-axis. "In order to understand the
definition of a limit, a student must coordinate an entire interval of output values,
imagine reversing the function process and determine the corresponding region of
input values" (Carlson, Oehrtman, & Thompson, 2007, p.160). Thus, the process of
validating a candidate requires a student to recognize that his/her customary ritual
of first considering input values is no longer appropriate. Instead the student must
consider first a range of output values around the candidate, project back to the x-
axis, and subsequently determine an interval around the limit value that will
produce outputs within the pre-selected y-interval. Larsen's research (2001)
suggests that the intricacies involved in this y-first process are arguably far more
complex for students than merely formalizing an x-first process, as Cottrill et al.
(1996) conjectured. The very complex nature of the formal definition makes it
highly unlikely that a student with a strong x-first view of functions would be able
to conceive of a new concept in such a y-first way, particularly when the focus
during a first term calculus course is on finding limits, not validating them.

An Initial x-First Characterization

Evidence from the first teaching experiment supports Larsen's (2001)
conjectures regarding the type of thinking students are likely to display in their
initial forays into formal limit reasoning. Indeed, Amy and Mike showed a strong
preference for reasoning from an x-first perspective, in a manner conducive to
finding limits. Mike and Amy's preference for an x-first perspective was evident in
their attempts to justify their responses to limit tasks I engaged them in towards the end of the third session. I drew a function with a removable discontinuity at the point (5, 7) (see Figure 5.6) and asked Mike and Amy how they might justify that the limit of the function was 7.

![Removable Discontinuity Graph](image)

**Figure 5.6 – Removable Discontinuity Graph**

For the first time in the teaching experiment, the roots of a justification process appeared in Amy's verbal reasoning.

Amy: I would be like, pick a point, any point. And...I'll show you that, that for any x-value you can give me, I'll give you a y-value that, that as your x-values get closer to 5, my y-values get closer to 7.

It is worth noting that the exhaustive process that Amy described here was focused on the x-axis first, in a manner consistent with how Larsen (ibid) describes students' informal understandings of limit. Mike displayed similar reasoning. His remarks towards the end of the third session indicated that, if pressed, he would justify the existence of a limit graphically by first considering x-values.

Craig: Okay so, how would you, your life depends on it and you have to convince someone...that the limit is 2. Tell me about the process that you would go through in that case, Mike, to convince them that the limit is 2.

Mike: Okay, well I would do as Amy did earlier and tell them give me any x-value close to 0, as close as you can get to 0. I will plug it in and I will give you a y-value that's just about 2.
Evidence of Amy and Mike's preference for an x-first perspective was abundant during Phase 2 of the experiment, as they made initial attempts at precisely characterizing limit at a point. As they discussed the prototypical graphical examples of limit they had generated at the outset of the fourth session, it was evident that they were continuing to think about limits in the same informal manner described by Larsen (ibid), with their focus residing first on the x-axis.

Craig: Under what conditions would you say that the graph of a function has a limit of 2 at x=5? What would have to be true about that function? Or what would have to be true about that graph?

Amy: ....[W]hat would have to be true of the graph, like, would be that umm, from both sides, as x-values get closer to 2, y-values get closer to 5. 41

Thus, Amy and Mike's first characterization of limit, constructed during Session 4, could be summarized as follows:

**Definition #1:** $f$ has a limit $L$ at $x=a$ provided as $x$-values get closer to $a$, $y$-values get closer to $L$. (Session 4)

**Zooming as a Metaphor for the Limiting Process**

Efforts to elicit a shift in Amy and Mike's reasoning to a y-first perspective were unsuccessful during Session 4, despite engaging them with a jump discontinuity task designed to illustrate for them the insufficiency of their initial x-first characterization. The jump discontinuity graph I drew for this task is seen in Figure 5.7.

41 Here, Amy inadvertently reversed her x-value and y-value, stating that as x-values get closer to 2, y-values get closer to 5, when, in fact, she means that as the x-values get closer to 5, y-values get closer to 2. Her hand gestures during this comment, as well as her subsequent reasoning, confirm that she merely misspoke.
Although both students had noted prior to this jump discontinuity task that the concept of limit describes local functional behavior, neither of them had yet explicated local functional behavior in any concrete way. As the conversation continued, however, the jump discontinuity graph in Figure 5.7 provided fertile ground for articulating the subtleties involved in describing local functional behavior. I pointed out to Amy and Mike that their initial definition of limit would incorrectly conclude that a function with a jump discontinuity like the one in Figure 5.7 has a limit, for as \( x \)-values get closer to \( x=4 \), corresponding \( y \)-values get closer to, say, 8. When asked if such a small jump (7.99 to 8.01) would constitute being close enough to 8 to lead one to conclude that a limit exists, Amy employed, for the first time, a zooming metaphor to describe the nature of the graphical inspection they would undertake to establish the existence of a limit.

Amy: Then I would say let's just zoom in a lot more and all of a sudden [7.99 and 8.01] start to look pretty dang different.

Amy and Mike subsequently discussed how zooming in on a graph, in a manner consistent with how one might zoom in on a graphing calculator, would help one...
determine the existence of a limit. This discussion yielded the following revised definition of limit, constructed at the end of Session 4.

**Definition #2:** If you could zoom forever and always get closer to \( a \) and \( L \), then you have a limit. (*End of Session 4*)

Although Amy and Mike used zooming as a metaphor for inspecting a function's local behavior during the fourth session, they were not specific about what it means to zoom. At the outset of the fifth session, I felt that such explication might lead them to realize that determining the existence of a limit requires one to zoom along the \( y \)-axis, not the \( x \)-axis. Through a sequence of questions, Amy and Mike discussed the effect that zooming along each axis has on a function's graph. In sum, Amy and Mike both consistently expressed the opinion that zooming along the \( y \)-axis pronounces the existence of a vertical jump discontinuity, whereas zooming along the \( x \)-axis does not. Thus, it appeared, by the end of the fifth session, that Amy and Mike were beginning to focus their attention on the \( y \)-axis. However, it was evident at the beginning of the sixth session that despite the time we had spent unpacking the language associated with their zooming metaphor, Mike's description of limits was no more from a \( y \)-first perspective than in previous sessions, as illustrated in the following excerpt.

Craig: *What is it that a limit describes? We've been trying to pin down and decide what a limit describes, but, what does it describe?*

Mike: *...As you approach an \( a \)-value it's the, the \( y \)-value at that point.*

Further, efforts during the sixth session to elicit a shift to a \( y \)-first perspective by having them explore a function with infinite oscillations, \( y = \sin \left( \frac{1}{x} \right) + 5 \), backfired.
As they explored the graph of this function on the calculator, they began by zooming on the x-axis so as to get a better sense of the function's behavior around the origin. Unfortunately, successive zooms on the x-axis resulted in the calculator graphing the function in a manner suggestive of a jump discontinuity. This led them to conclude that they had incorrectly assumed that to locate a jump discontinuity, one must zoom on the y-axis.

Amy: Yeah, I mean if I, if there is a vertical gap, there is a certain point at which zooming in on the x-axis is going to make it show up.
Craig: ....Now if we kept the y-axis the way that it was and just zoomed in on the x-axis would we see that vertical jump, or no?
Amy: Yeah, we'd see it.

Thus, as Phase 2 ended, Amy and Mike's x-first perspective persisted. Even after much discussion and refinement, they continued to define limit in an x-first fashion. The following formulation, constructed during Session 6, illustrates the persistence of their x-first perspective:

**Definition #4:** The limit $L$ of a function at $x=a$ exists if every time we look at the function more closely as we get infinitely close to $x=a$, it bears out the same pattern of behavior, i.e., looks to be approaching some $y$-value $L$ w/no vertical gaps in the graph.

*Persistence of an x-First Perspective – Phases 3 and 4*

Amy and Mike's continued use of an x-first perspective over the course of Sessions 4-6 prompted me to initiate a second pedagogical shift prior to the seventh session. By the end of the sixth session, Amy and Mike's efforts to reinvent the definition of limit at a point was stalled partially by their disinclination to assume a

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42 The reader will note that this discussion skips from Definition #2 to Definition #4. The evolution from Definition #2 to Definition #3 relates more closely to Theme #3, and thus, is described there.
y-first perspective. In response, my instructional agenda shifted to focus their attention to defining \textit{limit at infinity}, a context in which one need only consider infinite closeness on the y-axis. Despite the transition to a less cognitively-complex domain during Phase 3, Amy and Mike’s definition of \textit{limit at infinity} during the seventh session initially continued to employ an x-first perspective:

\textbf{Limit at Infinity:} For \( \lim_{x \to \infty} f(x) = L \), there must be some interval \((a, \infty)\) on which \( f \) is continuous and the maximum distances between y-values and \( L \) show a pattern of decreasing as \( x \) increases.\(^{43}\)

A serendipitous observation by Amy towards the end of the seventh session led to an eventual shift to a y-first perspective. The details of Amy’s observation and their subsequent shift in perspective will be detailed later in this chapter. However, for now it is worth noting that although Amy and Mike did appear aware of the usefulness of their refined definition of \textit{limit at infinity}, they nevertheless appeared initially unaware of its y-first perspective. As they returned their focus to \textit{limit at a point} during Phase 4, and began discussing how best to characterize infinite closeness on both the \( x \) and \( y \)-axes, their articulations, at least temporarily, regressed to an x-first perspective. For example, Amy’s initial characterization during the ninth session was stated in an x-first manner.

Amy: \textit{...as you take} \( x \)-values wherein umm, the absolute value of the distance between \( x \) and \( a \), umm, gets arbitrarily small, \( y \) gets arbitrarily close to \( L \).

\(^{43}\) To avoid confusion, Amy and Mike’s characterizations of \textit{limit at infinity} are not numbered, as I have only numbered their evolving definitions of \textit{limit at a point}.

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Shortly thereafter, following a discussion of how best to notate the idea that the
distance between the function values and $L$ get arbitrarily small in the interval
around $x=a$, Amy suggested the following definition of limit.

Amy: Why don't we just say that as you, as $x$ gets arbitrarily close to $a$, the
difference between $f(x)$ and $L$ gets arbitrarily small?

As she said this, she pointed to the absolute value statement, $|L-f(x)|$, written on the
board. Mike followed by writing the following definition, to which I will refer as
Amy and Mike's fifth definition of limit at a point, on the board.

**Definition #5:** As $x$ gets arbitrarily close to $a$, $|L-f(x)|$ gets arbitrarily small.

The reader should note that while this revised definition of limit at a point did
contain additional absolute value notation, it nevertheless had no more of a $y$-first
perspective than Amy and Mike's fourth definition of limit at a point.

Amy and Mike made significant progress over the course of the ninth session,
refining their definition of limit at a point so that it resembled something very
much like the conventional $\varepsilon$-$\delta$ definition.

**Definition #9:**  \[
\lim_{x \to a} f(x) = L \text{ provided that: given any arbitrarily small } \# \lambda, \\
\text{we can find an } (a\pm\theta) \text{ such that } |L-f(x)| \leq \lambda \text{ for all } x \text{ in that} \\
\text{interval except possibly } x=a.
\]
Despite their progress, however, it was evident that reasoning from a \( y \)-first perspective continued to cause them discomfort. As they were finalizing their characterization of \textit{limit at a point}, they discussed the order in which \( \lambda \) and \( \theta \) should be presented in their reinvented definition.

Mike: Correct me if I'm wrong, but didn't our \( \lambda \) give us our \( \theta \)?
Amy: [F]rankly I find it kind of counterintuitive to come at it from the \( \lambda \) perspective because we don't usually think about functions...as being defined in terms of an \( x \) coming from a certain \( y \). We usually think about it in the other direction, where we plug in a given \( x \)..., where \( x \) is the given and \( y \) is the dependent variable, you know?

Amy's comments here suggest that students' prior mathematical experiences do not lend themselves to reasoning about functions from a \( y \)-first perspective. Amy's reasoning during the final individual interview corroborates this finding. I presented Amy with an altered version of their final definition, with the quantifiers reversed as follows:

\[
\lim_{x \to a} f(x) = L \quad \text{provided that: } \exists \theta > 0, \forall \lambda > 0, \exists \delta > 0, \forall x, |x-a| < \delta \implies |f(x)-L| < \lambda
\]

Our ensuing conversation illustrates the extent to which Amy's previous mathematical experiences affected her reasoning about the formal definition.

Craig: Now looking at that definition, that second statement there,...how would you describe it as being different in terms of how it's written out?
Amy: I would say that it was...predicating the definition on \( x \) rather than on \( y \). Which is exactly what I would think you were supposed to do. But turns out, it doesn't work.
Craig: What do you mean by it seems like it should, like why does it seem like it should start with the \( x \)'s first?

\[46\] Mike and Amy spontaneously used \( \lambda \) in place of the conventionally-used \( s \), and \( \theta \) in place of the conventionally-used \( \delta \).
Amy: Because the y is the dependent value.
Craig: And you’re used to dependent variables coming second? Is that what you’re saying?
Amy: Yeah.
Craig: So that second statement there differs from yours in that it’s predicated on x’s.
Amy: Yeah. So I don’t know. So yeah, so I would be inclined to make it one predicated on the x’s because it feels like working backwards to say well like okay, this is what we’ve got for y, so what does that say about what we can have for x?

Amy’s comments underscore just how counterintuitive a y-first perspective can be for students. It is worth noting that Amy was my top student throughout the Calculus sequence. I mention this so that the reader can appreciate the likely pervasiveness of an x-first perspective among Calculus students.

Summary

To illustrate Amy and Mike’s inclination towards an x-first perspective, I conclude this section by again providing a summary of their reinvention of limit at a point, seen in Figure 5.8. The session number during which each definition was developed is shown in parentheses next to each definition so as to provide a chronological sense of progression.

| Definition #1: | f has a limit L at x=a provided as x-values get closer to a, y-values get closer to L. (Session 4) |
| Definition #2: | If you could zoom forever and always get closer to a and L, then you have a limit. (End of Session 4) |
| Definition #3: | A function has a limit L at a when zooming in FOREVER both horizontally and vertically yields no gaps that have length > 0 AND that it looks like it approaches a finite number L. (Session 5) |
| Definition #4: | The limit L of a function at x=a exists if every time we look at the function more closely as we get infinitely close to x=a, it |
bears out the same pattern of behavior, i.e., looks to be approaching some y value $L$ w/no vertical gaps in the graph.

| Definition #5: | As $x$ gets arbitrarily close to $a$, $|L-f(x)|$ gets arbitrarily small. (Session 6) |
| Definition #6: | As $x$ gets arbitrarily close to $a$, $|L-f(x)|$ gets arbitrarily small. For any arbitrarily small $\lambda$ you can find an $x$-value that satisfies $|L-f(x)| \leq \lambda$. (Session 9) |
| Definition #7: | For any arbitrarily small $\lambda$, we can find a value of $x$ arbitrarily close to $a$ such that $|L-f(x)| \leq \lambda$. (Session 9) |
| Definition #8: | For any arbitrarily small $\lambda$, we can find a value of $x$ arbitrarily close to $a$, i.e. $|x-a|<\theta$, such that $|L-f(x)| \leq \lambda$. Note: $\theta$ is an arbitrarily small #. (Session 9) |
| Definition #9: | $\lim_{x \to a} f(x) = L$ provided that: given any arbitrarily small $\lambda$, we can find an $(a\pm\theta)$ such that $|L-f(x)| \leq \lambda$ for all $x$ in that interval except possibly $x=a$. (Session 9 – Final Definition) |

**Figure 5.8 – Amy and Mike’s Evolving Definition of Limit**

It is noteworthy that Amy and Mike’s descriptions of limit mirror the $x$-first perspective characteristic of finding limits, a mathematical activity they had extensive experience with prior to this teaching experiment. The predominance of an $x$-first perspective in Amy’s and Mike’s reasoning suggests that students may benefit from being led to recognize that making coherent sense of the formal definition requires a type of thinking that may initially feel counterintuitive, given their prior mathematical experiences. It also appears that *limit at infinity* may be a context conducive to initiating a shift from an $x$-first perspective to a $y$-first perspective. This finding will be elaborated in subsequent sections of this chapter.
5.2.3 Theme 3: Potential Infinity as a Hindrance for Characterizing Infinite Closeness

Beginning in Phase 2, Amy and Mike invested a great deal of energy attempting to characterize what it means for \( x \)-values to get infinitely close to \( a \) and for \( y \)-values to get infinitely close to \( L \) along the \( x \)- and \( y \)-axes, respectively. Later in this chapter I will expound upon these efforts, focusing on how, ultimately, Amy and Mike were able to operationalize infinite closeness. For now, however, I will note that characterizing infinite closeness was a non-trivial task for Amy and Mike, with their attempts seemingly hindered by a notion of potential infinity.

Tirosh (1991) describes potential infinity and actual infinity, in relation to the history of mathematical development, as follows - "[T]he two competing ideas of infinity were potential infinity in which a mathematical process can be carried out for as long as required to approach a desired objective, and actual infinity in which one contemplates the totality of infinity, through, for example, conceiving the totality of all natural numbers at one time" (p.200). Williams (2001) reports that students' unwillingness to adopt a more formal model of limit appears to be related to the struggle students experience with the notion of actual infinity. Williams notes that students' understanding of limit is contingent upon their ability to make the jump from finite to infinite - "Actual infinity may thus be the most important cognitive obstacle to learning the formal definition" (p.364). Evidence from the first teaching experiment suggests that while students' ability to reason from an actual infinity perspective may help reduce the cognitive conflict that arises from
trying to imagine the incremental completion of an infinite process, there is an additional cognitive shift that students must make if they are to reinvent a formal definition of limit which captures the intended meaning of the conventional $\varepsilon$-$\delta$ definition. I will discuss the details of this additional cognitive shift later in this chapter. In this section, I provide evidence that reasoning from a potential infinity perspective hindered Amy and Mike's progress towards reinventing a formal definition of limit. I also note that while Amy and Mike grew to recognize the deficiencies of a potential infinity perspective, they appeared unable to establish a suitably alternative perspective.

*The Impossibility of Making an Exhaustive Case*

Evidence throughout the first and second phase of the teaching experiment suggests that Amy and Mike, indeed, reasoned from a potential infinity perspective. In particular, Amy expressed a potential infinity viewpoint of limit that led her to believe that one could never establish the existence of a limit with certainty. This viewpoint emerged as early as the third session, when I asked Amy to justify her claim that the limit of a function with a removable discontinuity at $(5, 7)$ was 7.

Craig: [H]ow would you go about convincing [someone] that that limit is 7?...
Amy: I don't think you can make an exhaustive case. I think you can just tire out your opponent with examples of, of cases in which you're correct.

As discussion of the removable discontinuity graph continued, Amy elaborated her position.
Amy: I can’t make a unilateral, exhaustive case, but I can exhaust you with my attempts at making this case. With, with instantiations... of my, umm, argument being true but... at the end of the day, you know, we don’t actually have any kind of principle on which to base that.... I feel like, you know, when you’re talking about... this like it was a trial and the burden of proof and I keep like, thinking of an analogy where this is like a murder trial or something.... I can give you like, you know, if, if he’s trying to prove that my client murdered someone at 3:04 p.m., and I’m going to show you how he was not murdering someone at 3:01 p.m. and 3:02 and 3:03 (laughing) and I can give you all of these things that he was doing instead of murdering people but I can’t actually show you what he was doing at 3:04 when the murder took place, you know?...[T]hat’s what really counts at the end of the day if we’re going to make an... assertion about the factuality.

Elements of Amy’s comments suggest that justification of a limit’s existence, for Amy, relies on the acceptance of one’s argument via a finite amount of evidence. The reader will note, however, that the limiting process is infinite in nature – that is, for a limit to exist, one must imagine that for every one of an infinite number of specified error tolerances about a proposed limit $L$, there exists a corresponding interval containing $a$ for which all $x$-values within that interval (except possibly $x=a$) have corresponding function values within the specified error tolerance of $L$.

Comments Amy made towards the end of Session 3 make it evident that Amy’s perception of how one would validate the candidacy of a limit was ultimately finite, not infinite.

Amy: ‘Cause I’m like talking about trying to like win an argument with another person. Eventually a reasonable human being after, after being thrown a gazillion examples of something being true might throw up their hands and be like, alright, based off of sheer statistical probability, I’m gonna give you this one. You know?
Zooming as a Tool for Carrying Out an Infinite Process

The reader will recall that Amy and Mike initially characterized limit in the following manner:

**Definition #1:** \( f \) has a limit \( L \) at \( x=a \) provided as \( x \)-values get closer to \( a \), \( y \)-values get closer to \( L \). (*Session 4*)

As the fourth session progressed, Amy and Mike began employing a zooming metaphor to describe the act of inspecting the local behavior of the function in greater detail. Their use of a zooming metaphor was in direct response to discussion about a jump discontinuity graph which had raised their awareness of insufficiencies in their first definition. By “zooming in on the graph,” they believed they would eventually be able to detect jump discontinuities that would prohibit the existence of a limit. At one point, Amy drew a zoomed-in version of the original jump discontinuity graph (see Figure 5.9), noting that such a function could either look like a straight, continuous line if one was zoomed-out far enough, or look like a significant vertical jump discontinuity if one was zoomed-in far enough.

![Figure 5.9 - Zoomed-in Jump Discontinuity Graph](image)

While the zooming metaphor appeared fruitful for describing a procedure by which one might establish a limit *not* existing, Amy and Mike were less explicit about
how zooming might allow them to establish when a limit *does* exist. At the end of the fourth session, I asked Amy and Mike what would serve as sufficient conditions for proving graphically that a limit exists. Their response implied that neither one of them could imagine the completion of such a procedure.

Craig: What would have to happen in a hypothetical world - you can do anything you want - what would have to happen for that limit to actually be 8? Under what conditions would that limit be 8?
Mike: I would have to be able to zoom in infinitely. I can't really comprehend what that would be but,
Craig: Oh, like keep this process going forever, or whatever?
Mike: Yeah.
Craig: Like this zooming-in process?
Mike: And the function would have to approach that height, from both sides, a specific one height from both sides, as I did this infinitely. That's the only way I could be sure.

Craig: Okay. So you're saying that if I could do that forever.
Mike: Umm-hmm.
Craig: And, as I do that forever, what's happening?
Mike: The function is approaching a single value from both sides.
Craig: Okay. Anything to add to that Amy?
Amy: I like it. Yeah. Do it forever and then I'll be happy with the graph.

As the fourth session ended, although Amy and Mike agreed that a graph could indicate when a limit *fails* to exist, they felt that it would not be a useful indicator of when a limit *does* exist because the zooming process would never end. This perspective was evident in their refined definition of limit.

**Definition #2:** If you could zoom forever and always get closer to *a* and *L*, then you have a limit. *(End of Session 4)*

The infinite nature of the limiting process would be an ongoing concern for both of them, particularly Amy, as the teaching experiment progressed. For instance, when asked at the end of the fifth session to characterize what it means graphically for a
function to have a limit $L$ at $x=a$, Amy insisted that their characterization include the notion that the limiting process can never be completed. Amy’s concern regarding the infinite nature of the limiting process was evident in their third definition.

**Definition #3:** A function has a limit $L$ at $a$ when zooming in FOREVER both horizontally and vertically yields no gaps that have length $> 0$ AND that it looks like it approaches a finite number $L$. *(Session 5)*

Indeed, Amy’s insistence on capitalizing the word *forever* highlighted her ongoing concern. At the outset of Session 6, Amy and Mike continued to convey a potential infinity perspective. In the excerpt below, “do it” refers to zooming-in on the graph.

Craig: [H]ypothetically, to determine if there is a limit, what would you have to do?
Amy: Do it forever.
Craig: Do it forever, okay. Any comments about that Mike?
Mike: I completely agree.

As the sixth session continued, Amy spoke at length about the fundamental issues that were problematic for her and Mike as they attempted to define limit. Foremost among Amy’s concerns was the seemingly impossible task of describing how one might carry out an infinite process.

Amy: I don’t know, it seems like we keep dancing around some kind of concept that we have to talk about in a series of, of, of analogies or hypothetical situations, you know? Like if we had a graph that we knew was a perfect representation, you know? This like...hypothetical, like, graph. Or the, or the hypothetical situation in which you are doing something forever....I guess like the first thing that leaps to mind for me is that we’re trying to parse out what we mean by, by these impossible processes that we’re describing for getting out what we think might, um, knowing whether we have a limit.
Craig: And you're saying impossible there why?
Amy: Because you can't zoom in forever.

As the conversation continued, Amy, in a manner consistent with the reasoning she communicated during Session 5, once again noted that the best one could do is establish when a limit fails to exist.

Amy: You can't do something an infinite number of times...we keep getting back to...we can't prove it, we can only not be disproven through, um, yeah, we only cannot be disproven....

Craig: And each time that you're zooming, either...that gap shows up or doesn't, is what you're saying. Is that, is that what you mean by we can only not be disproven?

Amy: Yeah, yeah, exactly. That we can, you know, we can have our, our assertion that the limit at x=a is 6. Um, but through the methods that we've been talking about, all we can do is you know just like, you know grind through endless iterations until we get tired of it and like I give up. And you know like all you can do is find, you, you can either find the, the level of examination which disproves your idea but you can't ever get to where you can conclusively prove it through the methods we've been discussing.

It was evident, then, that Amy was cognizant that reasoning from a potential infinity perspective would not ultimately allow them to establish the existence of a limit. However, as of the sixth session, it did not appear as though Amy had determined another way of reasoning that would free her and Mike from reasoning from a potential infinity perspective. The absence of a more productive perspective appeared to keep her from feeling motivated to increase the precision of the language she was using to describe \textit{limit at a point}.

Amy: I just, I have a hard time getting too worked up over the language about what it means to zoom and what we're looking for when we zoom when we have lurking in the back this presupposition that whatever that means to zoom, whatever we are looking for when we
zoom, we have to do, we have to repeat that process an infinite number of times.

In an effort to support Amy and Mike’s attempts to reason coherently about limit, I suggested that it might be easier for them to imagine the limiting process in a finite manner, focusing instead on what the process entails at each step. I also encouraged them to try to describe the procedure they would use to determine whether something is or is not a limit, setting aside the issue of having to potentially carry out this procedure forever. It was evident, however, that Amy did not feel comfortable ignoring the infinite nature of the limiting process.

Amy: I guess I feel like if we’re willing to accept that, um, we can, we can do something forever, we can repeat this process forever, looking for an exception to our hypothesis, and not finding one, we, you know, we’re gonna get pretty cozy with our hypothesis....I don’t know, it just seems like a really big leap to take.

In summary, through the end of the second phase of the experiment, neither Amy nor Mike recognized an alternative to reasoning from a potential infinity perspective. Their inclination to reason from an x-first perspective, and their fundamental concern about the possibility of completing an infinite process, led me to initiate a pedagogical shift at the outset of Session 7.

5.2.4 Theme 4: Limit at Infinity as a Context Conducive for Initiating Necessary Cognitive Shifts

Amy and Mike’s reasoning over the course of Sessions 4-6 prompted me to initiate a second pedagogical shift prior to the seventh session. By the end of the sixth session, Amy and Mike’s efforts to reinvent the definition of limit at a point
was stalled by their struggles to explicitly characterize *infinite closeness* and their disinclination to assume a y-first perspective. I have already described Amy and Mike’s preference for an x-first perspective in this chapter. Before proceeding, I will summarize their struggles to explicitly characterize *infinite closeness*.

*Difficulties Characterizing Infinite Closeness*

When first charged with the task of defining limit, Amy and Mike almost immediately began wrestling with how best to articulate local functional behavior. In response to their first attempt at defining limit ("As you take $x$’s closer to $a$, your $y$’s will get closer to $L"), I noted for them that their definition suggested that a limit could exist in the case of a jump discontinuity, as seen in Figure 5.10.

![Figure 5.10 - Jump Discontinuity Graph](image)

This observation spurred an important conversation, wherein Amy commented, for the first time, on the difference between being *close* and being *infinitely close*.

Craig: I’m a student and I come up and I say, I think the limit is 8 because, based on what you told me, I want to take $x$’s closer and closer to 4. So I did that and I looked at my $y$’s and they seemed like they were getting closer and closer to 8. So that’s why I think the limit is 8.

Amy: But aren’t you being kind of injudicious with your use of the word infinite? Because it’s not getting infinitely close to 8, it’s just getting, you know like, according to this graphical representation,
looking sort of close to 8. But that’s a far cry from infinitely close.

As the teaching experiment continued, Amy and Mike made a concerted effort to operationalize what it means to be infinitely close on both the x- and y-axes. Their efforts, however, were compromised by the potential infinity perspective they maintained, which I elaborated in the previous section of this chapter. Indeed, Amy and Mike felt that characterizing infinite closeness was an impossible task, for they believed that regardless of how close they found, say, x-values to a, there would always be a closer x-value. As Phase 2 progressed, in hopes of providing them cognitive support, I encouraged Amy and Mike to set aside the task of defining infinite closeness by increasing the rigor of their definition of closeness. Unfortunately, they resisted this approach, noting, as Amy did in the previous excerpt, that close is “a far cry from infinitely close.” Hence, at the end of the sixth session, I felt a change in the instructional trajectory was necessary.
a point. First, I believed that defining \textit{limit at infinity} would initiate a shift to a \textit{y}-first perspective, as they would be unlikely to zoom in on the \textit{x}-axis since limits at infinity require one to imagine the \textit{x}-values increasing without bound. Second, I felt that \textit{limit at infinity} would be a less cognitively taxing context in which to characterize \textit{infinite closeness}, because one need only imagine infinite closeness on one of the two axes. Finally, I anticipated that successful reinvention of the definition of \textit{limit at infinity} might provide support for defining \textit{limit at a point}, as the two definitions are similar structurally:

\begin{itemize}
  \item \textbf{Limit at Infinity:} \( \lim_{x \to \infty} f(x) = L \) provided: for every \( \varepsilon > 0 \), there exists an \( a \) such that \( x > a \rightarrow |f(x) - L| < \varepsilon \)
  \item \textbf{Limit at a Point:} \( \lim_{x \to a} f(x) = L \) provided: for every \( \varepsilon > 0 \), there exists a \( \delta > 0 \), such that \( 0 < |x - a| < \delta \rightarrow |f(x) - L| < \varepsilon \)
\end{itemize}

I will elaborate this final point in the next section of this chapter. For these reasons, reinventing the definition of \textit{limit at infinity} was the central focus of Phase 3. Instructionally, this phase was similar to Phase 2; Amy and Mike first generated prototypical examples and counterexamples of \textit{limit at infinity}, and then they used these as sources of motivation to iteratively refine their characterization of \textit{limit at infinity}.

As was the case with \textit{limit at a point}, their efforts to define \textit{limit at infinity} stalled during Session 7 as they used a variety of vague, and mathematically invalid, characterizations to describe the functional behavior seen in the graph in Figure 5.11.
“On the interval \((b, \infty)\) the function needs to approach some finite value \(L\).”
“Some interval \((b, \infty)\) on which as \(x\)-values increase, their corresponding \(y\)-values get closer to \(L\)”
“As \(x\) gets larger, the distance \(|L-y|\) between \(L\) and your corresponding \(y\)-values continues to decrease”

Indeed, Amy and Mike had reached a point where they were going in circles, exchanging phrases like getting closer for approaching and zeroing in. In response to their struggles to capture the notion of infinite closeness, I once again encouraged them to consider what it would mean for the function to be close to a proposed limit \(L\). Since Amy had earlier introduced absolute value notation during the seventh session, I thought defining closeness might lead them to think about progressively restrictive definitions of closeness, and that they might subsequently shrink \(y\)-bands around the limit \(L\) and use absolute value statements to notate those increasingly rigorous definitions. With this in mind, I asked Amy what difference, if any, she saw between the function getting closer to \(L\) and getting close to \(L\).

Craig: How do you see the two of those as being different? The difference between being closer and being close?
Amy: Well, the first one is...a comparative measure of one thing to the thing that came before it, whereas the second thing is completely meaningless to me.
Craig: The close part?
Amy: Yeah, close.
Craig: And why is it meaningless to you?
Amy: Because, uh, because it's completely subjective.
Craig: Okay, so it depends on what someone means by "close"?
Amy: Yeah. It's like, it's not a measurement of anything.

Interestingly, although the key to reinventing the definition of limit at infinity would eventually be increasingly restrictive definitions of closeness, Amy saw the notion of closeness as subjective and meaningless and continued to place much greater value on the notion of closer. This was an important theme in Amy and Mike's reinvention efforts - they felt compelled to focus on infinite closeness and were reluctant to take a step back and first define closeness, yet as the seventh session continued, it became evident that defining closeness was the catalyst for defining infinite closeness.

A Serendipitous Observation – The Importance of Bounding the Function

As the seventh session neared its conclusion, and as Amy and Mike continued in vain to try and pin down infinite closeness, Amy suggested the following necessary condition for limit at infinity: "The maximum distances between y-values and L show a pattern of decreasing as x increases." In response, I encouraged them to consolidate their conditions into one, concise definition. Amy responded as follows:

Amy: Yeah, so...there needs to be some interval from a to oo where the function is continuous and, um, where the maximum distance between the y-values and L show a pattern of decreasing as x increases.

Next, Amy began to write what she had just said aloud, transcribing the following words on the board: "For \( \lim_{x \to \infty} f(x) = L \), some interval \((a, \infty)\) on which \( f \) is

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continuous and the max distances between y-values and some finite number $L$ show a pat-". It is safe to assume that Amy was going to conclude their written definition with "a pattern of decreasing as $x$ increases," given that she had just ended her spoken definition with that phrase. Instead, though, she stopped writing and verbalized a significant observation.

Amy: Is this going to be enough? I mean, because what if, if we leave it at that, then isn’t that true for like, for $L$...like bigger than or equal to 4? You know? Like doesn’t that make it true for like, every single,

Mike: Hmm? Say it again?

Amy: So, okay, we need some interval from $a$ to $\infty$ on which $f$ is continuous and the maximum distances between y-values and some finite number $L$ show a pattern of decreasing as $x$ increases....[W]hat I’m having trouble with is just, is this specific enough to, like, to $L$ being 4? I mean, isn’t this thing that we just said also true for 5 and 6 and 9.2, because -

Craig: Oh, because why?

Amy: Because that’s, because on that interval, $f$ is continuous, and the maximum distances between y-values...and 10 are also decreasing.

This conversation marked a watershed moment in the teaching experiment. Amy recognized that their definition was not specific enough to a particular $y$-value, 4, and that while the distance between the $y$-values of the dampened sinusoidal function and 4 did show a pattern of decreasing, that was also true for the distance between $y$-values and other potential limits greater than 4. Indeed, their definition failed to exclude other possibilities for $L$. Amy’s observation motivated Amy and Mike to refine their definition of limit at infinity in such a way so as to eliminate all other potential limit candidates not equal to 4. In an effort to capitalize on Amy’s observation, I shifted Amy and Mike’s attention back to defining closeness.
Craig: You're saying your definition..., you don't want it to be such that someone could conclude...this limit could be anything other than 4.
Amy: Yeah.
Craig: So, what if we were to, just as a suggestion,...let's just back off a second..., so let's just say we want to maybe not show that 4 is the limit, but at least that this function gets close to 4. Because you guys were saying it's got to get how close to 4 to be the limit?
Amy: Infinite.
Craig: Infinitely close, but let's – infinitely is kind of tough, so let's back off of infinitely for a second. Let's say we just want to be close to 4. If we were able to show that this thing gets within 1 of 4, then that would keep the limit from being, say, 6 or 2.
Amy: Mm-hmm.
Craig: Does that make sense? So let me ask you instead, describe what it would mean for this function, instead of having a limit of 4, let's say, let's describe what it would mean for this function to get within 1 of 4. How would you write that out?...
Amy: Well, I feel like it would be useful to talk about it being, being bounded....[W]hat if we were to say that there is some y-value that this function will never exceed, and there's some y-value that it will never get bigger than and it will never get smaller than, you know?

Amy's suggestion to bound the function was important, as the idea of boundedness is central to the notion of limit at infinity. Encouraged by Amy's suggestion to bound the function, I asked Amy how she would go about describing whether the dampened sinusoidal function was bounded.

Craig: So if we wanted to be, um, within 1 of – we wanted to show that this function is within 1 of 4, what would your bounds be, then, Amy?
Amy: Uh, within 1, well, I guess that then it would be between 3 and 5, or would it be between 3.5 and 4.5? I guess 3 and 5.
Craig: Okay, could you draw those bounds up there?
Amy: Yeah.

Following this discussion, Amy drew a vertical line from the dampened sinusoidal function down to the x-axis, indicating a point past which the function would always be within the y-interval (3, 5). She then, for the first time, drew horizontal
bounds around \( L \) on the \( y \)-axis at \( y = 3 \) and \( y = 5 \) to indicate the interval in which \( f \) would have fall within 1 of \( L \) (see Figure 5.12)\(^{47}\).

![Dampened Sinusoidal Function with Close = 1](image.png)

**Figure 5.12 – Dampened Sinusoidal Function with Close = 1**

As the conversation continued, it was evident that Mike saw great value in the use of horizontal bounds around the limit \( L \). As the excerpt below suggests, Mike recognized that while being within 1 of \( 4 \) was not enough to ensure that the limit of \( f \) as \( x \to \infty \) was 4, it was sufficient for ensuring that the limit not be a value greater than 5 or less than 3.

Craig: [F]rom what point on is it between 3 and 5?
Mike: Maybe, there.
Craig: Okay. Now is that enough for the limit to be 4?
Mike: No.
Craig: Because what could it do?
Mike: Head toward 3 or 5.

Thus, Mike and Amy had now found a way to eliminate every potential limit candidate except those between 3 and 5. It was evident in Mike’s subsequent comments that he realized that being within 1 of \( L \) was a far cry from being infinitely close, which is the precision he desired. His remarks below suggest that

\(^{47}\) The reader will note that the horizontal bounds Amy drew are not such that the function \( f \) is always within 1 unit of \( L = 4 \) beyond the vertical line seen on the graph, which she had drawn previously. Amy’s spoken reasoning was such, however, that it was evident that this inaccuracy was merely an oversight on her part.
he had played out in his head the process of defining closeness along the y-axis with increasing rigor and had recognized that to eliminate all possibilities but y=4, the y-interval would need to basically be a single number.

Craig: Well, the way that we've drawn this here, would you agree that — you said that this function does what? It gets within

Mike: 1.

Craig: of 4. And I asked you is that enough for the limit to be 4? And you said what?

Mike: No.

Craig: Okay, so being within 1 of 4 isn’t enough. How close do you need to be to 4? One’s pretty close.

Mike: No, it’s not. You need to be infinitely close to 4.

Craig: So 1’s not enough?

Mike: No....The interval, like, I mean, the interval needs to be pretty much, like, 4, it needs to be from as close as you can below 4, as close as you can above 4, infinitely, and that will be — and you’ll see that the limit satisfies that.

Craig: Satisfies. What do you mean satisfies that?

Mike: That the limit is then 4 because the function is within those bounds, of infinitely close to 4 and 4, as you go toward ∞.

As Mike described the infinitely small interval around y=4, he used his fingers to denote an increasingly smaller interval on the y-axis. Importantly, Mike’s last comment contained elements of the conventional definition of limit at infinity — that f has a limit L as x→∞ provided f can be made to fall within any positive distance of L as x→∞. The excerpts above suggest that Amy’s introduction of the notion of boundedness resulted in a significant shift in her and Mike’s reasoning — now, instead of utilizing an x-first perspective, as they had in their previous definitions of limit at infinity (and limit at a point), they instead had shifted to a y-first perspective.
Transitioning from Closeness to Infinite Closeness

At this point in the seventh session, I felt the stage was set for Amy and Mike to begin making a transition from defining *closeness* to defining *infinite closeness*. I anticipated that if I were to have them define *closeness* in an increasingly restrictive manner, they would likely recognize how to capture the notion of *infinite closeness* in their definition of *limit at infinity*. To accomplish this, I asked Mike what the bounds around $y=4$ would be for the function to lie within $\frac{1}{2}$ of 4.

Craig: Okay, if I wanted to be within $\frac{1}{2}$ of 4, then what would my bounds be? Can you draw them?
Mike: Yup. So you'd have 4 $\frac{1}{2}$ and 3 $\frac{1}{2}$.

![Figure 5.13 - Dampened Sinusoidal Function with Close = $\frac{1}{2}$](image)

After drawing the new bounds (see Figure 5.13), Mike noted that yet more limit candidates had been eliminated from consideration.

Mike: So now the function – we know the limit isn’t 5 anymore, because it’s bounded by 4 $\frac{1}{2}$ and 3 $\frac{1}{2}$.

Mike also drew a new vertical line (see Figure 5.14) to signify the point on the $x$-axis past which the function would be within $\frac{1}{2}$ of 4.
Admittedly, the vertical line to which Mike was pointing is not at the smallest $x$-value beyond which the function is always within $\frac{1}{2}$ of 4. Nevertheless, it is significant that he recognized a more restrictive definition of *closeness* might subsequently affect the starting point for the interval $(a, \infty)$, on which $f$ falls within the desired bounds of the limit $L$.

Craig: And you drew a new vertical line there. Why did you draw – what’s significant about that new vertical line that you drew? This one right here?
Mike: That’s where the function is now defined within our new bounds.
Craig: For how many values of $x$?
Mike: From there to infinity.

The excerpts above suggest that defining *closeness* in an increasingly restrictive manner supported Mike in developing a keen sense of the limiting process. In fact, his subsequent comments suggest that he recognized that he and Amy could adequately characterize *limit at infinity* by eliminating all limit candidates other than $L$ via a process of choosing progressively tighter bounds around $L$.

Mike: And we can keep doing that.
Craig: What do you mean we can keep doing that?
Mike: We can keep getting closer until we get, I mean, like I said, it has to be infinitely close to 4....
Craig: Okay, so we need to keep doing that. Can you describe what we can keep doing? Articulate that.
Mike: We can keep making our bounds closer and closer to 4, and the function will keep lying within those new bounds that we make.

