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Yang-Baxter Equations

by

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Abstract

Multiple equations in math, physics, quantum information, and elsewhere are referred to as "the" Yang-Baxter equation, in spite of being a broad family of equations. Most of the equations are nonlinear matrix equations, where the unknown variable is a matrix. This is the case for the so called braided, algebraic, and generalized forms of "the" equation, which are the primary focus of this dissertation. Finding solutions to the various forms of these equations has been the subject of much research. The equations in all their forms are largely considered intractable in high dimensions, and only in dimension 2 have the solutions been fully classified.

We begin with an introduction to quantum computation, with a focus on the topological model and its connection to the braid group. Next, we introduce the braided, algebraic, and generalized forms of "the" Yang-Baxter equation. We provide a full classification of diagonal solutions to each form. In particular, we show that any diagonal matrix is a solution to the algebraic form in any dimension, and each instance of the braided and generalized forms only have diagonal solutions that are scalar multiples of the identity. We exploit the relationship between the algebraic and braided forms to construct a solution in any dimension that is applicable to topological quantum computation as a universal gate. The generalized form of the equation is parameterized by three natural numbers, (d, m, l), and we show that the only invertible solutions when $l \geq m$ are scalar multiples of the identity. We completely classify all solutions arising from an X-shaped ansatz for five different choices of (d, m, l), and provide a complete classification of X-shaped solutions to every odd dimensional braided equation, where there are no X-shaped solutions in any dimension. We fully classify permutation solutions to each instance of the braided and algebraic equations that can be written as a product of 3 or fewer transpositions. We show that the problem of classifying all invertible upper triangular solutions to the 4-dimensional algebraic Yang-Baxter equation can be split into 48 cases, and fully classify one of the cases.

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Table of Contents

List of Tables

List of Figures

Chapter 1

Introduction

> "Let's start at the very beginning.
> A very good place to start"
>
> The Sound of Music

In 1967, C. N. Yang was working on a one-dimensional N-body problem. After assuming the wave function takes on a specific form (called the Bethe hypothesis or Bethe ansatz) the wave function is shown to depend on a unitary matrix which satisfies what we would now refer to the quantum Yang-Baxter equation with spectral parameter [44]. McGuire had arrived at a very similar result appearing in [58] three years before Yang. A similar idea referred to as the star-triangle relation from statistical mechanics appeared as early as 1944 in [65]. In 1972 Rodney J. Baxter was working on a 2 dimensional lattice model of ice. He found that the wave function can be computed exactly and depends on a matrix which satisfies the quantum Yang-Baxter equation with spectral parameter [44]. In the late 1970's the connection was made to quantum field theory when Faddeev, Sklyani, and Takhtajan proposed the quantum inverse method, and the phrase "Yang-Baxter equation" was coined by them [44]. In 1982 the field of quantum groups began with a paper by Belavin and Drinfel'd [7] which describes a connection between particular Hopf Algebras and solutions to the quantum Yang-Baxter equation. Shortly thereafter, a connection was made to the braid group and its representation theory. Many other connections exist, and in particular, obtaining representations of the braid group is important for knot theory, 3-manifold invariants, quantum computation, and elsewhere.

We begin by recalling the classical braid group and its relation to topological quantum computation to provide context for the understanding of the main results of this dissertation. We next discuss the various forms of the Yang-Baxter equations and how they arise in this context. For some of these forms we provide a complete classification of diagonal solutions. We next classify low dimensional X-shaped solutions of a generalized form of the equation and fully classify the X-shaped solutions to the so-called braided Yang-Baxter equation in odd dimensions. Further classifications follow. For example, we fully classify permutations to particular forms of the Yang-Baxter equation that can be written either as a single transposition or a product of two or three transpositions. Finally, we address the invertible upper triangular solutions to the so-called algebraic Yang-Baxter equation in dimension 4. We note that many of these 48 cases remain computationally intractable with current technology. A case that is tractable with current technology is identified and is fully classified. With the impending implementation of quantum technologies and the increased computational power that they bring, future directions in this area include using this technology to address the remaining cases.

Chapter 2

Quantum computation

> "I think I can safely say that nobody understands quantum mechanics."
>
> Richard Feynman

2.1 Quantum mechanics

We begin with a brief introduction to quantum physics and quantum information. For a more detailed introduction see [63], [22], or [66]. Quantum mechanics describes phenomena observed at the atomic and subatomic level. All elementary subatomic particles are classified as either bosons or fermions. This classification depends on the *spin quantum number* of the particle which describes the angular momentum or spin of the particle. A *boson* is a subatomic particle whose spin quantum number is a positive integer (with 0 included) [66]. A *fermion* is a subatomic particle whose spin quantum number is an odd multiple of $\frac{1}{2}$ [66]. Photons (light particles) are examples of bosons and electrons are an example of fermions. Quasiparticles arise from multiple particle systems. An example arising from a 2 dimensional system of electrons exposed to an orthogonal magnetic field is the fractional quantum Hall effect [78].

In quantum mechanics, the *quantum state* (for example position or momentum) of both elementary particles and quasiparticles is modeled by a *wave function*, typically denoted ψ. The wave functions of a system are not directly observable. Instead, *observation or measurement* corresponds to the projection of the wave function onto a set of orthogonal basis states, called the *observational basis*. In the observational basis, states appear as vectors in a possibly infinite dimensional Hilbert Space. One interpretation of the square modulus of ψ is as a probability distribution over three dimensional physical space combined with a single time dimension. In this interpretation, $|\psi|^2$ is proportional to the probability of obtaining a specific observational basis state at a specific time. For a given physical property, an observation for that property corresponds to projecting onto the corresponding orthonormal basis.

In general, the Hilbert space of states is infinite dimensional. Due to stability and the challenges of realizing and working with general quantum systems, the quantum systems utilized in quantum computation are typically restricted to finite dimensional subsystems. For example, a two-level quantum system corresponds to when the observational bases for the system each have two basis vectors, and the result of a measurement is binary. These observational basis states are also called pure states and the particles in a two-level system are called *qubits*. Similarly, a d-level system corresponds to when the observational bases have d basis vectors, resulting in d possible measurement outcomes for each physical property. The particles in a d-level system are called *qudits*. After choosing an observational basis, a single qudit quantum state can be represented as a vector in $\mathbb{C}^d$. These vectors are only determined up to multiplication by an arbitrary nonzero complex number called the *phase*. Phase equivalent vectors thus represent the same state. In the Dirac notation [20] the phase equivalence classes of the observational basis vectors are denoted by $\{|0\rangle, \ldots, |d-1\rangle\}$. An arbitrary quantum state is then represented by a complex projective linear combination, or *superposition*, of the basis states:

$$|\psi\rangle = \alpha_0 |0\rangle + \cdots + \alpha_{d-1} |d-1\rangle$$

where each amplitude α_j is a complex number up to a global phase. Up to phase, the superposition can be assumed to be normalized by dividing by its length so that the sum of the square modulus of the coefficients adds to 1. For a state given in bra-ket notation by the ket $|\psi\rangle$ the conjugate transpose of a state vector $|\psi\rangle$ in bra-ket notation is given by the bra $\langle\psi|$. By the Born rule [66] a quantum system in the state $|\psi\rangle$ will be measured in the observational basis state $|j\rangle$ with probability:

$$| \langle j|\psi\rangle |^2$$

After normalization, the probability of obtaining the outcome $|j\rangle$ after measuring a normalized superposition is $|\alpha_j|^2$. The sum of the probabilities of all possible measurement outcomes must add to 1, and upon measurement the only information obtained is one of the observational basis states.

$$\sum_{i=0}^{d-1} |\alpha_i|^2 = 1$$

In particular the state space of a single qudit corresponds to the set of all complex lines through the origin in $\mathbb{C}^{d+1}$ which is the d-dimensional *complex projective space* $\mathbb{C}P^d$.

2.2 Models of quantum computation

Many different physical systems exhibit quantum phenomena, however, not all systems are amenable to the task of quantum computing. Quantum mechanical properties are only observed at the nanoscale (at the level of electrons and photons for example), making precise control and isolation from the environment difficult [32]. It is still uncertain which hardware model and qubit system is the best candidate for a scalable quantum computer. In the year 2000, David DiVincenzo wrote a paper outlining 5 essential requirements for the physical implementation of a quantum computer [23]. Since then lots of progress has been made and many implementations have been tried. Here we give a brief overview of some of the most promising models for quantum computation. The current models include the following types of qubits:

- Trapped Ion

  A trapped ion quantum computer uses atomic ions confined to radiofrequency traps as it's qubits [80]. Trapped ion systems have already been used to implement algorithms with a small number of qubits. Challenges include increasing the number of qubits while also being able to manipulate and measure qubits individually [80].