Amy also expressed awareness that the key to defining limit at infinity was to continue to bound the function $f$ around the limit $L$ with increasingly tighter bounds. In fact, in the excerpt below, it appears as though Amy had considered our prior conversations, wherein close had been defined as being within 1, or $\frac{1}{2}$, of 4, and had projected those ideas to capture any infinitesimal definition of closeness.

Amy: Well, if you would say that you could make the bounds as close to 4 as you want. And you – as long as you take big enough $x$-values, you will find a point after which that function stays within those bounds.

Albeit loosely phrased, Amy’s verbal description of limit at infinity contained all of the fundamental elements of the conventional definition. Recognizing this, I asked Amy to write out a revised definition of limit at infinity on the board. She wrote the following definition:

**Limit at Infinity:** It is possible to make bounds arbitrarily close to 4 and by taking large enough $x$-values we will find an interval $(a, \infty)$ on which $f(x)$ is within those bounds.

Amy’s revised articulation of limit at infinity marked a significant moment during the first teaching experiment. Specifically, it seemed as though an important shift had taken place for Amy – she appeared to no longer be reasoning from a potential infinity perspective. The phrase “arbitrarily close” represented a different way of reasoning about the limiting process than Amy had previously utilized. Instead of trying to imagine the satisfaction of every incrementally smaller criterion for closeness, Amy instead appeared to be imagining the definition holding for any
arbitrary criterion for closeness. Indeed, Amy was allowing a single, albeit arbitrary, criterion for closeness to represent every incrementally smaller criterion for closeness. In so doing, Amy had encapsulated the limiting process. This was a subtle, yet crucially important shift in Amy’s reasoning. Amy’s utilization of the notion of arbitrary closeness to encapsulate the limiting process was what ultimately led her and Mike to a coherent formal definition. This cognitive shift is distinct from that offered by Williams (2001). Indeed, one could reason about the limiting process from an actual infinity perspective, yet not make the cognitive shift that Amy made from infinite closeness to arbitrary closeness. As an example, one’s recognition that a limit at infinity would only exist provided every criterion of closeness around $L$ was satisfied would be indicative of that person’s ability to imagine the totality of infinity, and thus, according to Tirosh (1991), reason from an actual infinity perspective. Yet such a perspective would not necessarily resolve the dilemma of having to address an infinite number of criterions of closeness. I suggest, then, that it is not necessarily one’s ability to reason from an actual infinity perspective that supports him or her in developing coherent understanding of the formal definition of limit, as Williams (ibid) suggests. Rather it is one’s ability to encapsulate the infinite limiting process with the notion of arbitrary closeness that most directly provides leverage for productive progress towards reinventing the formal definition. Evidence in the next subsection of this chapter substantiates this point.
During the eighth session, Amy and Mike incorporated absolute value notation, and in so doing, found a way to quantitatively operationalize infinite closeness. As the following excerpt illustrates, the notion of arbitrary closeness was still at the forefront of Amy's thinking during Session 8.

Craig: Okay, so, how might you incorporate...that absolute value statement or that inequality into your articulation?
Amy: Well, I don’t know. I mean, you’re going to, you could, you could take an arbitrary, you know, I mean, you could get arbitrarily close to 0. You could pick a, a value as close to 0 as you wanted, and you could find an interval, uh, on which that would be true.
Craig: On which what would be true?
Amy: ...That this \( |L-y| \), absolute value of \( L-y \) is less than or equal to umm, a really, an arbitrarily tiny positive number.

Although Amy struggled at first to articulate her ideas, she ultimately suggested that no matter how restrictive their definition of closeness, they would be able to find an interval \( (a, \infty) \) along the x-axis for which the function would be close to the limit \( L \). Amy’s comments are noteworthy. First, Amy appeared to have become aware of a way to transition from describing the function being close to \( L \) to describing the function being infinitely close to \( L \). This suggests that first defining closeness in a precise manner, and quantifying that definition with the use of absolute value statements, subsequently allowed Amy to precisely define infinite closeness, something neither she nor Mike had previously been able to do. Second, her description summarizes the sequenced selection of an arbitrary closeness along the y-axis and a corresponding interval along the x-axis for which the function falls within the pre-selected bounds around \( L \). Amy’s description here was significant, for it was clear that she was now reasoning from a y-first perspective, a sign of
marked progress from earlier sessions. Third, and finally, her use of the word *arbitrarily* in relation to closeness to $L$ suggests that she had discovered what she viewed as a reliable way to encapsulate the infinite process of reflexively choosing tighter and tighter intervals around $L$. This was significant, in that in previous sessions Amy had repeatedly expressed concern over whether or not the limiting process, in its infinite nature, could ever be completed.

*Arbitrarily Close vs. Infinitely Close*

My analysis of Amy and Mike's reasoning during the eighth session revealed that whereas they had previously talked about getting *infinitely close* to either $L$ or $a$ in the context of *limit at a point*, that terminology was noticeably absent from their discussions and definition of *limit at infinity*. Instead, in reference to *limit at infinity*, their discussions of *infinite closeness* to $L$ were cast from the perspective of *arbitrary closeness* to $L$. For the reasons mentioned in the preceding paragraphs, this appeared to be an important cognitive shift. Prior to shifting their attention back to defining *limit at a point*, I was interested as to whether this had been a conscious choice. In reference to *limit at infinity*, I provided Amy and Mike the following prompt.

**Prompt:** Do you see the phrases *arbitrarily close* and *infinitely close* as meaning the same thing or is there a distinction between these two phrases for you?

Although Amy did express the opinion that the phrases *arbitrarily close* and *infinitely close* are similar in meaning, she did note an important distinction
between the two. The excerpt below suggests that Amy did, in fact, see the former
as a way to operationalize the latter.

Craig: I was going to ask if you guys see any difference between being
infinitely close and being arbitrarily close. I noticed in your revised
articulation you talked about being arbitrarily close but there have
been other times when you’ve talked about being infinitely close.
And I just wanted to check in and see for you if there’s any
difference between those two phrases, and if so what that difference
would be....

Amy: Well, umm, for me, the first thing, the reason why I like the phrasing
arbitrarily close better than infinitely close is because we can’t get
infinitely close ‘cause it’s a kind of, this is, this is an abstract idea
but it’s not something that we can practically do. Arbitrarily close
means that like you pick a number as close as you, as you want, you
know? And, and that’s something you can actually do and actually
test, you know? Whereas you can’t test getting infinitely close to
something.

Craig: So what would that testing look like for arbitrarily close? And what
are you picturing in your mind?

Amy: Umm, arbitrarily close or arbitrarily small, you know like, i.e., I
could actually take like the smallest real number that I can think of
off the top of my head and throw it at it and test it and see whether
the thing works, you know? Whereas like an infinitely small number
I can’t come up with because that’s, you know, that’s an abstraction.

It appears, then, that at least for Amy, infinitely close was an abstraction that could
be operationalized via the notion of arbitrarily close. Amy saw the smallest real
number that she could think of off the top of her head as an arbitrarily small
number, instead of as an infinitely small number, an entity she did not believe
exists. This suggests that the practical use of concepts and ideas related to the
notion of infinity may be difficult, or impossible, for students to imagine. Here
Amy had, by way of the concept of arbitrarily, devised a practical approach to
assess the behavior of infinitesimally small numbers.
So as to understand better how Amy was conceptualizing the use of arbitrarily small numbers in relation to limit, I asked her to walk through an example on the board of how she would test to see if their definition of limit at infinity would be satisfied by an arbitrarily small number. In response, she drew the function
\[ f(x) = \arctan(x) \] on the board, and noted that the limit, as \( x \to \infty \) would be \( \frac{\pi}{2} \). She then provided the following verbal explanation of how one might test an arbitrarily small positive number for \( \lambda \).

Amy: [S]ay like the smallest number I could think of off the top of my head was .01.
Craig: Okay.
Amy: Then I could start plugging in values of \( x \), you know? Say I start with just like 1, you know, and then like 10, stuff like that. And eventually that, that value that would be output would be, would... get within .01 of \( \pi/2 \) as long as I just keep cruising out here along the \( x \)-axis far enough....And then after that point it would remain within .01 of \( \pi/2 \).
Craig: Okay.
Amy: Forever.
Craig: Forever, okay.
Amy: Or for arbitrarily large values of \( x \).
Craig: Okay, and...if someone thought of a smaller number, what, how would that affect your picture then? Or what happens in terms of testing to see if this function stays within some bound? So let's say someone comes along and says .001.
Amy: Okay,...so somebody gave me .001 and then I would just continue plugging in values of \( x \) until, until uh, I got an output of the function that was within .001 of \( \pi/2 \). And not only was that an output that was within...\( \pi/2 \), but that was the starting of an interval on which all \( x \) values have corresponding outputs that are within that distance.

The excerpt above suggests, then, that Amy felt that any small number could represent all small numbers. Here she appeared flexible in her thinking about arbitrarily small numbers, recognizing that an arbitrarily small number need not
necessarily equate to any particular number, like, say, .01. Discussing and utilizing arbitrarily small numbers appeared to provide Amy a way in which to assess the existence of a limit at infinity without having to address the troublesome conceptual implications of notions related to infinity.

Summary

In conclusion, then, limit at infinity proved to be an appropriate context for initiating necessary cognitive shifts in Amy and Mike. Although Amy and Mike at first continued to reason from both an x-first and potential infinity perspective during Phase 3, the limit at infinity context supported them in eventually shifting to a y-first perspective and utilizing the notion of arbitrary closeness to encapsulate the limiting process. These shifts culminated in a joint characterization of limit at infinity that is conceptually synonymous to the conventional definition.

Final Definition:  \[ \lim_{x \to \infty} f(x) = L \]  provided for any arbitrarily small positive number \( \lambda \), we can find an interval \((a, \infty)\) such that for all \( x \) in \((a, \infty)\), \(|f(x) - L| \leq \lambda\).

Amy and Mike’s experience also suggests that first defining closeness in a precise manner, using absolute value statements to quantitatively notate closeness, and utilizing the notion of arbitrary closeness to encapsulate the limiting process may support students in resolving the cognitive difficulties that arise from trying to define infinite closeness while reasoning from a potential infinity perspective.
5.2.5 Theme 5: Reinventing *Limit at Infinity* as Support for Reinventing *Limit at a Point*

Evidence emerged during Phase 4 of the teaching experiment to suggest that reinventing the definition of *limit at infinity* provided Amy and Mike important support for reinventing the definition of *limit at a point*. I have already described in the previous section how *limit at infinity* provided Amy and Mike a context conducive to inducing cognitive shifts from an *x*-first and potential infinity perspective. These shifts appear to have been beneficial—Amy and Mike's ability to characterize *limit at a point* during the fourth phase of the experiment was greatly improved because, in general, they did not attempt to reason from the *x*-first and potential infinity perspectives that had hindered their progress during Phase 2. In addition, Amy and Mike's definition of *limit at infinity* appeared to be a useful tool that they implemented to make significant refinements to their definition of *limit at a point* during the fourth phase. In this section, I will provide evidence drawn from Session 9 of the teaching experiment of Amy and Mike's use of their definition of *limit at infinity* as a structural template to aid their reinvention efforts.

*Transitioning from Limit at Infinity to Limit at a Point*

The central instructional goal of the ninth session was for Amy and Mike to transition back to refining their definition of *limit at a point*. Prior to the ninth session, their definition of *limit at a point* was neither precise nor mathematically valid. However, the work Amy and Mike had done during the previous two
sessions (Sessions 7 and 8), had seemingly provided necessary support for them to successfully refine their definition of *limit at a point*. Having them make the transition from defining *limit at infinity* to defining *limit at a point* was the central instructional goal of the ninth session. At the outset of the ninth session, I provided Amy and Mike with four articulations they had constructed of *limit at infinity* during Sessions 7 and 8, pointing out the difference in specificity between their first articulation (which was mathematically invalid) and their final articulation. Their final articulation was actually the third one in the list below, but given that Amy had noted that the “by taking sufficiently large values of *x*” part of their final articulation was redundant and was subsumed in the latter part of their articulation, I took the liberty to offer them a “pruned” version of their final articulation.

**Earlier Articulation:** “As *x* gets larger, the distance |*L*- *y*| between *L* and your corresponding *y*-values continues to decrease.”

**Revised Articulation:** “It is possible to make bounds arbitrarily close to 4 and by taking large enough *x*-values we will find an interval (*a*, ∞) on which *f*(*x*) is within those bounds.”

**Final Articulation:** “\( \lim_{{x \to \infty}} f(x) = L \) provided for any arbitrarily small positive number \( \lambda \), by taking sufficiently large values of *x*, we can find an interval (*a*, ∞) such that for all *x* in (*a*, ∞), \(|f(x)| \leq \lambda\).”

**Final Articulation (pruned):** “\( \lim_{{x \to \infty}} f(x) = L \) provided for any arbitrarily small positive number \( \lambda \), we can find an interval (*a*, ∞) such that for all *x* in (*a*, ∞), \(|f(x)| \leq \lambda\).”

My rationale for recalling for Amy and Mike their evolving articulations of *limit at infinity* was to make evident how precisely their final articulation described the
necessary conditions for a function to have a limit at infinity. Having presented the
four articulations of limit at infinity shown above, I next wrote Amy and Mike’s
most recent definition of limit at a point, constructed during Session 6, on the
board.

**Definition #4:** The limit $L$ of a function at $x=a$ exists if every time we look
at the function more closely as we get infinitely close to $x=a$, it bears out the same pattern of behavior, i.e., looks to be
approaching some $y$ value $L$ w/ no gaps in the graph.

My pedagogical aim in presenting Amy and Mike with both their final definition of
limit at infinity and their most recent definition of limit at a point was to contrast
the difference in specificity and precision between their two definitions. The
excerpt that follows suggests that this pedagogical strategy was effective—we
being prompted, Amy noted how “unwieldy” their definition of limit at a point
seemed in contrast to their definition of limit at infinity.

Craig: So this was a couple weeks ago, this was the most, umm, recent
articulation we had for limit at a point. Now you’re laughing. Amy, why are you laughing?
Amy: ‘Cause it’s so unwieldy. Convoluted.

It appears, then, that following their initial attempts to define limit at a point, Amy
and Mike may have benefited by shifting their focus to reinventing the definition of
limit at infinity. Specifically, at the end of Session 8, they expressed awareness that
their definition of limit at infinity adequately addressed the prototypical examples
and counterexamples of limit at infinity they had previously generated, in that their
definition correctly validated existing limits and invalidated non-existing limits.
This was a success they had not experienced during Sessions 4-6 with their
characterizations of \textit{limit at a point}. At the outset of Session 9, the rigor and mathematical power of Amy and Mike's definition of \textit{limit at infinity}, then, appeared to contrast their current definition of \textit{limit at a point} in a manner that served to motivate them to make further revisions to their definition of \textit{limit at a point}. Further, this presentation of their two definitions set the stage for them to subsequently make use of some of the structure and notation in their precise definition of \textit{limit at infinity} as they refined their definition of \textit{limit at a point}.

\textit{Using Limit at Infinity as a Template for Limit at a Point}

Following Amy and Mike's regeneration of prototypical examples, I charged them with the task of more precisely characterizing \textit{limit at a point}. Almost immediately, Amy and Mike began making spontaneous use of their precise definition of \textit{limit at infinity}. It appears to have been important that I had written their final articulation of \textit{limit at infinity} on the board, for Amy was looking at it as she said the following:

Amy: I wonder if it would be useful to invoke some of the same language that we used in uh, that definition.
Mike: Hmm.
Amy: Like along the lines of umm, say as you take $x$-values wherein umm, the absolute value of the distance between $x$ and $a$, umm, gets arbitrarily small, $y$ gets arbitrarily close to $L$.

From a pedagogical standpoint, Amy's response is significant. The reader will recall that prior to defining \textit{limit at infinity}, Amy and Mike had experienced difficulty describing precisely what it would mean for $x$ to get infinitely close to $a$ on the $x$-axis. However, now that they had precisely described infinite closeness
along the y-axis in the context of a limit at infinity, they had a foundation from which to work as they returned to the task of describing infinite closeness along the x-axis in the context of a finite limit at a point. As the conversation continued, it became evident that Mike also saw the potential for using their definition of \textit{limit at infinity} as a structural template for precisely defining \textit{limit at a point}.

Amy: [W]e’re interested in minimizing the, the distance between $x$ and $a$.

Mike: So basically we’re doing the same thing. We’re making that interval on the x-axis instead of the y-axis.

Amy: Yeah.

Mike: Okay.

Amy: Same thing as what?

Mike: Like how we had up here, for the infinite limit, we had this interval getting closer to a value. And we now want that interval to be here. Get closer and closer to, so as you get closer and closer to $a$, like between .01 and $a$.

Amy: Yeah.

As he said “And we now want that interval to be here,” Mike pointed to $x=a$ on the x-axis. It is evident, then, that both Amy and Mike made use of their definition of \textit{limit at infinity} as a model for notating infinite (or arbitrary) closeness along the x-axis with absolute value statements. Further, it is reasonable to conclude that Mike viewed the value “.01” as representative of an arbitrarily small positive number around $a$, given that during Session 8, Amy had similarly used .01 to represent an arbitrarily small positive number around $L$.

\textit{Amy and Mike’s Definition Evolves}

The evolution of Amy and Mike’s definition of \textit{limit at a point} throughout the ninth session was not always fluid. Indeed, their attention was devoted to multiple
aspects of the concept simultaneously. In particular, their focus rapidly shifted back and forth between characterizing behavior along the x- and y-axes. Perhaps not surprisingly, Amy and Mike initially showed more comfort in precisely describing closeness along the y-axis, a notion to which they had needed to attend during the seventh and eighth sessions when they defined limit at infinity. Their familiarity with describing closeness along the y-axis is evident in the first refinement they made to their definition\(^{48}\) during Session #9.

**Definition #5:** As \(x\) gets arbitrarily close to \(a\), \(|L-f(x)|\) gets arbitrarily small.

Following this refinement, Amy again made use of their definition of limit at infinity to more precisely describe what it would mean for \(|L-f(x)|\) to get arbitrarily small.

Amy: And so if you wanted to phrase that in the similar way that we phrased the earlier one, you know, you could say something along the lines of by, by taking values of \(x\) sufficiently close to \(a\), you could satisfy an inequality like that wherein this distance is smaller than any small number \(\lambda\) you can think of.

In the excerpt above, “the earlier one” refers to their definition of limit at infinity, and “an inequality like that” refers to the inequality statement \(|L-f(x)| \leq \lambda\). After Amy reintroduced the notion of an arbitrarily small number \(\lambda\), and noted that “for any arbitrarily small number \(\lambda\), you can find an \(x\)-value that will satisfy that inequality,” Mike wrote a revised definition that included the same notation for characterizing

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\(^{48}\) The reader will recall that their previous definition, constructed during Session 6, was as follows: **Definition #4:** The limit \(L\) of a function at \(x=a\) exists if every time we look at the function more closely as we get infinitely close to \(x=a\), it bears out the same pattern of behavior, i.e., looks to be approaching some \(y\) value \(L\) w/no gaps in the graph.
infinite closeness on the y-axis that they had used in their definition of \textit{limit at infinity}.

\textbf{Definition #6:} As x gets arbitrarily close to a, |L-f(x)| gets arbitrarily small. For any arbitrarily small \# \lambda you can find an x-value that satisfies |L-f(x)|<\lambda.

The reader will note that although the first sentence in Amy and Mike’s sixth definition was from an x-first perspective, the second sentence was from a y-first perspective. Also, Definition #6 included the use of the notion of \textit{arbitrary closeness}, a notion fundamental to Amy and Mike’s success in reinventing the definition of \textit{limit at infinity}. Hence, some of the reasoning captured in Amy and Mike’s definition of \textit{limit at infinity} was beginning to make its way into their evolving definition of \textit{limit at a point}. As Amy and Mike discussed how best to describe infinite closeness along the x-axis, Mike once again looked to their definition of \textit{limit at infinity} for inspiration.

Amy: [\textit{Referring to Definition #6}] That doesn’t seem to get at the idea that like x is getting closer to a....You could find an x-value that’s just like way out there or something like that, you know?
Craig: So you, you’re wanting to talk about x-values...
Amy: At or around a.
Mike: Yeah, I feel like this, this is the same thing we had for our other interval for limits at infinity so I feel like we want to, we can just switch some things or change some things because our interval is now to a specific point.
As Mike said “because our interval is now to a specific point,” he looked at the graph seen in Figure 5.15, and used his hands to illustrate increasingly tighter bounds around $a$, suggesting both that he understood the fundamental difference between \textit{limit at infinity} and \textit{limit at a point}, and that he was trying to incorporate ideas from their definition of the former to precisely specify the latter. Interestingly, Amy did not appear aware that they could use an absolute value inequality statement to represent infinite closeness on the $x$-axis, even though they had done so for the $y$-axis.

Amy: [S]o I’m missing the chunk about being around $a$. How do we put that in?
Craig: So now you’ve pinned down what it means to be arbitrarily close to
Amy: Close to $L$, yeah, but, but now I’m missing the bit about, about, about $x$ being around $a$. I don’t know how…

As they thought how best to describe $x$ being near $a$, Mike refined their articulation:

\textbf{Definition #7:} For any arbitrarily small \( \lambda \) we can find a value of \( x \) arbitrarily close to \( a \) such that \(|L-f(x)| \leq \lambda\).

As I read their definition back to them aloud, Amy recognized how to characterize infinite closeness along the $x$-axis.

Craig: Okay, now, so you’re saying [a limit exists] provided for any arbitrarily small number $\lambda$, we can find a value of $x$ arbitrarily close
to $a$, so not way out there, but close to $a$ such that this [inequality holds].

Amy: Yeah. I mean,...I guess there's no reason why if we wanted to we could put, put the arbitrarily, $x$ being arbitrarily close to $a$ in, you know, couch it in the terms of a similar inequality wherein, wherein there's an arbitrarily small number $\rho$ that is, you know, such that $x-a$ is less than.

The excerpt above provides evidence that Amy made a connection to their definition of limit at infinity and saw the opportunity, as Mike had earlier, for parallel structure in their definition of limit at a point. Here, Amy not only suggested using an inequality statement with absolute value notation, but also, in a manner consistent with how they first operationalized infinite closeness in the context of limit at infinity, suggested a corresponding variable $\rho$ to represent closeness\(^{49}\). This suggestion led to a further refinement of their definition.

**Definition #8:** For any arbitrarily small number $\lambda$, we can find a value of $x$ arbitrarily close to $a$, i.e., $|x-a|<\theta$, such that $|L-f(x)|<\lambda$. Note: $\theta$ is an arbitrarily small number.

After some discussion regarding the number of $x$-values around $a$ for which the inequality $|L-f(x)|<\lambda$ must hold for each choice of $\lambda$, Amy and Mike arrived at their final articulation of limit at a point.

**Definition #9:** \[ \lim_{x \to a} f(x) = L \] provided that: given any arbitrarily small number $\lambda$, we can find an $(a+\theta)$ such that $|L-f(x)| \leq \lambda$ for all $x$ in that interval except possibly $x=a$. (Session 9 – Final Definition)

\(^{49}\) As Amy made this suggestion, Mike wrote “i.e., $|x-a|<\theta$” next to the phrase “arbitrarily close to $a$” in their definition. Amy offered no reluctance to the introduction of this notation, and from that point forth, closeness along the $x$-axis was quantified in terms of $\theta$. 

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Summary

In sum, there was ample evidence during Phase 4 of the teaching experiment to suggest that reinventing the definition of \textit{limit at infinity} provided Amy and Mike important support for reinventing the definition of \textit{limit at a point}. Indeed, Amy and Mike used their definition of \textit{limit at infinity} as a structural template to aid their reinvention efforts – on multiple occasions during the ninth session, they compared their precise definition of \textit{limit at infinity} with their vague characterizations of \textit{limit at a point} and subsequently evoked some of the same notation and language they had used for the former to refine their definition of the latter. Examples of this include the use of absolute value notation to characterize infinite closeness, first on the $y$-axis and then eventually on the $x$-axis. Also, the $y$-first structure of their definition of \textit{limit at infinity} appeared to guide them in shifting their characterization of \textit{limit at a point} from an $x$-first perspective to a $y$-first perspective. It is worth noting that from the outset of Session 9, Amy and Mike both expressed sentiments indicating they were pleased with the precision of their definition of \textit{limit at infinity}. The success they had realized in defining \textit{limit at infinity} appeared to raise their expectations of what they were capable of in regards to precisely characterizing \textit{limit at a point}. Pedagogically, this is noteworthy, as it suggests that students attempting to reinvent and/or understand the definition of \textit{limit at a point} may be well-served by first coming to understand the definition of \textit{limit at infinity}, a seemingly less cognitively complex concept. An inspection of Amy and Mike’s characterizations of \textit{limit at a point} indeed suggests that
reinventing limit at infinity during Sessions 7 and 8 had a positive effect on Amy and Mike’s efforts to precisely characterize limit at a point. Figure 5.16 captures the key formulations in the evolution of their definition.

<table>
<thead>
<tr>
<th>Amy and Mike’s Evolving Definition of Limit</th>
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<tbody>
<tr>
<td><strong>Definition #1:</strong> f has a limit L at x=a provided as x-values get closer to a, y-values get closer to L. (Session 4)</td>
</tr>
<tr>
<td><strong>Definition #2:</strong> If you could zoom forever and always get closer to a and L, then you have a limit. (End of Session 4)</td>
</tr>
<tr>
<td><strong>Definition #3:</strong> A function has a limit L at a when zooming in FOREVER both horizontally and vertically yields no gaps that have length &gt; 0 AND that it looks like it approaches a finite number L. (Session 5)</td>
</tr>
<tr>
<td><strong>Definition #4:</strong> The limit L of a function at x=a exists if every time we look at the function more closely as we get infinitely close to x=a, it bears out the same pattern of behavior, i.e., looks to be approaching some y value L w/no gaps in the graph. (Session 6)</td>
</tr>
<tr>
<td><strong>Definition #5:</strong> As x gets arbitrarily close to a,</td>
</tr>
<tr>
<td><strong>Definition #6:</strong> As x gets arbitrarily close to a,</td>
</tr>
<tr>
<td><strong>Definition #7:</strong> For any arbitrarily small # λ we can find a value of x arbitrarily close to a such that</td>
</tr>
<tr>
<td><strong>Definition #8:</strong> For any arbitrarily small # λ we can find a value of x arbitrarily close to a, i.e.,</td>
</tr>
<tr>
<td><strong>Definition #9:</strong> [ \lim_{x \to a} f(x) = L ] provided that: given any arbitrarily small # λ, we can find an (a±0) such that</td>
</tr>
</tbody>
</table>

Figure 5.16 – Amy and Mike’s Evolving Definition of Limit

5.2.6 Retrospective Findings

The first three themes presented in this chapter describe cognitive difficulties Amy and Mike experienced that hindered their attempts to characterize limit at a point. Themes 4 and 5 address a beneficial pedagogical decision that helped to
alleviate Amy and Mike’s cognitive difficulties and stimulate progress in their characterization of *limit at a point*. In the pages that follow, I describe three final themes that point to insights I gained only after retrospectively reflecting on the entire reinvention process.

**Theme 6: Reinvention of the Formal Definition of Limit: An Existence Proof**

The findings discussed in this chapter have by and large been focused on Amy and Mike’s *reasoning* and subsequent *pedagogical decisions* I made which were supportive of their efforts to reinvent the definition of limit. The finding I discuss here, however, pertains neither to reasoning nor to pedagogy, but rather to the potential students have to reinvent sophisticated mathematical ideas. While other studies (e.g., Larsen, 2001; Fernandez, 2004) have sought to describe how students *interpret* the formal definition of limit, this dissertation study is unique in that the students who participated in the teaching experiment were posed the challenge of *reinventing* the formal definition. The result of Amy and Mike’s efforts is indeed noteworthy – both students had neither seen nor were aware of the formal definition of limit, yet they were ultimately able to characterize *limit at a point* in a manner synonymous to that of the conventional $\varepsilon$-$\delta$ definition. Amy and Mike’s end product even captures the complex quantification structure and subtleties of the $\varepsilon$-$\delta$ definition.

**Final Definition:** $\lim_{x \to a} f(x) = L$ provided that: given any arbitrarily small $\lambda$, we can find an $(a \pm \theta)$ such that $|L-f(x)| \leq \lambda$ for all $x$ in that interval except possibly $x=a$. 

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Amy and Mike's success in reinventing the definition of limit is significant—I am unaware of other studies indicating such potential for students who had not previously seen the definition. Amy and Mike's successful characterization of limit establishes reinvention as a possible avenue to developing coherent understanding of the formal $\varepsilon$-$\delta$ definition. Evidence of this was apparent in Amy's reasoning during the tenth session. Despite twelve days having passed between Amy and Mike's successful characterization of limit during Session 9 and the tenth session, Amy nevertheless was able to coherently reason about what it would mean for a function $f$ to have a limit $L$ at $x=a$. In the following excerpt, the function being discussed is the one seen in Figure 5.17.

![Figure 5.17 - Generalized Removable Discontinuity Graph](image)

Craig: Amy, do you have anything that you want to add...in light of this function and what it means for the function to have a limit $L$?
Amy: ...No, not really..., just as long as we're placing due emphasis on, on, you know, yeah, that we're not just like picking a single $\lambda$ that, you know, you think of as being small or something,...but that the pattern bears out for any...successively smaller value that you could possibly want to pick for $\lambda$.
Craig: So,...I pick a progressively smaller number for $\lambda$ and hand it to you, then what do you do with that in terms of the, in terms of the picture I guess?
Amy: I can show you the interval around $a$ for which it's true that all the corresponding values of $f(x)$ with, that for all values of $x$ within that interval around $a$, all of their corresponding values of $f(x)$, umm,
produce, uh, distances from $L$ that are smaller than that $\lambda$ that you give me...[W]hat we can say based off of this formulation is that with the possible exception of $f(x)$ at $a$ specifically, it is true that for every $x$ single value, every single of that infinite set of $x$-values within that interval, that they are going to correspond to heights that are within that distance of $L$.

Hence, evidence from the first teaching experiment suggests that students who have never encountered the formal definition of limit have the potential to reinvent it by building upon their informal understandings through engagement in purposefully designed tasks. Further, in so doing, they stand to develop sophisticated understanding of what is a complex mathematical idea. In the last two sections of this chapter, I will elucidate two specific by-products of reinvention I observed during the first teaching experiment.

*Theme 7: Reinvention as Support for Coherently Interpreting Conventional Formulations of the Definition*

One of my conjectures prior to the first teaching experiment was that students would be able to coherently interpret the conventional $\varepsilon-\delta$ definition of limit if they first constructed their own definition. Following Amy and Mike's successful reinvention of the definition of limit, I tested that conjecture during the tenth session by asking them to respond to the following task:

**Written Prompt:** Please consider the following statements:

1) $\lim_{x \to a} f(x) = L$ provided that: Given any arbitrarily small $\# \lambda$, we can find an $(a \pm \delta)$ such that $|L-f(x)| \leq \lambda$ for all $x$ in that interval except possibly $x=a$.

2) $\lim_{x \to a} f(x) = L$ provided that: For every $\lambda>0$, there exists a $\theta>0$, such that $0<|x-a|<\theta \Rightarrow |f(x)-L| < \lambda$
The first of these statements is your own articulation of what it means for a function $f$ to have a finite limit $L$ at $x=a$. Consider the second statement. Does it capture the intended meaning of your own articulation? Comment on the similarities and differences in meaning in relation to your own articulation.

The reader will note that I did not indicate for Amy and Mike that the second statement was the conventional definition of limit at a point. In fact, throughout the teaching experiment, Amy and Mike were unaware that such a definition even existed. I did not want Amy and Mike to know that the second statement was the definition accepted by the mathematics community for fear that such knowledge would negatively impact the way they viewed their own definition. Indeed, their definition was accurate and coherent in its own right.

In response to this task, Amy and Mike did recognize that the second statement captured the intended meaning of their definition. In the excerpt below, it is evident that Amy saw their definition and the conventional definition as synonymous in meaning, although she expressed a preference for the conventional one because of its conciseness and, in particular, its strategy for handling every $\lambda$ simultaneously.

Craig: Okay, Amy what do you think about that second one?...
Amy: I like that one a lot better than ours.
Craig: Oh.
Amy: I wish I'd put it that way.
Craig: Why's that?
Amy: Because, because that umm, that includes any arbitrarily small $\lambda$, and it's without getting into notional concepts like smallness.
Craig: Okay.
Amy: It just covers it.
Craig: So you like that one because it, it covers smallness. What else does it cover?
Amy: It's rigorous. Every $\lambda$. 

Craig: Okay. Does, does it capture though the intended meaning of your
guys' articulation?
Amy: Yeah. Yeah.
Craig: Okay, why else do you like that one?
Amy: Well, 'cause it just seems like a concise way of saying, you know,
the same thing that, that I thought we were trying to get at, you
know?
Craig: Okay.
Amy: That for, no matter how closely you squeeze in around your
proposed limit you will find an interval around $x$ that, that
corresponds to $y$-values for which that's true.

Amy's comments above are significant, for they indicate that students have the
potential to coherently interpret the conventional definition of limit at a point if
they have their own definition from which to reason. Not only did Amy indicate
that the conventional definition captured the intended meaning of their definition,
she also noted how the conventional definition handled some of the subtle details of
their definition.

Craig: Okay. Umm, now I noticed in your guys's articulation you have this
piece about umm, except possibly $x=a$.
Amy: Umm-hmm.
Craig: Is that piece included in the second articulation? Or does that second
articulation fail to capture that idea?
Amy: Well, I think that it actually captures it nicely 'cause it says that $x-a$
is greater than 0, so that precludes the possibility of $x$ equaling $a$.

Attempts to develop coherent understanding of the formal definition have proven
difficult (Cornu, 1991; Dorier, 1995; Gass, 1992; Tall, 1992; Tall & Vinner, 1981;
Williams, 1991). While other efforts (Gass, ibid; Larsen, 2001; Fernandez, 2004;
Steinmetz, 1977) have been made to elicit coherent interpretations of conventional
formulations of the definition, Amy's reflections suggest that reinvention supports
students in coherently interpreting and understanding the formal definition.
Theme 8: Reinvention as Motivation for Need for Formal Definition

One of two primary pedagogical goals at the outset of the teaching experiment was to motivate in Amy and Mike a sense of necessity for a rigorous justification process for limit. Prior to the teaching experiment, their experience with limits had been primarily focused on finding limit candidates by using algebraic techniques to employ direct substitution, or by inspecting tables and/or graphs. They had not, however, been asked to justify or validate those limit candidates. Thus, it was my aim to generate cognitive conflict within Amy and Mike that might, in turn, motivate them to begin questioning whether the validation of proposed limits requires a process inherently distinct from that used to find limit candidates. I conjectured that successful reinvention of the definition of limit would be unlikely if Amy and Mike did not first see the necessity for such a definition. To motivate such necessity, the reader will recall that I provided Amy and Mike limit tasks for which it was possible to propose limit candidates, but not possible to subsequently justify those candidates with certainty using the algebraic techniques to which they had become accustomed. However, none of the tasks that Amy and Mike engaged with during the first phase of the experiment elicited in them any outwardly expressed sense of a distinction between the acts of finding and validating. As such, for Amy and Mike, reinventing the definition of limit did not appear to be motivated by an internal need to respond to a mathematical necessity. In fact, it was not until after reinventing the definition of limit that they expressed the opinion that
the mathematical role of the definition of limit is not that of finding limits, but rather validating proposed limits. Thus, in contrast to my initial conjecture that students must first become aware of the distinction between finding and validating so as to be properly motivated to reinvent the definition, Amy and Mike displayed evidence suggesting reinvention may provide students an experience that subsequently evokes awareness of that distinction.

During the tenth and final paired session, both Amy and Mike conveyed, for the first time, what they each saw as the purpose of the definition of limit. In both cases, their comments were unprompted. The following excerpt illustrates Amy's perspective.

Amy: [W]hat I was thinking about when he was doing that was...so we have this sort of like, you know, presupposition that...the limit of \( f(x) \) at \( a \) is \( L \), and then so, and then, so this is the like, this assertion that we're going to, you know, see whether that's true or not. And if it is true then it means that it satisfies these criteria...And I mean, 'cause this whole thing presupposes that we already have a pretty damn good guess about what \( L \) is. We already think we know, you know, we, I mean we do know...We know we have the right one. We just, umm, you know, we're trying to prove it, right?...I mean, this, this, you know,...is a fairly concise way of, of summing up the idea of like what this limit is. But by the time we get there to prove it we've already figured out what the limit was, based off of some kind of intuitive means, or something like that, you know?

This appeared to be a significant moment for Amy. Indeed, prior to this point in the teaching experiment, Amy had not made any mention regarding the definition's mathematical role. However, from this point forth, she noted on multiple occasions that the definition presupposes a given limit candidate, as the following excerpt suggests.
Amy: I feel like this is useful to use as like, uh, a test for, for once you’ve already got a hypothesis that you like feel is, you know, in your guts, is like you’re pretty sure that it’s correct. But it’s not good as a diagnostic, you know? I mean…if you didn’t really know where \( L \) was, I mean if it wasn’t one of these like straightforward functions…., if it was some kind of weird thing where like the limit of, you know, of \( f \) of .196blah-blah-blah, you know, is like…some equally hairy decimal, you know, that like it wouldn’t be revealed by some straightforward familiar function. This [definition] isn’t going to do us a lot of good in terms of finding \( L \) in the first place. But once, through some other means, we found it, you know, then we can use this as a test.

Thus, through the process of precisely characterizing limit at a point, Amy became cognizant that the definition is not a technique by which one might find a limit.

Here, Amy appears to reference a two-step process, wherein “through some other means” one finds \( L \) and then subsequently uses the definition to “test” \( L \)’s validity.

Given students’ extensive experience finding limit candidates during their introductory calculus experience, it is not surprising that they might be delayed in recognizing that the conventional definition of limit is not a technique by which they might similarly find limit candidates. Mike, too, displayed reasoning during the tenth and final paired session that suggests he had come to view the definition as an arbitrator among various limit candidates for a particular function.

Mike: But, I mean it seems like the point of why like this was brought up though, like for this one and for this one, I mean it says the limit as \( x \) approaches \( a \) is \( L \) if this is satisfied. Well neither of these are true, so that’s, I mean that’s why we have our definition. So we know that those aren’t, uh, limits.

From a guided-reinvention perspective, Amy and Mike’s verbalizations of the definition’s purpose are noteworthy. In theory, students’ motivation to invent (or,
reinvent) a mathematical notion is born out of necessity. Interestingly though, Amy and Mike did not appear to fully appreciate the purpose of their precise definition until they had completed the reinvention process. This suggests, at least in the case of limit, that the process of constructing a precise definition for the concept simultaneously increases one’s recognition of the need for such a precise definition.

5.3 – Summary

The reader will recall that the intent of this dissertation study was to engage students in an instructional sequence with two objectives: 1) To develop insight into students’ reasoning in relation to their engagement in instruction designed to support their reinventing the formal definition of limit, and; 2) To inform the design of principled instruction in relation to limit. In the preceding pages, I have elaborated eight themes that emerged from the first teaching experiment (Figure 5.18) that address these two research goals. These themes are presented both as results of the first teaching experiment and as a lens for understanding key issues that were implicated in Amy and Mike’s reinvention process.

<table>
<thead>
<tr>
<th>Emergent Themes</th>
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<tbody>
<tr>
<td><strong>Theme 1:</strong> Reliance on Algebraic Representations and Distrust of Graphs <em>(Sessions 3-6)</em></td>
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<tr>
<td><strong>Theme 2:</strong> Predominance of an x-First Perspective and the Counterintuitiveness of a y-First Perspective <em>(Sessions 3-10)</em></td>
</tr>
<tr>
<td><strong>Theme 3:</strong> <em>Potential Infinity</em> as a Hindrance to Characterizing <em>Infinite Closeness</em> <em>(Sessions 3-6)</em></td>
</tr>
<tr>
<td><strong>Theme 4:</strong> <em>Limit at Infinity</em> as a Context Conducive for Initiating Necessary Cognitive Shifts <em>(Session 7)</em></td>
</tr>
<tr>
<td><strong>Theme 5:</strong> Reinventing <em>Limit at Infinity</em> as Support for Reinventing <em>Limit at a Point</em> <em>(Sessions 9-10)</em></td>
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</table>
The first three themes point to conceptual difficulties students may experience as they attempt to characterize the limit concept. Indeed, Amy and Mike’s reliance on algebraic representations, distrust of graphs, and propensity towards reasoning from an *x-first* and *potential infinity perspective* limited their ability to articulate a precise definition. Figure 5.19 captures the key formulations in the evolution of their definition prior to resolving these conceptual difficulties.

<table>
<thead>
<tr>
<th>Definition</th>
<th>Description</th>
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<tbody>
<tr>
<td>Definition #1:</td>
<td><em>f</em> has a limit <em>L</em> at <em>x</em> = <em>a</em> provided as <em>x</em>-values get closer to <em>a</em>, <em>y</em>-values get closer to <em>L</em>. (Session 4)</td>
</tr>
<tr>
<td>Definition #2:</td>
<td>If you could zoom forever and always get closer to <em>a</em> and <em>L</em>, then you have a limit. (End of Session 4)</td>
</tr>
<tr>
<td>Definition #3:</td>
<td>A function has a limit <em>L</em> at <em>a</em> when zooming in FOREVER both horizontally and vertically yields no gaps that have length &gt; 0 AND that it looks like it approaches a finite number <em>L</em>. (Session 5)</td>
</tr>
<tr>
<td>Definition #4:</td>
<td>The limit <em>L</em> of a function at <em>x</em> = <em>a</em> exists if every time we look at the function more closely as we get infinitely close to <em>x</em> = <em>a</em>, it bears out the same pattern of behavior, i.e., looks to be approaching some <em>y</em> value <em>L</em> w/no gaps in the graph. (Session 6)</td>
</tr>
</tbody>
</table>

Themes 4 and 5 address the second of my two research goals. Shifting Amy and Mike’s focus at the outset of Session 7 to characterizing *limit at infinity* proved to be beneficial pedagogically, for it provided them a context conducive for resolving...
the conceptual difficulties they had experienced during previous sessions. Further, reinventing *limit at infinity* appeared to serve as a template from which Amy and Mike were able to work as they refined their characterization of *limit at a point* during Session 9. Figure 5.20 captures the key formulations in the evolution of their definition following their reinvention of *limit at infinity* during Sessions 7 and 8.

<table>
<thead>
<tr>
<th>Amy and Mike’s Evolving Definition of Limit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Definition #5: As (x) gets arbitrarily close to (a), (</td>
</tr>
<tr>
<td>Definition #6: As (x) gets arbitrarily close to (a), (</td>
</tr>
<tr>
<td>Definition #7: For any arbitrarily small (#\lambda) we can find a value of (x) arbitrarily close to (a) such that (</td>
</tr>
<tr>
<td>Definition #8: For any arbitrarily small (#\lambda) we can find a value of (x) arbitrarily close to (a), i.e. (</td>
</tr>
</tbody>
</table>
| Definition #9: \[
\lim_{{x \to a}} f(x) = L
\]
provided that: given any arbitrarily small \(#\lambda\), we can find an \((a+\theta)\) such that \(|L-f(x)|\leq\lambda\) for all \(x\) in that interval except possibly \(x=a\). (Session 9 – Final Definition) |

Figure 5.20 – Amy and Mike’s Evolving Definition of Limit: Formulations 5-9

The final three themes point to insights I gained only after retrospectively reflecting on the entire reinvention process. Themes 7 and 8 address both research objectives – Amy and Mike’s ability to coherently interpret and reason about the conventional \(\varepsilon-\delta\) definition of limit appeared to be supported by the experience of first constructing their own precise articulation. This not only offers insight into what might support coherent student reasoning in a complex domain, but also underscores the value in implementing instructional tasks designed to capitalize on
students' informal reasoning. The eighth and final theme suggests that students may derive the perspective needed to appreciate the need for a rigorous definition of limit only after having taken part in the reinvention process. This finding has both psychological and pedagogical ramifications as well. First, it supports a previously held belief (Cornu, 1991; Dorier, 1995; Gass, 1992) that students are not likely to recognize that the formal definition was conceived for solving sophisticated mathematical problems until they have themselves wrestled with the many subtleties and details of the concept. Second, the eighth theme suggests that discussions of the formal definition's mathematical role might be better suited as a reflective and concluding activity in an instructional sequence designed to develop students' understanding of the limit concept.

In closing, it is worth noting that given the emergent nature of the design of instruction in the experiment, the phases of the experiment are themselves at once a result of the research and a lens for understanding the tandem unfolding of reinvention and instruction in the experiment, on a global level, in four main phases:

**Phase 1:** Assessment/Attempts to Motivate Necessity

**Phase 2:** Initial Attempts to Define *Limit at a Point* via Graphical Conversations

**Phase 3:** Defining *Limit at Infinity*

**Phase 4:** Revisiting *Limit at a Point*
These phases of coupled reinvention and instruction informed pedagogical
decisions I subsequently made prior to, and during, the second teaching
experiment. I begin the next chapter with a discussion of how these phases
influenced pedagogical decisions in the second teaching experiment.
Chapter 6 – The Second Teaching Experiment

In this chapter I detail the results of the second teaching experiment, highlighting three products of the experiment which emerged in tandem: the phases of instruction, the evolution of the students’ characterization of limit, and finally, seven emergent themes which characterize student reasoning and point to subsequent pedagogical implications. The chapter consists of two main parts. In Part 1, I delineate the five instructional phases of the second teaching experiment, painting in broad strokes the unfolding of instruction across the experiment and highlighting instructional goals and tasks. In Part 2, as I describe in greater detail the evolution of the students’ characterization of limit, I discuss seven themes which emerged from my analysis of the data. Parts 1 and 2 are preceded by a brief explanation of two significant pedagogical changes that I made prior to the second teaching experiment, based on my analysis of the first teaching experiment.

6.0 – Preamble to the Second Teaching Experiment: Two Significant Pedagogical Changes

Contributing to an epistemological analysis involves the reiteration of the research cycle illustrated in Figure 3.1. Instructional design is shaped by analyses and modeling of students’ reasoning as they engaged with instructional tasks.

50 In Part 2, I do occasionally include the description of particular tasks that marked significant moments in the second teaching experiment. I do not, however, provide a complete listing of the instructional tasks employed in the second teaching experiment within this sixth chapter. The reader can find a complete description of the sequence of instructional tasks, as well as the rationale for those tasks, in Appendix C.
Analyses of the data generated in the first teaching experiment informed pedagogical decisions in the second teaching experiment. In particular, the phases of instruction described in Chapter 5 provided a structural template from which to proceed during the second teaching experiment. Following an analysis of the first experiment, two significant changes were envisioned for the second experiment. First, I felt that the first phase of instruction could be significantly shortened. During the first experiment, my efforts to assess the first pair of students' (Amy and Mike) informal understanding of limit and to motivate the necessity for a rigorous definition spanned the first three sessions. In hindsight, the second of these two focal points could have been addressed in much less time had I not left it to the students to recognize for themselves the need for a rigorous definition. Further, as the eighth theme in Chapter 5 (see Section 5.2.6) suggests, evidence emerged in the first experiment indicating that reinvention served as a motivation for the need for a formal definition, as opposed to the other way around. Hence, extensive efforts to instill in the second pair of students (Chris and Jason) a sense of motivation for the need for a formal definition at the outset of the second experiment seemed ill-founded. Also, in analyzing the first experiment, I found that the most fruitful data emerged during subsequent phases of instruction. As such, I aimed to shorten the first phase of instruction from three sessions to a single session during the second experiment.

The second significant change I envisioned for the second experiment involved defining limit at infinity. As I elaborated in Chapter 5, limit at infinity served as a
context conducive for initiating necessary cognitive shifts in the process of reinvention. Further, reinventing \textit{limit at infinity} appeared to support reinventing \textit{limit at a point}. In the second experiment, I was interested to learn whether Chris and Jason could successfully reinvent the definition of \textit{limit at a point} without first defining \textit{limit at infinity}. I viewed the second experiment, in part, as an opportunity to explore whether reinvention was possible under different conditions. Defining \textit{limit at infinity} had clearly been an important element in Amy and Mike's reinvention process. However, I also felt that defining what it means to be \textit{close} had provided Amy and Mike a vehicle for operationalizing what it means to be \textit{infinitely close}, a notion fundamental to the limit concept. Thus, I conjectured that the second pair of students might not need to first define \textit{limit at infinity}, and instead might benefit from characterizing what it means to be \textit{close}. This conjecture influenced my pedagogical decisions in the second experiment, resulting in phases of instruction distinct from those found in the first experiment, as the following discussion will illustrate.

6.1 – Part 1: Overview of Instructional Sequence

6.1.0 Introduction

The second experiment consisted of ten paired sessions\(^{51}\) and an individual exit interview with each student. The experiment unfolded in five distinct instructional phases, which collectively suggest the following emergent instructional trajectory:

---

\(^{51}\) Each session was separated by a span of approximately seven days with a few notable exceptions. Due to scheduling conflicts, eleven days separated Sessions 1 and 2, and fourteen days separated...
Phase 1: Assessment of Students' Informal Understanding and Attempts to Motivate Necessity (Session 1)

Phase 2: Initial Attempts to Define Limit via Graphical Conversations (Sessions 2-5)

Phase 3: Explicit Attempts to Define Closeness Using a Step Function (Session 6)

Phase 4: Refinement of Definition of Limit at a Point with Increased Notational Precision (Sessions 7-9)

Phase 5: Attempted Resolution of Central Issues and Completion of Reinvention Process (Session 10)

As was the case with the first teaching experiment, the transitions between phases were marked by instructional “sign posts” – shifts in instruction based on pedagogical decisions to alter the instructional trajectory in response to the students’ reasoning. In a manner similar to Chapter 5, in Part 1 of this chapter, I will delineate the five phases of the second experiment, painting in broad strokes the unfolding of instruction across the ten paired sessions and highlighting instructional goals and tasks.

Sessions 9 and 10. For pedagogical reasons that will be explained in Section 6.2.5, Sessions 7 and 8 occurred on consecutive days.
6.1.1 Phase 1: Assessment of Students' Informal Understanding and Attempts to Motivate Necessity (Session 1)

Two central pedagogical goals drove the instructional decisions that defined and demarcated the first phase of the teaching experiment. The first goal was to assess Chris and Jason's informal understandings of limit, and so I engaged them in tasks designed to assess and leverage their informal notions of limit. Assessment tasks included having Chris and Jason summarize the distinct strategies a student could employ to determine a limit and discussing the extent to which different representations of functions provide conclusive evidence regarding the existence of a limit.

The second pedagogical goal of this phase was to help motivate within Chris and Jason a sense for the necessity of a rigorous justification process for limit. As a departure from the first experiment, and in an effort to shorten the first phase of instruction, I did not generate this sense of necessity within Chris and Jason by having them experience cognitive conflict that might in turn motivate them to begin questioning whether the validation of proposed limits requires a process inherently distinct from that used to find limit candidates. Instead, in response to their claim during Session 1 that one could always use algebraic techniques to determine a limit with certainty, I informed them that numerous cases exist in which such a strategy is not employable, stressing for them the need for a more "all-
encompassing" definition of limit that is not reliant on algebraic techniques. I noted that this would be the focal point of subsequent discussions.

6.1.2 Phase 2: Initial Attempts to Define Limit via Graphical Conversations (Sessions 2-5)

My analysis of Chris and Jason's reasoning during Session 1 prompted me to make a pedagogical shift prior to Session 2. Throughout Session 1, although both students appeared more willing to engage in graphical conversations than had Amy and Mike, they nevertheless expressed a clear reliance on algebraic representations when asked how one might determine with certainty the existence of a limit. With the aim of having Chris and Jason precisely characterize the visual aspects of limits as related to the conventional $\varepsilon$-$\delta$ illustration (see Figure 5.1), I made a pedagogical shift at the outset of the second session. Tasks and activities during Sessions 2-5 were primarily focused on discussing limits in a graphical setting, in hopes that the absence of analytic expressions might support the enrichment of the visual aspects of Chris and Jason's respective concept-images. Tasks included generating prototypical examples and counterexamples of limit, which subsequently served as sources of motivation for Chris and Jason as they made initial attempts at characterizing what it means for a function to have a limit. Phase 2 of the experiment, then, constituted a period of iterative refinement for Chris and Jason; as they attempted to define limit, the examples and counterexamples they encountered created cognitive conflict for them, which they sought to relieve by
refining their definition. The focus of this phase of the experiment was on having Chris and Jason incorporate explicit language in their characterization of limit as they mulled over and wrestled with the essential characteristics and subtleties associated with the concept.

As was the case with Amy and Mike during the first experiment, Chris and Jason’s initial characterizations of limit were from an x-first perspective, a perspective coherent with the act of finding limits. One of my central instructional goals during the second phase of the teaching experiment was to elicit a shift in Chris and Jason’s reasoning towards a y-first perspective.

6.1.3 Phase 3: Explicit Attempts to Define Closeness Using a Step Function
(Session 6)

Chris and Jason’s reasoning over the course of Sessions 2-5 prompted me to initiate a second pedagogical shift prior to Session 6. By the end of Session 5, Chris and Jason’s efforts to reinvent the definition of limit was stalled by struggles also experienced by Amy and Mike during the first experiment. Specifically, Chris and Jason struggled to explicitly characterize infinite closeness and were disinclined to assume a y-first perspective. In response, and based on analysis of the first experiment, my instructional agenda shifted to focus their attention on defining what it means for a function’s output values to be close to a particular y-value, anticipating that their efforts to characterize and operationalize closeness might provide necessary support for characterizing infinite closeness, a notion they had
attempted in vain to characterize during the first five sessions. To facilitate their progress, I directed their attention to the step function seen in Figure 6.1\textsuperscript{52} and asked them to state sufficient conditions for the function to be within a pre-specified distance of a particular y-value (3, in this case).

![Figure 6.1 - Graph of Step Function](image)

The motivation for using a step function was that the functional behavior was such that it was clear which values of \( x \) produced function values that were close (based on a pre-specified definition) to the chosen y-value. In response to this task, the students first characterized closeness to 3, and used that definition as a tool in their attempts to operationalize infinite closeness to 3. Although they were ultimately unable to characterize infinite closeness during Session 6, their efforts nevertheless induced a noticeable shift to a y-first perspective, as was evident in their subsequent characterizations of limit. The step function context, then, supported their development of an increasingly precise definition of closeness, and thus appeared

\textsuperscript{52} In this chapter, Figures 6.1, 6.4-6.11, 6.13, 6.15, 6.17, and 6.21-6.24 are exact recreations of the graphs that were constructed during the actual teaching experiment. These recreations are provided solely for the purpose of improving readability, as the original images of these graphs were not adequately captured by video. All other graphs presented in this chapter are video images of the originally constructed graphs.
to be a context conducive to initiating a shift to a y-first perspective. As Session 6 concluded, Chris and Jason shifted their attention back to characterizing limit, demonstrating verbal understanding of a potentially infinite process by which one could determine whether $L$ is the limit of a function $f$ at $x=a$. In reference to prototypical examples and counterexamples of limit Chris and Jason had generated during Phase 2 of the experiment, they both explained why the procedure they explicated verbally would appropriately validate or invalidate the existence of a limit.

6.1.4 Phase 4: Refinement of Definition of Limit at a Point with Increased Notational Precision (Sessions 7-9)

Phase 4 of the second experiment marked a transition for Chris and Jason back to refining their written definition of limit. Prior to Phase 3, their characterizations of limit were neither precise nor mathematically valid. In particular, their formulations were from an $x$-first perspective and were function-dependent, in that they were based on the assumption that the functions they were characterizing were increasing. However, their exploration of the step function in Phase 3 appeared to provide necessary support for shifting their reasoning to a $y$-first perspective and formulating a more all-encompassing definition of limit. Much like Phase 2 of the experiment, Phase 4 constituted a period of iterative refinement for Chris and Jason. Sessions 7-9 unfolded with Chris and Jason making notable progress.
towards a coherent written definition, but ultimately coming up against two significant obstacles to completing the reinvention process.

6.1.5 Phase 5: Attempted Resolution of Central Issues and Completion of Reinvention Process (Session 10)

Chris and Jason’s progress towards precisely characterizing limit was hindered during Session 7-9 by their insistence on describing the largest x-interval on which the function f would fall within a pre-specified distance of the limit L, and by their struggle to find a suitable alternative perspective to the potential infinity perspective that had proven problematic for them during earlier sessions. At the outset of Session 10, I initiated a final pedagogical shift with the aim of dealing directly with these two remaining issues. In regards to the first issue described above, there were times during Sessions 8 and 9 when both Chris and Jason had intimated that they may not need to concern themselves with describing the largest x-interval, but merely determine whether or not any x-interval exists that would correspond to a pre-specified y-interval around L. During Session 10, I had Chris and Jason read interview excerpts from Sessions 8 and 9 in which they had suggested such an approach, and subsequently highlighted the value of pursuing such a direction. I addressed the second issue described above in a similar manner. Hence, the fifth and final phase entailed me taking on a more active role in the reinvention process, as I guided Chris and Jason towards resolving the two remaining issues that appeared to be hindering their progress. The resolution of the
first of these two remaining issues led Chris and Jason to arrive at a characterization of *limit* capturing much of the intended meaning of the conventional ε-δ definition. The second experiment concluded with Chris and Jason interpreting both the conventional ε-δ definition of limit, and Amy and Mike’s characterization of limit, in light of their own reinvention, and verbalizing their opinions regarding the mathematical role of the definition of limit.

6.2 – Part 2: Emergent Themes

6.2.0 Introduction

The overview I provide in Part 1 of this chapter describes in broad strokes the evolution of Chris and Jason’s definition of *limit at a point*. The purpose of the second part of this chapter is to describe in greater detail their reinvention of this complex definition. Figure 6.2 displays the key formulations developed by Chris and Jason in the reinvention process.

<table>
<thead>
<tr>
<th>Evolution of Chris and Jason's Definition of Limit</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Definition #1:</strong> y takes on values closer to the limit in question as you take x-values closer to the point at which you’re evaluating the limit. <em>(Session 2)</em></td>
</tr>
<tr>
<td><strong>Definition #2:</strong> When evaluating a limit, y takes on values closer to L, the limit in question as you take x-values closer to the point at which you’re evaluating the limit. <em>The limit need not equal the function’s value at that point.</em> <em>(Session 3)</em></td>
</tr>
<tr>
<td><strong>Definition #3:</strong> For some function y=f(x) a limit L exists at a point x=a when: 1) On some interval [b, a] such that b&lt;a, as x approaches a in the interval, y approaches some value . 2) On some interval [a, c] such that a&lt;c, as x approaches a within that interval, y approaches some value . 3) M=N <em>(Session 4)</em></td>
</tr>
<tr>
<td><strong>Definition #4:</strong> For some function y=f(x) a limit L exists at a point x=a when: 1) On some interval [b, a] such that b&lt;a, as x approaches the point a in the</td>
</tr>
</tbody>
</table>

53 In instances where one definition is similar to the one preceding it, additions or changes have been bolded and italicized so as to alert the reader to the refinements that were made.
interval, \( f(x) \) approaches \( f(a) \). 2) On some interval \([a, c]\) such that \( a < c \), as \( x \) approaches \( a \) within that interval, \( f(x) \) approaches \( f(a) \). (Session 4)

**Definition #5:**
For some function \( y = f(x) \), a limit \( L \) exists at a point \( x = a \) when: 1) On some interval \([b, a)\) when \( f \) is increasing, such that \( b < a < c \), as \( x \) approaches \( a \), \( f(x) \) approaches the max value on \([b, a)\). (Session 5)

**Definition #6:**
\( \text{CEnter} < a < \text{CExit} \). If \( \text{CEnter} = a \) but \( \text{CExit} \neq a \) or \( \text{CExit} = a \) but \( \text{CEnter} \neq a \), we do not have a limit \( L \) at \( a \). If \( \text{CEnter} = \text{CExit} \) then we do have a limit and \( L = \text{CTop} = \text{CBottom} \) (and) \( a = \text{CEnter} = \text{CExit} \). (Session 7)

**Definition #7:**
\( \text{CEnter} < a < \text{CExit} \). If \( \text{CEnter} = a \) but \( \text{CExit} \neq a \) or \( \text{CExit} = a \) but \( \text{CEnter} \neq a \), \( L \) is not the limit at \( a \). Doesn’t necessarily mean there is no limit, just that you guessed wrong. If \( \text{CEnter} = \text{CExit} \) then we do have a limit and \( L = \text{CTop} = \text{CBottom} \) (and) \( a = \text{CEnter} = \text{CExit} \). (Session 7)

**Definition #8:**
1) **Come up with a guess.**
2) **Determine a closeness interval around your guess.**
3) **Let CEnter equal the last x-value [before ‘a’] for which we become close. Let CExit equal the first x-value after ‘a’ for which we are no longer close.**
4) i) if CTop=CBot=L, then L is your limit
   ii) if CEnt=a and CExit≠a or CExit=a and CEnt≠a then L is not the limit
   iii) if CEnt=a<CExit then shrink your closeness interval and retry at Step 2. (Session 8)

**Definition #9:**
1) **Come up with a guess, \( L \).**
2) **Determine a closeness interval \( L \pm z \) around your guess.**
3) **If:** there exists an \( x_1 < a \) such that \( L + z > f(x) > L - z \) is true for all \( x \) between \( x_1 \) and \( a \) AND an \( x_2 > a \) such that \( L + z > f(x) > L - z \) is true for all \( x \) between \( x_2 \) and \( a \) then shrink your closeness interval and try again. If you can’t shrink your interval anymore, then \( L \) is your limit.
   **If not:** then \( L \) is not your limit. (Session 10 – Final Definition)

**Figure 6.2 – Evolution of Chris and Jason’s Definition of Limit**

The reader may recall that the intent of this dissertation study was to engage students in an instructional sequence with two objectives: 1) To develop insight into students’ reasoning in relation to their engagement in instruction designed to support their reinventing the formal definition of limit, and; 2) To inform the
design of principled instruction in relation to limit. The first research objective is focused on student cognition, while the second is focused on instruction. In a manner similar to the first experiment, the nature of the second experiment was such that student cognition and pedagogical decisions were intimately intertwined. The interconnectedness of student reasoning and instruction is also seen in the themes that emerged during the second experiment. My description of the evolution of Chris and Jason’s definition in Part 2 of this chapter is situated among seven themes that emerged during the second experiment. Each of these themes has both cognitive and pedagogical elements – the former characterize Chris and Jason’s reasoning about limit in the context of reinvention, while the latter address instructional findings related to student cognition. Figure 6.3 gives a listing of the seven themes.

<table>
<thead>
<tr>
<th><strong>Emergent Themes</strong></th>
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</thead>
<tbody>
<tr>
<td><strong>Theme 1:</strong> Predominance of an x-First Perspective and the Counterintuitiveness of a y-First Perspective (<a href="#">Sessions 2-7</a>)</td>
</tr>
<tr>
<td><strong>Theme 2:</strong> Potential Infinity as a Hindrance to Characterizing Infinite Closeness (<a href="#">Sessions 3-10</a>)</td>
</tr>
<tr>
<td><strong>Theme 3:</strong> Using a Step Function as a Context Conducive for Initiating Necessary Cognitive Shifts (<a href="#">Sessions 6-7</a>)</td>
</tr>
<tr>
<td><strong>Theme 4:</strong> Reinvention as Motivation for the Need for a Formal Definition (<a href="#">Sessions 7-10</a>)</td>
</tr>
<tr>
<td><strong>Theme 5:</strong> Desire for Precision as a Basis for Function-Dependent Characterizations (<a href="#">Sessions 7-9</a>)</td>
</tr>
<tr>
<td><strong>Theme 6:</strong> Reinvention of the Formal Definition of Limit: Corroborating Evidence of the Potential for Student Success (<a href="#">Sessions 2-10</a>)</td>
</tr>
<tr>
<td><strong>Theme 7:</strong> Reinvention as Support for Coherently Interpreting Other Mathematically Valid Formulations of the Definition (<a href="#">Session 10</a>)</td>
</tr>
</tbody>
</table>

**Figure 6.3 – Emergent Themes**
Evidence of the themes listed in Figure 6.3 emerged and re-emerged throughout the second experiment. My description of themes in this chapter is generally chronological, although some themes are discussed and then later revisited, in a manner that allows me to trace the chronological evolution of Chris and Jason's key formulations of limit. In the pages that follow, I situate the unfolding of Chris and Jason's characterization of limit seen in Figure 6.2 among the seven themes presented in Figure 6.3.

6.2.1 Theme 1: Predominance of an x-First Perspective and the Counterintuitiveness of a y-First Perspective

Larsen’s research (2001) suggests that the intricacies involved in using a y-first perspective are arguably far more complex for students than merely formalizing an x-first process, as Cottrill et al. (1996) conjectured. Evidence from the first experiment in this study corroborates Larsen's claims, and suggests that students are likely to initially employ an x-first perspective when attempting to formulate precise characterizations of limit and may find reasoning from a y-first perspective counterintuitive. Likewise, in the second experiment, Chris and Jason utilized an x-first perspective in their initial characterizations of limit.

Initial x-First Characterizations

Tasks and activities employed during Sessions 2-5 were focused primarily on discussing limits in a graphical setting. At the outset of Session 2, Chris and Jason

54 For a more complete description of what is meant by students utilizing an x-first and/or y-first perspective, please see Chapter 5.
generated prototypical examples and counterexamples of limit, seen in Figure 6.4, which subsequently served as sources of motivation as they made initial attempts at formulating a definition of limit.

![Figure 6.4 - Examples and Counterexamples of Limit](image)

Following their generation of prototypical examples and counterexamples, I asked them how they might characterize what it means for a function to have a limit of 2 at $x=5$.

Craig: So how could you...characterize for that person what it means for a function's limit to be 2?

Jason: ...Uh, what is the line doing?...And hopefully someone would say, well, it looks like it's going toward this point right here....That's the only way it clicked for me, is what is this line doing? What is the ant doing who's climbing the hill?

Chris: What is the height that...the function intends to reach? That's one of those things that sticks.

In sum, then, Chris and Jason's initial metaphors for limit were as follows:

(Chris) The limit is the height the function intends to reach.

(Jason) The limit is the point on the function to which the ants are marching.

These two metaphors were used repeatedly in our introductory Calculus course during the fall of 2006. Hence, it is not surprising that these served as grounding
metaphors for Chris and Jason throughout the second experiment as they attempted to define *limit*.

As Session 2 progressed, Chris and Jason recognized the lack of specificity offered by these two metaphors and began making efforts to define *limit* more precisely. As they did so, their initial descriptions were solely from an *x*-first perspective.

Jason: *y* takes on values closer to the limit in question as you take *x*-values closer to the, closer to whatever that is. Closer to
Chris: The actual limit?
Jason: Yeah, the point at which you’re evaluating the limit.

At the end of Session 2, they provided their first written characterization of *limit*.

**Definition #1:** *y* takes on values closer to the limit in question as you take *x*-values closer to the point at which you’re evaluating the limit. *(Session 2)*

It is important to note that the mention of *y*-values first in Definition #1 is not an indication that Chris and Jason were reasoning from a *y*-first perspective. Indeed, their actions and words during Session 2 were such that Definition #1 could just as easily have been expressed as follows: “As you take *x*-values closer to the point at which you’re evaluating the limit, *y* takes on values closer to the limit in question.”

As they made efforts to refine their characterization during Session 3, this *x*-first perspective was evident, as illustrated by the following excerpt, in which Jason is describing what is fundamental to him in characterizing *limit*.

Jason: Okay. The first one coming to mind, uh, that this is happening — it takes...*x*-values closer. That, that’s, like that’s pivotal, fundamental. That’s the first thing you have to have.
Jason’s focus on the $x$-axis is not surprising, given that his initial introduction to limits during introductory Calculus was focused primarily on finding limit candidates, an action which utilizes an $x$-first perspective.

**Worthwhile Additions Made in the Context of an $x$-First Perspective**

Although the $x$-first perspective Chris and Jason employed during Phase 2 of the experiment was ultimately incongruent with the act of validating limits, there nevertheless emerged notions and ideas about limit that later were important elements of their final $y$-first characterization. For instance, during Session 3, Chris repeatedly commented that a function’s value at the limiting point, $x=a$, need not equal the limit. This important subtlety of the conventional definition was captured in the first significant revision Chris and Jason made to their characterization of limit.

**Definition #2:** *When evaluating a limit, $y$ takes on values closer to $L$ the limit in question as you take $x$-values closer to the point at which you’re evaluating the limit. The limit need not equal the function’s value at that point.* (Session 3)

For the reader’s benefit, I have italicized and bolded the changes Chris and Jason made from their first definition to their second definition. Chris’s insistence that $f(a)$ need not equal $L$ is addressed in the final sentence of Definition #2. One other change is worth mentioning. Chris and Jason felt that their initial characterization (Definition #1) needed to be preempted by a phrase that would describe the action being taken. It is telling, therefore, that they decided to add the phrase “When evaluating a limit” to the beginning of their characterization. This addition lends
credence to the finding that Chris and Jason were employing an $x$-first perspective during Phase 2 of the experiment. Indeed, their extensive experience finding limit candidates (or, evaluating limits) during introductory Calculus appeared to influence the way in which they chose to characterize limit.

A Three Step Strategy

At the end of Session 3, in response to Chris and Jason’s second formulation, I directed their attention to the graph of a jump discontinuity (see Figure 6.5) that they had provided in Session 2 as a prototypical counterexample of limit.

Figure 6.5 – Graph of a Jump Discontinuity

My purpose for doing so was to note for them that their second characterization of limit was invalid, in that it would suggest that a function with a jump discontinuity at $x=a$ has a limit, since as $x$-values approach $a$, $y$-values approach the $L$ indicated in Figure 6.5. Their response seemed to not only indicate that they understood the deficiency in their second characterization, but also that they recognized that their definition needed to be phrased in such a way so as to appropriately exclude the existence of a limit at a jump discontinuity. The following excerpt captures my
acknowledgement that their second characterization was invalid, as well as their awareness that their definition needed refinement.

Craig: Now in reference to this first central thing that you guys are talking about, I think I would argue that those $y$'s are taking on values closer to $L$....

Jason: Yeah, I think I see where you're going to go with that.

Craig: But like, if this were coming [from the right], say

Jason: Totally. Even from the right, it's still getting closer.

Craig: Yeah.

Chris: So we have to add in that, it has to approach the same point from both sides....And yeah, they don't just have to be getting closer from here and here, they got to be getting closer to the same, the same spot. So can we work that in there?

Jason: Like it has to know that it's, where it's heading. Whereas, like right here, it could just be kind of head-, heading in the general correct direction, that's what Craig's pointing out with that picture up there. Like, they're both

Chris: Sure, sure.

Jason: taking on $y$-values and $x$-values are getting closer. And it seems like now we're trying to say okay, in addition to that, it also has to be going to the right point. Umm, let's say, how can I say this?...[T]he next place that the ant is going to travel from the left once it leaves this point is going to be the open circle. And at the same time the ant coming from this direction also has to be trying to get into the same spot.

Chris: So, the ants have to meet at the same spot?

Jason: Yeah....

Chris: Which is what we're trying to say.

Jason: Yeah....It's still really blurry though. The function intends to reach the same value from both sides.

Following this conversation, Chris and Jason worked to capture what it means for a function to intend “to reach the same value from both sides.” As they reflected back on introductory Calculus, they recalled that an oft-used rule of thumb they had employed for establishing the existence of a limit was checking to see whether the left-hand and right-hand limits each individually existed and were equal.
Chris: I'm trying to remember how we did this in class. Did we evaluate the limits separately for each side and then see if they matched?

Jason: And then determine if they're equal? Like see what the left-hand limit is and see what the right-hand limit is and then compare those? Yeah.

Chris: What do you think about that?

Jason: Yeah, 'cause then, umm, okay.

Chris: So that should exclude [the jump discontinuity graph] but keep [the examples of limit] in, right?

Craig: Can you say that again? I'm sorry, I missed that.

Chris: I'm throwing out there that maybe we should evaluate the limit on the left-hand side first and then evaluate the limit on the right-hand side and then see if they match.

Jason: I think what we're trying to do in there is...looking at the left-hand limit, looking at the right-hand limit, and then comparing those and see if we would call them equal.

Over the course of Sessions 4 and 5, Chris and Jason pursued a three-step defining strategy that can be described as follows: 1) Precisely describe what it means to have a left-hand limit; 2) Using the left-hand limit description as a model, alter the language slightly to precisely describe what it means to have a right-hand limit; and, 3) Note that if the left-hand and right-hand limits are equal, then a general limit exists. Their decision to abandon their previous characterizations and pursue this three-step strategy, as well as their description of the strategy, is captured in the following excerpt.

Jason: I'm kind of thinking we could pretty much like forget that's up there. Just start from scratch and come at it a different way.

Chris: I agree.

Jason: Umm, so something to the effect of, get a rough draft here. Part a, umm, as x approaches some value a from the left side, uh, the height it's intending to reach again will equal L. And then part b would be as it approaches.

Chris: Approaches from the right side.

Jason: from the right. Uh, it equals,

Chris: If it equals L.
Jason: Or we could say it equals $K$. And if $L$ equals $K$, then we have a limit.  
Chris: Okay.  
Jason: A general limit.

As Chris and Jason began attempting to formulate this three-step approach, the predominance of their $x$-first perspective remained evident. Indeed, it was clear how difficult it was for them to move away from the $x$-first characterization of limit that had served them so well during introductory Calculus.

Jason: I'm picturing the definition box in a calculus book. It would look like this. Uh, you couldn't just say a limit is, fill in the blank. It would say it exists when all these things are [trails off]. So here, we'll do it. Uh, #1, uh, $x$, let's see here, uh, as $x$

Chris: As $x$ approaches $a$, as $x$

Jason: It's hard to get away from as $x$ approaches $a$, then $y$ is approaching $L$.

Evidence of their desire to first place emphasis on describing behavior along the $x$-axis was abundant during Session 4, as the following excerpt suggests.

Jason: I'm thinking, as $x$ approaches, would it be as $x$ approaches $a$...

Chris: As $x$ approaches $a$ within the interval.

Jason: There you go. Yeah, because we've already specified it can only be coming from one way.

Chris: Within that interval?

Jason: So within some interval $b$ to $a$, such that $b<a$, as $x$ approaches $a$.

Chris: And here we're definitely approaching. What happens when we get there?

Jason: Yeah. Now we've described $x$. Now we've got to describe $y$, right?...As $x$ approaches $a$ within that interval, uh, $y$ approaches $L$?

Working from this $x$-first perspective, Chris and Jason attempted to formulate a precise characterization of when a limit $L$ would exist. As they worked, the counterexample to Definition #2 shown in Figure 6.6 served as motivation.
Figure 6.6 – Graph of a Jump Discontinuity

As Session 4 progressed, Chris and Jason eventually agreed upon a way to capture the existence of a left-hand limit. Using that same language, they described the existence of a right-hand limit and then noted that a limit would exist if the two one-sided limits agreed. In sum, then, the graph of a jump discontinuity, shown in Figure 6.6, motivated Chris and Jason to refine Definition #2. They subsequently employed a three step strategy designed to describe equality between a function’s left-hand and right-hand limit. After much discussion, they arrived at the following characterization of limit.

**Definition #3:** For some function \( y=f(x) \) a limit \( L \) exists at a point \( x=a \) when: 1) On some interval \([b, a]\) such that \( b<a \), as \( x \) approaches \( a \) within that interval, \( y \) approaches some value \( M \). 2) On some interval \([a, c]\) such that \( a<c \), as \( x \) approaches \( a \) within that interval, \( y \) approaches some value \( N \). 3) \( M=N \). (Session 4)

A couple of things are noteworthy about Definition #3. First, both of the first two parts of the definition were cast from an \( x \)-first perspective, in a manner congruent with Definitions #1 and #2. Thus, there was nothing to suggest that Chris and Jason had shifted to using a \( y \)-first perspective at this point in the teaching experiment. Second, despite being markedly more sophisticated in terms of specificity and notation than the two definitions that preceded it, Definition #3 nevertheless was an
insufficient characterization of limit. In particular, Chris and Jason had not yet clarified what was meant by “x approaching a” or “y approaching M.”

Summary

In a manner similar to Amy and Mike during the first teaching experiment, Chris and Jason reasoned from an x-first perspective throughout the second phase of the teaching experiment. Evidence in this section illustrates Chris and Jason’s use of an x-first perspective in formulating their first three definitions of limit. As the second phase of the experiment proceeded, Chris and Jason continued to reason from an x-first perspective; Definitions #4 and #5, to be discussed in Section 6.2.2, are evidence of this finding. In Session 7, following an important cognitive shift that resulted from their engagement with a task in Session 6, they explicated the counterintuitiveness of a y-first perspective. Their thoughts regarding the use of a y-first perspective will be discussed in Section 6.2.4. In the next section of this chapter, however, I will describe how Chris and Jason’s third characterization of limit led to some informative discussions about how they viewed infinite closeness along the x- and y-axes.

6.2.2 Theme 2: Potential Infinity as a Hindrance to Characterizing Infinite Closeness

Closeness

My a priori mathematical analysis of limit (see Section 3.2), as well as evidence from the first experiment, suggests that adequately characterizing infinite closeness on both axes is a fundamental component of reinventing the definition of limit. As
was the case with Amy and Mike, Chris and Jason invested a great deal of energy during Phase 2 attempting to characterize what it means for $x$-values to get \textit{infinitely close} to $a$ and for $y$-values to get \textit{infinitely close} to $L$ along the $x$- and $y$-axes, respectively. In Chapter 5, I detailed how a \textit{potential infinity} perspective hindered Amy and Mike’s efforts to characterize precisely the notion of \textit{infinite closeness}. I also noted that it was not necessarily a shift to an \textit{actual infinity} perspective that resolved Amy and Mike’s concerns regarding the infinite nature of the limiting process, as Williams (2001) suggests, but rather it was utilizing the notion of \textit{arbitrary closeness} to operationalize \textit{infinite closeness} that ultimately supported them in reinventing the formal definition of limit. During the second teaching experiment, characterizing \textit{infinite closeness} was a non-trivial task for Chris and Jason as well, with their attempts seemingly hindered by the use of a \textit{potential infinity} perspective. While there was some evidence of Chris and Jason reasoning from an \textit{actual infinity} perspective towards the latter part of the teaching experiment, they did not ever encapsulate the limiting process by utilizing the notion of \textit{arbitrary closeness}, as Amy and Mike had during the first teaching experiment. Not implementing the notion of \textit{arbitrary closeness} appeared to keep Chris and Jason from reinventing a formal definition of limit which mirrors exactly the intended meaning of the conventional $\varepsilon$-$\delta$ definition. I will elaborate these discussions later in the chapter. In this section, however, I describe how a \textit{potential infinity} perspective led Chris and Jason to use a function-dependent and mathematically invalid approach to defining limit during Sessions 4 and 5.
How Close is Close?

As early as Session 3, Chris and Jason recognized that a characteristic fundamental to the limit concept is the notion of infinite closeness. In their discussions of how best to characterize limit, their attention quickly turned to describing local functional behavior.

Jason: And what we’re trying to do with all this confusing language is describe in words what, what that function does in the vicinity if \( a \).
Chris: As \( x \) gets closer to \( a \),… approaches from both sides. I don’t want to say close, because how close is close?
Jason: Fantastic point.

The excerpt above captures Chris and Jason’s recognition of the challenge involved with describing local functional behavior. Chris’s question, “How close is close?,” became a focal point for them during the second phase of the experiment. As their conversation continued during Session 3, Chris and Jason’s comments suggested that thinking about and addressing infinite processes like the ones inherent to the limit concept is a challenge to which students taking introductory Calculus for the first time are not likely accustomed.

Chris: I have a feeling that’s a big concept for me – it doesn’t really matter what [the limit is] equal to at that point. Because everything you do in math up to that point, it’s just all about what does it equal here, what does it equal there? Do you know what I mean?
Jason: So, you, when you say everything in math, like prior to coming to calculus class.
Chris: Well, yeah, prior. Exactly.
Jason: ‘Cause it’s like every other class up until then we were talking about what stuff was, and now we’re talking about what stuff would be….
It appears possible, then, that the introduction of the limit concept may mark a significant moment in the mathematical career of a student, for embedded in the concept is the notion of describing not what something (a function, in this case) is, but rather what it would be. As the fourth session progressed, Chris and Jason’s efforts to capture the essence of local functional behavior (i.e., what height the function would be) became a focal point of their conversation, as the following excerpt suggests.

Craig: [T]he question I wanted to ask was what do you mean by close?
Chris: I think that’s what we’re trying to iron out.
Craig: Oh.
Chris: How we can describe what the line’s doing just before a, but not necessarily at a....
Jason: Yeah, except that just, for me that’s opening up the can of worms that, that how close is closer?
Chris: Sure.
Jason: You could always get closer.
Craig: So in reference to the how close is close enough or whatever, I, I have a prompt there that says ‘In reference to your discussions do you see the phrases “being close” and “being close enough” as synonymous or different? If they’re different, in what distinctive ways are they different?’
Jason: Well, the easy one is, they’re not the same because, uh, being close you could always get closer and
Chris: Being close enough implies that you’re there.
Jason: Yeah. Yeah, there’s an infinite amount of closers between close and close enough. Wow! That right there is a bumper sticker.

Jason’s claim that there are “an infinite number of closers between close and close enough,” along with his belief that “you could always get closer” anticipated subsequent struggles Chris and Jason had in conceiving of what it would mean for a function $f$ to be infinitely close to a particular $y$-value, $L$. As their conversation
continued, Jason became more emphatic about his suspicions of completing an infinite process.

Jason: I'm raising an objection now to the idea of close enough. I don’t think that there is close enough, because of the idea that there’s always a closer. So if there’s an infinite number of closers between close and close enough, how can close enough even exist?...Umm, ‘cause it would, okay, the idea of close enough means we’re getting ever so much closer to it, and I don’t know if I think there’s ever a close enough to say okay, now you’re there. You’re at the limit. You’re close enough now. I’m not sure about that.

Jason’s comments suggest he was experiencing conflict about defining a construct (limit) which he felt needed to include the notion of being “close enough,” an idea he simultaneously suspected could not exist.

The Emergence of an Elimination Scheme

The reader will recall that Chris and Jason had decided at the outset of Session 4 to take a three-step approach to defining limit, in which they would precisely characterize what it would mean to be a left-hand limit, subsequently alter their language to precisely characterize what it would mean to be a right-hand limit, and then claim that if those two one-sided limits existed and were equal, then a general limit would exist. Their first attempt at articulating this strategy was seen in Definition #3, presented initially in Section 6.2.1, and shown again here for the reader’s benefit:
**Definition #3:** For some function \( y = f(x) \) a limit \( L \) exists at a point \( x = a \) when: 1) On some interval \([b, a]\) such that \( b < a \), as \( x \) approaches \( a \) within that interval, \( y \) approaches some value \( M \). 2) On some interval \([a, c]\) such that \( a < c \), as \( x \) approaches \( a \) within that interval, \( y \) approaches some value \( N \). 3) \( M = N \). (Session 4)

While Chris and Jason's three-step strategy was certainly a plausible one, their characterization lacked the specificity needed to appropriately conclude that the function having a jump discontinuity at \( x = a \) shown in Figure 6.7 would not, in fact, have a limit at \( x = a \).