- Superconducting

  Superconducting qubits are based on pairs of electrons referred to as Cooper pairs [50], which are an example of a bosonic system. Superconducting qubits (for example Bose-Einstein condensates) currently require temperatures close to absolute zero to operate. While superconducting qubits also suffer from the problem of outside noise, they may be a good platform for performing noisy intermediate-scale quantum (NISQ) computing [50].

- Silicon

  Silicon qubit systems utilize the valence electrons from silicon to encode quantum information. Silicon provides an environment with very little magnetic interference and construction of silicon qubits could potentially take advantage of the large amount of infrastructure already inplace for manufacturing classical silicon computing chips [34].

- Photonic

  Photonic qubits utilize one or more of the optical degrees of freedom, for example the polarization, of photons to encode quantum information [75]. Photons may be particularly suited for multi-level quantum computing using qudits [75]. The challenges facing photonic hardware include noise reduction and reducing the optical infrastructure required to manipulating multiple qubit systems [75].

- Topological

  Topological qubits are based on quasiparticles called anyons. Anyons have been theorized to exist for decades and relatively recently have been observed by Google researchers [3]. While anyonic systems are extremely difficult to physically realize, they are potentially resistant to outside noise due to their topological properties when manipulated [61]. The next section gives a more detailed introduction to topological quantum computation.

2.2.1 Topological quantum computation

A topological quantum computer is a theoretical machine that manipulates topological phases of matter to perform computation. Topological phases of matter have

Figure 2.1: The 4-punctured disk

been observed experimentally as the fractional quantum Hall effect [5]. In particular, quasi-particles called *anyons*, which arise when electrons are exposed to a magnetic field, have been observed to exhibit a nontrivial phase change when exchanged [55]. When two bosons are physically exchanged their wave function is scaled by 1 (a +1 phase change). When two fermions are physically exchanged the wave function is scaled by -1 (a -1 phase change). The phase change on an anyon depends only on the number of particles exchanged and whether the exchanges were clockwise or counterclockwise, and not on the specific path taken [60], [61]. Nontrivial manipulation of a quantum state corresponds to a unitary operator acting on the state's wave function. Unitary operators acting on the Hilbert space of states are called *gates*. When restricted to 2 dimensions the evolution of the state of an anyon depends only on the number of particles exchanged, the clockwise or counterclockwise manner of exchange, and not on the specific path taken [60]. This topological resilience is what makes a topological quantum computer potentially resistant to local perturbation and noise [74]. This means that the topological characteristics of the path of exchange is all that is needed to know how the state evolves. For a system of n anyons this corresponds to the mapping class group of a n-punctured disk, or the n strand *braid group* [8]. An example of the 4-punctured disk is shown in figure 2.2.1.

The topological nature of evolving the quantum state of an anyonic system potentially makes it more resilient against environmental noise or decoherence [29]. A *quantum computation* can be thought of abstractly as the ability to manufacture, manipulate and measure quantum states [29]. Since the unitary evolution of anyons corresponds to braiding it is not enough to track the permutations of the anyons. Instead, the matrices representing the gates of an anyonic quantum system must also be representations of the Braid group, which means they must satisfy the Yang-Baxter relation, as described in section 4.1. The above is summarized in the following definitions.

Definition 2.2.1 (Quantum Computation)**.** A *quantum computation* is any computational model based upon the theoretical ability to manufacture, manipulate, and measure quantum states.[29]

Definition 2.2.2 (Quantum state)**.** The *state* of a quantum particle, denoted ψ or $|\psi\rangle$, is described by a vector (or wave function). The collection of all possible states form a Hilbert space.

Definition 2.2.3 (Observational basis)**.** In quantum information the Hilbert space of all possible states is typically finite dimensional. An *observational basis* is an orthonormal basis for the Hilbert space of states.

Definition 2.2.4 (Observation or measurement)**.** The result of *observing or measuring* a quantum system results in one of the basis vectors from the observational basis. The result of a measurement is probabilistic.

Definition 2.2.5 (Bra-Ket notation/Dirac notation and superposition)**.** Observational basis vectors or are denoted by *kets*: $\{|0\rangle, \ldots, |d-1\rangle\}$. The state $|\psi\rangle$ could be a linear combination or *superposition* of basis states. The conjugate transpose of a ket is a *bra*: $|\psi\rangle^\dagger = \langle\psi|$

Definition 2.2.6 (Qudit)**.** A quantum particle which is associated to a d dimensional state space is referred to as a *qudit*. When $d = 2$ the state is referred to as a *qubit*.

Definition 2.2.7 (Global phase)**.** Scaling by a unit complex number $e^{i\theta}$ does not affect the measurement outcome of a state. The number θ is referred to as the *global phase*. In particular, $|\psi\rangle$ and $e^{i\theta}|\psi\rangle$ are considered equivalent for any θ.

Example 2.2.1. States are considered equivalent if they result in the same measurement outcome. For example the states $\pi|0\rangle$, $|0\rangle$, $2024|0\rangle$ will all be measured in state $|0\rangle$ with probability 1.

Definition 2.2.8 (Anyon)**.** An *Anyon* is a quasiparticle which can undergo nontrivial evolutions when exchanged in two dimensions.

Definition 2.2.9 (Tensor and Kronecker Product)**.** The symbol $\otimes$ denotes the *tensor product*, when acting on two vector spaces, and the aB-convention *Kronecker product* when acting on two matrices. For example in the Dirac notation if $V = Span\{|j\rangle \mid j = 0 \dots d-1\}$, then $V \otimes V = Span\{|j\rangle \otimes |k\rangle = |j\rangle|k\rangle = |jk\rangle \mid j,k = 0 \dots d-1\}$

Definition 2.2.10 (Braid group)**.** The *braid group* with n-strands, B_n, is the group generated by: $\{I, \sigma_1, \dots \sigma_{n-1}\}$ with the relations: $\sigma_i\sigma_j = \sigma_j\sigma_i$ whenever $|i-j| > 1$, these relations are sometimes referred to as *far commutativity* and the so called Yang-Baxter relations: $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for $i = 1, \dots, n-1$. This group is further explained with examples in section 4.2.

2.3 Universal quantum gates

Recall as in [10] that a unitary matrix (acting on n qudits) is *universal* for quantum computation if it, together with all *local* unitary transformations from $V \to V$ (single qudit gates), generate a dense subgroup of $U(d^n)$. A unitary matrix is *exactly* *universal* if the full group $U(d^n)$ is generated. Brylinski showed in [10] that a two qudit gate U is universal if and only if it is entangling. That is, if there is a state $|ij\rangle \in V \otimes V$ such that $U|ij\rangle$ cannot be written as the tensor product of two qubits. An X-shaped solution will be universal since it entangles the state $|0\rangle$. The CNOT gate defined by $|ij\rangle \to |i, i \oplus j\rangle$ is exactly universal since it sends the state $|00\rangle + |10\rangle$ to $|00\rangle + |11\rangle$ [10]. In particular:

$$CNOT(|00\rangle + |10\rangle) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = |00\rangle + |11\rangle\,. \tag{2.1}$$

The entangling condition also provides a connection between topological entanglement and quantum entanglement. Any R matrix gives rise to a knot and link invariant [81], and if R is not entangling, it cannot be used to distinguish between two knots [2]. While almost every unitary gate is universal [19], finding matrices that are both universal and provide a braid group representation is a complex task. One way to construct a matrix is via the braided form of the Yang-Baxter equation, as described in section 4. There are other methods of constructing gates for topological quantum computation. For example gates based on the behavior of specific anyonic systems such as metaplectic anyons [18]. Some systems are particularly suited for qutrit computation, for example weakly-integral anyons [9] can form a universal system when supplemented with measurements. Some examples of universal gate sets include the Clifford set consisting of the CNOT, Hadamard, and phase gate, which is universal when combined with the phase shift gate. Another example is the Toffoli gate and Hadamard gate [1]. Some other examples of universal gates for topological quantum computing can be found in [16], [53], [86], [68], [79], [51]. Other gates have been found via braid group representations [59]. For example it is shown by Kauffman and Lomonaco in [48] that the following unitary solutions to the 2-dimensional braided Yang-Baxter equation are exactly universal as two qubit gates:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

There are also many known examples of matrices that are universal for quantum computation with qudits. For example the conditions under which a diagonal matrix is universal are listed in [10]. Another example is the CNOT gate. The CNOT gate generalizes to the controlled increment gate $C^n_{X,d}$, which is defined recursively in [42] and is recalled next. Let n be the number of qudits being acted on. Let X_d denote the $d \times d$ increment gate (INC):

$$X_d = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix} \tag{2.2}$$

CNOT (or sometimes CINC, the controlled increment gate) is then defined as in [42] by letting $C^1_{X,d} = X_d$ and then recursively constructing

$$C^n_{X,d} = \begin{pmatrix} I & 0 & 0 & \dots & 0 \\ 0 & C^{n-1}_{X,d} & 0 & \dots & 0 \\ 0 & 0 & I & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & I \end{pmatrix} \tag{2.3}$$

Note that the identity matrix block is repeated $n-2$ times in the lower right. Induction on n shows that $C^n_{X,d}$ is a real unitary matrix for all n.