![Figure 6.7 - Graph of a Jump Discontinuity](image)

This counterexample subsequently spurred them to adopt notation that expressed the idea of eliminating \( y \)-values not equal to \( L \) as limit candidates under consideration.

Craig: Now when you said \( y \) approaches, say, 7.99, what did you mean by “approach”? What do you mean by “approaches”?

Chris: I think we both mean that this line is getting closer and closer to 7.99 and not everything else that’s up here, but it technically is...So we’re trying to get rid of all this stuff up here and make it just talk about that point.

Craig: Okay, so with that in mind, you, you want to be able to describe getting closer and closer to 7.99 and not getting closer and closer to 9.

Chris: 9.

Craig: to 9 or whatever else, okay.
In an attempt to characterize left-hand limit in a manner that would appropriately “get rid of” y-values greater than$^{55}$ the proposed limit $L$, they replaced the phrase “$y$ approaches $L$” in their third definition with “$f(x)$ approaches $f(a)$” to capture the idea that all y-values in the Cartesian plane not associated with the function would be eliminated from consideration as limit candidates. These revisions were captured in their fourth definition.

**Definition #4:** For some function $y=f(x)$ a limit $L$ exists at a point $x=a$ when: 1) On some interval $[b, a]$ such that $b<a$, as $x$ approaches the point $a$ in the interval, $f(x)$ approaches $f(a)$. 2) On some interval $[a, c]$ such that $a<c$, as $x$ approaches $a$ within that interval, $f(x)$ approaches $f(a)$. (Session 4)

Interestingly, prior to this revision, Chris and Jason had repeatedly agreed that a limit need not equal its function value, $f(a)$, yet the desire to eliminate from consideration y-values not equal to the limit appeared to obscure this detail temporarily. As the following excerpt suggests, Chris and Jason felt that by tying the interval on the y-axis to the function $f$, they would eliminate all y-values not on the function, thus significantly reducing the number of y-values under consideration for the limit.

Craig: I noticed that not only did you replace some value $M$ with $f(a)$ but you also...replaced $y$ with $f(x)$. Was there a reason for that?

Jason: Yeah, the reason for that was $y$ approaching could be, $y$ could be doing anything but if it’s $f(x)$ that we’re talking about, that bars it onto this rule. Trying to disallow the whole universe of all possible things it could be doing.

Chris: That limits us to what the function’s actually doing.

Jason: $y$ is only allowed to do what $f(x)$ tells it to do.

$^{55}$ Greater than the proposed limit $L$, here, since the prototypical examples and counterexamples had all been drawn such that the respective functions were increasing.
After further consideration, however, Chris and Jason recalled that in the case of a removable discontinuity, a limit need not equal its function value. Jason’s comments at the end of Session 4 suggest he was aware of the shortcomings of a definition reliant on use of the phrase “f(a).”

Jason: I was really pleased with using f(x) approaching f(a), just for the idea that now we’re saying, well it’s not just doing whatever it wants, it’s specifically obeying the rule, y=f(x). But by including it in that term, couching it in that term, or whatever, if you will, then that right there [the graph seen in Figure 6.8] just becomes so problematic, ... because this, the f(a), that does obey f(x), it’s in the rules because that’s how it’s defined.

![Figure 6.8 - Graph of a Removable Discontinuity](image)

Their focus subsequently shifted to characterizing limit in such a way so as to avoid using f(a) in their definition, yet eliminate all y-values from consideration not equal to the proposed limit L. At the outset of Session 5, Jason summarized the challenges he and Chris were facing in defining limit.

Jason: I’m keenly aware that until we figure out a way to specify that hole that exists at a as either M or L or whatever we decide to call it, until we can do that, then we’re still including an infinite number of possibilities.
Chris: Umm-hmm.
Jason: Which is the problem. So it’s not even like we can, we can’t even weed out some of the possibilities. We have to weed out all of the possibilities except for it approaching L.
Evident in Jason’s comments were his and Chris’s ongoing concerns of how they might define limit in such a way so as to describe the function getting infinitely close to a single y-value.

**Emergence of a Maximum Value Approach**

As Session 5 unfolded, Chris and Jason devised a “maximum value” strategy for attending to the concerns Jason raised in the preceding excerpt. The maximum value strategy evolved as follows. Chris and Jason defined limit in a case-dependent manner, focusing their attention first on the increasing function shown in Figure 6.9.

![Figure 6.9 - Graph of an Increasing Function with a Removable Discontinuity](image)

**Figure 6.9 – Graph of an Increasing Function with a Removable Discontinuity**

With the intention of first defining left-hand limit (in a manner consistent with their previously described three-step strategy), Chris and Jason decided to imagine taking a point $b < a$ on the x-axis. They then agreed that this x-interval, $[b, a)$, would, in turn, produce a corresponding interval of function values along the y-axis, $[f(b), f(a))$ (demarcated in Figure 6.9). Their rationale for creating this interval on the y-

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56 In the midst of formulating this strategy, Chris and Jason recognized that they did not want to include $f(a)$ in the interval of y-values under consideration for the limit, given that $f(a)$ would not equal the limit in the removable discontinuity graph in Figure 6.9. To account for this exclusion, they denoted the x-interval $[b, a)$ and the corresponding y-interval $[f(b), f(a))$. 

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axis was that this allowed them to bound the y-axis, so as to restrict, or “weed out”
the possibilities for the limit\textsuperscript{57}.

Chris: When we say some value $M$ between $f(b)$ and $f(a)$, does that only
include those values to you?

Jason: Yeah.

Chris: So wouldn’t that get rid of everything above it?...

Craig: You’re saying you don’t have to worry about the stuff above it now?
Is that?

Chris: Maybe.

Jason: Right, we don’t have to worry about this infinite or this infinite.
(bitpointing to y-values below $f(b)$ and above $f(a)$ respectively]

This approach appeared promising, for I anticipated that they would continue to
eliminate possibilities for the limit in a manner similar to the technique Amy and
Mike employed during the first experiment, wherein they iteratively restricted their
attention on the y-axis about the proposed limit $L$. Unfortunately, however, Chris
and Jason’s potential infinity perspective continued to serve as a hindrance –
instead of continuing to restrict their attention on the y-axis, they appeared
paralyzed by the awareness of the never-ending nature of incrementally eliminating
y-values that remained in the interval $(f(b), f(a))$.

Jason: Yeah, except, you know, same kind of problem though. You got
infinity there and there. So we barred those. That’s great. That gets,
that eliminates some of, that’s what I was saying. We can eliminate,
we can take steps to eliminate this and this, but we still got all those
right there, right?

Indeed, they were reluctant to iterate the process of choosing smaller intervals on
the y-axis in the hopes of excluding more y-values from consideration, because they
felt that this process would be never-ending – there would always be an infinite

\textsuperscript{57} Throughout Session 5, Chris and Jason used the letters $L$ and $M$ interchangeably to represent the
limit.
number of possible $M$ values left. Instead of pursuing the iteration of an infinite process, they instead decided to describe the selection of the single value $M$ (i.e., the limit) from their original $y$-interval $[f(b), f(a)]$ by defining $M$ to be the maximum value of the function on the interval $[b, a)$. The following excerpt captures the essence of their "maximum value" strategy, and illustrates the potential infinity perspective that motivated such an approach.

Craig: You had bounded your $y$ stuff because you have all this stuff up here that you wanted to get rid of.
Jason: Umm-hmm.
Craig: So you had bounded it and you said...I'd gotten rid of an infinite number of stuff up here, and an infinite amount of stuff down here. That's pretty good. I still got, though, an infinite amount of things that it could be, right? It could be anything now between here and here [referring to the bounds on the $y$-axis].
Chris: That's okay because we've got it on an interval.
Craig: ....So what would you do with that interval then? Because someone could say well okay, now your value $M$ or whatever you're calling it, $L$, could be anywhere [between $f(b)$ and $f(a)$].
Jason: Umm-hmm.
Craig: Right? Now, well it's good that it's not up here anymore, but
Chris: That's where we're going to try and get a specific point with our maximum or minimum.
Jason: We're gonna say, you know, the maximum point on, in that case
Craig: Oh, I see...And that will be your $L$.
Jason: Umm-hmm. Right....
Craig: So you bound it, then you say, let's look at all the function values in that interval.
Jason: Umm-hmm.
Craig: Whichever one is the biggest one, that's your limit.
Jason: Yeah. And then that does an even better job then just getting rid of all that and all that. As...opposed to trying to eliminate the infinite, now we're specifying the one case that will solve it. See the idea of trying to eliminate the, you know, let's get rid of that, let's get rid of that. We can keep getting rid of stuff, but as long as we're getting rid of something from an infinite pool, you know, what's infinity minus one, you know? We're never gonna actually get to the, we're never going to get a finite value, or a finite concept, just trying to

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eliminate all the infinite concepts. So taking this approach is gonna let us say well, let’s not eliminate stuff. Let’s just state what the condition would be.

This maximum value strategy led to a case-dependent approach to defining limit that resulted in the following characterization.

**Definition #5:** For some function \( y=f(x) \), a limit \( L \) exists at a point \( x=a \) when: 1) On some interval \( (b, a) \) when \( f \) is increasing, such that \( b<a<c \), as \( x \) approaches \( a \), \( f(x) \) approaches the max value on \( (b, a) \). (Session 5)

Definition #5 has obvious limitations. First, the nature of Definition #5 anticipated what would have been a case-by-case approach to characterizing limit. Evidence suggests that Chris and Jason were aware that Definition #5 would not serve as a definition that would generalize to all functions.

Chris: So we’ll call this case #1....
Jason: What I’m thinking now is how do we make that one sentence work for all the various cases?

Second, in the case of an increasing function with a removable discontinuity (see Figure 6.9), there would be no explicitly defined maximum value on the interval \( [b, a) \). As Session 5 concluded, Chris and Jason appeared to recognize this limitation as well.

Craig: [T]he limit is the hole, right? That’s what you’re saying?
Jason: Yeah.
Craig: Is that the largest value the function takes on in that interval?
Jason: Oh.
Chris: Well, there’s no value there.
Jason: You’re right, this language is now saying that the limit isn’t right in there. But just to the left of it, and that’s not the case.
Craig: And a scrinch lower.
Jason: Yeah, a scrinch lower, right.
Craig: That, do you see what my question is there?
Jason: I totally see it, yeah.
Craig: So I like the, I like the approach you guys have. 'Cause now you've bounded it, and you're saying pick the largest one. But now my question is, what is the largest one?
Jason: You've got to take the largest one and then move one over. Yeah, I know that doesn't make it.

What is significant here is not that Chris and Jason characterized limit in an imprecise, mathematically invalid manner. In fact, evidence from the first teaching experiment suggests that imprecise, mathematically invalid characterizations appear to be expected phenomena (in the sense of Ferrini-Mundy & Graham, 1994) that are a fundamental component to the reinvention process. Indeed, what is significant about Chris and Jason's actions during Session 5 is that their utilization of a potential infinity perspective kept them from entertaining the notion of an iterative process by which they might restrict the size of the interval on the y-axis around the limit, and thus, led them to pursue an approach they believed would allow them to avoid completely the dilemma of precisely characterizing infinite closeness.

Summary

During Phase 2 of the experiment, Chris and Jason's utilization of a potential infinity perspective hindered them from precisely characterizing infinite closeness, a construct fundamental to the limit concept. Further, a potential infinity perspective motivated the development of a case-dependent defining strategy which they ultimately recognized as limited. It is also worth mentioning that the two
characterizations shared in this section are further evidence of the predominance of an *x*-first perspective in Chris and Jason’s reasoning.

**Definition #4:** For some function $y = f(x)$ a limit $L$ exists at a point $x = a$ when:

1) On some interval $[b, a]$ such that $b < a$, as $x$ approaches the point $a$ in the interval, $f(x)$ approaches $f(a)$.

2) On some interval $[a, c]$ such that $a < c$, as $x$ approaches $a$ within that interval, $f(x)$ approaches $f(a)$. (Session 4)

**Definition #5:** For some function $y = f(x)$, a limit $L$ exists at a point $x = a$ when:

1) On some interval $[b, a)$ when $f$ is increasing, such that $b < a < c$, as $x$ approaches $a$, $f(x)$ approaches the max value on $[b, a)$. (Session 5)

Both Definition #4 and Definition #5 place initial emphasis on the $x$-axis, in a manner consistent with the act of finding limits.

In Sections 6.2.1 and 6.2.2 I outlined Chris and Jason’s predominant use during the first five sessions of the experiment of both an *x*-first and potential infinity perspective. In Section 6.2.3, I will discuss my implementation during the sixth session of an instructional task that was designed to initiate a cognitive shift away from the problematic *x*-first and potential infinity perspectives Chris and Jason utilized during the first five sessions of the teaching experiment.

6.2.3 Theme 3: Using a Step Function as a Context Conducive for Initiating Necessary Cognitive Shifts

In the preceding two sections of this chapter, I described how Chris and Jason’s *x*-first and potential infinity perspectives hindered their progress in reinventing the definition of *limit*. Generally speaking, Chris and Jason appeared to be experiencing the same cognitive difficulties after Session 5 that Amy and Mike had
experienced through the first six sessions of the first experiment. In Chapter 5, I explicited how defining closeness in the context of limit at infinity proved to be a watershed moment in Amy and Mike’s reinvention of limit at a point. Indeed, doing so provided them a lens through which to characterize precisely what it means for a function \( f \) to get infinitely close to a proposed limit \( L \). After Session 5 in the second experiment, I felt an activity designed to elicit from Chris and Jason a definition of closeness was similarly needed. In contrast to the first experiment, however, I was interested to learn whether Chris and Jason could successfully reinvent the definition of limit at a point without first defining limit at infinity. As I mentioned at the outset of this chapter, I viewed the second experiment as an opportunity to explore whether reinvention was possible using an altered instructional trajectory. Defining limit at infinity had clearly been an important element in Amy and Mike’s reinvention process. However, I also felt that defining what it means to be close had provided Amy and Mike a vehicle for operationalizing what it means to be infinitely close, a notion fundamental to the limit concept. Thus, I conjectured that Chris and Jason might not need to first define limit at infinity, but might instead realize the same cognitive shifts Amy and Mike experienced by characterizing what it means to be close in the context of limit at a point. This conjecture influenced my decision in Session 6 to implement an instructional task designed to initiate in Chris and Jason cognitive shifts away from the \( x \)-first and potential infinity perspectives that had hindered their progress towards a coherent formal definition of limit.
Defining Closeness

As Session 6 began, Chris and Jason reflected on the limitations of their maximum value strategy from Session 5 and agreed that an unresolved issue was how they might characterize what it means for a function \( f \) to be infinitely close to a particular \( y \)-value, \( L \). Evidence suggests they were still reasoning from a potential infinity perspective.

Chris: [I]f we used the getting closer logic..., I don’t think we’ll ever get there because you can never get there....
Craig: Okay, but in essence we’re saying...for the limit to be \( L \), those function values have to be getting infinitesimally close to \( L \), whatever that means.
Jason: Right....
Chris: You’re not there but you’re as close as you’re gonna get....
Jason: [E]ven if it’s infinitesimally close,
Craig: Okay.
Jason: infinitesimally close, you’re still not there. So I think the extension of that is there’s no limits. ‘Cause you can’t ever get there.
Craig: Is that what you’re saying? If we’re talking about close, we’re never going to be close enough?
Chris: Yeah, basically.

In response to their ongoing concerns regarding the possibility of a function ever being “close enough” to a proposed limit, I began shifting their attention to defining close, as the following excerpt illustrates.

Craig: We keep coming back to what was your original question, how close is close enough? And all that stuff. And I wonder if, if we don’t worry about infinitesimally close for a second.
Jason: Yeah.
Craig: Before we’re even going to have a chance to talk about infinitesimally close, which we all have a picture of but can’t quite pin down.
Chris: Umm-hmm.
Craig: I think we should try, let’s back off and try to just describe what close would mean. Which feels a little weird because close is, that depends on what your perspective is, right?

Jason: Right...Your resolution.

Craig: So close for me, let’s say close means 10 units. So under my definition of close,...how would you write out what it means for a function $f(x)$, any function $f(x)$, to be “close” under my definition to a particular pre-determined value $L$ for every $x$ on the function? So I got some function, for every single one of its $x$-values, how would you write out what it means for that function to be close to a pre-determined value $L$?

In response to my encouragement to define close, Chris and Jason drew the function shown in Figure 6.10, and introduced the idea of bounds on the $y$-axis, both above and below $L$.

Jason: So for all $x$'s we want it to be close to $L$.

Craig: ....And my definition of close is within, say, 10 units.

Chris: 10 units.

Jason: Okay, okay. Here we go. We’ll say, uh, let’s say, uh, this $L$, $L=10$. Then we’ll have 15, and 5. There you go. In both directions, there you go. That is, for all $x$-values, always close to $L$. ‘Cause it’s al-, well actually this could be, this would actually be 0 and 20. It’s always going to be within 10 units.

Chris: So you’re just

Jason: Yeah, I’m just bounding it.

Chris: So everything in here [pointing to the $y$-interval $[0, 20]$ shown in Figure 6.10].

![Figure 6.10 – Illustration of Bounded Function](image-url)
This was a significant moment for Chris and Jason in the reinvention process, for it was the first time they had employed symmetric bounds on the y-axis about a particular y-value. Out of this discussion came their initial definition of *close*.

Chris: [A] function \( f(x) \) is close to \( L \) if and only if \( f(x) \) is within 10 units of \( L \).

The reader may recall that the prompt for this *closeness* task had been for Chris and Jason to define what it would mean for a function \( f \) to be close to a particular y-value, \( L \), for *all* values of \( x \). Unprompted, Jason recognized that their definition of close would need to be revised if they were to apply it to the limit concept – he reasoned that in the case of limit, they care only about closeness on an interval around the limiting point, \( x=a \).

Jason: With limit we're talking about a very small portion of the domain and this close idea sounds like we're trying to say, let's be close for the whole function. And I don't think that's a requirement.

Jason’s comments suggest he was trying to make a connection between the task with which they were currently engaged, and the original task – defining *limit*. Jason’s observation anticipated my planned pedagogical trajectory.

**The Step Function Task**

With the aim of shifting Chris and Jason’s attention back to defining limit, I directed their attention to the step function shown in Figure 6.11 and asked them what the limit would be at the removable discontinuity located at the coordinate-pair (3.5, 3).
They agreed that the limit would be 3. I next asked them when the function $f$ would be within 2.5 units of the limit.

Craig: [W]hen would you, under [the new] definition, be close to your limit? Can you show me on the graph, and maybe describe?

Chris: Whenever we're within two and a half.

Jason: Yeah, you can add two and a half to this and subtract it.

Chris: So anywhere between ...., .5 and five and a half?

Jason: Yeah.

In response to this more restrictive definition of close (2.5 as opposed to 10), Jason utilized the same approach he used in the previous task – he added and subtracted the specified error tolerance to the limit to form a $y$-interval around the proposed limit. Chris and Jason subsequently represented this $y$-interval with closed-interval notation, although they did so vertically, as seen in Figure 6.12, so as to denote that the interval was a $y$-interval, as opposed to an $x$-interval.

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58 My motivation for using a non-integer value (2.5) was that I did not want Chris and Jason to have to attend to whether being exactly within the specified distance of the limit would constitute being close. The step-function in Figure 6.11 is such that non-integer definitions of close subsequently make it easy to determine which values of $x$ result in corresponding function values within the specified distance of $L$. 
This was significant for it was the first time they had used notation of any kind to denote bounds on the y-axis. I next asked them which x-values would result in corresponding function values close to the limit. They agreed that x-values in the interval [1,6) would satisfy the specified definition of close. As the conversation continued, Chris and Jason began to coordinate x- and y-intervals as they described the set of x-values for which the function would be close under iteratively more restrictive definitions of close.

Craig: Okay. Now, I think we would agree that being within 2.5 of your limit, that’s not enough to mean that you have a limit....Let’s say that we wanted to be within, uh, 1.5,...or .5.

Jason: Then that shrinks [the y-interval] and then shrinks [the x-interval].

Chris: So as [the y-interval] gets smaller you’re including less stuff.

Craig: Including less stuff, what do you mean by that?

Chris: Well like, when the range was 2 ½, we went, we included 1, 2, 3, 4, 5, anything that gave a value [between 1 and 5]. When it was 1 ½, it would move up and we would not include anything that equates to 1 or 5 anymore....

Craig: Okay. And you said you included less stuff. Less y-values or less x-values?

Chris: Less x-values....[W]e’re shrinking by picking smaller definitions of close.

It is worth noting that defining close in an incrementally restrictive fashion (i.e., 10, 2.5, 1.5, .5, etc.) led Chris and Jason to focus their attention first on the y-axis,

59 Strictly speaking, it is perhaps incoherent to define closeness by assigning a particular number (e.g., 10) to the concept. More accurately, the numerical values (10, 2.5, 1.5, and .5) discussed in...
in contrast to the x-first perspective they had used in previous sessions to talk about limits.

Following their inspection of the step function, I encouraged them to consider the function in Figure 6.13, and construct a definition that would clearly state the conditions under which a limit would exist.

![Figure 6.13 - Graph of a Continuous Function](image)

As they began discussing how to refine their definition, it appeared that the two tasks they had engaged with during Session 6 had helped shift their focus to imagining a limiting process reliant on bounding the function within a predetermined proximity to the limit \( L \) on the y-axis.

Craig: So maybe to end today, let's, let's have you guys just take a shot at a definition.
Jason: Okay.
Craig: You know, like a function \( f(x) \) has a limit \( L \) at \( a \) under certain conditions.
Jason: Alright.
Chris: It means we're close.
Jason: I think where this approach was taking us to about shrinking the, our vertical interval or whatever, is that uh, [the function is] going to have to be within a predetermined amount....

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this task served as error tolerances, or conditions of proximity, that needed to be met for the function to be considered close to a particular y-value. However, in an effort to remain consistent with the language used by both me and the students during the teaching experiment, I use the phrase "defining close" to describe the establishment of criteria for closeness.
Thus, defining close, and subsequently discussing closeness in the context of the step function task, appeared to initiate important cognitive shifts for Chris and Jason. Although they were ultimately unable to precisely notate infinite closeness during Session 6, these tasks nevertheless induced a noticeable shift among Chris and Jason to a y-first perspective, as was evident in their subsequent characterizations of limit. There are a few possible explanations for the tasks inducing such a shift. First, both tasks were designed to focus the students’ attention on the y-axis. Specifically, the students were given a specific error tolerance along the y-axis and were asked how they might characterize what it means for a function to be within that error tolerance of a pre-determined y-value, \( L \). Further, in the first task, the students were asked to assume that the function need be “close” to \( L \) for all values of \( x \). Indeed, the first task was purposely designed to deemphasize the x-axis – instead of being asked what was true about the function for particular \( x \)-values (or particular \( x \)-intervals), the students were asked to characterize functional behavior uniformly across the \( x \)-axis. Second, both tasks were designed so that issues that had previously served as sticking points would be temporarily obscured. For instance, prior to Session 6, Chris and Jason had focused much of their attention on characterizing infinite closeness. However, their efforts to characterize infinite closeness had been hindered largely by their utilization of a potential infinity perspective. Resolving their concerns about whether an infinite process could ever terminate subsequently monopolized much of their attention. However, defining closeness, instead of infinite closeness,
allowed the students to temporarily set aside their concerns related to the concept of infinity, which appeared to free them to consider the benefits of reasoning about limits from a y-first perspective. A third possible explanation for the tasks inducing a shift from an x-first to a y-first perspective is that Chris and Jason were aware that their prior characterizations of limit were deficient. At the end of Session 5, they had both acknowledged that Definition #5 was an inadequate definition. Thus, at the outset of Session 6, both students were likely agreeable to alternative approaches to defining limit. Further, as they began employing a y-first perspective, Chris and Jason may have recognized that such a perspective allowed them to address issues that their x-first characterizations had been unable to resolve.

Towards the end of Session 6, Chris and Jason demonstrated verbal understanding of a potentially infinite process by which one could determine whether \( L \) is the limit of a function \( f \) at \( x = a \).

Chris: Hypothetically you could shrink your closeness interval... around the limit to where all you're left with is the limit.

Craig: ...And you're saying if you can keep doing that, and each, at each point at which you're doing that you are able to do what along the x-axis? You kept talking about these intervals. But what specifically are you saying about that interval around \( a \)?...

Chris: I think this was just as, as long as your interval encompasses \( a \),

Jason: As long as \( a \) is in the interval.

Chris: you're good. Keep going.

Craig: And as long as \( a \) is in the interval, and what's true about the interval? I mean the interval is just like randomly picked? Or,

Chris: No, and that interval's the interval that you're close.

Jason: You have a closeness interval that corresponds to an x-value interval that includes \( a \).
As Session 6 concluded, Chris and Jason made initial attempts at constructing a written articulation of the verbal reasoning they expressed in the preceding excerpt.

Summary

Prior to Session 6, Chris and Jason displayed evidence of reasoning from an x-first perspective and repeatedly voiced concerns regarding whether an infinite process could ever terminate. The latter of these two issues made them reluctant to pursue a defining strategy that included an iterative process of shrinking intervals about a proposed limit along the y-axis. This reluctance motivated them to employ the function-dependent maximum-value strategy to characterizing limit that I described in Section 6.2.2. This approach culminated in their fifth characterization of limit, constructed during Session 5.

Definition #5: For some function \( y = f(x) \), a limit \( L \) exists at a point \( x = a \) when: 1) On some interval \([b, a)\) when \( f \) is increasing, such that \( b < a < c \), as \( x \) approaches \( a \), \( f(x) \) approaches the max value on \([b, a)\). (Session 5)

In response to Chris and Jason’s inclination to reason from an x-first and potential infinity perspective, I implemented two instructional tasks during Session 6 designed to initiate in Chris and Jason cognitive shifts away the perspectives that had hindered their progress. The tasks in Session 6 did not appear to elicit in Chris and Jason an immediate shift away from a potential infinity perspective. Although many of their subsequent written characterizations intimated that Chris and Jason believed that the end of the limiting process were possible, there were numerous verbal exchanges during Sessions 7-9 between Chris and Jason which suggested
that they were still reasoning predominantly from a potential infinity perspective. Neither student consistently expressed reasoning reminiscent of an actual infinity perspective until Session 10. More importantly, unlike Amy and Mike, neither Chris nor Jason ever utilized the notion of *arbitrary closeness* to operationalize *infinite closeness*. In Section 6.2.9, I will elaborate Chris and Jason’s skepticism of the notion of *arbitrary closeness* in the context of limit.

The two central tasks in Session 6 did appear to initiate in Chris and Jason a shift from an *x*-first perspective to a *y*-first perspective. Indeed, following these two tasks, evidence emerged which suggests that Chris and Jason benefited from first defining *closeness* outside of the context of limit, and then incorporating that definition of *closeness* as they attempted to characterize local functional behavior for a step function. In particular, their subsequent verbal descriptions of the limiting process focused first on constructing what they referred to as a “closeness interval” around *L* on the *y*-axis, and only then on a corresponding “nearness interval” along the *x*-axis. For the duration of the experiment, Chris and Jason’s subsequent written characterizations of limit were from a *y*-first perspective, suggesting that Session 6 marked a profoundly transitional phase in the reinvention process. A few possible explanations for the tasks inducing such a shift are: 1) The tasks were designed to focus the students’ attention on the *y*-axis; 2) Both tasks were designed to set aside issues that had partially obstructed the students from previously considering the benefits of reasoning from a *y*-first perspective; and, 3) The students viewed their existing characterization of limit as deficient and thus were open to alternative
approaches that would support productive refinements of their definition. These refinements of their definition, as well as further evidence illustrating this shift to a y-first perspective, will be shared in subsequent sections of this chapter as I continue to describe emergent themes from the second experiment.

Chris and Jason's shift from an x-first to a y-first perspective did not come without some skepticism, however. In Session 7, Jason repeatedly noted the counterintuitive nature of reasoning about limits from a y-first perspective. In the following section, I will revisit Theme 1 (Predominance of an x-First Perspective and the Counterintuitiveness of a y-First Perspective) as I summarize Jason's comments.

6.2.4 Theme 1 Revisited: Predominance of an x-First Perspective and the Counterintuitiveness of a y-First Perspective

In Section 6.2.1, I described the predominance of an x-first perspective in the reasoning Chris and Jason initially employed as they attempted to precisely characterize what it means for a limit to exist. Indeed, their first five characterizations of limit (see Figure 6.3) focused attention first on the x-axis, in a manner consistent with the action of finding limit candidates. In Section 6.2.3, I described two instructional tasks implemented during Session 6 which were designed, in part, to elicit in Chris and Jason a shift to a y-first perspective. Although their verbal characterizations of limit towards the end of Session 6 appeared to be stated from a y-first perspective, comments Jason made during
Session 7 suggests that students may find it counterintuitive to reason in such a fashion.

"You don’t...pick a y-value and see what corresponding x-value comes from it"

At the outset of Session 7, I asked Chris and Jason to write out individually an articulation of the conditions they had stated verbally during Session 6 that would need to be met to ensure a limit exists. I also encouraged them to note any ongoing, unresolved concerns they each had about their current characterization of limit. The purpose of this task was for me to learn what aspects of their characterization were most salient for them, as well as what cognitive difficulties they were still experiencing.

As Jason subsequently reflected on the y-first approach he and Chris had verbalized at the end of Session 6, he expressed surprise that they had focused their attention first on the y-axis.

Jason: I remember feeling like, okay well why are they letting the vertical axis kind of run the show? I think Chris was talking about as we shrink [the y-interval]..., then [the x-interval] is at the same time shrinking...Well, and this is why I was kind of confused....‘cause you’re talking about what’s the limit at a, so we were setting it up to say, okay in this scenario, what’s the limit at a? Well, rather than just going to a and looking at it, what we do, what it appeared that we were doing was say, well let’s start up here on the y-axis and make a, you know, a beginning interval. And then see what the corresponding interval is [marking an interval on the x-axis].

As the conversation continued, Jason became more explicit about why he found it counterintuitive to form an interval along the y-axis first when establishing the existence of a limit.
Craig: You said you found it strange that the y-axis was dictating. What did you mean by that?

Jason: Um, at, at a point in your discussion I was kind of looking back and just listening to Chris state what he was saying and...the conversation was framed in, well as we shrink [the y-interval], ...that seemed to be kind of what was like driving the whole thought process. And I remember kind of being a little concerned, thinking like why, why are we, why would you do that? Why, letting this [pointing to the y-axis] be your independent variable, if you will....

Craig: ...[A]re you saying that's odd to you to have the y-axis be the independent variable and if so why is that odd to you?

Jason: Uh, just because of all the years of math training where you put your independent variable, you're taking, uh, you're taking an assumption with an independent variable. We're going to start, we're going to arbitrarily pick a value there and then see what logic or behavior follows as a result of it. And on an x-y graph you're saying, well what's going on here? You've got a curve, you say, oh, well this is what's going on. What you don't do is go, uh, pick a y-value and see what corresponding x-value comes from it. That's the, you know, just the, the language of independent and dependent variables.

Jason’s comments lend credence to Larsen’s (2001) conjectures that validating limits requires a type of reasoning distinct from the reasoning students employ when finding limits. Further, Jason’s sentiments, which corroborate viewpoints expressed by Amy and Mike during the first experiment, suggest that students’ previous mathematical experiences with functions, and specifically, with independent and dependent variables, do not appear to include sufficient focus on inverse processes. Carlson, Oehrtman, and Thompson (2007) note that understanding the definition of limit relies on such facility: Yet, given students’ extensive experience and success applying x-first reasoning in functional contexts prior to Calculus, y-first reasoning may not only feel counterintuitive, but mathematically invalid.
Jason: Yeah, and just to be clear about it, I'm not saying that starting, you know, let this be your independent and then see what happens along the x-axis, I'm not saying there's a problem with doing that. Just that it, it is foreign, uncomfortable, ....and I'm not sure if it's going to be in the realm of mathematical allowance.

Despite Jason's initial skepticism, he increasingly came to support their subsequent y-first characterizations of limit as the teaching experiment proceeded. One reason for this growing acceptance may have been the recognition that their y-first descriptions correctly characterized examples and counterexamples of limits, whereas their x-first descriptions had failed to do so. In Section 6.2.5, I will discuss the first of these y-first descriptions as I describe the notational scheme that Chris introduced during the seventh session to articulate his and Jason's y-first characterization of limit.

6.2.5 The Emergence of a proto ε-δ Coordination Scheme

The purpose of this section is to describe the proto ε-δ coordination scheme Chris constructed during Session 7 that subsequently served as a foundation for the characterizations of limit he and Jason formulated during the remainder of the experiment. In response to being asked at the beginning of Session 7 to individually compose a written articulation of the verbal characterization of limit they had provided at the end of Session 6, Chris spontaneously introduced four symbols (CTop, CBottom, CEnter, and CExit) meant to denote the endpoints of the y- and x-intervals foundational to the conventional ε-δ illustration shown in Figure 6.14 (Stewart, 2001).
As Chris introduced these four words into their lexicon, he drew the graph shown in Figure 6.15 and defined each of the terms.

**Figure 6.14 - \( \varepsilon-\delta \) Illustration of Limit**

Chris: [T]his [pointing to the y-interval around \( L \)] is our closeness threshold or range that we were talking about last time.

Craig: Okay.

Chris: So I called this CTop and this CBOTTOM.

Craig: The \( C \) standing for closeness?

Chris: Yeah. Alright, so our, our whole idea was we would shrink our closeness threshold and eventually come up to a point...

Jason: Right, within 10 units, within 6 units,

Chris: Yeah.

Jason: within 3 units. Okay.

Chris: That’s what we were doing last time....So here I called this point CENTER....And then I called this CEEXIT....So this [referring to CENTER] is the point at which we were [not] close, and now we become close. Right here. And this [referring to CEEXIT] is the point where we are close and now we’re no longer close. And now we had said as long as \( a \) is in between CENTER and CEEXIT, so as long as...there’s some interval where...\( a \) is between the two, does that make sense?
Chris and Jason's comments in the preceding excerpt point to their understanding of the limiting process as one that is iterative in nature, in which closeness intervals (i.e., intervals about $L$ on the $y$-axis) are iteratively shrunk until all that remains is a single point, $L$. It is worth noting that the manner in which Chris drew the bounds seen in Figure 6.15, as well as the order in which he articulated the four constructs, places emphasis first on the selection of an interval along the $y$-axis. For the duration of the experiment, Chris and Jason both described limits in this fashion, with the selection of a $y$-interval about $L$ subsequently dictating the formation of an $x$-interval about $a$. This $y$-first approach was a noticeable departure from the $x$-first reasoning they had utilized in Phases 1 and 2, implying that the tasks with which they engaged in Session 6 may have initiated a lasting shift to a $y$-first perspective.

As Session 7 progressed, Chris and Jason made efforts to distinguish between characteristics particular to the graph shown in Figure 6.15, where a limit does exist, and the jump discontinuity graph shown in Figure 6.16, where a limit does not exist.

\[\text{Figure 6.16 – Graph of a Jump Discontinuity}\]

Based on Chris's definition of CEnter and CExit, he and Jason concluded that a jump discontinuity would result in either CEnter or CExit, but not both, equaling 259
the limiting point, \( a \). They subsequently used this criterion as the basis for a limit failing to exist. Further, they concluded that a limit would exist if the process of shrinking the \( y \)-interval about \( L \) resulted in \( C_{\text{Top}} \) equaling \( C_{\text{Bottom}} \).

Craig: \( L \) would be your limit provided you could always do what?...
Chris: \( L \) would be your limit provided you could always...
Jason: Yeah, umm, that you can go through the process of shrinking the \( y \)-interval and
Chris: End up with \( C_{\text{Top}} \) which is equal to \( C_{\text{Bottom}} \) which is equal at the limit.

This discussion culminated with the following formulation for the definition of limit.

**Definition #6:** \( C_{\text{Enter}} < a < C_{\text{Exit}} \). If \( C_{\text{Enter}} = a \) but \( C_{\text{Exit}} \neq a \), or \( C_{\text{Exit}} = a \) but \( C_{\text{Enter}} \neq a \), we do not have a limit \( L \) at \( a \). If \( C_{\text{Enter}} = C_{\text{Exit}} \) then we do have a limit and \( L = C_{\text{Top}} = C_{\text{Bottom}} \) [and] \( a = C_{\text{Enter}} = C_{\text{Exit}} \). (*Session 7*)

The first statement in Definition #6 designates \( a \) as an \( x \)-value between \( C_{\text{Enter}} \) and \( C_{\text{Exit}} \). The second statement summarizes Chris and Jason's characterization of how the limiting process would end in a case in which the limit fails to exist, while the third statement summarizes their characterization of how the limiting process would end in a case where the limit does exist. Definition #6 lacked the explicit description that some of the subsequent refinements of their definition included regarding the order in which intervals were chosen along the respective axes, as well as the iterative nature of the limiting process. Nevertheless, Definition #6 served as a foundation for the characterizations of limit Chris and Jason formulated during the remainder of the teaching experiment. In Section 6.2.6, I discuss a significant realization Chris and Jason experienced regarding the mathematical role...
of the definition of limit, and how that recognition led to an important refinement of their characterization.

6.2.6 Theme 4: Reinvention as Motivation for the Need for a Formal Definition

Prior to the first experiment, I conjectured that successful reinvention of the definition of limit would be unlikely if students did not first see the necessity for such a definition. However, as I described in Chapter 5, it was not until after reinventing the definition of limit that Amy and Mike expressed the opinion that the mathematical role of the definition is not that of finding limits, but rather validating them. Thus, in contrast to my initial conjecture that students must first become aware of the distinction between finding and validating so as to be properly motivated to reinvent the definition, Amy and Mike displayed evidence suggesting reinvention may provide students an experience that subsequently evokes awareness of that distinction. The second experiment revealed similar evidence – it appears that the act of reinvention elicited in Chris and Jason a gradual awareness of the distinction between finding and validating limits, and subsequently, an understanding of the necessity for a rigorous definition of limit. During Phase 1, Chris and Jason viewed algebraic techniques that led to the use of direct substitution as simultaneously a method for finding limits and validating them. Indeed, there was no indication that they believed something more rigorous would be needed to validate limits than simply applying direct substitution via algebraic techniques.
Craig: It sounds like [algebra] is the golden, the golden ticket.
Chris: ....You gotta do that one every time.
Craig: To, to prove it?
Chris: Yeah.

... 
Craig: Given any algebraic function that someone hands you, will you always be able to do algebraic tricks to get to the point where you can use direct substitution to get the limit?
Jason: Really seems like the point of Calculus 1 was to enumerate all the different families of functions and then come up with a means to solve all of them and do that. So I think, just my gut reaction is, uh, yeah. Yeah, you, you should be able to take any objective function and take any limit on that function and

Craig: At any point?
Jason: At any point. And provided you know all of the many mathematical methods of doing it,
Chris: That's my gut reaction, too.
Jason: you should be able to.

Chris and Jason’s comments during Session 1 suggest that students may not be initially aware of the necessity of validating limit candidates – for both students, it appeared that there was no reason to question limit candidates found via the application of rules or “tricks” handed down from the instructor. In the following excerpt, Jason’s comments provide insight into how students might view the rules they learn for determining limits.

Jason: This is a running pattern that I’ve always noticed with math classes, is that you get presented with a problem that you don’t, you can’t, you don’t know how to do anymore. You knew how to do everything up until that point, and now here’s something that you don’t know how to do. Then the instructor gives you a tool for how to do it. And you just have to take that on faith because it’s not going to be for another three or four classes down the road that you get to learn exactly...why L'Hospital’s Rule works. So you just gotta kinda take it on faith and memorize all the rules and make sure that you’re taking, you’re accounting for the rules as you go.
Jason's comments suggest that students may believe there is a certain amount of "taking things on faith" accompanied with the initial exploration of limits. With this in mind, it is possible that students may not be as likely to question the need for a formal definition of limit if they are already taking on faith that they have the techniques necessary to solve limit problems and that if those techniques should fail, their teacher will provide other more sophisticated techniques that will suffice.

As Chris and Jason began to formulate characterizations of limit in Phase 2, they exhibited an ongoing focus on finding limits.

"When evaluating a limit"

During Session 3, Chris and Jason constructed their second characterization of limit, discussed previously in Section 6.2.1.

Definition #2: When evaluating a limit, \( y \) takes on values closer to \( L \) the limit in question as you take \( x \)-values closer to the point at which you're evaluating the limit. The limit need not equal the function's value at that point. (Session 3)

The phrase "when evaluating a limit" was an addition to their first characterization, and was motivated by their desire to include in their definition a description of the action being taken.

Chris: What do you think about having like 'when evaluating a limit' or, just to give us a start?...Just to say, yeah, to, to further describe what we're doing. Like I, I feel like if that's the definition [referring to Definition #1],

Jason: Umm-hmm.

Chris: we just kind of jump right into it. Not really say what we're doing.
Chris's suggestion is illustrative of his and Jason's perspective during the first two phases of the experiment – neither student explicitly expressed awareness of the distinction between finding limit candidates and subsequently validating them. To be clear, they did express the understanding that their central task was to construct a general characterization of what it means for a function $f$ to have a limit $L$ at $x=a$. However, they did not express reasoning suggestive of an explicit awareness that their characterizations presuppose the proposal of a particular $y$-value, $L$, believed to be the limit. During Session 5, I suggested to them that their characterizations did not appear to be a means by which one would find, or determine, a limit. I contrasted their current definition of limit with the algebraic techniques they had described in Sessions 1 and 2 of the experiment. My intention was that they might become more explicitly aware of the difference between finding and validating. The following excerpt, however, reveals just how robust their belief was that their definition was a vehicle through which they might find limits.

Chris: $L$ is, I don't know, to me $L$ is something out there and with our definition we're trying to find it.

Jason: So like at the end of, if we were to have these three conditions to be satisfied then by virtue of those three conditions,

Chris: Yeah.

Jason: that would call $L$ into existence?

Chris: Uh-huh.

Chris and Jason's lack of awareness of the distinction between finding and validating limits appeared to parallel their propensity for reasoning from an $x$-first perspective. Following the step function task during Session 6, however, their cognitive shift to a $y$-first perspective was accompanied by a burgeoning
recognition of the mathematical role of the definition. Phase 4 of the experiment (Sessions 7-9) included abundant discussion between Chris and Jason about a two-step limit process – Step 1 referring to the act of finding limits, and Step 2 referring to the act of validating limits.

A Two-Step Process

Following Chris’s introduction of CTop, CBottom, CEnter, and CExit, described in Section 6.2.5, he and Jason arrived at Definition #6, presented again here for the reader’s benefit.

**Definition #6:** CEnter < a < CExit. If CEnter = a but CExit ≠ a, or CExit = a but CEnter ≠ a, we do not have a limit L at a. If CEnter = CExit then we do have a limit and L = CTop = CBottom [and] a = CEnter = CExit. (Session 7)

After articulating this characterization in writing on the board, Chris presented the function shown in Figure 6.17, which he saw as a counterexample to their definition, noting that the point at which the function becomes close (CEnter) is equal to the point at which the function is no longer close (CExit).

![Figure 6.17 - Graph of a Jump Discontinuity](image)

**Figure 6.17 – Graph of a Jump Discontinuity**

As we discussed this counterexample, it was evident that Chris still did not view their definition as a vehicle through which one would validate a limit candidate.
Craig: If someone walked up and said I think this point is my limit, your definition would say yes it is....
Chris: I don't know that the definition requires you to pick an $L$. That was my other reservation....[H]ow do we pick where $L$ is? I mean, that's what we're trying to find is $L$.

This exchange was followed by a period of silent reflection for Chris, as Jason discussed issues unrelated to Chris's observations in the preceding excerpt. During this reflective period, Chris appeared to make an important realization as he stared at the graph shown in Figure 6.18.

![Figure 6.18 - Graph of a Jump Discontinuity](image)

The first line in the following excerpt captures Chris's realization.

Chris: Okay, so our definition requires that we know what the limit is....
Jason: Or at least have suspicion of where to start.
Chris: Yeah.
Jason: Which I, I'm not terribly uncomfortable with. I don't think it's bad that you have a suspicion, and then you begin your logic.
Craig: Why, why do you say that Jason?
Jason: Actually, well now I'm, well because, uh, in Calculus class when we need to find out what a limit is, we're allowed to go and look at the calculator and get an idea from it....
Chris: The problem is if our intuition on where to start is wrong we're kind of screwed.

It appears, then, that Chris and Jason were both beginning to recognize that their definition was reliant on a pre-selected candidate, $L$. This marked an important moment in the experiment, as it was the first time that either student has explicitly
suggested that their definition required a predetermined value for $L$. Further, Chris’s final comment anticipated the limitations of their definition – i.e., that the definition could validate a limit candidate but would not be useful for generating that candidate. Chris subsequently explored whether their definition would appropriately invalidate an incorrectly chosen candidate in the case where a limit did exist. This exploration revealed the need for a refinement to their definition.

Craig: Okay, so then let me ask you, this is the limit we’re interested in and we guessed too high. Will your approach of taking increasing degrees of closeness, ratcheting up your specificity, lead you to saying that $L$ really is the limit or lead you to realize that $L$ really isn’t the limit?

Chris: $L$ isn’t the limit. It, it’ll lead us to be-, maybe we have to tweak it a little bit, but we could end up saying that what we guessed for $L$ was not the limit, but not necessarily that there isn’t a limit…. [Instead of saying, well you don’t have a limit $L$ at $a$, we could just change that to, what we guessed as $L$ wasn’t right.

Chris’s suggestion that they refine their articulation led to the following characterization of limit. The bold, italicized writing in Definition #7 represents the addition Chris and Jason made to their previous articulation.

**Definition #7:** $\text{CEnter} < a < \text{CExit}$. If $\text{CEnter} = a$ but $\text{CExit} \neq a$ or $\text{CExit} = a$ but $\text{CEnter} \neq a$, $L$ is not the limit at $a$. Doesn’t necessarily mean there is no limit, just that you guessed wrong. If $\text{CEnter} = \text{CExit}$ then we do have a limit and $L = \text{CTop} = \text{CBottom}$ [and] $a = \text{CEnter} = \text{CExit}$. (Session 7)

This refinement anticipated what was to be a central topic of conversation for Chris and Jason as Session 7 ended. Their discussion culminated in the articulation of a two-step process which distinguished between finding limit candidates and validating those candidates.
Craig: So this process that you guys are describing, this thing \textit{[referring to their definition]}, let me ask this question, are you using this process to find the limit, to validate what you think the limit is, or are those two things the same thing?

Jason: They are not the same thing.

Chris: ...I think of [the definition] as validating what we think that the limit is....I think of this as, we have a guess, is it right?

Craig: But not how you’d go about

Chris: Yeah, right.

Craig: tracking down what you think \( L \) might be.

Chris: Right....

Jason: Yeah, this is definitely only validating a guess that we already have. So that’s like step 2. We can do step 2. We’ve almost got a definition that will work for step 2. But step 1 of, well where do we even start or guess? Umm, that, uh, the way I would go about doing that, if you handed me a table of values, just inspect it. And being a function, I’d start plugging in values, if you were to ask me what is, when...\( x=3 \), what’s the limit? Then I’d be looking at 2.9, 2.9999, doing that whole process....Step 1 is looking at the graph and going, I think the limit is at that circle.....So for [the definition] to work at all, your guess has to be correct. Like you have to guess the limit and then prove it. This only works as support evidence....I don’t know, I like the, I like the approach and definitely think it works when you already have a reason to suspect that \( L \) is correct.

The preceding excerpt captures Chris and Jason’s recognition that the definition of \textit{limit} is not a means by which one would determine a limit. This recognition appeared to be significant for them, for it continued to be a focal point at the beginning of Session 8\textsuperscript{60} as they looked for opportunities to further refine their definition.

Jason: So, you take a guess. You start with an assumed value for \( L \) and then formulate the closeness interval, right? Isn’t that kind of a first step?

\textsuperscript{60} It is worth noting that for pedagogical reasons, Sessions 7 and 8 took place on consecutive days. Chris and Jason had made significant progress during Session 7, but had needed thirty minutes to recollect what they had done during the previous session. With the aim of facilitating progress, at the end of Session 7 I suggested we reconvene the following day. They agreed that meeting on consecutive days would benefit their progress. Hence, Sessions 7 and 8 occurred on consecutive days.
Chris: Yeah.
Jason: Like assume some value $L$.
Chris: ...So now we have a guess.
Craig: You have a guess. I think the answer, I think the limit is $L$.
Jason: We’ve gone through L’Hospital’s, we’ve calculated an algebraic limit....So really this, if this were to be practical, this would just be a verification of something that we already know.

As Session 8 progressed, Chris and Jason’s refinements to their definition illustrated their awareness that their definition presupposes the selection of a limit candidate.

**Definition #8:**

1) *Come up with a guess.*
2) *Determine a closeness interval around your guess.*
3) *Let* $C_{\text{Enter}}$ *equal the last* $x$-*value* [before ‘$a$’] *for which we become close. Let* $C_{\text{Exit}}$ *equal the first* $x$-*value after ‘$a$’* for which we are no longer close.
4) i) if $C_{\text{Top}}=C_{\text{Bot}}=L$, then $L$ is your limit
   ii) if $C_{\text{Ent}}=a$ and $C_{\text{Exit}}\neq a$ or $C_{\text{Exit}}=a$ and $C_{\text{Ent}}\neq a$ then $L$ is not the limit
   iii) if $C_{\text{Ent}}<a<C_{\text{Exit}}$ *then shrink your closeness interval and retry at Step 2.* (Session 8)

Definition #8 was noteworthy for multiple reasons. First, Chris and Jason’s explication of an iterative process was more apparent – the last part of Step 4 indicated a looping characteristic in their definition that was absent from earlier articulations. Second, their characterization included a precise definition for two constructs central to their definition – $C_{\text{Enter}}$ and $C_{\text{Exit}}$. Third, and most relevant to the theme presented in this section, Definition #8 captured Chris and Jason’s recognition that the validation of a limit candidate is preceded by the selection of a candidate (or, equivalently, the formulation of a “guess” for the limit). To be clear, comments Chris and Jason made previously during the teaching experiment suggest
that they were aware that the employment of algebraic techniques, the consultation
of tables, and the inspection of graphs are all means by which someone might
search for the value of a limit. These prior comments did not, however, suggest that
Chris and Jason viewed the numeric result of such a search as merely a candidate
for a limit. To the contrary, prior to Session 7, neither Chris nor Jason had
acknowledged even a need for a validation process. This realization during Session
7 was significant, and appeared to be a result of their engagement in the reinvention
process.

**Summary**

Evidence from the second teaching experiment suggests that the experience of
contemplating the subtleties inherent to the limit concept while attempting to
formulate a precise characterization may support students in coming to understand
the need for a rigorous definition. This suggests, at least in the case of limit, that the
process of constructing a precise definition for the concept might simultaneously
increases one's recognition of the need for such a precise definition.

I further conjecture that the shift to a y-first perspective in Session 6 facilitated
Chris and Jason's subsequent recognition of the distinction between finding and
validating limits. Specifically, their focus on the y-axis appeared to foreground the
presence of a y-value, \( L \), about which they were constructing progressively tighter
bounds. This led them to wonder explicitly why a particular \( L \) was the focus of
their graphical exploration, which, in turn, appeared to spur their recognition of the
two-step process described in this section. Thus, evidence in the second experiment suggests a relationship between a student shifting to a y-first perspective and delineating between the actions of finding and validating limits.

The distinction between finding and validating limits, as well as an understanding of the mathematical role of the definition of limit, became increasingly unambiguous to Chris and Jason during the final three sessions of the experiment. I will revisit Theme 4 at the end of this chapter, highlighting insights Chris and Jason provided as they reflected upon the reinvention process. In Section 6.2.7, however, I will describe how the notational scheme Chris proposed in Session 7 became ultimately problematic in his and Jason's attempts to precisely characterize limit.

6.2.7 Theme 5: Desire for Precision as a Basis for Function-Dependent Characterizations (Sessions 7-9)

In the mathematical-conceptual analysis of limit elaborated in Chapter 3, I note that proximity along both the x- and y-axes is a fundamental component of the formal definition. Describing closeness to $a$ and $L$, respectively, was a focal point for Chris and Jason from the outset of their efforts to characterize precisely what it means for a limit to exist. The two tasks I implemented during Session 6, described in Section 6.2.3, appeared to support Chris and Jason in reasoning from a y-first perspective, imagining the coordination of x- and y-intervals, and beginning to articulate the iterative nature of the limiting process. Subsequent to the tasks with
which they engaged in Session 6, Chris introduced constructs (CEnter, CExit, CTop, and CBottom) designed to define the bounds of the x- and y-intervals essential to their characterization of limit. Phase 4 of the experiment (Sessions 7-9) consisted primarily of Chris and Jason’s efforts to implement Chris’s constructs as they made refinements to their definition. In Section 6.2.6, I described the evolution of thought that led to the formulation of Definition #8. Chris’s proposed notational scheme was initially beneficial, in that it provided him and Jason a means for capturing the visual aspects of limit as related to the conventional $\varepsilon$-$\delta$ illustration (see Figure 6.14). Specifically, CTop and CBottom were appropriate descriptors for the upper and lower endpoints of the shrinking closeness intervals about the proposed limit $L$. Similarly, CEnter and CExit helped express the requirement of locating an interval containing $a$ on which the function falls within the previously chosen y-interval, (CBottom, CTop). However, while the use of CEnter and CExit was helpful for describing the bounds of the x-interval about $a$, their use ultimately proved to be problematic for Chris and Jason.

Do CEnter and CExit always exist?

Upon analyzing Chris and Jason’s reasoning during Sessions 7 and 8, I realized that in their attempt to characterize one of the fundamental components of the definition of limit – that the interval constructed on the x-axis containing $a$ is such that the function $f$ is close on that interval – they had inadvertently introduced two function-dependent constructs (CEnter and CExit). For example, for the graph
shown in Figure 6.19, with removable discontinuities on either side of the coordinate pair \((a, L)\), there would not technically exist either a CEnter or a CExit for the designated CTop and CBottom. Based on the definition of CEnter and CExit provided by Chris and Jason in Definition #8, CEnter equals the last \(x\)-value prior to \(a\) such that the function “becomes close” (i.e., enters the closeness interval defined by CTop and CBottom). However, the graph shown in Figure 6.19 has no such \(x\)-value – because of the density of the real number line, there is no way to determine the first \(x\)-value to the right of the first removable discontinuity (looking from left to right) in the graph in Figure 6.19.

![Figure 6.19 - Counterexample to the Existence of CEnter and CExit](image)

As we discussed this graph during Session 9, Chris appeared to become aware of this problem.

Chris: Like, CEnter isn’t the hole, it’s like right next to the hole.
Craig: Why, because of the way you’ve defined CEnter?
Chris: Because of, well yeah, because you’re not defined at the hole....At the hole you’re not close. But the very, as close as you can possibly get to the hole, you are close.

Indeed, as Chris realized in the preceding excerpt, based on their definition, CEnter would be the smallest \(x\)-value greater than the first removable discontinuity (in the positive \(x\)-direction) seen in Figure 6.19. The denseness of the real number line,
however, makes it impossible to identify such an $x$-value. Likewise, Chris and Jason recognized that their definition of CExit was problematic. Chris’s comments in the following excerpt, in reference to the graph shown in Figure 6.20, suggest that he was aware of the inability, in certain cases, to identify CExit.

Chris: Well, how can we say that CExit is $a$ when it was the first point at which you’re no longer close?
Jason: Uh-huh.

Figure 6.20 – Graph Illustrating the Difficulty in Identifying CExit

As Session 9 progressed, resolving the issue of how to define CEnter and CExit became a central focus.

Craig: I feel like there’s two issues out on the table right now….I want to just make sure I verbalize what they are. One issue is CEnter, it’s the last point at which you became close. But what does that mean? Like if there’s a hole there, like what’s the last point at which you became close? Well, it’s just maybe a little bit to the left of the hole? But how are you going to find that? It’s kind of getting back to the max value idea.
Jason: Umm-hmm.
Craig: And likewise CExit’s the first point at which you’re no longer close. Well how are you going to define that if it happens to take place at a discontinuity? So that’s one issue is we kind of know what we mean here but, uh, how are we going to define these things?
It is worth noting that CEnter and CExit had been helpful constructs for capturing fundamental elements of Chris and Jason's concept image. Thus, it was not surprising that they subsequently attempted to frame their characterization of limit around these constructs. In their attempts to characterize the limiting process during the latter part of Session 6, Chris and Jason recognized the need for determining a corresponding $x$-interval for each predetermined $y$-interval. This recognition arose when Chris drew bounds around $L$ on the $y$-axis, extended those bounds to meet the function, and subsequently drew vertical lines originating at the intersection points and terminating at the $x$-axis, as shown in Figure 6.21.

![Figure 6.21 - First Illustration of Bounds](image)

The process just described became the sole means by which Chris and Jason constructed $x$-intervals during Sessions 7-9. Given my repeated requests for them to be more specific and precise in their characterization of limit, it is understandable that they attempted to describe the $x$-intervals by way of defining their endpoints. Unfortunately, they were so focused on defining the largest $x$-interval, that they lost sight of the characteristic they had initially intended to capture – that the interval contains $a$ and has corresponding function values that are close everywhere on the interval, except possibly at $a$. As a result of their focus on defining the
largest $x$-interval, Definitions 7-9 were reliant on the existence of CEnter and CExit, and thus were function-dependent.

Resistance to Abandoning Their Initial Approach

As Session 9 progressed, I made repeated efforts to evoke awareness in Chris and Jason that for each predetermined $y$-interval, more than one corresponding $x$-interval exists that would satisfy their stated conditions for closeness. My rationale for these efforts was that Chris and Jason might be more likely to abandon their use of CEnter and CExit in their definition if they understood that they were not restricted to describing the bounds of the largest corresponding $x$-interval, which had become the focus of their attention. As the following excerpt suggests, Chris and Jason did appear to be aware that more than one $x$-interval existed. Interestingly though, Jason’s final comment seems to indicate that he believed that the construction of their $y$-interval about $L$ dictated that they must use the largest $x$-interval about $a$.

Craig: You went looking for an $x$-interval such that what was true? What was true about that $x$-interval?
Jason: That on that $x$-interval, all the points in between are close....except possibly at $a$.
Craig: Okay, now let me ask you this....Is that the only interval, the only $x$-interval that is true? Is this the only $x$-interval on which we are close based on that definition of close?
Chris: ...No.
Craig: ...But the key here is that you were able to find one that worked. You happened to pick the largest one, right?
Chris: Yeah.
Jason: Well, and I’m still thinking of that as we didn’t just pick it, it must be so because those are the points on the function that correspond to CTop and CBottom.
Later, as we discussed a horizontal line function, I suggested that every $x$-value contained in an arbitrary interval about $a$ would have a corresponding function value within the predetermined closeness of $L$. I subsequently marked an arbitrary $x$-interval about $a$ on the $x$-axis, as shown in Figure 6.22.

![Graph of a Horizontal Line](image1)

**Figure 6.22 – Graph of a Horizontal Line**

Chris and Jason's response again suggested that while they appeared open to the idea that more than one $x$-interval might exist for each predetermined closeness interval, they nevertheless believed that only the establishment of the largest such $x$-interval would allow one to subsequently choose a smaller closeness interval and iterate the process.

Craig: So for instance, I could, you're saying I could pick here and here for all I care, right?...And that would be CEnter and CExit?
Chris: But how do you just arbitrarily pick those?
Jason: ...The, the one on the inside you have arbitrarily picked,....I'm not cool with that.
Craig: ...I'm just wondering is this interval a legal interval?
Chris: Just saying that there's some interval that exists within the one that we chose?
Craig: I guess my question was...given some closeness interval, is there one and only one $x$-interval such that what you said is true is true?
Chris: I think that's the largest one.
Craig: That's the largest one, but there are smaller ones that would
Chris: There could be.
Craig: There could be. Okay, but the thing that allows you to reiterate is that you've found this largest one? Or that you've found one?
Chris: ....The largest one.
Craig: The largest one? Okay. And if you found the largest such interval you’d get to
Chris: Go again.

As Session 9 concluded, Chris and Jason repeatedly acknowledged the possibility that more than one $x$-interval might exist for each predetermined $y$-interval, but maintained that the selection of an arbitrary $x$-interval was not congruent with their characterization.

Craig: [D]o we have to find the largest [x-interval] for which that’s true?
Chris: No, it’s just that’s the one that we come up with when we use the last point, first point [referring to CEnter and CExit, respectively].
Craig: Okay. I, okay, so with that in mind, so for this closeness interval, before we can shrink that closeness interval, you’re saying we have to find an interval along the $x$-axis on which we are close, except possibly at $a$. And my question was, we seem to be really focused on trying to find the largest one on the $x$-axis for which that is true, but I’m suggesting that maybe there are some that are smaller.
Jason: Well, it might be in there but it doesn’t correspond, it’s not playing by the rules that we have set up.

Chris and Jason’s reluctance to discontinue the use of CEnter and CExit was understandable. These constructs had been self-created and self-implemented and had successfully captured imagistic elements fundamental to their concept image of limit. It is no surprise, then, that despite their acknowledgment that incorporating the two constructs in their characterization of limit would lead to a function-dependent definition, they nevertheless resisted abandoning these two ideas.

Summary

Evidence from the second experiment suggests that in the context of reinvention, students may characterize $limit$ in a manner that specifies the bounds
of the $x$-interval corresponding to the predetermined $y$-interval. This desire for specificity in regards to proximity along the $x$-axis is understandable, given that a coherent, mathematically valid definition of limit requires the precise articulation of numerous other details and subtleties. It is reasonable, therefore, that students may balk at the notion of establishing the existence of an $x$-interval containing $a$ without also having to define its endpoints.

Despite their recognition during Session 9 that the inclusion of CEnter and CExit in their characterization was problematic, Chris and Jason were nevertheless reluctant to discontinue the use of these ideas. This was not surprising because the implementation of CEnter and CExit had allowed Chris and Jason to characterize limit in a manner that more successfully validated and invalidated examples and counterexamples of limit, respectively, than had their previous defining strategies. The lack of recognition that they need only establish the existence of any $x$-interval containing $a$ (not necessarily the largest $x$-interval) for each predetermined $y$-interval was one of two remaining issues for Chris and Jason in their efforts to reinvent the definition of limit. In Section 6.2.10, I describe how they ultimately overcame this issue. First, however, I revisit the second theme described in this chapter – potential infinity as a hindrance to characterizing infinite closeness.
6.2.8 Theme 2 Revisited: Potential Infinity as a Hindrance to Characterizing Infinite Closeness

In Section 6.2.2, I discussed Chris and Jason's inclination to reason from a potential infinity perspective during the first two phases of the experiment, and how this perspective hindered them from characterizing infinite closeness. In Section 6.2.3, I described the implementation of two instructional tasks that were designed to initiate in Chris and Jason a cognitive shift away from a potential infinity perspective. The reader may recall that tasks implemented during Session 6 did not appear to elicit an immediate shift away from a potential infinity perspective. Indeed, although many of their subsequent written characterizations appeared to indicate that Chris and Jason believed it possible to complete the limiting process\(^{61}\), there were numerous verbal exchanges during Sessions 7-9 which suggested that they were still trying to describe the incremental completion of an infinite process in a finite amount of time. Incrementally carrying out the entirety of the limiting process in one's mind is impossible because of the process's infinite nature. The conventional \(\varepsilon-\delta\) definition of limit circumvents such logical dilemmas by encapsulating the infinite limiting process via the notion of an arbitrary small number, \(\varepsilon\), designed to represent any and all error tolerances around a proposed limit \(L\). In Chapter 5, I described Amy and Mike's implementation of the notion of arbitrary closeness as a means of operationalizing infinite closeness. It is worth noting that their implementation of arbitrary closeness was spontaneous and not

\(^{61}\) Definition #8, presented in Section 6.2.6, is one such example.
targeted by me based on a priori instructional design decisions. Further, it was not until I conducted a retrospective analysis that I recognized the significant role that the notion of arbitrary closeness played in supporting Amy and Mike’s efforts to reinvent a definition of limit capturing the intended meaning of the conventional ε-δ definition. As such, I did not target the implementation of the notion of arbitrary closeness during the second teaching experiment.

Although Chris and Jason never made use of the notion of arbitrary closeness, evidence suggests that they did come to recognize the limitations of reasoning from a potential infinity perspective. For example, during Session 9, their concerns regarding how the limiting process could be incrementally completed became evident.

Jason: [A]t some point we’re jumping away from dealing with an interval, a beginning and an ending, to the finite point. So like...I’m just, yeah, I’m, I’ve been growing concerned that, well what happens, okay, well we got, let [the width of the closeness interval] be .00001, then we’re effectively right on, right on either side of the line....

Chris: And it’s as close as you can possibly get to L without being L.

Jason: Umm-hmm. Then when you make that next step,

Chris: I’m concerned...there too.

Later, Chris made his concerns even more explicit.

Jason: I’m starting to wonder, that’s almost our definition of a limit. It’s starting to weed out all of the erroneous language that we might not need.

Chris: It, it is, but I still have a problem with, like how do you say that this is the limit? Or how do you say that anything we come up with is the limit? Like, ’cause we’re not necessarily looking at x=a. And so at some point there’s, we’re infinitesimally close to the hole....

Craig: How does the process end?

Chris: Yeah. How does it end?
In summary, then, evidence from both teaching experiments suggests the following trajectory of student reasoning in regards to the role infinity plays in the limit concept: first, students may reason about the limiting process from a potential infinity perspective, trying to imagine the incremental completion of an infinite process. Next, students appear to recognize the limitations of such a perspective and, in turn, seek a new perspective which will allow them to reconcile the cognitive dilemma which arises when one attempts to imagine the completion of an infinite process in a finite amount of time. In the first teaching experiment, this period of dissatisfaction was resolved by Amy and Mike by implementing the notion of *arbitrary closeness*, not merely by shifting to an actual infinity perspective, as Williams (2001) suggests. In the second teaching experiment, this period of dissatisfaction was superficially resolved by Chris and Jason by imagining that the infinite limiting process could hypothetically end in a finite amount of time. Details of this superficial resolution are described in Section 6.2.9.

Two central unresolved issues remained for Chris and Jason prior to Session 10. First, as I described in Section 6.2.7, they lacked recognition that they need only establish the existence of *any* x-interval containing a (not necessarily the largest x-interval) for each predetermined y-interval. Second, as I just discussed, they had not found a suitable alternative perspective to address the limitations offered by a potential infinity perspective. Section 6.2.9 details the attempted resolution of these two central issues and the completion of the reinvention process.
6.2.9 Phase 5: Attempted Resolution of Central Issues and Completion of Reinvention Process (Session 10)

My central pedagogical goal for Session 10 was to guide Chris and Jason to a resolution of the two central issues I described in the preceding two sections. In analyzing Session 9, I found evidence that Chris and Jason at times appeared to be reasoning in ways conducive to resolving these two issues. As an example, both Chris and Jason appeared to acknowledge that for each predetermined y-interval, there were existing x-intervals other than the one produced by their characterization. Comments they made also suggested that they understood the problematic nature of attempting to define the largest x-interval for each predetermined y-interval – specifically, that the density of the real number line would make it impossible to identify a CEnter and/or CExit in certain cases. They appeared reluctant, however, to diverge from the use of the notation with which they had been working during Phase 4. My aim during Session 10 was to direct their attention to moments during Session 9 when they had expressed ideas which I felt anticipated the resolution of the two remaining cognitive hurdles they were experiencing, and to subsequently highlight the benefits of pursuing those ideas. Although encouraging them to pursue particular ideas they had previously expressed forced me to take on a more active guiding role in the reinvention process, such a pedagogical approach still kept the focus of reinvention on Chris and Jason's ideas, as opposed to my own agenda. This distinction is important to make, as it highlights the theoretical perspective which guided my study. In the
following pages, I briefly describe the nature of Chris and Jason's attempts to resolve their two remaining issues.

Large x-Interval versus Any x-Interval

At the outset of Session 10, I wrote Chris and Jason's most recent characterization of limit on the board, and I noted that during Session 9 they had expressed general satisfaction with this articulation but had acknowledged the problematic nature of using CEnter and CExit. I pointed out that these two constructs had been helpful for them, in that they had descriptively captured some of the ideas fundamental to their understanding of limit. I acknowledged, though, that defining CEnter and CExit had been problematic. I reminded Chris and Jason that they had expressed confidence in the first two steps of their definition of limit – the definition requires a proposed $L$, as well as the initial selection of a closeness interval about $L$. I then noted that they had expressed concern about how to describe the establishment of a corresponding $x$-interval about $a$. Next, I asked them to read excerpts I provided them from Session 9 during which their comments had pointed to ways they could characterize limit without having to utilize CEnter and CExit. The following is an example of one such excerpt.

Craig: You went looking for an $x$-interval such that what was true? What was true about that $x$-interval?
Jason: That on that $x$-interval, all the points in between are close...except possibly at $a$.

... 
Craig: For, for me to be allowed to shrink this definition of closeness,...I have to find some interval such that [$a$] is between...
Jason: Uh-huh.
Craig: And I'm close on that whole interval, right?
Jason: Yeah.
Craig: I'm just asking does that necessarily, do we have to find the largest [x-interval] for which that's true?
Chris: No.

Having them read through these excerpts seemed pedagogically effective. In particular, the excerpts provided them a basis for conversation that was grounded in their own words and ideas, and provided a way for me to credit them with the recognition that they only care about the existence of any x-interval, not necessarily the largest one. After they read the excerpts from Session 9, we discussed the graphs shown in Figure 6.23, noting that in cases in which the limit exists, they would always be able to find an x-interval about a on which the function would be close (based on a pre-specified definition of closeness) at every x-value, except possibly at a.

Figure 6.23 – Four Graphs: Examples and Counterexamples of Limit

The reading of and reflecting on excerpts and subsequent exploration of graphical examples of limits appeared to provide Chris and Jason insight as to how they might refine their definition.
Jason: So we just need to couch it in the idea that, uh, you have to have an x-interval that includes points, plural, on either side that are close. I mean that’s, that’s the nutshell....

Chris: We just need to say well, if one exists, okay, shrink your interval and check again.

Jason: Yeah. Then... denote that CEnter, or are we going to abandon the concept of CEnter and CExit?

Chris: Do, do we even need to say it, what it is? Do we need to give it a value?

Jason: Probably not if we’re just talking about some interval. We don’t have to be explicit about what, what we’ll call it.

There are a couple of possible explanations for Chris and Jason’s decision to abandon the use of CEnter and CExit in their characterization. First, they had previously acknowledged that their definition of CEnter and CExit did not allow them to properly classify all examples and counterexamples of limit. Hence, Chris and Jason were somewhat aware that there were problematic aspects of utilizing these two constructs, and thus, they were likely motivated to pursue other approaches to defining limit. Second, the nature of the task described previously, wherein Chris and Jason were asked to read specific excerpts from Session 9, was strongly suggestive of the need to abandon these two constructs. It is a strong possibility, then, that Chris and Jason inferred the intended outcome of the task, and subsequently decided to discontinue the use of CEnter and CExit.

After discontinuing the use of CEnter and CExit, the students discussed ways to characterize the existence of an x-interval and ultimately arrived at the notation seen in their final definition.

**Definition #9:**

1) Come up with a guess, \( L \).
2) Determine a closeness interval \( L \pm z \) around your guess.
3) If: 
there exists an \( x_1 < a \) such that \( L+z>f(x)>L-z \) is true for all \( x \) between \( x_1 \) and \( a \) AND an \( x_2 > a \) such that \( L+z>f(x)>L-z \) is true for all \( x \) between \( x_2 \) and \( a \), then shrink your closeness interval and try again. If you can’t shrink your interval anymore, then \( L \) is your limit.

If not: then \( L \) is not your limit.

(Session 10 – Final Definition)

**Shifting From Potential Infinity**

Having resolved how best to characterize the existence of an \( x \)-interval about \( a \), Chris and Jason continued to express concern as to how an infinite limiting process could ever actually validate the candidacy of a proposed limit – i.e., how could incrementally restricting closeness intervals around a proposed limit candidate \( L \) result in the validation of that candidate? To be clear, there was evidence of Chris and Jason’s dissatisfaction with a potential infinity perspective and interest in finding an alternative perspective that would adequately resolve the cognitive dilemma related to infinity described previously. To respond to this concern, I once again directed Chris and Jason’s attention to comments they had made during previous sessions, this time highlighting verbal exchanges in which they had speculated that the limiting process would terminate if they could no longer shrink the interval on the \( y \)-axis. The following is one such example.

Jason: It’s a matter of restricting what’s outside of, like, what you’re not concerned with. You’re not concerned with all the stuff above your limit in question or below it. It’s a matter of restricting the, I guess the range of where you’re examining it, to a point where you only have one point, and then that will be your limit.
After they had read the excerpts I provided, I noted that Jason’s characterization, seen in the preceding excerpt, effectively characterized the intentions of their limiting process – to eliminate from consideration every y-value except the actual limit. I then suggested that imagining the actual carrying out of an infinite process in one’s mind is impossible, but that they might benefit from shifting to a hypothetical perspective, wherein instead of imagining the termination of an infinite process of incrementally shrinking their closeness interval, they could imagine a limit existing provided they could establish the existence of an x-interval for any arbitrary closeness interval about $L$.

Craig: [Y]ou guys have both brought this up, what you just said like, how do you ever get to $L$? Like, what is, if you can’t shrink your interval anymore, what does that even look like? Maybe what it takes is just making a hypothetical shift and saying like, alright, fine, $L$ is your limit if you could always find this [referring to an x-interval] for whatever closeness interval that you had.

Jason: Okay.

My intention in the preceding excerpt was to encourage Chris and Jason to adequately address the “end” of the limiting process they were trying to describe. Although they had expressed displeasure with a potential infinity perspective, they appeared unaware that their definition of limit continued to be stated from a potential infinity perspective, in that they required the continued shrinking of closeness intervals about $L$ “to a point where you only have one point, and then that will be your limit.” My aim was that they would resolve the dilemma of adequately addressing an infinite number of closeness intervals by imagining a single arbitrary closeness interval, as Amy and Mike had. Unfortunately, although my suggestion
that Chris and Jason adopt a more hypothetical perspective did appear to elicit some kind of cognitive shift, their final definition of limit, described in the previous subsection of this chapter, was still stated from a potential infinity perspective. Indeed, the cognitive shift that occurred for Chris and Jason appeared to be distinct from that which occurred for Amy and Mike. Specifically, whereas Amy and Mike employed the notion of arbitrary closeness to operationalize infinite closeness, Chris and Jason’s shift might best be described as sweeping the cognitive issues of a potential infinity perspective “under the rug.” Jason’s comments below illustrate the cognitive shift he appeared to make during the tenth session – he seemingly resolved the cognitive dilemma of imagining the carrying out of an infinite process by simply accepting that the end of the process must somehow (mysteriously, perhaps) happen.

Jason: Well that, the sentence, if you can’t shrink your interval anymore then, then that, that is capturing the $z$ going to 0, right? Okay, so from the very outset when we wanted to come up with a method, it seemed like we were trying to, uh, come up with a point by point description of what $x$ goes to $a$ means. And, it’s like, I’m starting to feel like, okay, well, we did a really good job of talking about as $x$ starts going to $a$, but then, you know, it gets so far and then all of a sudden we have to say, okay, (inaudible), it jumps in a rocket and it shoots down to 0. You know? It’s not, it’s not approaching, it’s not going anymore. We’re just saying, okay, well now take it to 0. Just go there.

Jason’s description of jumping in a rocket and shooting “down to 0” is reminiscent of the actual infinity perspective Williams (2001) claims is essential to students’ understanding of limit. During the individual exit interviews that concluded the
experiment, Chris and Jason each independently reasoned in a manner suggestive of an actual infinity perspective.

Jason: You can’t discretely ever get to the limit 'cause there’s always going to be a smaller increment that you can add on. That’s the closer and closer problem. So at some point you’re going to have to jump to just letting \( z \) equal 0....I don’t think you can actually get to 0, letting \( z=0 \), by just incrementally shrinking \( z \)....[Y]ou could do that over and over, letting \( z \) get smaller and smaller and see that it’s going to continue to hold until you get to the point where \( z \) is 0. Barring again that you can’t incrementally get there – you’re going to have to make a jump at some point.

To suggest that Jason no longer saw value in reasoning from a potential infinity perspective would be inaccurate. To the contrary, Jason was clear in articulating the value of being able to reason from a potential infinity perspective when describing limits.

Craig: When we finished last time then, did you feel like the, this thing that you guys had at the end of the day took care of everything? To your knowledge anyway?

Jason: To my knowledge, everything. The only caveat, and again, this is what I was saying earlier, the barring is the, the necessity of jumping to letting \( z=0 \). Somebody could let \( z \) equal all of those incremental steps and they could do that all day long if they really wanted to push the issue. You know, 6.9 with fifty 9’s after it. Uh, at some point I think that would actually prove valuable because they would, that, that in itself would, would show someone...that okay, that this is what the approaching behavior is all about. Like 6.99 is closer than 6.9 to 7. And that’s really, at least to me, that’s like the heart of, of what the concept was.

Jason’s comments here are significant, for they suggest that a potential infinity perspective is necessary for understanding “the heart of the concept” – i.e., “the approaching behavior.” Yet, as was evident in both of the teaching experiments conducted in this study, students’ reinvention and understanding of the definition of
limit appears reliant on their ability to move beyond reasoning solely from a potential infinity perspective. In the first teaching experiment, Amy’s utilization of the notion of arbitrary closeness to encapsulate the limiting process was what ultimately led her and Mike to a coherent formal definition. In the second teaching experiment, Chris and Jason did not choose to employ the notion of arbitrary closeness. In fact, when asked to interpret the definition of limit proposed by the first pair of students, Chris and Jason expressed disapproval for the notion of arbitrary closeness, noting that Amy and Mike’s definition only addresses a single closeness interval and does not include a provision for checking smaller and smaller closeness intervals.

Chris: They keep \( \lambda \) fixed but call it arbitrarily small.
Jason: Okay.
Craig: Oh, I see what you’re saying. Like yours is getting smaller and smaller and smaller.
Chris: Yeah.
Craig: They’re saying like
Jason: It’s already, it’s already there. It’s already small.
Chris: It’s static there....So they’re basically doing the same thing we are except they pick a closeness interval, say it’s really arbitrarily small, and if you find points within that where it’s within the closeness interval, then that’s the limit. Whereas we’re saying, shrink your closeness interval and keep checking. So here they have a problem if you pick too large of a closeness interval.

It seems, then, that Chris and Jason interpreted the notion of arbitrarily small as representing a single definition of closeness, as opposed to all definitions of closeness. This interpretation ultimately hindered them from reinventing a definition of limit which encapsulated the infinite limiting process.
In summary, I suggest that it is not necessarily one's ability to reason from an actual infinity perspective that supports him or her in developing coherent understanding of the formal definition of limit, as Williams (ibid) suggests. Rather it is one's ability to encapsulate the infinite limiting process with the notion of *arbitrary closeness* that most directly provides leverage for productive progress towards reinventing the formal definition. To be clear, the definition of limit constructed by Chris and Jason did capture many sophisticated elements of the conventional $\varepsilon$-$\delta$ definition. However, it did not resolve the cognitive dilemma which arises from utilizing a potential infinity perspective.

### 6.2.10 Retrospective Findings

Themes 1-5 presented in this chapter address cognitive and pedagogical themes which emerged *during* the reinvention process. Themes 1 and 2 (see Sections 6.2.1 and 6.2.2, respectively) characterize cognitive difficulties Chris and Jason experienced that hindered their attempts to characterize *limit*. Theme 3 (see Section 6.2.3) describes an instructional task implemented during Session 6 which appeared to initiate an important shift in Chris and Jason's reasoning that, in turn, spurred them to make significant progress in their reinvention efforts. Theme 4 (see Section 6.2.6) points to reinvention as a means by which students might come to see the need for a rigorous validation process for limits, while Theme 5 (see Section 6.2.7) characterizes the motivation behind a student-proposed notation scheme. In the pages that follow, I describe two final themes that point to insights I gained only
after retrospectively reflecting on the second experiment. To conclude the chapter, I describe evidence which emerged during the individual exit interviews which provides corroborating evidence for Theme 4.

**Theme 6: Reinvention of the Formal Definition of Limit: Corroborating Evidence of the Potential for Student Success**

In Chapter 5, I presented evidence from the first experiment suggestive of students' potential to reinvent and reason about sophisticated mathematical ideas. Evidence from the second experiment corroborates this finding. Chris and Jason had neither seen nor were aware of the formal definition of limit, yet they were able to characterize limit in a manner which captured many of the fundamental elements of the conventional ε-δ definition.

**Final Definition:**

1) Come up with a guess, \( L \).
2) Determine a closeness interval \( L \pm \varepsilon \) around your guess.
3) If: \( \exists x_1 < a \) such that \( L + \varepsilon > f(x) > L - \varepsilon \) is true for all \( x \) between \( x_1 \) and \( a \) AND \( \exists x_2 > a \) such that \( L - \varepsilon < f(x) < L + \varepsilon \) is true for all \( x \) between \( x_2 \) and \( a \), then shrink your closeness interval and try again. If you can't shrink your interval anymore, then \( L \) is your limit. If not: then \( L \) is not your limit.

As I mentioned in Chapter 5, I am unaware of other studies indicating such potential for students who had not previously seen the definition.

The depth with which Chris and Jason reasoned about non-trivial aspects and subtleties of the limit concept towards the latter phases of the experiment suggests
that reinvention supports the development of advanced mathematical thinking (in the sense of Tall, 1991). Chris and Jason’s construction of a definition of limit capturing much of the intended meaning of the conventional $\varepsilon$-$\delta$ definition lends credence to my claim in Chapter 5 that students who have never encountered the formal definition of limit have the potential to reinvent it by building upon their informal understandings through engagement in purposefully designed tasks. At the end of Chapter 5, I noted that one byproduct of reinventing the definition appears to be the ability to coherently interpret conventional formulations of the definition. In the following pages, I provide further evidence of this finding.

**Theme 7: Reinvention as Support for Coherently Interpreting Other Mathematically Valid Formulations of the Definition**

One of my conjectures prior to this dissertation study was that reinvention would support students’ efforts to subsequently interpret and reason about the conventional $\varepsilon$-$\delta$ definition of limit. Evidence from the first experiment suggests that my conjecture was well-founded. At the end of the second experiment, I sought to further test that conjecture by asking Chris and Jason to respond to the following task:

**Written Prompt:** Please consider the following two statements:

1) $\lim_{x \to a} f(x) = L$ provided that: Given any arbitrarily small $\# \lambda$, we can find an $(a \pm \theta)$ such that $|L-f(x)| \leq \lambda$ for all $x$ in that interval except possibly $x=a$.

2) $\lim_{x \to a} f(x) = L$ provided that: For every $\lambda>0$, there exists a $\theta>0$, such that $0<|x-a|<\theta \Rightarrow |f(x)-L|<\lambda$. 

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These statements are alternative articulations of what it means for a function \( f \) to have a finite limit \( L \) at \( x=a \). Consider both of these statements. Does each statement capture the intended meaning of your own articulation? Comment on the similarities and differences in meaning of each of these statements in relation to your own articulation.