Proof. When $n = 1$, we have $C^1_{X,d} = X_d$ and $X_d X_d^T = I$. Now suppose $C^n_{X,d}$ is unitary up to some $n > 1$.

$$C^n_{X,d}(C^n_{X,d})^\dagger = \begin{pmatrix} I & 0 & 0 & \dots & 0 \\ 0 & C^{n-1}_{X,d} & 0 & \dots & 0 \\ 0 & 0 & I & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & I \end{pmatrix} \begin{pmatrix} I & 0 & 0 & \dots & 0 \\ 0 & (C^{n-1}_{X,d})^\dagger & 0 & \dots & 0 \\ 0 & 0 & I & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & I \end{pmatrix} \tag{2.4}$$

$$= \begin{pmatrix} I & 0 & 0 & \dots & 0 \\ 0 & C^{n-1}_{X,d}(C^{n-1}_{X,d})^\dagger & 0 & \dots & 0 \\ 0 & 0 & I & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & I \end{pmatrix} = \begin{pmatrix} I & 0 & 0 & \dots & 0 \\ 0 & I & 0 & \dots & 0 \\ 0 & 0 & I & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & I \end{pmatrix} \tag{2.5}$$

□

The CNOT gate appears in this family as $C^2_{X,d}$. The examples from [48] above may be the only known unitary braid group representations that are also universal for quantum computation. In section 5 we construct an example of a braid group representation that is also universal for quantum computation in all dimensions.

Chapter 3

Methodology

"... he who seeks for methods without having a definite problem in mind seeks for the most part in vain."

David Hilbert

What follows is a summary of some of the techniques, useful properties, and constructions used in this dissertation.

3.1 Properties of the Kronecker Product

The aB convention Kronecker product of a $m \times n$ matrix A and a $k \times l$ matrix B is the $mk \times nl$ block matrix given by:

$$A \otimes B = \begin{pmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{pmatrix}$$

The Kronecker product satisfies many useful properties including the following [41]:

- $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$ when all matrices are a appropriately sized. Note that this also applies to vectors $(A \otimes B)(v \otimes w) = (Av \otimes Bw)$.

- $(A \otimes B)^{-1} = (A^{-1} \otimes B^{-1})$

- $(A \otimes B)^* = (A^* \otimes B^*)$ where $*$ denotes complex conjugation.

- $(A \otimes B)^T = (A^T \otimes B^T)$

- There exist permutation matrices P_1 and P_2 such that $P_1(A \otimes B)P_2 = (B \otimes A)$. When A and B are both $n \times n$ then the $n^2 \times n^2$ *swap operator* P defined in the Dirac notation by $P : |ij\rangle \to |ji\rangle$ has the property that $P(A \otimes B)P = (B \otimes A)$.

Proof. Here we provide a proof of the last property since it is fundamental to the relationship between two forms of the Yang-Baxter equation. Let e_i and e_j be two basis vectors of V. Then $P(A \otimes B)P(e_i \otimes e_j) = P(A \otimes B)(e_j \otimes e_i) = P(Ae_j \otimes Be_i) = (Be_i \otimes Ae_j) = (B \otimes A)(e_i \otimes e_j)$ □

3.2 Gröbner Bases

Gröbner Basis methods are roughly a generalization of Gaussian elimination and polynomial long division for the purpose of solving systems of multivariate polynomial equations. For a given set of multivariate polynomials (an ideal) a Gröbner Basis is another set of multivariate polynomials which can be computed from the original set, and has properties which can help solve original system. Here we recall the most relevant definitions and properties from [17].

Definition 3.2.1 (Linear Combination)**.** A *linear combination* of a set of polynomials is a weighted sum of those polynomials where the weights are themselves arbitrary polynomials in the same variables.

Definition 3.2.2 (Ordering)**.** An admissible *ordering* is one in which 1 is always considered minimal and $u \prec v \Rightarrow ut \prec vt$ for any power products u, v, t.

Definition 3.2.3 (Leading coefficient)**.** Given an ordering and a polynomial P, the greatest term is called the *leading term* and the corresponding coefficient the *leading coefficient*.

Definition 3.2.4 (Power product)**.** Any product of the form $x_1^{\alpha_1} \dots x_n^{\alpha_n}$ with no leading coefficient is called a *power product*.

Definition 3.2.5 (Monomial)**.** Any product of the variables $x_1, \dots, x_n$ with a leading coefficient is called a *monomial*.

Definition 3.2.6 (Leading power product)**.** The largest power product appearing in a given polynomial under a particular ordering is called the *leading power product*. Denoted $LPP(f)$. Denote by $LM(f)$ the leading monomial which includes the leading coefficient.

Definition 3.2.7 (Reduction)**.** The polynomial *reduction* of a polynomial f by a set of polynomials $\{g_1, \dots g_n\}$ is a process that results in the expression of f as a linear combination of the polynomials $g_1, \dots g_n$ and a remainder r such that $f = q_1 g_1 + q_2 g_2 + \cdots + q_n g_n + r$

Definition 3.2.8 (S-Polynomial)**.** The *S-Polynomial* of two polynomials f and g under a particular ordering is defined by:

$$SP(f, g) = LCM(LPP(f), LPP(g)) \left(\frac{f}{LM(f)} - \frac{g}{LM(g)} \right)$$

where LCM is the least common multiple.

Definition 3.2.9 (Gröbner basis)**.** A basis B is a *Gröbner basis* if and only if the S-polynomials between each pairs of basis elements reduces to 0 (Buchberger's theorem).

A Gröbner basis GB for a polynomial ideal I has many nice properties including:

- Given a monomial ordering a GB can be found using Buchberger's algorithm.
- The *variety* (that is the set of common zeroes, or solutions to a polynomial system) generated by I and GB are the same.
- *Hilbert's nullstellensatz* The variety generated by I is empty if and only if $GB = \{1\}$

Buchberger's algorithm [11] is guaranteed to result in a Gröbner basis, however, in practice the computational complexity can be very large. In particular if the number of variables is n and d is the maximum degree of any monomial appearing in the set of polynomials than the degree of the polynomials in GB is bounded by $2(\frac{d^2}{2}+d)^{2^{n-1}}$.

Example 3.2.1. As an example consider the following system from [82]:

$$x^2 + y + z - 1 = 0,$$
$$x + y^2 + z - 1 = 0,$$
$$x + y + z^2 - 1 = 0.$$

These form an ideal $I = \{x^2 + y + z - 1, x + y^2 + z - 1, x + y + z^2 - 1\}$. A Gröbner basis for this system under the lexicographic monomial ordering can be computed using Buchberger's algorithm [17], which is implemented in most computer algebra systems. The Gröbner basis consists of the following polynomials:

$$z^6 - 4z^4 + 4z^3 - z^2,$$

$$
\begin{aligned}
2yz^2 + z^4 - z^2, \\
y^2 - y - z^2 + z, \\
x + y + z^2 - 1.
\end{aligned}
$$

The first polynomial contains only the variable z and can be factored:

$$
\begin{aligned}
z^6 - 4z^4 + 4z^3 - z^2 &= z^2(z^4 - 4z^2 + 4z - 1) \\
&= z^2(z-1)(z^3 + z^2 - 3z + 1) \\
&= z^2(z-1)^2(z^2 + 2z - 1)
\end{aligned}
$$

The original system can now be solved using back substitution into the remaining basis polynomials to obtain the five solutions:

$$
\begin{array}{lll}
z = 0 & y = 0 & x = 1 \\
z = 0 & y = 1 & x = 0 \\
z = 1 & y = 0 & x = 0 \\
z = -1 - \sqrt{2} & y = -1 - \sqrt{2} & x = -1 - \sqrt{2} \\
z = -1 + \sqrt{2} & y = -1 + \sqrt{2} & x = -1 + \sqrt{2}
\end{array}
$$

This example illustrates both the elimination and extension theorems for the purpose of solving multivariate systems. These theorems are explained in more detail in [17].

3.3 Automated subsystem solver

In practice computing a Gröbner basis for a large system of multivariate polynomials may require a prohibitive amount of memory due to the number of polynomials in the Gröbner basis and the number of terms in each basis polynomials. Computing a Gröbner basis was therefore only possible for solving the algebraic Yang-Baxter equation in dimension 2 [37]. Mathematica's Solve and Reduce functions automatically utilize Gröbner basis methods for solving multivariate polynomial systems. Therefore these functions will fail when the Gröbner basis is too large to compute. Some polynomial systems which arise when considering certain special cases of the Yang-Baxter equation(s) have the property that it is possible to compute a Gröbner basis for a subset of the polynomials. This observation inspired the following algorithm:

Algorithm 1 Subsystem Solver

```
Attempt to solve a subsystem of size q for a fixed length of time. If time runs out
reduce q.
for each partial solution s do
    Compute the matrix M determined by s and add M to a hash table
    if M is already in the hash table then
        return
    end if
    Substitute s into the remaining polynomial system
    if s is a solution then
        save s and return
    else if s is not a solution then
        Recursively apply the subsystem solver on the new system
    end if
end for
```

When a subsystem of the equations involved in one of the Yang-Baxter equation(s) can be solved using Gröbner Basis methods the main limitation to finding all possible solutions is usually the number of additional sub-cases that need to be considered. Hashing the matrix obtained at each iteration of the algorithm ensures (up to potential collisions of the hashing algorithm) that we do not consider duplicate partial solutions. There are many other approaches to the Yang-Baxter equations, including choosing a particular ansatz, or initial guess about the form of the solution, see for example [56], [40], [15], [39], differential approaches [84], [85], and solutions that arise from particular Lie algebras [6], [64] [77].

Chapter 4

The Yang-Baxter equations

> "Mathematics is the art of giving the same name to different things."
>
> Henri Poincaré

4.1 Yang-Baxter equations in the context of topological quantum computation

The gates in a topological quantum computer correspond to physically braiding the underlying anyonic system. There are several approaches to constructing these gates. One method is a bottom-up style method such as the one described in [35] and [18], which starts with a physical anyonic system which will satisfy the braid relations due to its innate physical properties, the matrices arising from these representations must satisfy the braided form of the Yang-Baxter equation. The other method is more top-down, as described in [48], starting with a braid group representation and then engineering a physical system to realize the corresponding gate. What follows is motivated by the top-down approach. The changes in phase of a quantum system consisting of n ordered anyons corresponds to the n strand braid group in topology, denoted by B_n. The next section is a brief introduction to this group, from which the braided Yang-Baxter equation naturally arises from the group's representation theory. Forms of the Yang-Baxter equations also appear in statistical mechanics, quantum groups, and elsewhere.

4.2 The braid group

The *braid group* B_n was first introduced by E. Artin in 1925 [4]. The group B_n can be visualized as n vertical strands which are braided over and under one another, while maintaining that each vertical strand must pass the horizontal line test. That is, each strand must strictly travel from top to bottom without any local maxima or minima. Braids are considered equivalent if they are ambient isotopic with fixed endpoints. Some particular ambient isotopies give rise to the relators of the braid group. Multiplication between two braids is defined by placing the braids vertically above each other, and gluing them together as indicated in Figure 4.1.

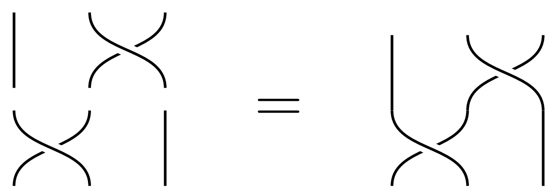

Figure 4.1: The multiplication of two 3 strand braids.

A set of generators for the braid group consists of the set $\{I, \sigma_1, \ldots \sigma_{n-1}\}$, where σ_i is the braid with strand i crossing over and to the right of strand $i+1$, and I is the braid with no crossings. There are two types of relations in the braid group, which we will refer to as *braid relations*. The first type of relation requires that two generators commute as long as they are at least two strands apart: $\sigma_i \sigma_j = \sigma_j \sigma_i$ whenever $|i - j| > 1$, these relations are sometimes referred to as *far commutativity*. The generators must also satisfy the second type of relations referred to as the Yang-Baxter relations:

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \tag{4.1}$$

This relation can be seen in Figure 4.2 which shows two braids that are ambient isotopic with fixed endpoints. A group *representation* is a homomorphism from one

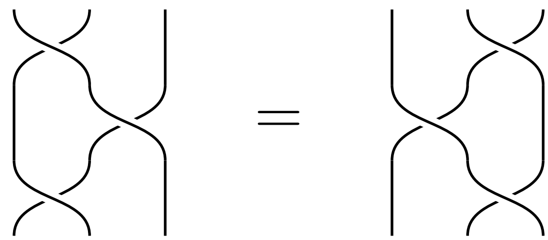

Figure 4.2: Two ambient isotopic braids.

group to another. A *linear representation* is a homomorphism from one group into a group of invertible matrices. A *faithful* representation is representation a one-one homomorphism. Linear representations of the braid group have historically been of interest for studying the braid group.

If V is a d dimensional vector space over a field $\mathbb{F}$, denote by $V^{\otimes n}$ the tensor product of V with itself n times. It is then natural to look for representations of B_n in $\mathrm{Aut}(V^{\otimes n})$. One way to define such a braid group representation is by mapping each generator as follows:

$$\sigma_i \to I^{\otimes i-1} \otimes R \otimes I^{\otimes n-i-1} \tag{4.2}$$

where $R : V \otimes V \to V \otimes V$ is an invertible linear map. Here $\otimes$ denotes the aB convention Kronecker product between two linear maps. In particular, the ij block of $A \otimes B$ is given by $(a_{ij}B)$ and $A^{\otimes k}$ denotes the Kronecker product of the matrix A with itself k times. This defines a representation of the braid group as long as the mapping R is chosen so that all of the braid relations are satisfied.

4.3 The braided Yang-Baxter equations

Definition 4.3.1. Let V be a d-dimensional vector space over $\mathbb{C}$. Let I be the $d \times d$ identity matrix on the vector space V, and $R : V \otimes V \to V \otimes V$ an invertible linear transformation. The matrix R satisfies the d-dimensional *braided Yang-Baxter equation* (bYBE) when:

$$(R \otimes I)(I \otimes R)(R \otimes I) = (I \otimes R)(R \otimes I)(I \otimes R) \tag{4.3}$$

Any matrix satisfying equation 4.3 is referred to in the literature as an R-matrix [54]. In the representation defined in equation 4.2, the matrix R, must satisfy the matrix form of the appropriate bYBE, defined here or in [45], [47], and elsewhere.

4.4 Generalized Yang-Baxter equations

Definition 4.4.1 (Generalized Yang-Baxter equations)**.** Let d, m, and l be natural numbers. Let V be a vector space over $\mathbb{C}$ of dimension d, and $R : V^{\otimes m} \to V^{\otimes m}$ be an invertible matrix. Denote the identity on V by I_V. The matrix R is a solution to the (d, m, l)-*generalized Yang-Baxter equation* (gYBE) whenever

$$(R \otimes I_V^{\otimes l})(I_V^{\otimes l} \otimes R)(R \otimes I_V^{\otimes l}) = (I_V^{\otimes l} \otimes R)(R \otimes I_V^{\otimes l})(I_V^{\otimes l} \otimes R) \tag{4.4}$$

This gives another way to represent the braid group as introduced by Rowell, Zhang, Wu, and Ge in [70]. For any fixed d when $m = 2$ and $l = 1$ this expression is equivalent to the d-dimensional bYBE. Any bYBE solution gives rise to a braid group representation while a solution to a gYBE gives rise to a braid group representation whenever the far commutativity relations are also satisfied. This is guaranteed when

$l > m/2$ [15]. The corresponding representation of B_n is given by the homomorphism:

$$\sigma_i \to I_{d^l}^{\otimes i-1} \otimes R \otimes I_{d^l}^{\otimes n-i-1} \tag{4.5}$$

where I_{d^l} is the $d^l \times d^l$ identity matrix. Finding matrices which satisfy either a gYBE or bYBE is a difficult task and only a few examples are known, even in low dimensions. A full classification has been completed only for the (2,2,1)-gYBE (equivalent to the 2-dimensional bYBE), see [38]. The unitary solutions to this same set of equations in [26]. Other solutions have been obtained by picking an ansatz to make the problem more tractable. One example is the charge conserving ansatz, the solutions for this ansatz have been fully classified in [57].