Chris and Jason's response to the preceding task suggests that the experience of constructing their own definition of \textit{limit} provided them a perspective conducive to making sense of the two alternatively stated formulations. Upon inspection, they fairly quickly noted parallels between their definition and the first statement.

Jason: Yeah, there's all these symbols floating around. I'm trying to see how they all relate to each other to even be able to draw a picture.
Chris: I find it confusing that they use \( \lambda \) and \( \theta \), but that's just me.
Jason: It just occurred to me that the parentheses \( a \pm \theta \), that's the same language as our \( x_1 < a, x_2 > a \). Yeah, oh, okay. Right, right. So, them, us. Okay, for us this is \( a \), and we have some arbitrary \( x_1 \) and \( x_2 \). That's this guy and this guy....And for them, it's, what, \( a - \theta \), \( a + \theta \). So, this guy, \( = x_1, = x_2 \). And the id-, the idea of using a variable here rather than saying \( a - 5 \), \( a + 5 \), uh, that's the same, capturing the same thing that we were trying to do with \( x_1 \) and \( x_2 \). It's going to allow \( x_1 \) and \( x_2 \) to get closer to \( a \). In the same sense, that's going to happen as \( \theta \) takes on smaller and smaller values. That's what I'm trying to say. That's a parallel between, between our ideas.

The preceding excerpt illustrates Jason's recognition that the first statement captured the intended meaning of his and Chris's articulation of behavior along the \( x \)-axis, albeit with different notation. As he compared his and Chris's articulation with the first statement, he drew two identical graphs, labeled the graphs "us" and "them," and demarcated the notation used in each articulation. After Jason verbalized his comparison of notation along the \( x \)-axis, I asked him to similarly compare notation along the \( y \)-axis. Figure 6.24, as well as the excerpt that follows,
captures Chris and Jason’s interpretation of the notation Amy and Mike used to describe closeness along the y-axis.

Figure 6.24 – Us versus Them Comparison

Craig: Can you show me “us versus them” for \( \lambda \)?
Jason: Yeah, I was gonna just try to make sure that, I’m trying to see if our, like, is our \( L+z \) analogous to,
Chris: \( \lambda \)’s basically \( z \), I think.
Jason: That’s what I’m, what I’m suspecting. So, but what’s \( L \) minus, the absolute value of \( L-f(x) \), what are they trying to draw our attention to there?.... Is that a closeness interval?
Chris: More like a, what’s it called when you calculate the error? So they’re saying that \( f(x) \) is...something you calculate. And they’re saying that \( L \) minus whatever you calculate, if it’s less than \( \lambda \),
Jason: Yeah.
Chris: then you have a limit....
Jason: So I guess they still have, do they still have an \( L+\lambda \) and an \( L-\lambda \)?....
Chris: They combined them in the absolute value.

The experience of reinventing the definition also appeared to provide Chris and Jason a basis for making sense of the conventional \( \varepsilon-\delta \) definition of limit. After discussing Amy and Mike’s definition (Statement #1), their attention shifted to discussing the conventional formulation.

Jason: They’re still using the same idea where \( \lambda \) is equivalent to our \( z \)....
Chris: Just what does the first part mean? \( 0<|x-a| \), which is \( <\theta \). So what’s \( \theta \)? Where are they trying to use the \( \theta \)?...Is that the \( x \)?
Jason: Yeah. Umm, the, the interval is \( a-\theta \).
Chris: So you’re saying that there’s some interval.
Jason: That’s the same interval.

As the conversation continued, they agreed that each of the other two statements captured the intended meaning of their formulation. However, on multiple occasions they made a point of expressing preference for their own articulation, noting that for someone new to the limit concept, their definition was easier to interpret.

Jason: [T]his sentence [referring to the conventional \(\varepsilon-\delta\) definition of limit], limit, it’s not even a sentence. It’s cryptic hieroglyphics. This sentence is saying that this vertical concept here, this limit, equals \(L\) provided blah-blah-blah....[T]hey’re just using a, a different language for how to zero it in....
Chris: Based on these two definitions we could probably shrink ours into one sentence and it would pretty much look like both of these, I would think....It would be similar, but you lose readability....
Jason: Now I can safely say that if I were going to be introduced to the concept of limit, I would have a much better time dealing with that [referring to their definition] than either one of these [referring to the other two articulations].
Chris: That’s what I was trying to say.

Jason’s final comment in the preceding excerpt is significant, because it underscores sentiments expressed by Cornu (1991) regarding the inappropriateness of introducing the conventional \(\varepsilon-\delta\) definition of limit at the outset of someone’s exploration of limits. “[T]his unencapsulated pinnacle of difficulty occurs at the very beginning of a course on limits presented to a naïve student. No wonder they find it hard” (p.163). Indeed, the level of sophistication with which Chris and Jason reasoned about limits during the latter phases of the reinvention process, as well as the ability they demonstrated to coherently interpret other mathematically valid
formulations of the definition, suggests that reinvention may be a vehicle for understanding the conventional \(\varepsilon-\delta\) definition of limit.

**Theme 4 Revisited: Reinvention as Motivation for the Need for a Formal Definition**

In Section 6.2.6, I described revelations articulated by Chris and Jason during Session 7 which suggests that the experience of contemplating the subtleties inherent to the limit concept, while also attempting to formulate a precise characterization, aided them in coming to understand the need for a rigorous definition. In particular, I mentioned that the act of reinvention appeared to elicit in Chris and Jason a gradual awareness of the distinction between finding and validating limits. This awareness appeared to grow stronger over the course of the final three paired sessions. Upon reflecting on the reinvention process during their respective individual exit interviews, both Chris and Jason explicitly noted that the definition is not a tool one would use to find a limit candidate, but rather is a means by which someone could validate a limit candidate. In the excerpt below, Chris refers to the two-step process he and Jason first described in Session 7. The reader will recall that Chris and Jason viewed Step 1 as the act of finding a limit candidate, and Step 2 as the act of validating that candidate.

Chris: I see the role that our articulation accomplishes as the second part of a two step process. Ideally you’d want to get a function and be able to say, okay, well what’s the limit at \(x=a\)?...But that’s not exactly what we came up with. With our process we can’t tell you what the limit is. We can only tell you if the guess you come up with for the limit is correct or not....

In his individual exit interview, Jason echoed Chris’s sentiments.
Jason: Is [the definition] applicable for calculating or finding a limit? Or helping with your homework? Or anything like that? Probably not. It would probably impress, uh, a math teacher. That, that level of thinking.

The individual exit interviews also revealed that Chris and Jason each viewed the definition as a means of characterizing the conceptual underpinnings of limit. This was not a perspective either student had previously articulated. The following excerpt captures this perspective.

Jason: Now somebody who's in class and you hand this [referring to their definition] to them and say calculate the limit,...probably not going to be very fruitful. But as a matter of considering, conceptualizing what it is we’re doing when we’re talking about the abstract concepts of getting closer and infinitesimals and all this stuff. Umm, if they could make that connection that what, basically what your calculator is doing as you’re zooming in and out is the same process that you’re doing here....

Craig: The purpose of that definition in mathematics is

Jason: To describe, uh, to describe in specific terms the abs-, the abstract concept of limit for all functions of a single variable....And the operative word being “to describe.”...Not going to evaluate or find it or anything like that. It’s just going to, this will describe it.

Craig: Okay....Do you see that as different than doing limit problems, quote unquote?

Jason: Definitely. I think after, uh, after doing a hundred or two hundred limit problems in various functions, uh, that okay, well now you’ve got a loose wrap-around concept of it. Then you’re ready to take a look at this. This isn’t the first thing I’d present to a Calc 1 student. Give them, talk about ants and intending to reach. Give them problems. And then say okay, now look at this. And you’ll find this will describe any function, whether it be continuous or not. Uh, it’s going to describe for you when a limit exists and when it doesn’t.

Jason’s comments are noteworthy in that they are illustrative of the effect that reinvention can have on someone’s understanding of the limit concept (provided he/she fit the criteria on which selection for this study was based) – at the outset of
the experiment, Chris and Jason’s focus was predominantly on the act of finding limit candidates, and they showed no evidence of having made the distinction between finding limit candidates and validating those candidates. Following reinvention, however, both students were able to articulate differences between these two distinct processes. Further, they demonstrated an awareness that the definition serves the mathematical role of, as Jason put it, “conceptualizing what it is we’re doing when we’re talking about the abstract concepts of getting closer and infinitesimals....”

In sum, evidence from the second experiment corroborates one of the central findings from the first experiment – in contrast to my initial conjecture that students must first become aware of the distinction between finding and validating so as to be properly motivated to reinvent the definition of limit, both pairs of students displayed evidence suggesting reinvention may provide students an experience that supports the emergence of awareness of that distinction.

6.3 – Summary

In this chapter, I have elaborated seven themes that emerged from the second experiment (Figure 6.25) that address my two research goals. These themes are presented both as results of the second experiment and as a lens for understanding key issues that were implicated in Chris and Jason’s reinvention process.
Emergent Themes

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Figure 6.25 – Emergent Themes

Themes 1 and 2 point to conceptual difficulties students may experience as they attempt to characterize the limit concept. It is worth noting that these two themes also surfaced in the first experiment. Like Amy and Mike, Chris and Jason’s propensity towards reasoning from an $x$-first and potential infinity perspective hindered their initial attempts to articulate a precise definition. Figure 6.26 captures the key formulations in the evolution of their definition prior to resolving these conceptual difficulties.

Chris and Jason’s Evolving Definition of Limit

| Definition #1: | $y$ takes on values closer to the limit in question as you take $x$-values closer to the point at which you’re evaluating the limit. (Session 2) |
| Definition #2: | *When evaluating a limit, $y$ takes on values closer to $L$, the limit in question as you take $x$-values closer to the point at which you’re evaluating the limit. The limit need not equal the function’s value at that point.* (Session 3) |
### Definition #3:
For some function $y = f(x)$ a limit $L$ exists at a point $x = a$ when: 1) On some interval $[b, a]$ such that $b < a$, as $x$ approaches $a$ in the interval, $y$ approaches some value $M$. 2) On some interval $[a, c]$ such that $a < c$, as $x$ approaches $a$ within that interval, $y$ approaches some value $N$. 3) $M = N$ (Session 4)

### Definition #4:
For some function $y = f(x)$ a limit $L$ exists at a point $x = a$ when: 1) On some interval $[b, a]$ such that $b < a$, as $x$ approaches the point $a$ in the interval, $f(x)$ approaches $f(a)$. 2) On some interval $[a, c]$ such that $a < c$, as $x$ approaches $a$ within that interval, $f(x)$ approaches $f(a)$ (Session 4).

### Definition #5:
For some function $y = f(x)$, a limit $L$ exists at a point $x = a$ when: 1) On some interval $[b, a]$ when $f$ is increasing, such that $b < a < c$, as $x$ approaches $a$, $f(x)$ approaches the max value on $[b, a]$. (Session 5)

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**Figure 6.26 – Chris and Jason’s Evolving Definition of Limit:**

**Formulations 1-5**

Theme 3 addresses the second of my two research goals – to inform the design of principled instruction in relation to learning the concept of limit. Shifting Chris and Jason’s focus during Session 6 to defining *closeness* instead of *infinite closeness* was pedagogically beneficial, and it appeared that the use of a step function was a context conducive to eliciting a productive definition of *closeness*. Evidence from Sessions 6 and 7 suggests that the step function task elicited an important shift from an $x$-first to a $y$-first perspective, the result of which was noticeable progress towards a precise definition of limit. This shift to a $y$-first perspective also facilitated Chris and Jason’s subsequent recognition of the distinction between finding and validating limits. Specifically, their focus on the $y$-axis appeared to foreground the presence of a $y$-value, $L$, about which they were constructing.
progressively tighter bounds. This led them to explicitly wonder why a particular $L$ was the focus of their graphical exploration, which, in turn, appeared to spur their recognition that the definition is not a means by which one would find a limit candidate. Thus, evidence in the second experiment suggests a relationship between a student shifting to a $y$-first perspective and distinguishing between the actions of finding and validating limits. Theme 4, then, addresses cognitive accommodations Chris and Jason made as a result of their engagement with principled instruction.

Theme 5 points to problems that can arise when students attempt to precisely articulate one of the elemental components of the limit concept. Evidence from the second experiment suggests that in the context of reinvention, students may characterize limit in a manner that specifies the bounds of the $x$-interval corresponding to the predetermined $y$-interval. This desire for specificity in regards to proximity along the $x$-axis is understandable, given that a coherent, mathematically valid definition of limit requires the precise articulation of numerous other details and subtleties. It is not surprising, therefore, that students may balk at the notion of establishing the existence of an $x$-interval containing $a$ without also having to define its endpoints. Indeed, Chris and Jason repeatedly voiced reluctance to abandon the self-formulated constructs of CEnter and CExit. Figure 6.27 captures the key formulations in the evolution of their definition following their shift to a $y$-first perspective during Session 6 and their recognition during Session 7 of the distinction between finding and validating limits. The
reader will note that Definitions 6-8 predated the resolution of the issues addressed by Theme 5.

<table>
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<tr>
<th><strong>Chris and Jason’s Evolving Definition of Limit</strong></th>
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<tr>
<td><strong>Definition #6:</strong> CEnter&lt;a&lt;CExit. If CEnter=a but CExit≠a or CExit=a but CEnter≠a, we do not have a limit L at a. If CEnter=CExit then we do have a limit and L=CTop=CBottom [and] a=CEnter=CExit. <strong>(Session 7)</strong></td>
</tr>
<tr>
<td><strong>Definition #7:</strong> CEnter&lt;a&lt;CExit. If CEnter=a but CExit≠a or CExit=a but CEnter≠a, L is not the limit at a. Doesn’t necessarily mean there is no limit, just that you guessed wrong. If CEnter=CExit then we do have a limit and L=CTop=CBottom [and] a=CEnter=CExit. <strong>(Session 7)</strong></td>
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</table>
| **Definition #8:** 1) Come up with a guess.  
2) Determine a closeness interval around your guess.  
3) Let CEnter equal the last x-value [before ‘a’] for which we become close. Let CExit equal the first x-value after ‘a’ for which we are no longer close.  
4) i) if CTop=CBot=L, then L is your limit  
   ii) if CEnt=a and CExit≠a or CExit=a and CEnt≠a then L is not the limit  
   iii) if CEnt<a<CExit then shrink your closeness interval and retry at Step 2. **(Session 8)** |
| **Definition #9:** 1) Come up with a guess, L.  
2) Determine a closeness interval $L \pm z$ around your guess.  
3) If: there exists an $x_1<a$ such that $L+z>f(x)>L-z$ is true for all $x$ between $x_1$ and $a$ AND an $x_2>a$ such that $L+z>f(x)>L-z$ is true for all $x$ between $x_2$ and $a$ then shrink your closeness interval and try again. If you can’t shrink your interval anymore, then L is your limit.  
If not: then L is not your limit.  
***(Session 10 – Final Definition)*** |

Figure 6.27–Chris and Jason’s Evolving Definition of Limit: Formulations 6-9

The final two themes point to insights I gained only after retrospectively reflecting on the entire reinvention process. Theme 6 has pedagogical implications. Chris and Jason’s reinvention of the definition of limit suggests that students who have never
encountered the formal definition have the potential to reinvent it by building upon their informal understandings through engagement in purposefully designed tasks. Theme 7 addresses both research objectives – Chris and Jason’s ability to coherently interpret other mathematically valid formulations of the definition of limit appeared to be supported by the experience of first constructing their own precise articulation. Evidence of this arose in the first experiment as well. As I mentioned in my summary of the first experiment, this finding not only offers insight into what might support coherent student reasoning in a complex domain, but it also underscores the value in implementing instructional tasks designed to capitalize on students’ informal reasoning.

In closing, it is important to note that in addition to the seven themes and the evolving characterization of limit that emerged during the second teaching experiment, the phases of reinvention that materialized are themselves a result of the research. The five main phases identified in this chapter serve as a lens for understanding the dialectic unfolding of reinvention and instruction during the second experiment.

**Phase 1:** Assessment of Students’ Informal Understanding and Attempts to Motivate Necessity

**Phase 2:** Initial Attempts to Define Limit via Graphical Conversations

**Phase 3:** Explicit Attempts to Define Closeness Using a Step Function

**Phase 4:** Refinement of Definition of Limit at a Point with Increased Notational Precision

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Phase 5: Attempted Resolution of Central Issues and Completion of Reinvention Process

These phases of coupled reinvention and instruction, along with the four phases which characterized the first experiment, contribute to an epistemological analysis (in the sense of Thompson & Saldanha, 2000) of the concept of limit of a real-valued function and its formal definition.
Chapter 7 – Conclusions

This seventh and concluding chapter consists of five parts. In Section 7.1, I summarize the central findings of the study, focusing on thematic elements which emerged during the two teaching experiments. In Section 7.2, I discuss the pedagogical implications of the central findings presented in Section 7.1. In Section 7.3, I describe how the study helps to address a gap in the research base on students’ understanding of limit. In Section 7.4, I discuss three limitations of the study germane to the specific research objectives that guided my work. Finally, in Section 7.5, I suggest possibilities for future research, based partially on the implications of the limitations discussed in Section 7.4.

7.1 – Central Findings

7.1.0 Introduction

This dissertation study had two central research objectives:

1. To develop insight into students’ reasoning in relation to their engagement in instruction designed to support their reinventing the formal definition of limit, and;

2. To inform the design of principled instruction that might support students’ attempts to reinvent the formal definition of limit

The reader may recall that the first objective listed above was at the foreground of my study, and efforts to address it were guided by three central research questions:
Can students fitting the specified selection criteria, and in the context of guided reinvention, reinvent a definition of limit which captures the intended meaning of the conventional $\varepsilon$-$\delta$ definition?

In the process of reinvention, what cognitive difficulties do students experience which hinder their progress, and how are such difficulties resolved?

In their attempts to reinvent the definition of limit, do students reason from an $x$-first perspective initially and, if so, what types of tasks help initiate a shift to a $y$-first perspective?

Further, this first objective was set against the broader background goal of contributing to an epistemological analysis (in the sense of Thompson & Saldanha, 2000) of the concept of limit of a real-valued function and its formal definition. To contribute to an epistemological analysis of concept X means to develop insight, through empirical inquiry within instructional settings, into the question “What is involved in coming to understand concept X?” The eight themes presented in Chapter 5 and the seven themes presented in Chapter 6 collectively provide insights that inform how students may come to understand the concept of limit and its formal definition in relation to their engagement with purposefully designed task sequences. While the themes unique to each teaching experiment in this study are certainly noteworthy in their own right, I choose to focus my discussion in this chapter on those themes which emerged in both teaching experiments, and thus, speak to commonalities between the two case studies and participating pairs of
students. Such a discussion also allows me to address the three research questions which focused my study. In Section 7.1.1, I focus on four thematic findings which characterize student reasoning within the process of reinvention. In Section 7.1.2, in an effort to contribute to an epistemological analysis of limit, I provide my own genetic decomposition (in the sense of Cottrill et al., 1996). Finally, in Section 7.1.3, I make a case for the potential benefits of a Lakatosian (1976) approach to mathematics, based on two prominent retrospective findings from my study.

7.1.1 Thematic Findings within the Process of Reinvention

In this section, I focus on four thematic findings which characterize student reasoning within the process of reinvention and which help to address the second and third research questions. The first two thematic findings point to cognitive difficulties specific to the limit concept which hindered the students’ reinvention of the definition of limit. The third thematic finding describes how engagement in a particular instructional task appeared to have supported the students in resolving said cognitive difficulties\(^{62}\), while the fourth thematic finding characterizes the students’ motivation for the need for a formal definition of limit as a byproduct of their engagement of the reinvention process.

Although the two pairs of students in this study reinvented the definition of limit in two distinct ways, elements of their respective reinvention processes were

\(^{62}\) As indicated in the preceding chapters, while defining \textit{closeness} did eventually lead the first pair of students to reason from an \textit{arbitrary closeness} perspective, this was not the case for the second pair of students. Nevertheless, defining \textit{closeness} did support the second pair of students in resolving other identified cognitive difficulties.
strikingly similar. In previous chapters, I have discussed a distinction, set forth by Larsen (2001), between the thought process required to find limit candidates and the subsequent inverse thought process required to validate those candidates. Evidence from both teaching experiments addresses the third research question – it does, in fact, appear that students are likely to employ an x-first perspective in their initial attempts to define limit and to view the utilization of a y-first perspective as counterintuitive. An x-first perspective is consistent with the thought process required to find limit candidates. A likely explanation for the students’ initial utilization of an x-first perspective is that their prior experience with limits and exploration of functional behavior was overwhelmingly focused on mastering algebraic, tabular, and graphical techniques designed to help one determine a reasonable guess for the value of a limit. Such techniques all require one to reason from an x-first perspective. The counterintuitiveness of a y-first perspective appeared to arise out of the students’ expectation that the variable represented on the y-axis always serve as the dependent variable – both pairs of students expressed surprise that reasoning from a y-first perspective was not only mathematically valid, but furthermore, was productive for meeting their objectives. In sum, one of two central hindrances to each pair of students making progress towards reinventing the definition of limit was their initial insistence on employing an x-first perspective and their view of a y-first perspective as counterintuitive.

The second hindrance to the students’ efforts to reinvent the definition of limit was their struggle to find a suitable alternative to the potential infinity perspective
they initially utilized in reasoning about limit. I remind the reader here that Tirosh (1991) describes potential infinity and actual infinity, in relation to the history of mathematical development, as follows – "[T]he two competing ideas of infinity were potential infinity in which a mathematical process can be carried out for as long as required to approach a desired objective, and actual infinity in which one contemplates the totality of infinity, through, for example, conceiving the totality of all natural numbers at one time" (p.200). Evidence in both teaching experiments suggests that it is not, as Williams (2001) proposes, the acceptance of an actual infinity perspective that leads students to reason coherently about the conventional ε-δ definition, but rather the utilization of the notion of arbitrary closeness as a means of operationalizing infinite closeness which supports students in reinventing a definition of limit that captures the intended meaning of the conventional formulation. In both experiments, a focal point of the students' reinvention efforts was defining precisely what it means for a function to get infinitely close to a value along both the x- and y-axes. However, both pairs of students explicitly expressed frustration over trying to characterize infinite closeness, reasoning that doing so is impossible since "you can always get closer," and that therefore, there is no such thing as being infinitely close. It is worth noting that the students' recognition that one could "always get closer" is an indication of their awareness of the limitations of a potential infinity perspective. To be clear, then, both pairs of students appeared to initially follow the same reasoning trajectory in regards to issues related to infinity – both pairs began by reasoning from a potential infinity perspective, with
their focus on describing the incremental completion of the infinite limiting process. Both pairs of students subsequently recognized the limitations of such a perspective, noting in distinct ways the impossibility of completing an infinite process in a finite amount of time. This led both pairs of students to seek a suitable alternative perspective. At this point, the two pairs diverged in their reasoning. The first pair of students spontaneously employed the notion of *arbitrary closeness* to encapsulate the infinite limiting process. This decision to operationalize *infinite closeness* via the notion of *arbitrary closeness* marked a critical moment in the first pair's reinvention of the formal definition of limit. The second pair of students did not utilize the notion of *arbitrary closeness* and instead swept the cognitive issues of a potential infinity perspective "under the rug," in the sense that they seemingly resolved the cognitive dilemma of imagining the carrying out of an infinite process by simply accepting that the end of the process must somehow mysteriously happen. The reader may recall that the second pair of students interpreted the notion of *arbitrarily small* as representing a *single* definition of closeness, as opposed to *all* definitions of closeness. This interpretation ultimately hindered them in reinventing a definition of limit which encapsulated the infinite limiting process.

In reference to supporting the students' in overcoming the two hindrances previously described, defining *closeness* prior to defining *infinite closeness* proved to be a watershed moment in both teaching experiments. Albeit under different circumstances, both pairs of students defined *closeness* outside of the context of *limit at a point* and subsequently used that definition to operationalize *infinite*
closeness in the context of limit at a point. The first pair of students defined closeness in the context of limit at infinity, while the second pair of students defined closeness completely outside of the context of limit. In both cases, defining closeness in an incrementally restrictive fashion (i.e., 10, 2.5, 1.5, .5, etc.) appeared to initiate important cognitive shifts for the students. First, the iterative nature of this defining process gave the students a way to imagine how one might define closeness at any level of desired specificity, thus allowing them to think of infinite closeness as a notion that can be characterized in a hypothetical manner (i.e., as closeness at any level of desired specificity). While only the first pair of students subsequently encapsulated the limiting process by utilizing the notion of arbitrary closeness, operationalizing infinite closeness by first defining closeness appeared to support both pairs of students in making significant and profound refinements to their respective definitions of limit. Second, defining closeness also appeared to support the students in adopting a y-first perspective. There are a couple of possible explanations for why such a phenomenon occurred. One possible explanation for the students adopting a y-first perspective was that in both teaching experiments, the respective defining tasks were designed to focus the students' attention on the y-axis. Specifically, the students were given a specific error tolerance along the y-axis and were asked how they might characterize what it means for a function to be within that error tolerance of a pre-determined y-value, L. Further, in both cases, the defining task was purposely designed to deemphasize

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63 i.e., in response to the prompt, "For every single one of its x-values, how would you write out what it means for that function to be close to a pre-determined value L?"
the x-axis – the first pair of students defined *closeness* in the context of *limit at infinity*, which only has a condition of infinite closeness on the y-axis; the second pair of students, instead of being asked what was true about the function for *particular x-values* (or *particular x-intervals*), were asked to characterize functional behavior *uniformly* across the x-axis. So, in sum, the adoption of a y-first perspective by both pairs of students could be partially explained by particular design details of the respective defining tasks in which they were engaged. A second possible explanation for the students' adoption of a y-first perspective was that both pairs of students became aware that their prior characterizations of limit were deficient. Hence, both pairs of students became amenable to alternative approaches to defining limit, and further, as they began employing a y-first perspective, they may have recognized that such a perspective allowed them to address issues that their x-first characterizations had been unable to resolve.

On a related note, in both teaching experiments, the students' respective adoption of a y-first perspective appeared to facilitate a subsequent recognition of the distinction between finding limit candidates and subsequently validating those candidates. In both cases, the students' focus on the y-axis appeared to foreground the presence of a y-value, $L$, about which they were constructing progressively tighter bounds. This led them to wonder explicitly why a particular $L$ was the focus of their graphical exploration, which, in turn, appeared to spur their recognition that the definition they were constructing presupposes the existence of a limit candidate. Thus, evidence in both teaching experiments suggests a relationship between a
student adopting a y-first perspective and distinguishing between the actions of finding and validating limits. The ability to distinguish these two appears to be at least partially supported by the experience of contemplating the subtleties inherent to the limit concept while attempting to formulate a precise characterization of it. Further, recognizing the distinction between finding and validating limit candidates appears to support students in coming to see a need for a rigorous definition. This suggests, at least in the case of limit, that the necessity principle set forth by Harel (2001) might be addressed as students are in the process of constructing a precise definition for the concept, rather than prior to their engagement in guided reinvention. Put another way, the activity of attempting to construct a precise definition of limit might simultaneously increase learners' recognition of the need for such formality, and thereby constitute a medium propitious for the emergence of a necessity principle of sorts.

One might be tempted to infer from the central findings I have presented thus far that inducing a cognitive shift in students from an x-first and potential infinity perspective to a y-first and arbitrary closeness perspective subsequently places greater value on the latter and devalues the former. I would argue to the contrary, however, as evidence from both experiments suggests that it is the ability to employ both perspectives flexibly that allows someone to develop a rich and robust understanding of the limit concept and its formal definition. Reasoning from an x-first and potential infinity perspective might support someone in developing a sense for the essence of the limit concept — i.e., that limit describes the local behavior of a
function as its independent variable approaches a particular $x$-value. Meanwhile, as evidence from both teaching experiments suggests, reasoning from a $y$-first and arbitrary closeness perspective likely supports learners in understanding the intricacies and subtleties of the formal definition of limit. Thus, to be clear, evidence from the teaching experiments suggests that reasoning from both an $x$-first and potential infinity perspective and a $y$-first and arbitrary closeness perspective is fundamental in supporting students' efforts to develop a rich and robust understanding of the limit concept and its formal definition.

7.1.2 Contributing to an Epistemological Analysis

The reader may recall that to contribute to an epistemological analysis (in the sense of Thompson & Saldanha, 2000) is to gain insight into what is entailed in coming to understand a particular mathematical idea in relation to engagement in instruction designed to support the development of that understanding. The central objective of this dissertation study was to develop insight into students' reasoning in relation to their engagement in instruction designed to support their reinventing the formal definition of limit. This objective was set against the broader background goal of contributing to an epistemological analysis of the concept of limit of a real-valued function and its formal definition.

The genetic decomposition offered by Cottrill et al. (1996) can be thought of as a conjectured model of how students may come to formalize their understanding of limit. In this sense, a genetic decomposition can be thought of as a contribution to
an epistemological analysis. In Chapter 2, I noted the ways in which the genetic decomposition proposed by Cottrill et al. lacked empirical evidence that could inform the latter stages of their model of student reasoning. The aim of my research was to help elucidate those latter stages. In Figures 7.1 and 7.2, I provide my own genetic decomposition, based on data gathered during the two teaching experiments which formed this dissertation study. A few details are worth noting. First, unlike the genetic decomposition presented by Cottrill et al., the genetic decomposition presented here focuses only on the transition from informal to formal reasoning (i.e., stages 5-7 in the genetic decomposition offered by Cottrill et al.). Thus, this genetic decomposition is based on the assumption that students already have an informal understanding of limit. Specifically, this means that students are able to:

1) Discuss when a limit does exist and why
2) Discuss when a limit does not exist and why
3) Determine limits for both finite and infinite situations
4) Sketch graphs satisfying given conditions related to both finite and infinite limits
5) Provide an informal definition of limit that demonstrates viable conceptual understanding

64 It is worth noting that the methodology employed in this dissertation study was different than that utilized in the study conducted by Cottrill et al. (1996), in that student reasoning about limit in my study was in the context of reinvention, as opposed to interpretation, of the formal definition. Hence, the genetic decomposition presented here was based on data collected in an experimental setting distinct from that experienced by the students in the Cottrill et al. study.
Second, I choose to split the genetic decomposition into two parts. Part 1 characterizes student reasoning prior to the instructional intervention of encouraging the students to define \textit{closeness}. Conversely, Part 2 characterizes students' reasoning subsequent to this instructional intervention. Third, unlike the genetic decomposition proposed by Cottrill et al., the one presented here is not in a strict numeric stage format, but instead is presented in the form of a flow chart. This was done to maximize the explanatory power of the cognitive model. In particular, this form allows for the description of multiple cognitive difficulties being experienced by the students simultaneously. A description of each part of the genetic decomposition follows.
The first part of the genetic decomposition can be summarized as follows:

Evidence from this study suggests that in response to being charged with the task of defining what it means for a function to have a limit $L$ at $x=a$, students' initial characterizations are likely to be cast from an $x$-first perspective and include vague

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65 Shaded boxes and arrows denote noteworthy instructional interventions, and thus, are not, strictly speaking, part of the genetic decomposition. However, given the dialectic between student reasoning and instruction, it is reasonable, given the study's methodology, to include the initial task which situated student reasoning; and, in Part 2 of the genetic decomposition, the instructional intervention which initiated the resolution of students' cognitive difficulties. The un-shaded boxes and arrows in this diagrammatic representation represent the students' ways of reasoning in the context of reinventing the formal definition of limit, and thus, constitute the core of this genetic decomposition.
descriptions of infinite closeness. The first definition provided by Amy and Mike is one such example: \( f \) has a limit \( L \) at \( x = a \) provided as \( x \)-values get closer to \( a \), \( y \)-values get closer to \( L \). Upon recognizing that vague descriptions of infinite closeness mischaracterize particular functions as having limits at \( x \)-values for which no limit exists (e.g., functions with jump discontinuities), students attempt to flesh out what they mean by \( x \)-values getting closer to \( a \) and \( y \)-values getting closer to \( L \). In their attempts to describe infinite closeness with greater precision, students' focus appears to turn to describing the limiting process. Attempts to summarize the infinite limiting process appear to lead students to subsequently utilize a potential infinity perspective. The inability to describe the completion of an infinite process in a finite amount of time appear to raise students' awareness of the limitations of a potential infinity perspective, and in turn, motivate the students to seek a new perspective. However, despite the motivation to adopt a new perspective, all four students in this study had difficulty finding a suitable alternative to the potential infinity perspective they initially employed, and also continued to reason from an \( x \)-first perspective.

Evidence from the dissertation study suggests that in response to the difficulties described in the preceding paragraph, students may benefit greatly from being asked to define closeness in a concrete and increasingly restrictive manner. Part 2 of the genetic decomposition, shown in Figure 7.2, illustrates the continued evolution of student reasoning about limit in the context of reinvention.
Defining closeness appears to initiate two significant cognitive shifts in student reasoning. First, evidence suggests that defining closeness may help students shift from an x-first perspective to a y-first perspective. The shift from an x-first to a y-first perspective in this study can largely be explained by the nature of the instructional interventions employed. The adoption of a y-first perspective, in turn, appears to raise students' awareness to the fact that the definition of limit relies on a pre-existing limit candidate, \( L \). This awareness supports students in distinguishing between the act of finding limit candidates and validating limit candidates. In
particular, students appear to gain awareness that the definition of limit is not designed to find limit candidates.

Second, defining *closeness* in a concrete and increasingly restrictive manner appears to lead students to recognize how to operationalize *infinite closeness*. Whereas prior to the instructional intervention students in this study expressed frustration over trying to define *infinite closeness*, the act of defining *closeness* in a concrete and increasingly restrictive manner appeared to allow them to momentarily set aside the challenge of having to actually complete the infinite limiting process. Shifting their attention away from the insurmountable task of describing the incremental completion of an infinite limiting process appeared to provide the students a suitable mental environment for recognizing that they could use the notion of *arbitrary closeness* to encapsulate the infinite limiting process. The adoption of an arbitrary closeness and y-first perspective, along with the recognition of the distinction between finding and validating limit candidates, appears to support students in reinventing, and reasoning coherently about, a definition synonymous to the conventional e-δ definition.

### 7.1.3 Retrospective Findings

Numerous studies (Cornu, 1991; Dorier, 1995; Fernandez, 1994; Gass, 1992; Larsen, 2001; Tall, 1992; Tall & Vinner, 1981; Williams, 1991) have indicated that developing coherent understanding of the formal definition of limit is decidedly complex. This dissertation study is unique, however, in that inferences about how
students reason about the formal definition of limit are made in the context of reinvention, as opposed to interpretation, of the formal definition. I am not aware of other studies that have taken a similar approach.

Despite two different instructional trajectories, evidence from both teaching experiments suggests that reinvention may be a context conducive to developing students’ understanding of limit and its formal definition. Although neither pair of students had previously seen the formal definition, both pairs were able to construct a definition capturing much of the intended meaning and quantification structure of the conventional \(\varepsilon-\delta\) definition\(^{66}\), thus answering the first research question in the affirmative. Further, the process of reinvention appeared to support the students in coherently interpreting other mathematically valid formulations of the definition. For example, following the reinvention process, both pairs of students were presented the conventional \(\varepsilon-\delta\) definition of limit for the first time and subsequently were able to articulate explicit mappings between their definition and the conventional definition. The students’ successful reinvention of the definition and subsequent interpretation of other mathematically valid formulations of the definition suggests that in the context of limit, guided reinvention may indeed be an approach conducive to developing a depth of understanding not likely to be attained by students who are merely shown the formal definition and asked to interpret its meaning.

\(^{66}\) As noted in Chapter 5, the first pair of students constructed a definition of limit synonymous to the conventional \(\varepsilon-\delta\) definition.
It is worth noting that the nature of the discussions in which the students in this study engaged is reminiscent of the mathematical-social interactions characterized initially by Lakatos (1976) in his famous book, *Proofs and Refutations*, and more recently illustrated by Larsen and Zandieh (2007) in the context of abstract algebra. Although the instructional trajectory in this study was both heavily guided and strongly scaffolded, the students nevertheless were provided a medium in which they could be mathematically creative. In a manner consistent with the dynamic set forth by Lakatos, all four students each took on both the role of conjecturer and refuter, seeking to build upon their informal understandings of limit by iteratively refining a self-constructed definition of the concept. Interestingly, upon reflection, the students acknowledged the uniqueness of this type of active role in mathematical learning. One such particularly illustrative excerpt from the second teaching experiment highlights the potential benefits of engaging students in a learning environment that aims to stimulate authentic mathematizing, and provide them with an experience ostensibly akin, in spirit, to that of mathematicians such as Cauchy and Weierstrass in the development of analysis.

Craig: What did you guys accomplish in these past ten weeks?
Jason: ...What did we accomplish? A level of mental gymnastics not encountered in any of my studies ever...[C]lass is usually taught in the, “I’m [the] lecturer, I’m going to tell you, now you learn this. And then recite it back to me later when I ask.” Okay, and that’s not what we were doing in these interviews....Okay, so I know that the limit is this value that the function intends to reach. Okay, now deconstruct that into some fundamentals, some elements, and try to describe those elements. And try to put them together in a cohesive picture. So take what you know, and then figure out why you know that....Umm, yeah, so, first and foremost [it] was getting the
opportunity to, to do those mental gymnastics. Uh, start at the real general level – height function intends to reach – getting down to the nitty-gritty specifics. And then, okay, the limit is this. And then having Craig go, well what if I just do this? And erase one dot that makes a critical change to what we now know. Well, now we need to take what we know, well that’s now our top, you know, that’s our general level. Now we’ve got to get back into the nitty-gritty and get even more nitty and grittier. Restate, come back, and look for counterexamples....I’ve never had the experience of feeling like, uh, my study was more fruitful with others around....I can always get through the material more, uh, sensibly and more efficiently if it’s just me dealing with the material. But that’s probably because up until this point I’ve only ever been trying to understand the broad general level of any given subject.

All four students commented during the teaching experiments that the type of learning with which they were engaged was different than anything they had previously experienced in an academic setting. For instance, on more than one occasion, Jason commented that he felt as though he and Chris were acting like “real mathematicians.” The depth with which both pairs of students reasoned about non-trivial aspects and subtleties of the limit concept towards the latter phases of the respective teaching experiments suggests that a Lakatosian approach to learning mathematics may indeed support the development of advanced mathematical thinking (in the sense of Tall, 1991).

7.2 – Pedagogical Implications

The central findings presented in Section 7.1 inform pedagogy in some important ways. First, evidence in this study suggests that students may benefit from having their ability to reason from a y-first perspective developed prior to Calculus. Based on data gathered in this study, students’ experiences with functions
appear to be predominantly from an $x$-first viewpoint. Developing students' understanding of inverse functions may be one way of providing them an opportunity to reason from a $y$-first perspective. A significant exploration of inverse functions is often left to the end of a Pre-calculus course, and thus, many times only receives a superficial treatment. The structure of Pre-calculus courses could be changed to increase the opportunities for students to think flexibly about functions. Further, when students first encounter the formal definition of limit, they would likely benefit from the observation that the formal definition requires reasoning from a $y$-first perspective distinct from the $x$-first perspective commonly used when describing functional behavior.

Second, evidence from both teaching experiments underscores the value of having students define $closeness$ in a concrete and increasingly restrictive manner. In their attempts to define limit, the students in this study became paralyzed by the prospects of characterizing what it means to be infinitely close. However, when they were able to set aside the cognitive dilemma of incrementally completing an infinite process, and were asked only to define what it means to be close (in a concrete and finite sense) to a particular $y$-value, $L$, the students were then able to recognize how they might operationalize infinite closeness by use of their definition of $closeness$. Having students define what it means to be close to some pre-determined value $L$, either in the context of limits at infinity or in the context of a step function, may support them in reasoning coherently about infinite closeness.
Third, evidence from the first teaching experiment suggests that mathematics educators may be well advised to have their students reason about, and define limit at infinity prior to engaging with the notion of limit at a point. The definition of limit at infinity is less cognitively demanding than the definition of limit at a point, and, as the first teaching experiment suggests, is a definition which may serve as a template for reinventing the definition of limit at a point. Activities designed to develop students' understanding of the notion of limit at infinity, as well as their understanding of the related notion of the limit of a sequence, may provide the type of cognitive support necessary for subsequently developing students' understanding of the more complex definition of limit at a point.

A strong informal understanding of limit was a prerequisite for being selected for the teaching experiment phase of this study. Mike, Amy, Jason, and Chris each began the reinvention process with the ability to do each of the following:

1) Discuss when a limit does exist and why
2) Discuss when a limit does not exist and why
3) Determine limits for both finite and infinite situations
4) Sketch graphs satisfying given conditions related to both finite and infinite limits
5) Provide an informal definition of limit that demonstrates viable conceptual understanding

It is my belief that a student's ability to reinvent, and/or reason coherently about, the formal definition of limit would be severely compromised if he or she did not
possess each of the aforementioned skills. Thus, I believe an exploration of limits should start with a focus on the development of the informal understandings listed above; a study of the formal definition should not mark the beginning of a student's study of limits, but rather should arise more naturally in an effort to add greater precision to informal characterizations.

Before proceeding to Section 7.3, an important observation is worth making regarding the pedagogical implications I have presented above. Evidence from this dissertation study suggests that classroom, or school, mathematics, as it is currently conceived in the United States, is distinct from what it means to really do mathematics. Jason's response to the question, What did you guys accomplish in the past ten weeks?, underscores the incongruence between his experience as a participant in this study and his previous experiences with mathematics.

Craig: What did you guys accomplish in these past ten weeks?
Jason: ...[W]hat did we accomplish? A level of mental gymnastics not encountered in any of my studies ever....[C]lass is usually taught in the, "I'm [the] lecturer, I'm going to tell you, now you learn this. And then recite it back to me later when I ask." Okay, and that's not what we were doing in these interviews....I've never had the experience of feeling like, uh, my study was more fruitful with others around.

There is a bit of a paradox, then, in making suggestions for the mathematics classroom as it is currently conceived. The insights gained in this study about how students reason about the concept of limit were, I believe, a result of placing students in an environment distinct from the mathematics classroom. Indeed, the four students in this study reasoned about limit in an environment more
representative of that experienced by actual mathematicians. Understanding and progress was made as a result of a sequence of conjectures and refutations. The benefits of such an environment are evident in the findings presented in Section 7.1. I suggest, therefore, that the mathematics education community consider what the goal of classroom mathematics is. If the goal is truly for students to learn how to do mathematics, it seems, then, that the classroom environment may very well need to be reconceived. It would be easy for one to respond to this point by noting the logistical impossibilities (or at least, difficulties) in having students engage regularly in a classroom setting in the type of interactions chronicled in this dissertation. I acknowledge that both the number of students in the average American classroom, and the amount of content teachers are traditionally asked to cover make a pedagogical approach similar to the methodology employed in this study infeasible. This observation, however, only furthers my argument. A choice needs to be made: either the size of the American classroom can be decreased and the amount of mathematics addressed annually can be reduced so that students can engage authentically in mathematics, or we can continue to teach students something other than what it truly means to do mathematics.

7.3 – Contributions to the Field of Mathematics Education

The findings reported in the previous section, as well as in Chapters 5 and 6, serve both as novel and unique contributions to the field of mathematics education
and as evidence which corroborates previous findings related to the limit concept. In this section, I explicate the specific contributions of this dissertation study.

The review of the literature presented in Chapter 2, pertaining to students' learning and understanding of the limit concept, reveals that while extensive research (Bezuidenhout, 2001; Cornu, 1991; Davis & Vinner, 1986; Tall, 1992; Tall & Vinner, 1981; Williams, 1991) has delineated the misconceptions students commonly develop as they study the concept of limit informally, significantly less is known about how students might come to reason coherently about the formal definition of limit. The study reported here was designed to address this gap. Specifically, my central objective was to contribute to an epistemological analysis (in the sense of Thompson & Saldanha, 2000) of the concept of limit and its formal definition. The study was a success in this regard – the emergent themes presented in Chapters 5 and 6, the tracing of the two pairs of students' evolving definitions, and the phases of instruction which characterized the two teaching experiments all collectively inform what is entailed in coming to reason coherently about the limit concept and its formal definition. This contribution helps address the latter steps of the genetic decomposition proposed by Cottrill et al. (1996). The reader may recall that while Cottrill et al. found evidence suggestive of how students may come to find limit candidates, they lacked substantive evidence of how students may come to validate those candidates. The research presented here helps demarcate how students reason about both of these mental processes, and in particular, underscores
**guided reinvention** as a context which supports students in coming to make the distinction between these two processes.

The findings of this study are unique in that they are based on inferences about how students reason about the formal definition of limit in the context of *reinvention*, as opposed to *interpretation*, of the formal definition. While some studies (Cottrill et al., 1996; Fernandez, 2004; Larsen, 2001) have discussed students’ interpretations of the formal definition of limit, I am not aware of other studies that have modeled student reasoning about limit in the context of reinvention. The findings of this study contribute to others’ efforts (including Larsen, 2004; Rasmussen & King, 2000; Zandieh & Rasmussen, 2007) to develop local instructional theories for undergraduate mathematics in a manner consistent with the goals of developmental research and the realistic mathematics education (RME) program.

Finally, the research presented here serves to corroborate previous findings related to the limit concept. Most prominently, evidence emerged in both teaching experiments which lend credence to Larsen’s findings (2001) that students’ facility with the formal definition of limit is supported by their ability to employ a thought process opposite of that used to reason informally about limits (i.e., a y-first perspective, as opposed to an x-first perspective). Further, there was compelling evidence from this study that corroborates Williams’s claim (2001) that students reasoning from a potential infinity perspective may struggle to understand the formal definition of limit. It is worth noting that unlike the two studies mentioned
here, empirical support for both Larsen’s and Williams’s respective findings arose in this dissertation study in the context of *reinvention*, as opposed to *interpretation*, of the formal definition of limit. This suggests that the limitations of reasoning from an *x*-first and potential infinity perspective extend beyond the context of interpretation of the formal definition.

### 7.4 – Limitations of the Study

As is the case with all empirical research, the study reported here included some noteworthy limitations. My discussion in this section focuses on those limitations related to the initial goals of the study.\(^{67}\)

First, participants were selected on the basis of criteria\(^{68}\) that constrain me from extending the findings of the study beyond the type of student selected for the two teaching experiments. To be clear, the four students selected for the teaching experiment are not representative of the average student having completed a three-term introductory calculus sequence. Rather, the four students were each A-level students who had not previously been introduced to the formal definition of limit. Further, each student had previously demonstrated a propensity for articulating their reasoning process in both written and verbal form, and for being more “coherence-seeking,” relatively speaking, than others in regards to consistently make sense of their experiential world as it relates to complex mathematical ideas.

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\(^{67}\) The reader may recall that the study had two central goals: 1) To develop insight into students’ reasoning in relation to their engagement in instruction designed to support their reinventing the formal definition of limit, and; 2) To inform the design of principled instruction that might support students’ attempts to reinvent the formal definition of limit.

\(^{68}\) The specific criteria for participant selection are described in Chapter 4.
Selecting students fitting these criteria was sensible given the first objective of the study. Also, considering the limited body of evidence regarding students’ reasoning about the formal definition of limit, it seemed sensible to select students who had demonstrated above-average ability in calculus so that I might gain insight into the upper boundaries of students’ capabilities with the concept. Thus, while the criteria for participant selection was justified and in line with the study’s objectives, it nevertheless limits the applicability of the subsequent findings.

A second limitation of the study relates to the methodological design – each of the teaching experiments discussed here was conducted with a pair of students\(^{69}\). While pairs research does appear to foster the explication of reasoning unique from that which is characteristic of individual interviews, there are some resultant analytic limitations. In particular, pairs research makes it difficult for the researcher to attribute specific ideas and reasoning to any one student at any one time, as the articulation of ideas and reasoning by one individual are comprised partially of their response to ideas and reasoning expressed by their partner (Simon et al., 2008). As a result, establishing a clear unit of analysis is complicated – in this particular case, although at times there were ideas and notation attributable to a specific individual, it is unsafe to assume that such ideas and notation arose in a vacuum. The emergent themes presented in this document, therefore, ought to be interpreted with this analytic challenge in mind. Where possible, and when it felt appropriate to do so, I have ascribed student reasoning to a particular student. At

\(^{69}\) For the sake of being concise, I will use the phrase *pairs research* to characterize the methodological design of the teaching experiments.
other times, the themes have been presented in the context of how a pair of students collectively reasoned about a particular idea. A second, related limitation of pairs research is that findings may not be representative of what would emerge from a study conducted with students working to reinvent the definition of limit individually. It is certainly conceivable that students working individually might struggle to reinvent a definition of limit which captures the intended meaning of the conventional $\epsilon$-$\delta$ definition. Conversely, it is also possible that conducting a similar study with individual students might allow the researcher to better understand the origins and/or motivations for students employing particular defining strategies in the process of reinvention.

Finally, a third limitation of this study relates to its restricted explanatory power in regards to what contributed to the cognitive shifts students experienced as they worked to refine their definition of limit. The study was designed to be, and is, descriptive, in that it provides snap shots of the development of student reasoning in relation to the concept of limit. The Lakatosian environment which emerged during each teaching experiment as a result of working with pairs of students was both a powerful and limiting feature of the study. Such an environment fostered substantive reasoning unique from that which might have emerged had I conducted the teaching experiments with individual students. In this sense, the Lakatosian environment supported the explication of reasoning germane to my central research objective. However, the extent to which I was able to provide explanatory analysis of individual learners’ thinking and conceptions was limited by the inability to
consistently ascribe thoughts and ideas to a particular student. Thus, while pursuing the central research objectives supported me in coming to understand how students might reason about the concept of limit and its formal definition, the study is less revealing in regards to explaining what motivated the students to employ particular defining strategies at particular times or what processes might account for significant shifts in their ways of thinking and reasoning.

7.5 – Implications for Future Research

The limitations discussed in the preceding section of this chapter point to opportunities for potentially relevant research in the future. One potentially useful line of research would be a follow-up study designed to better understand the transitions in students' reasoning (in relation to the concept of limit) by focusing on the processes that may drive the development of the conceptions documented here. Specifically, the central objective of such a study would be to gain insight that would allow for a more explanatory account of: 1) Students' decisions to spontaneously employ varying defining strategies; and, 2) The motivation that underlies shifts in student cognition which result in the resolution of pertinent cognitive difficulties. It is likely that conducting such a study with individual students, as opposed to pairs of students, would allow the researcher to better identify the source and nature of the transitions in student reasoning (Simon et al., 2008).
A second potentially productive direction for research would be to conduct a study with objectives similar to the two which drove this dissertation study, but with the pedagogical objective at the foreground of the study and the cognitive modeling objective at the background of the study. Epistemological analysis and local instructional theory collectively form a framework which supports mathematics education researchers in gaining insight into how students reason about a particular mathematical idea, which, in turn, can allow for improved pedagogical practices in relation to that idea. While the design of this dissertation study supported the modeling of student reasoning about the concept of limit, contributing to an epistemological analysis supports only one piece of the framework. However, foregrounding the pedagogical objective would allow for a research design more conducive to producing a suggested local instructional theory for the concept of limit and its formal definition.

A third possibility for future research would be to conduct a study with objectives similar to this dissertation study, but with a focus on students’ reasoning in the context of algebraic representations of limit, as opposed to graphical representations of limit. An assumption underlying the mathematical-conceptual analysis of limit provided in Chapter 3 is that understanding the conventional ε-δ definition of limit relies on one’s ability to interpret imagistic features of a function’s graphical representation. In line with this assumption, the sequence of instructional tasks used in the two teaching experiments was focused predominantly on engaging students in conversations about limit from a graphical
perspective. It is conceivable, however, that students could reinvent the definition of limit from purely an algebraic perspective. Such reinvention would likely provide insight to student reasoning about limit distinct from the themes which emerged in this study.

As I noted in Section 7.4, the four participants in the teaching experiment phase of this study were hardly representative of the average student. Indeed, these students possessed a robust informal understanding of limit that supported them in their efforts to reinvent the formal definition. Existing research (Bezuidenhout, 2001; Cornu, 1991; Davis & Vinner, 1986; Monaghan, 1991; Tall, 1992; Tall & Vinner, 1981; Williams, 1991) has documented that many students, however, struggle to attain such understanding, and possess concept images containing persistent misconceptions. Research by Williams (ibid) reveals that despite the presence of discrepant events, calculus students are resistant to refining their informal understandings of limit. The instructional trajectory presented in this dissertation study appears reliant on students' ability to proficiently do each of the following:

1) Discuss when a limit does exist and why
2) Discuss when a limit does not exist and why
3) Determine limits for both finite and infinite situations
4) Sketch graphs satisfying given conditions related to both finite and infinite limits
5) Provide an informal definition of limit that demonstrates viable conceptual understanding

If this type of foundation is, in fact, a jumping off point for developing a formal understanding of limit, it would be beneficial to gain greater insight into what might move the average student from naïve conceptions of limit to the point where they have a strong informal understanding of limit. Such research would potentially allow the findings presented in this dissertation to extend to a greater population of students.

Finally, it is worth noting that the social interactions between me, as the researcher, and the student participants necessarily influenced their reasoning during the reinvention process. For instance, it is reasonable to assume that the manner in which particular tasks were presented influenced the types of responses given by the students. One such example is the closeness task employed in both teaching experiments. In both cases, I asked the students to describe what it means for a function, $f$, to get close to a particular $y$-value, $L$, and I defined closeness to be a particular finite value. My phrasing of the task assuredly has some influence on the subsequent cognitive shift the students made from an $x$-first to a $y$-first perspective, as they were asked to engage in a task that focused their attention on the $y$-axis. Further, it is also likely that the social dynamic unique to each pair of students played a role in the reasoning that emerged in each teaching experiment. This study did not have as one of its objectives the aim of describing the influence that social interactions played in the reinvention process. Research that sought to
explain the role of social interactions between the students, as well as between the students and the researcher, might present a more complete picture of the emergence of formal limit reasoning than what is presented here.
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Appendix A – Informal Limit Reasoning Survey

Name: ________________________________
Email Address: __________________________

Directions: Please answer all of the following questions completely. Please know that the description of the reasoning you used for each task is what is of greatest interest and importance to me. With this in mind, the more detail and description you provide regarding your thought process, the better. Also, please feel free to use the back of the paper for additional writing space if needed.

Part I
In Questions 1 and 2 on the following pages, five student responses have been provided for each question. These are responses from other students who have previously taken Calculus. For each student response, please indicate whether you agree or disagree. Also, please provide justification and/or further comment for why you agree or disagree with each student response. That is, explain the thought process you used to arrive at the conclusion you made. Also, for each question, please indicate the response you feel is most appropriate.
1. Evaluate \( \lim_{x \to 0} f(x) \), where \( f(x) = \begin{cases} 
  x, & x > 0 \\
  -x, & x < 0 \\
  17, & x = 0 
\end{cases} \)

a) The limit of the function keeps getting closer to 0 but never actually reaches 0.
   Agree/Disagree (circle one)
   Justification/Response:

b) The limit exists and equals 0
   Agree/Disagree (circle one)
   Justification/Response:

c) The limit does not exist because \( f(x) \) is not defined by a single formula.
   Agree/Disagree (circle one)
   Justification/Response:

d) The limit does not exist because \( f(x) \) changes slope at \( x = 0 \).
   Agree/Disagree (circle one)
   Justification/Response:
e) The limit exists and equals 17.

Agree/Disagree (circle one)

Justification/Response:

2. Consider the function whose table of values is given below. Notice that the domain values have been represented in decimal form, but you may assume that the domain of the function is all reals.

<table>
<thead>
<tr>
<th>X</th>
<th>f(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.7</td>
<td>1.973</td>
</tr>
<tr>
<td>3.8</td>
<td>1.992</td>
</tr>
<tr>
<td>3.9</td>
<td>1.999</td>
</tr>
<tr>
<td>4.0</td>
<td>6.000</td>
</tr>
<tr>
<td>4.1</td>
<td>2.001</td>
</tr>
<tr>
<td>4.2</td>
<td>2.008</td>
</tr>
<tr>
<td>4.3</td>
<td>2.027</td>
</tr>
</tbody>
</table>

Table 1

What is the \( \lim_{x \to 4} f(x) \) ?

a) \( \lim_{x \to 4} f(x) \) exists and is equal to 1.99998

Agree/Disagree (circle one)

Justification/Response:

b) \( \lim_{x \to 4} f(x) \) exists and is equal to 6.000

Agree/Disagree (circle one)

Justification/Response:

c) \( \lim_{x \to 4} f(x) \) exists and is equal to 2

Agree/Disagree (circle one)

Justification/Response:
d) \( \lim_{x \to 4} f(x) \) keeps getting closer to 2 but never actually reaches 2.

Agree/Disagree (circle one)

Justification/Response:

e) There is not enough information given to determine the limit with certainty.

Agree/Disagree (circle one)

Justification/Response:

Part II

1. Please provide an example of a function that has a limit of 6 as \( x \to 4 \).

Note: We say that a function has a "finite" limit as \( x \to a \) if a limit exists and equals a finite number \( L \). Hence, here your example would have a finite limit of 6.

2. In Question #1 above, you were asked to provide an example of a function that has a finite limit of 6 as \( x \to 4 \). Now, consider all of the different ways a function could have a finite limit 6 as \( x \to 4 \). Below, please provide examples of the different ways a function could have a finite limit 6 as \( x \to 4 \). Feel free to use any of the standard representations for a function, such as graphical, tabular, or algebraic.
3. Making reference to #2 above, how would you convince a classmate that the examples you provided do, in fact, have a finite limit 6 as $x \to 4$?

4. Consider all of the different ways a function could fail to have a finite limit $L$ as $x \to a$ (where $a$ is a finite number). Below, please provide examples of the different ways a function could fail to have a finite limit $L$ as $x \to a$. Feel free to use any of the standard representations for a function, such as graphical, tabular, or algebraic.

5. Making reference to #4 above, how would you convince a classmate that the examples you provided do, in fact, fail to have a finite limit $L$ as $x \to a$ (where $a$ is a finite number)?
Part III

1. Sketch a single graph of a function $f$ that satisfies all of the following conditions:

$$\lim_{x \to 3^+} f(x) = 4 \quad \lim_{x \to 3^-} f(x) = 2 \quad \lim_{x \to -2} f(x) = \frac{2}{-3} = -\frac{2}{3} = 1$$

$$\lim_{x \to -\infty} f(x) = 6 \quad \lim_{x \to \infty} f(x) = \infty \quad \lim_{x \to 5^+} f(x) = -\infty$$

2. Below are five statements about limits of functions provided by students. Please rate the extent to which you believe each of the statements is accurate (1 = not accurate at all...5 = completely accurate). After each rating, please provide a brief description of your thought process.

1. 1 2 3 4 5 A limit of a function is a number or y-value past which a function cannot go.

   **Rationale/Explanation for/of your rating:**

2. 1 2 3 4 5 A limit of a function is a number that the y-values of a function can be made arbitrarily close to by restricting x-values.

   **Rationale/Explanation for/of your rating:**

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70 These five statements are borrowed from research by Williams (1991).
3. 1 2 3 4 5 A limit of a function is a number or y-value the function values get close to but never reach.

Rationale/Explanation for/of your rating:

4. 1 2 3 4 5 A limit of a function is an approximation that can be made as accurate as you wish.

Rationale/Explanation for/of your rating:

5. 1  2  3  4  5 A limit of a function is determined by plugging in numbers closer and closer to a given number until the limit is reached.

Rationale/Explanation for/of your rating:

3. Please describe in a few sentences what you understand a limit to be. That is, describe what it means to say that the limit of a function $f$ as $x \to a$ is some number $L$. 

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Appendix B – Instructional Sequence for First Teaching Experiment

Phase 1: Assessment and Attempts to Motivate Necessity (Sessions 1-3)

Session 1 – Tasks and Rationale

Task 1: What is \( \lim_{x \to 4} \frac{3}{x + 3} \)?

Task 2: What is \( \lim_{x \to 4} \frac{x^2 - 2x - 8}{x - 4} \)?

Follow up questions:
Is your approach to determining these limits different in any way? With what degree of certainty can you say that the limit is what you say it is? Can you justify the limit is what you say it is?

Purpose: The purpose of these first two tasks was to establish the extent to which students believe algebraic techniques validate the candidacy of limits. These two tasks were designed to contrast with Task 3, wherein the students were provided a tabular representation of a function. Collectively, these three tasks were designed to raise students’ awareness of the need for a rigorous definition of limit.

Task 3: Consider the function whose table of values is given below. Notice that the domain values have been represented in decimal form, but you may assume that the domain of the function is all real numbers.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
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<tbody>
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Table 1
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a) \( \lim_{x \to 4} f(x) \) exists and is equal to 1.99998

Agree/Disagree (circle one)

Justification/Response:

b) \( \lim_{x \to 4} f(x) \) exists and is equal to 6.000

Agree/Disagree (circle one)
c) \( \lim_{x \to 4} f(x) \) exists and is equal to 2.
Agree/Disagree (circle one)

Justification/Response:

d) \( \lim_{x \to 4} f(x) \) keeps getting closer to 2 but never actually reaches 2.
Agree/Disagree (circle one)

Justification/Response:

e) There is not enough information given to determine the limit with certainty.
Agree/Disagree (circle one)

Justification/Response:

Purpose: The purpose of this task was to assess the extent to which students’
believe a tabular representation of a function can validate the
candidate of a limit. Task 3 was designed to contrast with Tasks 1
and 2, and to elicit the students’ awareness of the need for a rigorous
definition of limit.

Session 2 – Tasks and Rationale

Task 4: How was your approach to determining the limit in Tasks 1 and 2
the same or different?
Purpose: The purpose of this task was to bring the students’ attention to their
use of direct substitution in each of the two algebraic limit problems
they encountered during Session 1. In particular, the aim was to
assess whether the students believed that direct substitution is a
technique which can be employed anytime an algebraic
representation is available. This task was designed in anticipation of
Task 7, wherein the students were provided an algebraic
representation, yet the utilization of direct substitution is
mathematically invalid.

Task 5: What does it mean to have a finite limit?
Purpose: This task was designed to assess one students’ informal
understanding of the phrase “finite limit”. The particular student had
confused this notion on the Informal Limit Reasoning Survey taken
prior to the teaching experiment, as well as during the first session.
This purpose of this task was for the students to reach a consensus
about the intended meaning of the phrase “finite limit”, with the aim
of them recognizing that a limit is a y-value. The intention was that
such recognition might elicit the articulation of a y-first perspective.

Task 6: Can a formula misbehave really close to the limiting point in the
same way a table or graph could misbehave?
Purpose: This task was designed in response to one student’s claim that a tabular representation lacks the certainty provided by an algebraic representation because of its discrete nature. The student had previously voiced concern that a function could “misbehave” between the limiting point and the closest x-value for which functional behavior is described. The purpose of this task was to lead the student to realize that inspecting functional behavior by “plugging in points” to the algebraic representation provides no more certainty than a tabular representation.

Task 7: Evaluate \( \lim_{x \to 0} f(x) \), where \( f(x) = \begin{cases} x, & x > 0 \\ -x, & x < 0 \\ 17, & x = 0 \end{cases} \)

a) The limit of the function keeps getting closer to 0 but never actually reaches 0.
   Agree/Disagree (circle one)
   Justification/Response:

b) The limit exists and equals 0
   Agree/Disagree (circle one)
   Justification/Response:

c) The limit does not exist because \( f(x) \) is not defined by a single formula.
   Agree/Disagree (circle one)
   Justification/Response:

d) The limit does not exist because \( f(x) \) changes slope at \( x = 0 \).
   Agree/Disagree (circle one)
   Justification/Response:

e) The limit exists and equals 17.
   Agree/Disagree (circle one)
   Justification/Response:

Purpose: The purpose of this task was to assess students’ informal reasoning about limit, and to determine the extent to which they believe algebraic and graphical representations can validate the candidacy of a limit.

Task 3: Consider graphs, tables, and algebraic formulas. In each case, with what degree of certainty would you say a function has a limit, depending upon the representation of the function you had to work with (algebraic versus table versus graph)?

Purpose: The purpose of this task was to assess the extent to which the students relied on particular functional representations for validating the candidacy of a limit. The aim was to determine the students’ openness to engaging in graphical conversations.
Session 3 – Tasks and Rationale

Task 9: During the last session, in regards to the function \( f(x) = \begin{cases} x, & x > 0 \\ -x, & x < 0 \\ 17, & x = 0 \end{cases} \), you discussed the extent to which you agreed with the following statement: “The limit of the function keeps getting closer to 0 but never actually reaches 0.” Can you say again whether you agree or disagree with that statement? Further, the phrase “the function keeps getting closer” implies movement. What is it that’s moving?

Purpose: During the second session, in response to Task 7, the students had been inconsistent in the manner in which they talked about limits. In particular, they appeared unsure of whether a limit is a fixed y-value or an entity which includes the notion of movement. The purpose of this task is to assess the students’ reasoning in regards to how movement relates to the limit concept.

Task 10: In Tasks 1 and 2, what is it that you plugged in to get 6? In reference to Task 7, when you talked about plugging in exactly 0, before you plugged in exactly 0, had you altered that function in someway in your head? If so, how had you altered it?

Purpose: The purpose of this task was to understand better how Amy thinks of “direct substitution.” During the second session, it was unclear whether Amy was using direct substitution in the same manner in Task 7 as she had in Tasks 1 and 2. This task was designed to elicit any distinction Amy saw between her uses of direct substitution in each task. The ultimate aim was to lead Amy to recognize that direct substitution is not always an employable technique and to raise her awareness of the need for a rigorous definition of limit.

Task 11: Do you see the way in which you went about determining the limit in Tasks 1 and 2 as different than the technique(s) you used in Tasks 3 and 7?

Purpose: The purpose of this task was for the students to further delineate the difference between simple algebraic cases, wherein direct substitution is employable, and other cases, including graphs, tables, and most algebraic cases, in which direct substitution cannot be used. The aim was to develop their motivation for a rigorous definition of limit.
Task 12: Can the graph of a function exist in and of itself, or must it originate from either a table or an algebraic equation/formula?

Purpose: This task was designed to assess the extent to which the students believed that functions could exist solely as graphical representations. This task, in conjunction with Task 13, was designed to encourage the students to reason about limits from a graphical perspective, with the eventual aim of having them characterize the imagistic elements of limit contained in the conventional $\varepsilon-\delta$ illustration.

Task 13: How would you go about convincing someone, given only the graph of the function on the board (a function with a removable discontinuity graph at the coordinate pair $(5, 7)$) that the limit of the function at $x=5$ is $7$?

Purpose: In the first two sessions, the students' reliance on algebraic representations had obscured their reasoning about a potentially infinite process they might employ to justify the existence of a limit. The purpose of this task was to shift the focus of the students' discussions about limits to a graphical perspective.

Task 14: Consider the function $f(x) = \frac{\sin(2x^2)}{x^2}$. What is the $\lim_{x \to 0} \frac{\sin(2x^2)}{x^2}$?

How would you justify that the limit is what you say it is? Was that method the same or different as the other limit tasks you looked at in previous sessions?

Purpose: This task was designed to create cognitive conflict for the students. In the previous session, one of the students had claimed that direct substitution is always an employable technique given an algebraic representation. Task 7 (the piecewise task) was designed to raise the students' awareness to the fact that direct substitution is not always an employable technique. Analysis revealed, however, that the piecewise function in Task 7 was simplistic enough so as to allow the students to alter the function in their head into a form which allowed for direct substitution. In contrast, this task was designed so that the students could not similarly make a mental alteration to the function, and so that they might engage in conversations about limit from a graphical perspective.
Phase 2: Initial Attempts to Define Limit at a Point via Graphical Conversations (Sessions 4-6)

Session 4 – Tasks and Rationale

Task 15: Please generate as many distinct examples of how a function could have a limit of 2 at \( x=5 \). In other words, what are the different scenarios in which a function could have a limit of 2 at \( x=5 \)?

Purpose: The purpose of this task was to have the students generate their own graphical examples so that they later had activity to mathematize when asked in Task 16 what it means for a function to have a limit of 2 as \( x\to5 \). The rationale for having the students construct the prototypical examples themselves (as opposed to providing the prototypical examples for the students) was so that they would have some ownership of these graphs – in previous sessions they had expressed wariness regarding graphs that “come from someplace unknown.” Such ownership may more likely elicit engagement in graphical conversations.

Task 16: Consider each of the different examples you provided in the previous task. In each case, you claimed that the function has a limit of 2 at \( x=5 \).

Question 1: Suppose someone new to calculus was asking you about the limit of each of these functions. How would you explain to that person why the limit is 2 in each case?

Question 2: In general, under what conditions would you say that the graph of a function has a limit of 2 at \( x=5 \)? What would have to be true about that function? Or what would have to be true about that graph?

Purpose: This task was designed as a follow-up task to Task 16. The aim was to gain insight into how the students might justify the existence of a limit in the context of a graphical representation. The phrasing of the second question, “under what conditions would you say that the graph of a function has a limit of 2 at \( x=5 \),” was designed to support the students in reasoning about limits from the perspective of what \textit{would} have to be true for a limit to exist, as opposed to trying to establish what \textit{is} true (which is admittedly more difficult when one considers that the latter requires the completion of an infinite process).

Task 17: According to your current definition, the limit is 2 if, as I take \( x \)-values closer to 5, my \( y \)-values get closer to 2. However, consider
the graph on the board which has a removable discontinuity at $a$, but also has a single removable discontinuity on either side of $a$, where the function is defined at each point to be a different height. I notice that as I take $x$-values closer to 5, it isn't always the case that the $y$-values get closer to 2. Would you claim that the limit for the function on the board is 2?

**Purpose:** This task was designed “on the fly” in response to the students' first characterization of limit. The purpose of the task was to gain insight into the $x$-values on which the students' attention was focused, and to motivate in the students awareness that their definition needed to be refined so as to more precisely articulate what it means to “get closer.” This task also served the dual purpose of providing a graph with a removable discontinuity where the limit does exist, to contrast with the graph provided in the following task (Task 19), wherein a jump discontinuity exists and thus the limit fails to exist. The rationale was that providing the students with both an example and a counterexample might support them in developing necessary and sufficient conditions for the existence of a limit.

**Task 18:** Consider the graph on the board which has a very small jump discontinuity (*the graph is drawn with a jump, at $x=4$, from 7.99 to 8.01*). Given the graph of this function, what do you think the limit is as $x$ approaches 4?

**Follow-up Questions:**

Why does this graph fail to have a limit at $x=4$? Under what conditions would that limit be 8?

**Purpose:** The purpose of this task was to foreground for the students the necessity of precisely articulating what is meant by infinite closeness. Further, this task was designed with the aim of motivating them to refine their $x$-first characterization. In sum, this task was designed to spur the students to discuss what it means to be close, both in terms of closeness to $a$ and, more importantly, in terms of closeness to $L$.

**Session 5 – Tasks and Rationale**

**Task 19:** During the previous session, you talked about an infinite set of $y$-values between the two holes in the graph of the very small jump discontinuity (from Task 18). Please describe, as accurately as possible, that set of $y$-values.

**Purpose:** The purpose of this task was to help elicit in the students a shift to a $y$-first perspective, and to support them in talking about *intervals along the $y$-axis*. The intention was that talking about intervals along
the y-axis might introduce to the students’ vernacular a way to talk about getting close (or even infinitely close) to a particular y-value (say, $L$) along the y-axis.

**Tasks 20/21:** We talked a lot last week about zooming in. Consider the jump discontinuity case we looked at last week. Imagine zooming in once on the x-axis. Describe/draw what picture would result from zooming in once along the x-axis. (pause) Now zoom in a second time along the x-axis. Describe/draw what picture would result from zooming in a second time along the x-axis. (pause) Imagine you were to continue zooming in along the x-axis. Describe/draw what picture would result from continual zooming along just the x-axis. As this zooming process continues, would there ever be a resultant horizontal interval along the x-axis wherein you would no longer see the graph? (pause) Now imagine zooming in once on the y-axis. Describe/draw what picture would result from zooming in once along the y-axis. (pause) Now zoom in a second time along the y-axis. Describe/draw what picture would result from zooming in a second time along the y-axis. (pause) Imagine you were to continue zooming in along the y-axis. Describe/draw what picture would result from continual zooming along just the y-axis. As this zooming process continues, would there ever be a resultant vertical interval along the y-axis wherein you would no longer see the graph? (Task 21 was a repeat of the questions above but instead started with: Now consider the removable discontinuity case we looked at last week.)

**Purpose:** The notion of zooming surfaced in the previous session. The purpose of these two tasks was to unpack the idea of zooming, and to support the students in recognizing the effect of zooming in along the x and y-axis, respectively. In particular, these tasks were designed to initiate a shift to a y-first perspective. Further, it is worth noting that these tasks were posed as a sequence of questions in which the students were asked to zoom in once at a time and consider at each zoom what the respective graphs might look like. The rationale for posing the questions in this manner was that doing so might foreground the iterative nature of the limiting process. Finally, the purpose of coupling these two tasks was to highlight how the result of zooming along the y-axis differs when a limit does not exist (versus when it does), but that the result of zooming along the x-axis is unaffected by the presence of a vertical jump discontinuity.
Task 22: Please describe a procedure (potentially infinite) one could use to see if the y-values for a function were getting infinitely close to a specific y-value $L$.

**Purpose:** This task was designed to capitalize on discoveries students made during the previous two tasks. The purpose of this task was to provide the students an opportunity to characterize precisely *infinite closeness* along the y-axis, with the aim of further initiating a shift to a *y-first* perspective. The rationale for acknowledging for the students that the procedure they were being asked to describe might be infinite was that such acknowledgement might help alleviate their concerns about the infinite nature of the limiting process. Have the students complete this task individually (in writing) first, so as to better understand their individual reasoning regarding this procedure.

Task 23: Please describe what would happen procedurally if it turned out the y-values could not get infinitely close to a specific y-value $L$. Is there a difference procedurally between being able to get infinitely close to $L$ and *not* being able to get infinitely close to $L$?

**Purpose:** This task was designed to support the students in recognizing characteristics fundamental to the existence of a limit. The aim of this task was to contrast what happens procedurally in regards to zooming when one is able to get infinitely close to a y-value (say $L$) and when one is *not* able to get infinitely close to a y-value (say $\bar{L}$).

Task 24: How would you characterize graphically when a function $f(x)$ has a limit $L$ at $x=a$?

**Purpose:** This task was the culmination of a sequence of tasks designed to elicit a shift to a *y-first* perspective. The purpose was to assess the extent to which unpacking what it means to zoom along both axes had a positive effect on the students' ability to characterize what it means for a limit to exist. Further, this final task of Session 5 was designed to provide the students a jumping off point for the following session. In a manner similar to Task 22, have the students complete this task individually (in writing) prior to discussing their responses aloud.

Session 6 – Tasks and Rationale

Task 25: In reflecting back upon the last two sessions/interviews, what do you think, from your point of view, the mathematical goal has been for our discussions?
Purpose: This task was designed to assess the students' interpretation of the purpose of the preceding two sessions. The students are asked only about the last two sessions because a pedagogical shift was initiated at the outset of Session 4 to focus the students' attention on limits in the context of graphical representations. Have the students complete this task individually (in writing) prior to discussing their responses aloud. This task can be completed via email between sessions so that student responses can guide the subsequent formation of instruction.

Task 26: Consider the functions \( f(x) = \sin(1/x) + 5 \) and \( f(x) = \sin\left(\frac{10x^2}{x}\right) + 5 \). Imagine in both cases you are asked to explore what the general limit is, if it exists, for the function at \( x=0 \). Imagine also having to craft an explanation so that someone else would be able to decide, in the manner you did, what the limit is for these functions at \( x=0 \). In other words, articulate as precisely as you can a procedure for determining the limit of these functions, so that someone else might be able to replicate your procedure for these functions, and others.

Purpose: This task is designed with the aim of having the students develop their concept image of "not limit" beyond jump discontinuities, and with the purpose of having the students recognize that a limit fails to exist when a function fails to approach a single \( y \)-value. These two functions were presented in tandem so that the students might contrast the behavior of a rapid harmonic function where the oscillations "settle down" upon zooming-in with the behavior of a function like \( \sin(1/x)+5 \), where no matter how many times one zooms, the oscillations do not settle, but become more pronounced. Allow students ample time to explore the behavior of the first of these two functions, as its erratic nature is unlike the removable discontinuities and jump discontinuities to which students are likely accustomed. Note that both functions have been vertically shifted on the \( y \)-axis by 5 units so that student utterances about behavior around "0" will be easy to differentiate in terms of \( x \)-values and \( y \)-values. If time allows, encourage the students to refine their most recent definition of limit at a point based on the insights they gained from this task.
Phase 3: Characterizing *Limit at Infinity* (Sessions 7-8)

Session 7 – Tasks and Rationale

**Task 27:** Generate (draw) as many distinct examples of how a function $f$ could have a limit of 4 as $x \to \infty$. In other words, what are the different scenarios in which a function could have a limit of 4 as $x \to \infty$?

**Purpose:** This task was the first in a sequence of tasks designed to have the students reinvent the definition of *limit at infinity* so that they might: a) experience a shift to a *y-first* perspective; and, b) utilize their definition of *limit at infinity* as a template for refining their definition of *limit at a point*. This task was designed so that the students would later have activity to mathematize as they attempted to characterize what it means for a function to have a finite limit as $x \to \infty$. The rationale for having the students construct the prototypical examples themselves (as opposed to providing the prototypical examples for the students) was so that they would have some ownership of the graphs that were to become the focal point of subsequent discussion.

**Task 28:** Please generate/draw as many distinct ways you can think of in which a function could *fail* to have a limit of 4 as $x \to \infty$.

**Purpose:** Evidence from previous sessions suggested that students' concept image of "not limit" influenced their reasoning regarding what it means for a limit to exist. This task was designed to have the students produce prototypical counterexamples that might subsequently be used as tools for characterizing *limit at infinity*.

**Task 29:** Under what conditions would a function $f$ have a limit of 4 as $x \to \infty$? Design your description so that someone else who is going to use it can know what conditions to check in order to decide whether $f$ has a limit of 4 as $x \to \infty$.

**Purpose:** The purpose of this task was for the students to mathematize their activity from the preceding two tasks. The intention was for the students to construct a precise definition in a less cognitively complex context (*limit at infinity* versus *limit at a point*), with the central pedagogical aim of initiating a shift to a *y-first* perspective. Encourage the students to physically write the necessary and sufficient conditions on the board as they work, as they will likely be more precise with the language that they use when writing than if they are merely to discuss the conditions aloud.
Session 8 – Tasks and Rationale

Task 30: Recall the activities you worked on during the last session. With the work you did last session in mind, how would you explain to someone what it means for a function to have a limit $L$ as $x \to \infty$? Feel free to draw examples and talk through why those examples have a limit $L$ as $x \to \infty$.

Purpose: This task was designed to assess the robustness of the students’ understanding of their previous characterization of limit at infinity. Students may need ample time to recall the details of their previous characterization and to recreate a coherent definition. Encouraging the students to draw examples is important – evidence from previous sessions indicates that these examples can be used as tools to motivate precision in their subsequent articulations. Students will possibly restate their definition from the last session in the process of addressing this task. Students developed the following during Session 7:

**Final Articulation:** “It is possible to make bounds arbitrarily close to 4 and by taking large enough $x$-values we will find an interval $(a, \infty)$ on which $f(x)$ is within those bounds”

If the students struggle to restate this articulation (or something synonymous to it), state the articulation on the board, so as to provide the students a jumping off point for Task 31.

Task 31: Last week you discussed how best to characterize what it means to be “close.” Closeness appeared to be a subjective notion in your estimation. It seemed, however, that the related idea of distance had been helpful to you. Specifically, as you built your definition of limit at infinity last week, you talked about measuring distances in terms of absolute value: $|L-y|$. This appeared, at times, to be a notation that aided your discussion of the key ideas associated with limits at infinity. I noticed though that your final articulation from last session did not include such notation. How might you incorporate the absolute value statement you discussed last week, as well as the notion of distance, in your definition?

Purpose: This task was designed to encourage the students to incorporate the notion of distance and absolute value notation in their articulation. Operationalizing infinite closeness in a precise manner (using absolute value notation) was important as the students subsequently transitioned in Session 9 to refining their definition of limit at a point. To support the students in incorporating absolute value
statements in their most recent definition of limit at infinity, it may be helpful to remind them of an earlier articulation they constructed which used absolute value notation. Students developed the following during Session 7:

Earlier Articulation: “As x gets larger, the distance |L-y| between L and your corresponding y-values continues to decrease”.

Task 32: Consider the following three articulations:

“It is possible to make bounds arbitrarily close to L and by taking large enough x-values we will find an interval (a, ∞) on which there is an x for which f(x) is within those bounds”

“It is possible to make bounds arbitrarily close to L and by taking large enough x-values we will find an interval (a, ∞) on which there are an infinite # of x-values for which f(x) is within those bounds”

“It is possible to make bounds arbitrarily close to L and by taking large enough x-values we will find an interval (a, ∞) on which for every x, f(x) is within those bounds”

Which of these three statements captures the intended meaning of your characterization of limit at infinity? Are the three statements synonymous or do they differ in significant ways?

Purpose: This task was designed to assess the students’ understanding of the universal quantification on x in the definition of limit at infinity. Understanding the quantification structure for limit at infinity was important, for it anticipated much of the quantification structure for limit at a point. The three articulations provided in the task should mirror the language the students have used to that point in the session. Thus, if absolute value notation was incorporated in the previous task, the prompt for this task should reflect that. If the students claim that any of the three statements do not capture the intended meaning of their characterization, encourage them to provide counterexamples which justify their claim.

Task 33: Consider the following three articulations:

“There is a bound close to L such that by taking large enough x-values we will find an interval (a, ∞) on which for every x, f(x) is within those bounds”
“There are an infinite # of bounds close to $L$ such that by taking large enough $x$-values we will find an interval $(a, \infty)$ on which for every $x, f(x)$ is within those bounds”

“For any arbitrary bound close to $L$, by taking large enough $x$-values we will find an interval $(a, \infty)$ on which for every $x, f(x)$ is within those bounds”

Which of these three statements captures the intended meaning of your characterization of limit at infinity? Are the three statements synonymous or do they differ in significant ways?

**Purpose:**
Similar to Task 32, Task 33 was designed to assess the students’ understanding of quantification. In this particular case, the aim was to assess the students’ understanding of the universal quantification on the bounds around the limit $L$. Again, understanding the quantification structure for limit at infinity was important, for it anticipated much of the quantification structure for limit at a point. As was the case with Task 32, the three articulations provided in the task should mirror the language the students have used to that point in the session, and, if the students claim that any of the three statements do not capture the intended meaning of their characterization, encourage them to provide counterexamples which justify their claim.

**Task 34:** Consider the following two articulations, the first of which is the articulation of limit at infinity you have constructed over the course of this session and last week’s session.

(Provide the students with their most recent articulation of limit at infinity, as well as an articulation that has the order of quantification reversed (i.e., stated as an EA statement instead of as an AE statement).)

Does the second statement capture the intended meaning of your articulation? If not, how do the two statements differ in meaning?

**Purpose:** The purpose of this task was to assess whether the students noted a difference in meaning when the quantification structure was reversed.
Phase 4: Revisiting Limit at a Point (Sessions 9-10)

Session 9 – Tasks and Rationale

Task 35: Consider the following four articulations of limit at infinity you constructed during the previous two sessions.