Theorem 4.4.1. The only invertible diagonal solutions for the (d, m, l)-generalized Yang-Baxter equation are scalar multiples of the identity matrix.

Proof. Let (d, m, l) be given such that $l < m$ (the $l \geq m$ case is handled below). Let $R = diag(r_1, \ldots, r_{d^m})$ be an invertible diagonal matrix. Both $(R \otimes I)$ and $(I \otimes R)$ are diagonal and will commute. The left hand side of the (d, m, l)-gYBE can be written:

$$(R \otimes I_d^{\otimes l})(I_d^{\otimes l} \otimes R)(R \otimes I_d^{\otimes l}) = (I_d^{\otimes l} \otimes R)(R \otimes I_d^{\otimes l})(I_d^{\otimes l} \otimes R) \tag{4.6}$$

$$(R^2 \otimes I_d^{\otimes l})(I_d^{\otimes l} \otimes R) = (I_d^{\otimes l} \otimes R^2)(R \otimes I_d^{\otimes l}) \tag{4.7}$$

$$(R^2 \otimes I_d^{\otimes l})(I_d^{\otimes l} \otimes R) = (R \otimes I_d^{\otimes l})(I_d^{\otimes l} \otimes R^2) \tag{4.8}$$

$$R \otimes I_d^{\otimes l} = I_d^{\otimes l} \otimes R \tag{4.9}$$

On the left hand side of equation 4.9, each of the variables of R is repeated d^l times:

$$R \otimes I_d^{\otimes l} = diag(r_1, ..., r_1, r_2, ...r_2, ..., r_{d^m}, ..., r_{d^m}) \tag{4.10}$$

On the right hand side of equation 4.9, R is repeated d^l times.

$$I_d^{\otimes l} \otimes R = diag(r_1, r_2..., r_{d^m}, r_1, ...r_{d^m}, ..., r_1, ..., r_{d^m}) \tag{4.11}$$

To make $R \otimes I_d^{\otimes l} = I_d^{\otimes l} \otimes R$, the first d^l variables: $r_1, ..., r_{d^l}$ must be equal to r_1. Substituting r_1 for those variables in $R \otimes I_d^{\otimes l}$ results in r_1 being repeated d^{2l} times:

$$R \otimes I_d^{\otimes l} = diag(r_1, ..., r_1, r_1, ...r_1, ..., r_{d^m}, ..., r_{d^m}) \tag{4.12}$$

Examining $R \otimes I_d^{\otimes l} = I_d^{\otimes l} \otimes R$ shows that the next group of d^l variables, labeled $r_{d^{l+1}}, ..., r_{d^{2l+1}}$, are also equal to r_1. This process can be repeated until all d^m variables are shown to equal r_1.

□

Theorem 4.4.2. All invertible solutions to the (d, m, l)-gYBE are of the form λI_{d^m} whenever $l \geq m$.

Proof. Fix (d, m, l) with $l \geq m$, note that this makes R a $d^m \times d^m$ matrix, and I_d the $d \times d$ identity matrix. We can then write:

$$(R \otimes I_d^{\otimes l})(I_d^{\otimes l} \otimes R)(R \otimes I_d^{\otimes l}) = (I_d^{\otimes l} \otimes R)(R \otimes I_d^{\otimes l})(I_d^{\otimes l} \otimes R) \tag{4.13}$$

$$(R \otimes I_d^{\otimes l})(I_d^{\otimes m} \otimes I_d^{\otimes l-m} \otimes R)(R \otimes I_d^{\otimes l}) = (I_d^{\otimes m} \otimes I_d^{\otimes l-m} \otimes R)(R \otimes I_d^{\otimes l})(I_d^{\otimes l} \otimes R) \tag{4.14}$$

$$(R \otimes I_d^{\otimes l})(I_d^{\otimes m} R \otimes (I_d^{\otimes l-m} \otimes R) I_d^{\otimes l}) = (I_d^{\otimes m} R \otimes (I_d^{\otimes l-m} \otimes R) I_d^{\otimes l})(I_d^{\otimes l} \otimes R) \tag{4.15}$$

$$(R \otimes I_d^{\otimes l})(R \otimes I_d^{\otimes l-m} \otimes R) = (R \otimes I_d^{\otimes l-m} \otimes R)(I_d^{\otimes l} \otimes R) \tag{4.16}$$

$$(R^2 \otimes I_d^{\otimes l}(I_d^{\otimes l-m} \otimes R)) = ((R \otimes I_d^{\otimes l-m})I_d^{\otimes l} \otimes R^2) \tag{4.17}$$

$$(R^2 \otimes I_d^{\otimes l-m} \otimes R) = (R \otimes I_d^{\otimes l-m} \otimes R^2) \tag{4.18}$$

$$(R^{-1} \otimes I_d^{\otimes l})(R^2 \otimes I_d^{\otimes l-m} \otimes R) = (R^{-1} \otimes I_d^{\otimes l})(R \otimes I_d^{\otimes l-m} \otimes R^2) \tag{4.19}$$

$$(R \otimes I_d^{\otimes l-m} \otimes R) = (I_d^{\otimes m} \otimes I_d^{\otimes l-m} \otimes R^2) \tag{4.20}$$

$$(R \otimes I_d^{\otimes l-m} \otimes R)(I_d^{\otimes l} \otimes R^{-1}) = (I_d^{\otimes m} \otimes I_d^{\otimes l-m} \otimes R^2)(I_d^{\otimes l} \otimes R^{-1}) \tag{4.21}$$

$$(R \otimes I_d^{\otimes l-m} \otimes I_d^{\otimes m}) = (I_d^{\otimes m} \otimes I_d^{\otimes l-m} \otimes R) \tag{4.22}$$

$$R \otimes I_d^{\otimes l} = I_d^{\otimes l} \otimes R \tag{4.23}$$

It follows by the argument in the proof of theorem 4.4.1 that R must be a scalar multiple of the identity I_{d^m} in order to solve the (d, m, l)-gYBE whenever $l \geq m$. □

4.5 The algebraic Yang-Baxter equation

Definition 4.5.1 (Algebraic Yang-Baxter equations)**.** Let $V = \mathbb{C}^d$ with basis $\{|i\rangle = e_i \quad | \quad i = 1 \dots d\}$. Let $P : V \otimes V \to V \otimes V$ be the swap operator interchanging qudits denoted in the Dirac notation [21] by $P : |ij\rangle \to |ji\rangle$, let I be the $d \times d$ identity matrix, and $R : V \to V$ a $d^2 \times d^2$ matrix. Let R_{ij} denote the matrix which applies R only on the i and j factors of a vector in $V \otimes V \otimes V$:

$$R_{12} = (R \otimes I) \tag{4.24}$$

$$R_{13} = (I \otimes P)(R \otimes I)(I \otimes P) \tag{4.25}$$

$$R_{23} = (I \otimes R) \tag{4.26}$$

the *algebraic Yang-Baxter equation* (aYBE) is then defined by:

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \tag{4.27}$$

The d-dimensional aYBE and bYBE are closely related: if R solves the d-dimensional aYBE then RP solves the d-dimensional bYBE and vice versa. The matrix form of the d-dimensional aYBE can be written as a system of polynomial equations (see [38], [26] or the appendix):

$$R^{k_1k_2}_{j_1j_2}R^{l_1k_3}_{k_1j_3}R^{l_2l_3}_{k_2k_3} = R^{k_2k_3}_{j_2j_3}R^{k_1l_3}_{j_1k_3}R^{l_1l_2}_{k_1k_2} \tag{4.28}$$

where each equation is indexed by $(j_1, j_2, j_3, l_1, l_2, l_3)$, with each index ranging from 1 to d, and following the Einstein summation convention used in differential geometry [27], sums are taken over repeated indices. In this case, the indices denoted with a k (k_p for $p = 1, 2, 3$) are summed over. This indexing appears with some variation between references because equation 4.28 is invariant under certain index changes, as listed in [38]. We will use the lexicographic of basis vectors and the following convention:

$$R(e_i \otimes e_i) = \sum_{a,b} R^{ab}_{ij} e_a \otimes e_b \tag{4.29}$$

For example, when $d = 3$, the matrix R looks like:

$$
\begin{pmatrix}
R^{11}_{11} & R^{11}_{12} & R^{11}_{13} & R^{11}_{21} & R^{11}_{22} & R^{11}_{23} & R^{11}_{31} & R^{11}_{32} & R^{11}_{33} \\
R^{12}_{11} & R^{12}_{12} & R^{12}_{13} & R^{12}_{21} & R^{12}_{22} & R^{12}_{23} & R^{12}_{31} & R^{12}_{32} & R^{12}_{33} \\
R^{13}_{11} & R^{13}_{12} & R^{13}_{13} & R^{13}_{21} & R^{13}_{22} & R^{13}_{23} & R^{13}_{31} & R^{13}_{32} & R^{13}_{33} \\
R^{21}_{11} & R^{21}_{12} & R^{21}_{13} & R^{21}_{21} & R^{21}_{22} & R^{21}_{23} & R^{21}_{31} & R^{21}_{32} & R^{21}_{33} \\
R^{22}_{11} & R^{22}_{12} & R^{22}_{13} & R^{22}_{21} & R^{22}_{22} & R^{22}_{23} & R^{22}_{31} & R^{22}_{32} & R^{22}_{33} \\
R^{23}_{11} & R^{23}_{12} & R^{23}_{13} & R^{23}_{21} & R^{23}_{22} & R^{23}_{23} & R^{23}_{31} & R^{23}_{32} & R^{23}_{33} \\
R^{31}_{11} & R^{31}_{12} & R^{31}_{13} & R^{31}_{21} & R^{31}_{22} & R^{31}_{23} & R^{31}_{31} & R^{31}_{32} & R^{31}_{33} \\
R^{32}_{11} & R^{32}_{12} & R^{32}_{13} & R^{32}_{21} & R^{32}_{22} & R^{32}_{23} & R^{32}_{31} & R^{32}_{32} & R^{32}_{33} \\
R^{33}_{11} & R^{33}_{12} & R^{33}_{13} & R^{33}_{21} & R^{33}_{22} & R^{33}_{23} & R^{33}_{31} & R^{33}_{32} & R^{33}_{33}
\end{pmatrix} \tag{4.30}
$$

Theorem 4.5.1. All $d^2 \times d^2$ diagonal matrices are solutions to the d-dimensional aYBE.

Proof. Let R be a $d^2 \times d^2$ diagonal matrix. Using the indexing convention described above, a diagonal matrix will only have nonzero entries along the diagonal: R^{ab}_{ab} and zero entries elsewhere. Now consider equation 4.28 indexed by $(j_1, j_2, j_3, l_1, l_2, l_3)$. All terms in the sum on the left side of equation 4.28 will vanish unless each variable is from the diagonal of R, that is unless $k_1 = j_1$, $l_1 = k_1$, $l_2 = k_2$, $k_2 = j_2$, $k_3 = j_3$, $l_3 = k_3$. By the same reasoning, the terms in the sum on the right hand side will vanish unless $k_2 = j_2$, $k_1 = j_1$, $l_1 = k_1$, $l_2 = k_2$, $j_3 = k_3$, $k_3 = l_3$, $k_2 = l_2$. This results in the following equation which is satisfied regardless of the diagonal entries of R:

$$R^{j_1 l_2}_{j_1 l_2} R^{j_1 j_3}_{j_1 j_3} R^{j_2 j_3}_{j_2 j_3} = R^{j_2 j_3}_{j_2 j_3} R^{j_1 j_3}_{j_1 j_3} R^{j_1 l_2}_{j_1 l_2} \tag{4.31}$$

□

4.6 The set-theoretic Yang-Baxter equation

In 1992 Drinfeld posed the question of classifying permutation solutions to the algebraic and braided Yang-Baxter equations. This question can be generalized to the set-theoretic Yang-Baxter equation, of which the permutation solutions are a particular class of solution.

Classifying the set-theoretic (permutation) solutions to the Yang-Baxter equation is a problem posed by Drinfeld in 1992 [24]. The set theoretic Yang-Baxter equation when the cardiality of the set X is equal to d is equivalent to the d-dimensional bYBE. A d-dimensional set theoretic solution can be extended to a solution of the d-dimensional bYBE via linearization. In particular by letting the elements of X form a basis of a vector space we can convert a d-dimensional set theoretic solution to a permutation matrix solution of the d-dimensional bYBE. The field of quantum groups has historically studied algebraic structures arising from bYBE solutions over vector spaces. Set-theoretic solutions also give rise to interesting algebraic structures [31]. What follows is an introduction to the terminology of set-theoretic solutions.

The paper by Etingof, Schedler, and Solovieve in 1999 [28] initiated the theory of involutive solutions. Subsequently in 2003 Gateva-Ivanova and Van den Bergh in [30] further studied square-free involutive solutions (ones in which $r(x,x) = (x,x)$ for all $x \in X$). Since then, lots of work has been done on involutive solutions and associated algebraic structures including connections to radical rings, and homology [67], [12]. These connections have produced new families of set-theoretic solutions, however, the problem of classifying all solutions or constructing new families of solutions is still open [67]. Next we list some basic definitions as in [67] and elsewhere.

Definition 4.6.1 (Set-theoretic Yang-Baxter equation)**.** Let X be a nonempty set and $r : X \times X \to X \times X$, let I denote the identity on X, and let $\times$ denote the direct product. The *set-theoretic Yang-Baxter equation* (set-theoretic bYBE) is defined by:

$$(r \times I)(I \times r)(r \times I) = (I \times r)(r \times I)(I \times r) \tag{4.32}$$

The pair (X, r) is a *set-theoretic solution* to the Yang-Baxter equation.

Definition 4.6.2 (Set-theoretic aYBE)**.** Let X be a nonempty set and $r : X \times X \to X \times X$, let I denote the identity on X, and let $\times$ denote the direct product. The *set-theoretic quantum Yang-Baxter equation* (set-theoretic aYBE) is defined by:

$$r^{12}r^{13}r^{23} = r^{23}r^{13}r^{12} \tag{4.33}$$

The pair (X, r) is considered a set-theoretic solution to the quantum Yang-Baxter equation.

The map r can be written in terms of it's two components:

$$r(x, y) = (\sigma_x(y), \tau_y(x)) \tag{4.34}$$

Definition 4.6.3 (Non-degenerate)**.** The pair (X, r) is said to be *non-degenerate* if both σ_x and τ_y are bijective maps from X to itself, for each $x \in X$. If only σ_x is bijective then (X, r) is said to be left non-degenerate, and if only τ_y is bijective then (X, r) is said to be right non-degenerate.

Definition 4.6.4 (Involutive)**.** The pair (X, r) is an *involutive* solution if r^2 is the identity on $X \times X$.

Definition 4.6.5 (Square-free)**.** A solution (X, r) is *square-free* if $r(x, x) = (x, x)$ for all $x \in X$.

Definition 4.6.6 (Finite)**.** A solution (X, r) is *finite* if the set X is finite.

When a solution is non-degenerate and involutive one can write:

$$r^2(x, y) = r(\sigma_x(y), \tau_y(x)) = (\sigma_{\sigma_x(y)}(\tau_y(x)), \tau_{\tau_y(x)}(\sigma_x(y))) = (x, y) \tag{4.35}$$

and therefore:

$$\tau_y(x) = \sigma^{-1}_{\sigma_x(y)}(x) \tag{4.36}$$

$$\sigma_x(y) = \tau^{-1}_{\tau_y(x)}(y) \tag{4.37}$$

The majority of papers in this area focus on non-degenerate involutive solutions due to the connection, conjectured by Gateva-Ivanova [30]. Gateva-Ivanova conjectured that every non-degenerate, involutive, square-free, finite solution to the set-theoretic aYBE, comes from a binomial semigroup (defined in [30]). The converse of this conjecture was already known to be true [73]. This conjecture was proved by Rump in [71], leading many to study these semigroups and their associated solutions.

Definition 4.6.7 (Decomposable)**.** A solution (X, r) is *decomposable* if there is a disjoint partition $X = Y \sqcup Z$ such that both Y and Z are nonempty, $r(Y \times Y) \subseteq Y \times Y$, and $r(Z \times Z) \subseteq Z \times Z$

Gateva-Ivanova's conjecture can be equivalently stated in terms of decomposability [28]: Let $1 < |X| < \infty$, then there is a non-trivial decomposition $X = Y \sqcup Z$ such that $Y \times Y$ and $Z \times Z$ are invariant under R.

In [72] Rump introduces the idea of Braces to study involutive non-degenerate solutions. In 2017 Guarnieri and Vendramin introduced the idea of skew braces to study non-degenerate solutions that aren't necessarily involutive [33].

4.6.0.1 Algebraic structures associated to set theoretic solutions [83]

Skew braces correspond to non-degenerate solutions, and braces correspond to involutive non-degenerate solutions. Classifying braces and skew braces therefore gives a pathway to classifying these types of solutions. It is still unknown what algebraic structure corresponds to general set-theoretic solutions.

Definition 4.6.8 (Structure group)**.** The *structure group* [72], [33] of a solution (X, r) is the group

$$G(X, r) = \{X | xy = \sigma_x(y)\tau_y(x)\}$$

Definition 4.6.9 (Skew brace)**.** A (left) *skew brace* [72], [33] is a set B with two group operations, denoted $+$ and $\circ$, such that $(B, +)$ and $(B, \circ)$ are groups and for any $a, b, c \in B$ the following relation is satisfied:

$$a \circ (b + c) = (a \circ b) - a + (a \circ c)$$

Definition 4.6.10 (Brace)**.** A (left) *brace* [72], [33] is a skew brace, denoted by $(B, +, \circ)$, where $(B, +)$ is abelian.

Braces and skew braces are connected to set theoretic solutions [33] as follows. Let A be a skew left brace. Let $\sigma_a(b) = a^{-1} + (a \circ b)$. Then $r_A : A \times A \to A \times A$ defined by:

$$r_A(a, b) = (\sigma_a(b), \sigma^{-1}_{\sigma_a(b)}((a \circ b)^{-1} + a + (a \circ b)) \tag{4.38}$$

is a non-degenerate solution of the Yang-Baxter equation. The solution r_A is involutive if and only if $a + b = b + a$ for all $a, b \in A$. Skew braces have received a lot of focus recently due to the connection between skew braces and set-theoretic solutions, see [13], [14], [43], and the references therein.

Structure	Solution Type
Braces	Non-degenerate involutive
Skew braces	Non-degenerate
q-cycle sets	Left non-degenerate
?	Arbitrary solutions

Table 4.1: A summary of the algebraic structures associated to different set theoretic solution types [83]

4.7 Symmetries of the Yang-Baxter equations

Each solution of a gYBE generates more solutions under the following symmetries, to the same gYBE (and with the appropriate choice of (d, m, l) these symmetries also generate more solutions to the d-dimensional bYBE and d-dimensional aYBE).

Proposition 4.7.1. If R is an invertible solution to the (d, m, l)-gYBE then the following are also invertible solutions:

1. λR for any nonzero scalar λ
2. R^{-1}
3. The complex conjugate R^*
4. The transpose R^T and hence the complex adjoint $R^\dagger$
5. $Q^{\otimes m} R (Q^{-1})^{\otimes m}$ where Q is any complex non-singular $d \times d$ matrix.
6. PRP when $m = 2$, $l = 1$ and $P : |ij\rangle \to |ji\rangle$ is the swap matrix.

Proposition 4.7.2. If (X, r) is an invertible solution to the set-theoretic Yang-Baxter equation then so are the following:

1. (X, r^{-1})

2. $(X, (Q \times Q)r(Q \times Q))$ where $Q : X \to X$ is invertible.

These symmetries are well known and appear in [46] and [38], a proof that 1-5 apply to the gYBEs, and that 6 applies to the d-dimensional bYBE and d-dimensional aYBE is provided in appendix A.1. The proof for the symmetries of the set-theoretic equations is very similar after replacing $\otimes$ with $\times$.

Definition 4.7.1 (Diagonal dressing)**.** New solutions to the d-dimensional aYBE can be obtained from already known solutions via *diagonal dressing* as described in [39]. Let $\hat{R}$ be a solution to the m-dimensional aYBE. Then a solution in dimension $n > m$, denoted R, can be obtained as follows. Let $A \subset \{1, \ldots, n\}$ and define

$$R_{ij}^{kl} = \begin{cases} \hat{R}_{ij}^{kl} & i, j, k, l \in A \\ s_{ij}\delta_i^k \delta_j^l & \text{otherwise} \end{cases} \tag{4.39}$$

The numbers s_{ij} must satisfy:

$$\hat{R}_{ij}^{kl}(s_{mi}s_{mj} - s_{mk}s_{ml}) = 0 \tag{4.40}$$

$$\hat{R}_{ij}^{kl}(s_{im}s_{ml} - s_{km}s_{mj}) = 0 \tag{4.41}$$

$$\hat{R}_{ij}^{kl}(s_{im}s_{jm} - s_{km}s_{lm}) = 0 \tag{4.42}$$

for all $i, j, k, l \in A$ and $m \notin A$.

Definition 4.7.2 (Block dressing)**.** Alternatively R can be constructed using a *block dressing* [39]:

$$R_{ij}^{kl} = \begin{cases} \hat{R}_{ij}^{kl} & i,j,k,l \in A \\ \delta_i^k F_j^l & j,l \in A, i,k \notin A \\ G_i^k \delta_j^l & i,k \in A, j,l \notin A \\ \delta_i^k \delta_j^l & \text{otherwise} \end{cases} \tag{4.43}$$

Where F and G must satisfy:

$$(F \otimes F)\hat{R} = \hat{R}(F \otimes F) \tag{4.44}$$

$$(I \otimes F)\hat{R}(G \otimes I) = (G \otimes I)\hat{R}(I \otimes F) \tag{4.45}$$

$$(G \otimes G)\hat{R} = \hat{R}(G \otimes G) \tag{4.46}$$

$$FG = GF \tag{4.47}$$

Example 4.7.3. Next we describe solutions arising from a commutative set of matrices. Let $\{N(\alpha), M(\alpha)|\alpha \in S\}$ be a set of commuting matrices $d^2 \times d^2$ and S an indexing set. Then the following is a solution to the d-dimensional aYBE [39]:

$$R_{ij}^{kl} = \sum_{\alpha \in S} N(\alpha)_i^k M(\alpha)_j^l \tag{4.48}$$

Example 4.7.4. An example of a commutative ring of matrices in any dimension is the set of circulant matrices which take the form:

$$\begin{pmatrix} c_0 & c_{n-1} & \cdots & c_2 & c_1 \\ c_1 & c_0 & c_{n-1} & & c_2 \\ \vdots & c_1 & c_0 & \ddots & \vdots \\ c_{n-2} & & \ddots & \ddots & c_{n-1} \\ c_{n-1} & c_{n-2} & \cdots & c_1 & c_0 \end{pmatrix} \tag{4.49}$$

Chapter 5

A universal Yang-Baxter operator

> "It is possible to invent a single machine which can be used to compute any computable sequence."
>
> Alan Turing

5.1 Constructing the operator

Unitary representations of the braid group serve as the gates in a topological quantum computer [48]. It is therefore desirable to find unitary solutions the d-dimensional bYBE or gYBE, which are also universal for quantum computation. Finding solutions to any of the variation Yang-Baxter equation is generally a difficult task, which reduces to solving a large system of multivariate polynomial equations. In particular, the d-dimensional bYBE involves solving a system of d^6 cubic equations in d^4 variables. What follows is one of the primary contributions of this paper: a not previously noted universal unitary solution to the d-dimensional bYBE for any $d \geq 2$. To construct the solution, first recall the discrete quantum Fourier transform denoted

F_d and with $\omega = e^{i2\pi/d}$:

$$F_d = \frac{1}{\sqrt{d}} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \dots & \omega^{d-1} \\ 1 & \omega^2 & \omega^4 & \dots & \omega^{2(d-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{d-1} & \omega^{2(d-1)} & \dots & \omega^{(d-1)(d-1)} \end{pmatrix} \tag{5.1}$$

To construct the d-dimensional bYBE solution, we define the following unitary solution to the d-dimensional aYBE:

$$R_d = (I \otimes F_d) C^2_{X,d} (I \otimes F_d^\dagger) \tag{5.2}$$

In theorem 5.1.1, we show that this solution is part of a larger family of diagonal unitary solutions to the d-dimensional aYBE, any of which can be converted to a solution of the d-dimensional bYBE by composing with the swap operator P.

Theorem 5.1.1. All $d^2 \times d^2$ diagonal matrices are solutions to the d-dimensional aYBE, and in particular, R_d provides an example of an exactly universal unitary solution to the d-dimensional aYBE.