**Earlier Articulation:** “As \( x \) gets larger, the distance \(|L-y|\) between \( L \) and your corresponding \( y \)-values continues to decrease”.

**Revised Articulation:** “It is possible to make bounds arbitrarily close to 4 and by taking large enough \( x \)-values we will find an interval \((a, \infty)\) on which \( f(x) \) is within those bounds”

**Final Articulation:** “\( \lim_{x \to \infty} f(x) = L \) provided for any arbitrarily small positive number \( \lambda \), by taking sufficiently large values of \( x \), we can find an interval \((a, \infty)\) such that for all \( x \) in \((a, \infty)\), \(|L-f(x)| \leq \lambda\)”

**Final Articulation (pruned):** “\( \lim_{x \to \infty} f(x) = L \) provided for any arbitrarily small positive number \( \lambda \), we can find an interval \((a, \infty)\) such that for all \( x \) in \((a, \infty)\), \(|L-f(x)| \leq \lambda\)”

Note the difference in specificity between the first articulation and your final articulation. Consider also your most recent definition of limit at a point, which you constructed three sessions ago.

“The limit \( L \) of a function at \( x=a \) exists if every time we look at the function more closely as we get infinitely close to \( x=a \), it bears out the same pattern of behavior, i.e., looks to be approaching some \( y \)-value \( L \) with no gaps in the graph”

What differences, if any, do you see between your final definition of limit at infinity and your most recent definition of limit at a point?

**Purpose:** The purpose of this task was to elicit awareness in the students of the difference in specificity between their final definition of limit at infinity and their most recent definition of limit at a point. In particular, the aim was for the students to recognize how precisely their definition of limit at infinity described the necessary and sufficient conditions for a function to have a limit at infinity. If students view their definition of limit at infinity as efficient and as having mathematical utility, they may be likely to use it as a template for refining their definition of limit at a point.
Task 36: Do you see the phrases “arbitrarily close” and “infinitely close” as meaning the same thing, or is there a distinction between these two phrases for you?

Purpose: In the students’ definition of limit at infinity, they used absolute value notation to operationalize what it means to be arbitrarily close. In the students’ most recent articulation of limit at a point (see above), they had used the phrase infinitely close as opposed to arbitrarily close. The purpose of this task was to draw their attention to the fact that the phrases infinitely close and arbitrarily close are synonymous, so that they, in turn, might characterize infinitely close in the same manner that they previously characterized arbitrarily close.

Task 37: Please draw the different scenarios in which a function $f$ could have a finite limit $L$ at $x=a$.

Purpose: The purpose of this task was for the students to regenerate the different ways in which a function could have a finite limit $L$ at $x=a$ so that they would have specific examples to characterize (i.e., activity to mathematize) when asked in the next task to refine their definition of limit at a point. This task was important given that it has been three weeks since their focus had been on limit at a point, as opposed to limit at infinity. Another rationale for this task was to mirror the sequence of tasks that led to the students’ precise articulation of a definition of limit at infinity.

Task 38: Consider your most recent characterization of limit at a point. How might you more precisely articulate what it means for a function to have a finite limit at a finite point?

Purpose: The purpose of this task was for the students to continue their efforts to refine their definition of limit at a point. The placement of this task following the tasks that precede it was purposeful – the aim was for the students to use their definition of limit at infinity, as well as the examples of limit at a point they constructed in Task 40, as tools for refining their definition of limit at a point.

Session 10 – Tasks and Rationale

Task 39: Consider the function drawn on the board which has a removable discontinuity at $x=a$. How would you explain to a classmate from your calculus class why this function has the limit $L$ at $x=a$?

Purpose: The purpose of this task was to assess the robustness of students’ understanding of the definition they had constructed during the
previous session. Students will possibly restate their definition from the last session in the process of addressing this task. Students developed the following during Session 9:

$$\lim_{x \to a} f(x) = L$$ provided that: given any arbitrarily small \( \# \lambda \), we can find an \((a \pm \theta)\) such that \(|L-f(x)| \leq \lambda \) for all \( x \) in that interval except possibly \( x = a \).

**Task 40:**
At the end of last week’s session, you provided a clear articulation of what it means for a function to have a finite limit \( L \) at a finite point \( x = a \). Restate your definition on the board. It may be helpful to consider the explanation you just gave during Task #42 as you restate your definition.

**Purpose:**
This task was designed to again assess the robustness of students’ understanding of their preceding formulation. Students will possibly restate their definition from the last session in the process of addressing the preceding task. If they do otherwise, prompt them to restate their definition. Expect the process of reconstituting their definition to take some time, but that it will be facilitated by their working together. Further, having them talk through a specific example in Task 42 (above) will provide them with an activity to mathematize in this task.

**Task 41:**
Using the definition you restated during the previous task, please explain why the continuous function I have drawn on the board has a limit \( L \) at \( x = a \).

**Purpose:**
The purpose of this task was to once again assess the robustness of students’ understanding of their preceding formulation. This task allowed me to check how consistently the students’ reasoned about their definition in regards to particular functions. My reasoning for asking them to address a continuous function was to see if they might comment on how continuity affects the usage of their definition in establishing the existence of a limit.

**Task 42:**
Please explain why the following two functions fail to have a finite limit \( L \) at \( x = a \). (On the board I drew the vertical jump discontinuity graph (from 7.99 to 8.01) that we had discussed during Sessions 4 and 5, as well as the algebraic representation \( f(x) = \sin(1/x) + 5 \) that we had discussed during Session 6.)

**Purpose:**
The purpose of this task was to assess the robustness of the students’ understanding of their definition in relation to two counterexamples that had been prominent throughout the teaching experiment. In
particular, my aim was to see whether the students might provide a
description of what it means for a function to fail to have a limit.

**Task 43:** Please consider the following five statements:

1) \( \lim_{x \to a} f(x) = L \) provided that: Given any arbitrarily small \( \delta \), we can
find an \( (a \pm \theta) \) such that \( |L - f(x)| \leq \delta \) for all \( x \) in that interval except
possibly \( x = a \).

2) \( \lim_{x \to a} f(x) = L \) provided that: For every \( \lambda > 0 \), there exists a \( \theta > 0 \), such
that \( 0 < |x - a| < \theta \rightarrow |f(x) - L| < \lambda \)

3) \( \lim_{x \to a} f(x) = L \) provided that: There exists a \( \lambda > 0 \), such that there exists
\( x \rightarrow a \) that exists a \( \theta > 0 \), such that \( 0 < |x - a| < \theta \rightarrow |f(x) - L| < \lambda \)

4) \( \lim_{x \to a} f(x) = L \) provided that: There exists a \( \lambda > 0 \), such that for every
\( \lambda > 0 \), \( 0 < |x - a| < \theta \rightarrow |f(x) - L| < \lambda \)

5) \( \lim_{x \to a} f(x) = L \) provided that: For every \( \lambda > 0 \), there exists a \( \theta > 0 \), such
that \( |f(x) - L| < \lambda \rightarrow 0 < |x - a| < \theta \)

The first of these statements is your articulation of what it means for
a function \( f \) to have a finite limit \( L \) at \( x = a \). Consider each of the other
four statements. Does each of these other statements capture the
intended meaning of your own articulation? Comment on the
similarities and differences in meaning of each of these other four
statements in relation to your own articulation.

**Purpose:**
The purpose of this task was to further assess the robustness of the
students' understanding of their definition, as well as their ability to
coherently reason about the conventional \( \varepsilon-\delta \) definition (Statement
2) accepted by the mathematical community, and to reason about
other symbolic mathematical statements (Statements 3-5) that do not
characterize what it means to be a limit.

**Individual Exit Interview - Tasks and Rationale**

**Task 44:** Consider the precise articulation you came up with during these
sessions for what it means for a function to have a finite limit \( L \) at
\( x = a \):

\( \lim_{x \to a} f(x) = L \) provided that: given any arbitrarily small \( \delta \), we can
find an \( (a \pm \theta) \) such that \( |L - f(x)| \leq \lambda \) for all \( x \) in that interval except
possibly \( x = a \).
What does this statement accomplish? What does it allow you (or another mathematician) to do? What do you see as the role of this mathematical statement?

**Purpose:**
The purpose of this task was to gain insight into how the act of reinventing the definition of limit affects one's perspective on the purpose of a rigorous definition of limit.

**Task 45:**
Consider each of the following slightly erroneous articulations, provided by other calculus students, of what it means for a function $f(x)$ to have a finite limit $L$ at $x=a$.

1) $\lim_{x \to a} f(x) = L$ provided that: There exists a $\mu > 0$, such that for every $\rho > 0$, $0 < |x-a| < \rho \Rightarrow |f(x)-L| < \mu$

2) $\lim_{x \to a} f(x) = L$ provided that: There exists a $\theta > 0$, such that for every $\lambda > 0$, $0 < |x-a| < \theta \Rightarrow |f(x)-L| < \lambda$

3) $\lim_{x \to a} f(x) = L$ provided that: For every $\varepsilon > 0$, there exists a $\delta > 0$, such that $|x-a| < \delta \Rightarrow 0 < |f(x)-L| < \varepsilon$

Please diagnose these different student articulations, providing either an example of a function that has a limit $L$ at $x=a$ that doesn’t satisfy the provided definition, or an example of a function that doesn’t have a limit $L$ at $x=a$ but does satisfy the provided definition.

**Purpose:**
The purpose of this task was to assess the flexibility and robustness of the students' understanding of the formal definition they reinvented. The aim was for the students to produce counterexamples to each articulation, as I felt the ability to do so might indicate depth to their understanding of their formal definition. The rationale for using a variety of symbols was to see whether the students view the actual symbols used in their formulation as inconsequential.

**Task 46:**
Reflect back upon the past 10 weeks. With respect to limit, what did you accomplish in these past 10 weeks?

**Purpose:**
The purpose of this task was to understand better what the students felt was their role and purpose in this teaching experiment so as to inform the revision and restructuring of the instructional sequence for the second pair of students.
Appendix C – Instructional Sequence for Second Teaching Experiment

Phase 1: Assessment of Informal Understanding and Attempts to Motivate Necessity (Session 1)

Session 1 – Tasks and Rationale

Task 1: Consider the function whose table of values is given below. Notice that the domain values have been represented in decimal form, but you may assume that the domain of the function is all real numbers.

<table>
<thead>
<tr>
<th>X</th>
<th>f(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.7</td>
<td>1.973</td>
</tr>
<tr>
<td>3.8</td>
<td>1.992</td>
</tr>
<tr>
<td>3.9</td>
<td>1.999</td>
</tr>
<tr>
<td>4.0</td>
<td>6.000</td>
</tr>
<tr>
<td>4.1</td>
<td>2.001</td>
</tr>
<tr>
<td>4.2</td>
<td>2.008</td>
</tr>
<tr>
<td>4.3</td>
<td>2.027</td>
</tr>
</tbody>
</table>

Table 1

What is the \( \lim_{x \to 4} f(x) \)?

a) \( \lim_{x \to 4} f(x) \) exists and is equal to 1.99998
   Agree/Disagree (circle one)
   Justification/Response:

b) \( \lim_{x \to 4} f(x) \) exists and is equal to 6.000
   Agree/Disagree (circle one)
   Justification/Response:

c) \( \lim_{x \to 4} f(x) \) exists and is equal to 2
   Agree/Disagree (circle one)
   Justification/Response:

d) \( \lim_{x \to 4} f(x) \) keeps getting closer to 2 but never actually reaches 2.
   Agree/Disagree (circle one)
   Justification/Response:

e) There is not enough information given to determine the limit with certainty.
   Agree/Disagree (circle one)
   Justification/Response:
**Purpose:** The purpose of this task was to assess the extent to which students' believe a tabular representation of a function can validate the candidacy of a limit. Task 1 was designed to elicit the students' awareness of the need for a rigorous definition of limit.

**Task 2:** You mentioned in the previous task that you felt that the table did not provide you enough information to determine the limit of the given function with certainty. Please describe what information you would need in a table to be able to determine a limit with certainty.

**Purpose:** This task was designed to focus the students' attention on issues of closeness and local functional behavior. The aim was for the students to recognize the importance of precisely describing what it means to be infinitely close.

**Task 3:** Please summarize/describe the different strategies a student could use to determine the limit of a function. In other words, if someone gave you a function and asked you to find the limit of that function at, say, $x=4$, what are all the different strategies you might employ to determine the limit?

**Purpose:** This task was designed to foreground the different ways in which someone might find a limit candidate. The pedagogical aim was for the students to recognize that direct substitution is one of many employable options for finding limit candidates. This task may reveal students' viewpoints on the extent to which different functional representations validate the candidacy of limits. If not, proceed to Task 4.

**Task 4:** Consider the different strategies you discussed in Task 3. To what extent does each of these tasks provide certainty in regards to determining the limit of a function?

**Purpose:** This task was designed to motivate the students to seek a rigorous definition of limit. The aim was for the students to recognize that while direct substitution provides certainty at a level higher than tables and/or graphs, it is not always possible to employ such a technique.

**Task 5:** Please generate as many distinct examples of how a function could have a limit of 2 at $x=5$. In other words, what are the different scenarios in which a function could have a limit of 2 at $x=5$?

**Purpose:** The purpose of this task was to have the students generate their own graphical examples so that they later had activity to mathematize when asked in Session 2 what it means for a function to have a limit of 2 as $x \to 5$. The rationale for having the students construct the
prototypical examples themselves (as opposed to providing the prototypical examples for the students) was so that they would have some ownership of these graphs – in the first teaching experiment, the students had expressed wariness regarding graphs that “come from someplace unknown.” Such ownership may more likely elicit engagement in graphical conversations.

Phase 2: Initial Attempts to Define Limit via Graphical Conversations (Sessions 2-5)

Session 2 – Tasks and Rationale

Task 6: If \( \lim_{x \to 0} f(x) = f(0) \), what do we know about the function at \( x=0 \)?

Purpose: During the first session, the students had, at times, confounded the notions of limit, continuity, and differentiability. This task was designed to resolve such confusion so that the students would not be overly focused on issues unique to differentiability as they attempted to reinvent the definition of limit. If the students respond to this task by claiming the function must be differentiable at 0, ask them to discuss the function \( f(x) = |x| \) in terms of its limit at \( x=0 \), as well as its continuity and differentiability at \( x=0 \).

Task 7: Please determine the following limit: \( \lim_{x \to 0} \frac{\sin(2x^2)}{x^2} \).

Purpose: During the first session, both students had claimed that one could always use algebraic techniques to determine a limit with certainty. This task was designed to capitalize on students’ trust of algebraic approaches. Specifically, one can anticipate, in this case, that the students will utilize L'Hospital’s Rule to determine this particular limit, and that the result of using L’Hospital’s Rule will match the proposed limit that arises from a graphical inspection of the function’s behavior. Then, question the students as to their reasons for trusting the results they get from employing algebraic techniques like L’Hospital’s Rule (i.e., How are algebraic tricks supported by what it means to be a limit?). The aim of this task was to instill a motivation for the need for a rigorous description of what it means to be a limit.

Task 8: Please generate as many distinct examples of how a function could have a limit of 2 at \( x=5 \). In other words, what are the different scenarios in which a function could have a limit of 2 at \( x=5 \)?
Purpose: This task was a follow-up to Task 5. Session 1 ended with the students still responding to this task. Hence, it was repeated during Session 2. See Task 5 for a description of the purpose of this task.

Task 9: Please generate/describe/draw as many distinct examples of how a function could fail to have a limit of 2 at $x=5$. In other words, what are the different scenarios in which a function could fail to have a limit of 2 at $x=5$?

Purpose: Evidence from the first teaching experiment suggested that students’ concept image of “not limit” significantly influenced their reasoning regarding what it means for a limit to exist. This task was designed to have the students produce prototypical counterexamples that might subsequently be used as tools for characterizing limit. A byproduct of this task was increasing the students’ focus on graphical conversations as opposed to algebraic representations.

Task 10: Consider each of the prototypical examples and counterexamples you provided in Tasks 8 and 9. In each of the prototypical examples, you claimed that the function has a limit of 2 at $x=5$.

Question 1: Suppose someone new to calculus was asking you about the limit of each of these functions. How would you explain to that person why the limit is 2 in each case?

Question 2: In general, under what conditions would you say that the graph of a function has a limit of 2 at $x=5$? What would have to be true about that function? Or what would have to be true about that graph?

Purpose: This task was designed as a follow-up task to Tasks 8 and 9. The aim was to gain insight into how the students might justify the existence of a limit in the context of a graphical representation. The phrasing of the second question, “under what conditions would you say that the graph of a function has a limit of 2 at $x=5$?,” was designed to support the students in reasoning about limits from the perspective of what would have to be true for a limit to exist, as opposed to trying to establish what is true (which is admittedly more difficult when one considers that the latter requires the completion of an infinite process).

Session 3 – Tasks and Rationale

Task 11: Consider the three statements written on the board. (The following three articulations, constructed by the students during Session 2, were written on the board prior to Session 3):
Imagine someone who didn’t know what we were doing was to walk into this room and ask you, “Why are those statements on the board? What are you guys doing?” How would you answer them?

**Purpose:** The purpose of this task was to assess the students’ interpretation of their mathematical goal in the teaching experiment. A secondary aim was to direct their attention to the iterative nature of the defining process (i.e., the defining process is one which requires conjecture, analysis, and refinement).

**Task 12:** *(Continued discussions about a justification process)* There was no new prompt for this task. Rather, the prototypical examples and counterexamples generated by the students during Session 2 were provided on the board, and the same prompt given in Task 10 (see Session 2) was also provided. The students were then reminded that at the end of the last session, they had been in the process of refining their definition. The students were then encouraged to continue their efforts to characterize limit.

**Purpose:** The purpose of this task was to orient the students to the efforts they had previously made to characterize limit so that they might make continued progress in the defining process. Examples and counterexamples were provided on the board so that the students might be more apt to use them as tools for refinement.

**Task 13:** According to your current definition, the limit is 2 if, as I take \( x \)-values closer to 5, my \( y \)-values get closer to 2. However, consider the graph on the board which has a removable discontinuity at \( a \), but also has a single removable discontinuity on either side of \( a \), where the function is defined at each point to be a different height. I notice that as I take \( x \)-values closer to 5, it isn’t always the case that the \( y \)-values get closer to 2. Would you claim that the limit for the function on the board is 2?

**Purpose:** This task was designed “on the fly” in response to the first pair of students’ initial characterization of limit (during the first teaching experiment). This task was also appropriate during the second teaching experiment. The purpose of the task was to gain insight into the \( x \)-values on which the students’ attention was focused, and
to motivate in the students awareness that their definition needed to be refined so as to more precisely articulate what it means to "get closer." This task also served the dual purpose of providing a graph with a removable discontinuity where the limit does exist, to contrast with the graph provided in the following task (Task 14), wherein a jump discontinuity exists and thus the limit fails to exist. The rationale was that providing the students with both an example and a counterexample might support them in developing necessary and sufficient conditions for the existence of a limit.

Task 14: Consider the graph on the board which has a very small jump discontinuity (the graph is drawn with a jump, at x=4, from 7.99 to 8.01). Given the graph of this function, what do you think the limit is as x approaches 4?

Follow-up Questions:
  Why does this graph fail to have a limit at x=4? Under what conditions would that limit be 8?

Purpose: The purpose of this task was to foreground for the students the necessity of precisely articulating what is meant by infinite closeness. Further, this task was designed with the aim of motivating them to refine their x-first characterization, and recognize that an x-first perspective is problematic. In sum, this task was designed to spur the students to discuss what it means to be close, both in terms of closeness to a and, more importantly, in terms of closeness to L.

Session 4 – Tasks and Rationale

Task 15: Consider your most recent description of limit written on the board. (Students developed the following during Session 3):

When evaluating a limit, y takes on values closer to L/the limit in question as you take x values closer to the point at which you're evaluating the limit. The limit need not equal the function’s value at that point.

Recall that you had replaced “the limit in question” with “L”. Reflect back upon when you first included “the limit in question” in your description of limit. What idea(s) had you intended to capture with this phrase? To what does “the limit in question” refer?

Purpose: The purpose of this task was to assess whether the students’ use of the phrase “the limit in question” was intended to address a distinction between finding limit candidates and subsequently validating limit candidates.
Task 16: (Continued discussions about a justification process) There was no new prompt for this task. Rather, the prototypical examples and counterexamples generated by the students during Session 2 were again provided on the board, and the same prompt given in Task 10 (see Session 2) was also provided. The students were then reminded that at the end of the last session, they had been focused on trying to address the question “How close is close enough?” in the process of refining their definition. The students were then encouraged to continue their efforts to characterize limit.

Purpose: The purpose of this task was to orient the students to the efforts they had previously made to characterize limit so that they might make continued progress in the defining process. In particular, the aim was to generate continued discussion about how best to articulate proximity along the x- and y-axes. Examples and counterexamples were provided on the board so that the students might be more apt to use them as tools for refinement.

Task 17: In reference to your discussions, do you see the phrases “being close” and “being close enough” as synonymous or different? If they’re different, in what distinctive ways are they different?

Purpose: Evidence from the first teaching experiment suggested that defining closeness supported the students in operationalizing infinite closeness. Prior to defining closeness in the following task (Task 18), however, it was worthwhile to assess whether the students saw a distinction between close and close enough. This task was designed, then, to provide the students an opportunity to articulate explicitly this distinction, with the aim of subsequently supporting Task 18.

Task 18: Let’s set aside for a moment the question you raised in regards to proximity – “How close is close enough?” Instead, let’s back off and address the following question - How would you precisely describe what it means for y-values to be close to $L$?

Purpose: Evidence from the first teaching experiment suggested that defining closeness supported the students in operationalizing infinite closeness. The purpose of this task was for the students to define a less cognitively complex notion (closeness vs. infinite closeness), with the aim of having the students develop notation which could be extended to characterizing infinite closeness.
Interview 5 – Tasks and Rationale

Task 19: Consider your most recent description of limit written on the board. (Students developed the following during Session 4):

For some function y=f(x), A limit L exists at a point x=a when:
1) On some interval [b,a], such that b<a<c, as x approaches a within that interval, f(x) approaches some value M between f(b) and f(c)

What are, as you see it, the central issues you guys are currently trying to resolve in your description? Thinking back to last week, what barriers were you meeting up against in trying to refine your description of limit?

Purpose: This task was designed to assess whether the students identified the same central issues needing resolution as the issues identified by the researcher in his analysis between sessions. For this task, students responded in writing first, and then responded verbally. This format was motivated by the conjecture that the students would be more precise in their language if they were required to write out their ideas in writing, rather than just articulate them aloud.

Task 20: Last week you frequently talked about intervals along the x-axis. You mentioned repeatedly that you were trying to localize where you were taking the limit. One of you twice commented that you thought you would need intervals along the y-axis as well, yet I didn’t see intervals for the y-axis in your description at the end of the session. Can you comment on why you chose not to include intervals for the y-axis?

Purpose: During the previous session, the students intimated that they wanted to capture the notion of an interval along the y-axis, but showed reluctance to pursue the idea because they were not familiar with conventional notation designed to denote such an interval. This task was designed to encourage the students to construct their own notation for intervals along the y-axis, based on the conjecture that such notation might support the evolution of their definition of limit.

Task 21: Consider the function \( f(x) = \frac{\sin(2x^2)}{x^2} \). Recall that during the second session you were asked to find the limit of this function as \( x \to 0 \), and decided that the limit was 2. What strategy did you employ in finding this limit?

Follow-up Question: Where did the \( L \) in your definition come from?
The purpose of this task was to raise the students’ awareness to the distinction between techniques designed to find candidates for limits and justification methods/processes designed to validate such candidates.

**Task 22:** *(Continued discussions about a justification process)* There was no new prompt for this task. Rather, the prototypical examples and counterexamples generated by the students during Session 2 were again provided on the board, and the same prompt given in Task 10 (see Session 2) was also provided. The students were then encouraged to address the issues they raised in Task 19, and to continue their efforts to characterize limit.

The purpose of this task was to orient the students to the efforts they had previously made to characterize limit so that they might make continued progress in the defining process. In particular, the aim was to focus the students on the issues they had raised at the outset of the session, as well as direct their attention to the ongoing issue of how best to articulate proximity along the x- and y-axes. Examples and counterexamples were provided on the board so that the students might be more apt to use them as tools for refinement.

**Phase 3:** Explicit Attempts to Define Close Using a Step Function (Session 6)

**Session 6 – Tasks and Rationale**

**Task 23:** In the past few weeks, you have described the function approaching a particular value. For instance, last week you said that $f(x)$ approaches some specific value $M$. What is it that is being approached? *[Prior to this task, a function was drawn on the board with a removable discontinuity at $x=a$, with $f(a)$ undefined, and with $M$ as the limit at $x=a$.]*

The purpose of this task was to have the students acknowledge that they are trying to describe the function approaching the limit. The pedagogical aim is for the students to explicitly note that their focal point is a y-value. The overarching purpose of this task, and the succeeding tasks was to initiate a shift to a y-first perspective.

**Task 24:** Is the limit an x-value or a y-value?

This task was designed as a follow-up to Task 23, in anticipation of the students not specifying that the limit is a y-value. Again, the purpose of this task was to shift the students’ attention to the y-axis.
Task 25: For $M$ to be the limit of this function [referring to the function provided in Task 23], how close do the function values/y-values have to get to $M$?

Purpose: The purpose of this task was to reorient the students to what had been a focal point in the preceding sessions - characterizing infinite closeness. This task was designed as an anticipatory task to Task 26.

Task 26: Forget about limits for a moment. Instead, think about the idea of close. Close means different things to different people. Imagine that for me, close means being within 10 units of a particular y-value. Under my definition of close, how would you write out what it means for a function $f(x)$ to be close to a particular pre-determined value $L$ for every $x$?

Purpose: Evidence from the first teaching experiment suggested that defining closeness supported the students in operationalizing infinite closeness. Task 18 was the first attempt at having the students define closeness. However, the students did not explicitly respond to Task 18, as their attention got diverted by other issues. Hence, this was a second attempt at having the students define closeness. The purpose of this task was for the students to define a less cognitively complex notion (closeness vs. infinite closeness), and to do so outside of the context of limit. The overarching aim was that the students might develop notation which could be extended to characterizing infinite closeness.

Task 27: Consider the step function drawn on the board. [For this task, the floor function was drawn on the board, with one notable exception - the function was undefined at $x=3.5$, and thus, had a removable discontinuity at the coordinate pair $(3.5, 3)$.] What is the limit of this function as $x\to3.5$?

Purpose: The task was designed to familiarize the students with the floor function, and to reorient their attention to limits. This task was designed as an anticipatory task to Task 28.

Task 28: Recall that earlier I noted that close, for me, means being within 10 units of a particular y-value. Imagine that close, for someone else, means being within 2.5 units of a particular y-value. For what values of $x$ would the function be close to the limit, under this more rigorous definition of close?

Purpose: This task was designed to foreground three fundamental features of the formal definition - first, a y-first perspective which supports understanding the definition; second, that for each pre-determined specification of proximity along the y-axis, there exists a
corresponding set of $x$-values which satisfy that specification; and, three, the limiting process is iterative, and requires successively more rigorous specifications for proximity along the $y$-axis. It was helpful to choose a non-integer value for closeness (i.e., 2.5 units) so that students did not have to focus their attention on whether being exactly $n$ units away from $L$ constituted being close.

**Task 29:** Reflecting upon the past few tasks, define precisely what you think it would mean for a function to be close to a predetermined $y$-value, $L$.

**Purpose:** The purpose of this task was for the students to mathematize their activity from the previous three tasks by writing out a precise definition of closeness along the $y$-axis. It is worth nothing that this task did not require the students to define limit, per se. Rather, this task was designed with the aim of having the students develop notation for closeness which might extend to infinite closeness when, in Task 30, they were asked to define limit.

**Task 30:** Consider the function drawn on the board. [For this task, a function was drawn on the board which was continuous and differentiable, save for a removable discontinuity at the coordinate pair $(a, L)$.] Under what conditions would $L$ be the limit for that function at $x=a$?

**Purpose:** This task was the culmination of a sequence of tasks designed to elicit a shift to a $y$-first perspective. The purpose of this particular task was to provide the students an opportunity to make further progress in their efforts to characterize limit, with the aim of having them utilize their discoveries from this session as they made refinements.

**Phase 4:** Refinement of Definition of Limit at a Point with Increased Notational Precision (Sessions 7-9)

**Session 7 – Tasks and Rationale**

**Task 31:** Consider the discussion we had last week with regards to limit. At the end of the last session you had a reasonably precise verbal articulation of the conditions that would need to exist for $L$ to be the limit of a function at $x=a$. With this discussion in mind, please write down as precisely as possible a definition for limit. I recognize and acknowledge that you each may still have some ongoing, unresolved concerns about your current “definition” or “description” of limits. Please also list these concerns as precisely as possible.
The purpose of this task was to assess which ideas were most salient for the students from the previous session. In particular, this task was designed to elicit a written articulation of the definition the students had discussed verbally during the previous session. The latter part of the task was designed to foreground any unresolved cognitive conflict the students were experiencing. Collectively, the students' written responses to this task motivated their subsequent efforts to refine their definition. The format of having the students individually respond to this task in writing first was motivated by the conjecture that the students would be more precise in their language if they were required to write out their ideas in writing, rather than just articulate them aloud.

**Task 32:**

(Continued discussions about a justification process) There was no new prompt for this task. Rather, the students were encouraged to continue their efforts to characterize limit, and to use the concerns they identified in their responses to Task 31 as focal points for their discussion.

**Purpose:** The purpose of this task was to provide the students ample time to make necessary revisions to their characterization of limit based on their responses to the previous task.

**Session 8 – Tasks and Rationale**

**Task 33:**

(Continued discussions about a justification process) There was no new prompt for this task. Rather, the prototypical examples and counterexamples central to the students' discussions during Session 7 were provided on the board. The students were then encouraged to continue their efforts to characterize limit.

**Purpose:** The purpose of this task was to provide the students ample time during Session 8 to make necessary revisions to their characterization of limit. The students had been in the midst of pursuing a number of ideas at the end of Session 7 (one day earlier), and thus, needed little direction at the outset of this session. Once again, examples and counterexamples were provided on the board so that the students might be more apt to use them as tools for refinement.

**Session 9 – Tasks and Rationale**

**Task 34:** Consider the function drawn on the board. [For this task, a function was drawn on the board which had a jump discontinuity at x=a, with f(a) not defined. An L was demarcated on the y-axis in the
middle of the vertical jump.] Please describe whether your current definition of limit would characterize the chosen value for $L$ as the limit of the function on the board.

**Purpose:**

The purpose of this task was to create a perturbation for the students in regards to how they had previously defined two constructs central to their current definition of limit – CEnter and CExit. The aim was that they would recognize that their definition of these two constructs was such that their characterization of limit would validate the value chosen for $L$, despite that value clearly not being the actual limit. The eventual aim was that the students would abandon these two problematic constructs and instead base the existence of a limit on the condition that for every pre-determined $y$-interval about $L$, an $x$-interval exist consisting of $x$-values with corresponding $y$-values contained within the pre-determined $y$-interval.

**Task 35:** Consider the counterexample you have referenced repeatedly over the course of the past few sessions. [Previously, the students had spontaneously noted a counterexample to their characterization of limit. The counterexample was such that a jump discontinuity existed at $x=a$, with $f(a)$ defined to be a height within the vertical jump.] Please discuss how your current definition addresses this counterexample.

**Purpose:** The purpose of this task was to motivate the students to make further revisions to their definition. In particular, the aim was that the students might address two unresolved issues – the problematic nature of CEnter and CExit, and how the limiting process terminates. These had been two focal points throughout the previous session.

**Task 36:** Consider your most recent definition of limit that you have written on the board. [Previously during Session 9, students developed the following]:

1) *Come up with a guess.*
2) *Determine a closeness interval around your guess, $L \pm z$*
3) *Let CEnter equal the last $x$-value for which we become close, with CEnter$a$.
   Let CExit equal the first $x$-value for which we are no longer close, with CExit>a.*
4) i) if either CEnt or CExit $= a$ but not the other then $L$ is not the limit.
Imagine choosing a particular closeness interval in Step 2. Based on this closeness interval, how many distinct x-intervals would exist about $a$, such that the function would always be close on the interval, except possibly at $a$?

**Purpose:** This task was implemented when Task 35 failed to support the students in resolving the two issues discussed above. This task was designed to elicit recognition by the students that more than one x-interval exists for each pre-determined y-interval. Previously, their focus had been on defining the largest x-interval and they had appeared unaware that they need only establish the existence of any x-interval satisfying the necessary conditions.

**Task 37:** In your stated definition, how would the iterative process result in the conclusion that a limit fails to exist? Likewise, how would the iterative process result in the conclusion that a limit does exist?

**Purpose:** The purpose of this task was to assess the extent to which the students were still hindered by a potential infinity perspective. The aim was for the students to recognize the value in phrasing their definition from a hypothetical perspective – i.e., what would have to be true for a limit to exist?

**Phase 5:** Resolution of Central Issues and Completion of Reinvention Process (Session 10)

**Session 10 – Tasks and Rationale**

**Task 38:** Consider your most recent definition of limit. [During Session 9, students developed the following]:

1) *Come up with a guess.*
2) *Determine a closeness interval around your guess, $L \pm z$*
3) *Let CEnter equal the last x-value for which we become close, with CEnter<$a$. Let CExit equal the first x-value for which we are no longer close, with CExit>$a$.*
4) i) *if either CEnt or CExit = $a$ but not the other then L is not the limit.*

$ii)$ *if either CEnter or CExit (or both) fail to exist, then L is not the limit*

$iii)$ *if CTop=L=CBot, then L is the limit*

$iv)$ *if CEnt<$a$<CExit then shrink closeness interval and go back to #2.*
ii) if either CEnter or CExit (or both) fail to exist, then
L is not the limit

iii) if CTop=L=CBot, then L is the limit

iv) if CEnt<\(a\)<CExit then shrink closeness interval
and go back to #2.

Note that this is an articulation that you have expressed confidence
in, save for the issue of how the notions of CEnter and CExit are, at
times, problematic. You have consistently voiced agreement about
the first two steps of your definition – the definition requires a guess
for \(L\), as well as an initial closeness interval about your guess. You
have expressed concern, however with Step 3. Please read the
following five excerpts from Session 9, which summarize thoughts
you have conveyed about characterizing behavior along the \(x\)-axis.

[Students were then given the following excerpts]:

**Excerpt 1:**
Chris: \(a\) is still between CEnter
Jason: Umm-hmm.
Chris: and CExit. So there's some closeness interval which \(a\) is in
between.

**Excerpt 2:**
Jason: That you have an interval on which you're always close and
that CEnter and CExit both exist. CEnter's less than \(a\) and
CExit's more than \(a\). So we have all the, all the happy
conditions.

**Excerpt 3:**
Craig: You went looking for an x-interval such that what was true?
What was true about that x-interval?
Jason: That on that x-interval, all the points in between are close.
Craig: Except possibly
Jason: Except possibly at \(a\).

**Excerpt 4:**
Craig: For, for me to be allowed to shrink this definition of
closeness, what do I have to do?
Jason: Well, we have to satisfy 4c.
Craig: Okay, I have to find some interval such that this is between
there.
Jason: Uh-huh.
Craig: And I'm close on that whole interval, right?
Jason: Yeah.
Craig: I'm just asking does that necessarily, do we have to find the largest one for which that's true?
Chris: No, it's just that's the one that's, we come up with when we use the last point, first point.
Craig: Okay. I, okay, so for this closeness interval, before we can shrink that closeness interval, you're saying we have to find an interval along the x-axis on which we are close, except possibly at $a$. And my question was, we seem to be really focused on trying to find the largest one on the x-axis for which that is true, but I'm suggesting that maybe there are some that are smaller.
Jason: Well, it might be in there but it doesn't correspond, it's not playing by the rules that we have set up.
Craig: So I am thinking about that idea in relation to this. (the horizontal line function)
Jason: Uh-huh.
Craig: If we go down here and try to find the, before we're going to be able to shrink this closeness interval, we're going to have to find the x-interval down here that does that, but we're going to have a heck of a time finding the largest one.
Jason: Right.
Craig: Can we find one that
Chris: Which is kind of what I was going for.
Craig: So that's kind of what you were going for is try, try to find one that works, not
Chris: Yeah.
Craig: necessarily the largest one.
Chris: I guess I'm trying to encompass does one even exist?

Excerpt 5:
Jason: 'Cause the, the, the closeness idea was based on there being multiple points on either side that are close. And now, now we have a condition where only one point's close.
Craig: And it's not on either side.
Jason: Yeah.
Craig: Okay. Say, wait, say that again. You said the close-, the one we had was based on there being points on either side that were close?
Jason: Right.
Chris: That's the idea.
[Following the students’ reading of these excerpts, they were encouraged to pursue the idea that they need only establish the existence of any x-interval that contains a (on which f(x) is close except possibly at a). The students were also encouraged to abandon the notions of CEnter and CExit if they felt the notions were too problematic. The students were then encouraged to modify their definition, taking into account the ideas (captured in the excerpts) they had considered pursuing during Session 9]

Purpose: The purpose of this task was to support the students in resolving the first of two remaining issues, identified initially in this document in Task 35. The rationale for having the students read excerpts from previous sessions was that the students might be more likely to pursue ideas for which they received credit.

Task 39: Previously, you have expressed concern as to whether the limiting process can ever end. Specifically, you have discussed whether your definition accurately describes what would have to happen for a limit to exist. Please read the following two excerpts, the first of which is from Session 8, and the second of which is from Session 9. These excerpts summarize thoughts you have conveyed about how the limiting process might terminate. [Students were then given the following excerpts]:

Excerpt 1:
Jason: So what does this, uh, this vertical interval idea that we got going on, it’s not, the whole idea of closeness on it might not be as important as we were, as we were giving it. Like we’re just trying to come up with an, something to, again, just to like localize it. It doesn’t matter whether or not, maybe I can figure out a better. It’s not a matter of getting closer and closer and closer. It’s a matter of restricting what’s outside of, like, what you’re not concerned with. You’re not concerned with all the stuff above your limit in question or below it. It’s a matter of restricting the, I guess the range of where you’re examining it. To a point where you only have one point, and then that will be your limit. Or you’re able to reveal that gap discontinuity, the jump discontinuity.

Excerpt 2:
Craig: This case will, well, what will happen in this case? Well, as closeness gets smaller and smaller, will CEnter and CExit always exist?
Jason: Un-, until you get to, well because at some point we’re jumping away from dealing with an interval, a beginning and an ending, to the finite point. So like
Craig: Like, like $L$.
Jason: as, as $z$ goes to 0. Yeah, yeah.

[Following the students’ reading of these excerpts, they were encouraged to think about and pursue the idea expressed in the first excerpt, with the aim of answering the question, How does the process end in the case where a limit exists? The students were then encouraged to modify their definition]

Purpose: The purpose of this task was to initiate a shift in the students to an actual infinity perspective, and thus, to support the students in resolving the second of two remaining issues, identified initially in this document in Task 35. The rationale for having the students read excerpts from previous sessions was that the students might be more likely to pursue ideas for which they received credit.

Task 40: Please consider the following two statements:

1) $\lim_{x \to a} f(x) = L$ provided that: Given any arbitrarily small # $\lambda$, we can find an $(a\pm \theta)$ such that $|L-f(x)| \leq \lambda$ for all $x$ in that interval except possibly $x=a$.
2) $\lim_{x \to a} f(x) = L$ provided that: For every $\lambda >0$, there exists a $\theta >0$, such that $0<|x-a|<\theta \Rightarrow |f(x)-L| < \lambda$

These statements are alternative articulations of what it means for a function $f$ to have a finite limit $L$ at $x=a$. Consider both of these statements. Does each statement capture the intended meaning of your own articulation? Comment on the similarities and differences in meaning of each of these statements in relation to your own articulation.

Purpose: The purpose of this task was to further assess the robustness of the students’ understanding of their definition, as well as their ability to coherently reason about, and interpret, other mathematical valid formulations of the definition – i.e., the formulation (Statement 1) constructed during the first teaching experiment, and the conventional $\varepsilon\delta$ definition (Statement 2) accepted by the mathematical community.
Individual Exit Interview – Tasks and Rationale

Note: For both tasks below, the students (in their respective exit interview) responded to each task in writing first. Once both tasks have been addressed in writing, the students then responded verbally. This format was motivated by the conjecture that the students would be more precise in their language if they were required to write out their ideas in writing, rather than just articulate them aloud.

Task 41: Consider the precise articulation you came up with during these interviews for what it means for a function to have a finite limit $L$ at $x=a$:

1) Come up with a guess, $L$.
2) Determine a closeness interval $L\pm \zeta$ around your guess.
3) If
   a. There exists an $x_1<a$ such that $L+\zeta>f(x)>L-\zeta$ is true for all $x$ between $x_1$ and $a$ and there exists an $x_2>a$ such that $L+\zeta>f(x)>L-\zeta$ is true for all $x$ between $x_2$ and $a$, then shrink your closeness interval and try again.
   b. If you can’t shrink your interval anymore, then $L$ is your limit.
4) If not, then $L$ is not your limit.

What does this statement accomplish? What does it allow you (or another mathematician) to do? What do you see as the role of this mathematical statement?

Purpose: The purpose of this task was to gain insight into how the act of reinventing the definition of limit affects one’s perspective on the purpose of a rigorous definition of limit.

Task 42: Reflect back upon the past 10 weeks. With respect to limit, what did you accomplish in these past 10 weeks?

Purpose: The purpose of this task was to understand better what the students felt was their role and purpose in this teaching experiment. Such information served to provide comparable data for the first teaching experiment, and to inform the revision and restructuring of the instructional sequence for future implementations.