Proof. First note that R_d is unitary since it is the product of unitary matrices. Since the CNOT gate $C^2_{X,d}$ along with all single qudit gates is an exactly universal gate set, another way to prove a particular matrix is exactly universal is to show that CNOT can be expressed using that matrix and the Kronecker product of local unitary matrices [10]. To express $C^2_{X,d}$ in terms of R_d and the Kronecker product of local linear transformations, we can conjugate R_d by $(I \otimes F_d)$ as follows:

$$C^2_{X,d} = (I \otimes F_d^\dagger) R_d (I \otimes F_d)$$

To see that R_d is a solution to the d-dimensional algebraic Yang-Baxter equation, we can simplify the form of R_d using block matrix multiplication:

$$\begin{aligned}
R_d &= (I \otimes F_d) C^2_{X,d} (I \otimes F_d^\dagger) \\
&= \begin{pmatrix} F_d & 0 & 0 & \dots & 0 \\ 0 & F_d & 0 & \dots & 0 \\ 0 & 0 & F_d & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & F_d \end{pmatrix} \begin{pmatrix} I & 0 & 0 & \dots & 0 \\ 0 & X_d & 0 & \dots & 0 \\ 0 & 0 & I & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & I \end{pmatrix} \begin{pmatrix} F_d^\dagger & 0 & 0 & \dots & 0 \\ 0 & F_d^\dagger & 0 & \dots & 0 \\ 0 & 0 & F_d^\dagger & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & F_d^\dagger \end{pmatrix} \\
&= \begin{pmatrix} I & 0 & 0 & \dots & 0 \\ 0 & F_d X_d F_d^\dagger & 0 & \dots & 0 \\ 0 & 0 & I & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & I \end{pmatrix}
\end{aligned}$$

This matrix turns out to be diagonal since the Fourier transform diagonalizes X_d. The characteristic polynomial of X_d is $\lambda^d - 1$ and therefore the eigenvalues of X_d are the d roots of unity: $e^{\frac{k2\pi i}{d}}$ for $k = 1 \dots d$. It is then straightforward to check that the corresponding eigenvectors are given by the columns of F_d.

As a consequence of theorem 4.5.1 $R_d P$ is a unitary solution to the d-dimensional bYBE. and by the symmetries above $(Q \otimes Q) R_d (Q \otimes Q)^{-1}$ is also a universal unitary solution to the d-dimensional bYBE whenever Q is a complex $d \times d$ unitary matrix.

This provides a way to generate many non-trivial examples of unitary solutions to the d-dimensional bYBE, which are also universal as quantum gates, and provide an explicit decomposition of the CNOT gate. □

Example 5.1.2. When $d = 2$ we have $\omega = e^{i2\pi/2} = e^{\pi i} = -1$ and:

$$F_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{5.3}$$

$$I \otimes F_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} = I \otimes F_2^{\dagger} \tag{5.4}$$

$$C_{X,2}^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{5.5}$$

Therefore R_d is given by:

$$R_d = (I \otimes F_2) C_{X,2}^2 (I \otimes F_2^{\dagger}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{5.6}$$

This example is one of the universal gates found in dimension 2 in [48].

Example 5.1.3. Another non-diagonal example is when $Q = F_3^3$, then we obtain the following unitary solution to the 3-dimensional bYBE:

$$\frac{1}{6}\begin{pmatrix} 4 & 1-i\sqrt{3} & 1+i\sqrt{3} & 0 & 0 & 0 & 2 & -1+i\sqrt{3} & -1-i\sqrt{3} \\ 2 & -1+i\sqrt{3} & -1-i\sqrt{3} & 4 & 1-i\sqrt{3} & 1+i\sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & -1+i\sqrt{3} & -1-i\sqrt{3} & 4 & 1-i\sqrt{3} & 1+i\sqrt{3} \\ 1+i\sqrt{3} & 4 & 1-i\sqrt{3} & 0 & 0 & 0 & -1-i\sqrt{3} & 2 & -1+i\sqrt{3} \\ -1-i\sqrt{3} & 2 & -1+i\sqrt{3} & 1+i\sqrt{3} & 4 & 1-i\sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & -1-i\sqrt{3} & 2 & -1+i\sqrt{3} & 1+i\sqrt{3} & 4 & 1-i\sqrt{3} \\ 1-i\sqrt{3} & 1+i\sqrt{3} & 4 & 0 & 0 & 0 & -1+i\sqrt{3} & -1-i\sqrt{3} & 2 \\ -1+i\sqrt{3} & -1-i\sqrt{3} & 2 & 1-i\sqrt{3} & 1+i\sqrt{3} & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1+i\sqrt{3} & -1-i\sqrt{3} & 2 & 1-i\sqrt{3} & 1+i\sqrt{3} & 4 \end{pmatrix}$$

As any diagonal matrix solves the d-dimensional aYBE, other universal unitary solutions to the d-dimensional bYBE can be generated in a similar way. The conditions under which an arbitrary diagonal matrix is universal is provided in [10]. If a topological quantum computer is built, this result provides a method for constructing a qudit gate that is guaranteed to be universal and solve the d-dimensional bYBE.

Chapter 6

X-shaped solutions

> "Let us wonder at how X was just a rare letter until algebra came along and made it something special that can be unravelled to reveal inner value."
>
> Bernardine Evaristo

6.1 X-shaped solutions to the gYBE

The permutation solutions to a gYBE can be found by brute force computation in low dimensions. From a permutation matrix solution, one can construct a *monomial* solution by replacing the 1's with variables and solving for the conditions under which the new matrix is a solution. The $(2,3,1)$ and $(2,3,2)$ monomial solutions have been classified fully in [62]. In contrast to the bYBEs, the $(2,3,1)$ and $(2,3,2)$ gYBEs don't have any monomial solutions with d free parameters. Other than the monomial solutions in [62], there are currently only a few known solutions to the non-bYBE (d,m,l)-gYBE up to the symmetries in proposition 4.7.1. One well known solution is the *X-shape* solution to the $(2,3,2)$-gYBE that appears in [70]:

$$R_X = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{6.1}$$

After that solution was found, a handful of other solutions were found in [15], [69], [49]. The question of finding all unitary $(2, 3, 2)$-gYBE solutions with nonzero entries in the same position as the nonzero entries of R_X is posed in [15]. This question demonstrates another more general method used to find solutions: pick an ansatz or initial guess about the form of the matrix, and in some cases, this will simplify the system of polynomial equations enough that they can be fully solved. We compute all X-shaped solutions for 6 different instances of the gYBE, and the 3-dimensional aYBE. We were unable to find any solutions when d is odd, leading to the following conjecture.

6.1.1 Odd dimensional solutions

Conjecture 6.1.1. The (d,m,l)-gYBE has no X-shaped solutions when d is odd.

Lemma 6.1.1 (The determinant of an X-shaped matrix)**.** Let $d > 1$ be an odd integer. Let $a = \frac{d+1}{2}$, and $a_i = 2a - i$. Let $\prec$ denote the lexicographic ordering. Define the set of paired indices A by:

$$A = \{ij | i + j = \text{even and } ij \prec aa\} \cup \{ij | i + j = \text{odd and } ij \succ aa\} \tag{6.2}$$

where i and j both range from 1 to d. The determinant of a $d^2 \times d^2$ X-shaped matrix is given by:

$$R^{aa}_{aa} \prod_{ij \in A} R^{ij}_{ij} R^{a_i a_j}_{a_i a_j} - R^{ij}_{a_i a_j} R^{a_i a_j}_{ij} \tag{6.3}$$

Example 6.1.2. When $d = 3$ define the following two sets:

$$A_1 = \{12, 21\} \tag{6.4}$$

$$A_2 = \{33, 31\} \tag{6.5}$$

Define the permutation S such that $S(e_1 \otimes e_2) = e_3 \otimes e_3$, $S(e_2 \otimes e_1) = e_3 \otimes e_1$. Then the X-shaped matrix:

$$\begin{pmatrix}
R^{11}_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & R^{11}_{33} \\
0 & R^{12}_{12} & 0 & 0 & 0 & 0 & 0 & R^{12}_{32} & 0 \\
0 & 0 & R^{13}_{13} & 0 & 0 & 0 & R^{13}_{31} & 0 & 0 \\
0 & 0 & 0 & R^{21}_{21} & 0 & R^{21}_{23} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & R^{22}_{22} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & R^{23}_{21} & 0 & R^{23}_{23} & 0 & 0 & 0 \\
0 & 0 & R^{31}_{13} & 0 & 0 & 0 & R^{31}_{31} & 0 & 0 \\
0 & R^{32}_{12} & 0 & 0 & 0 & 0 & 0 & R^{32}_{32} & 0 \\
R^{33}_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & R^{33}_{33}
\end{pmatrix} \tag{6.6}$$

is transformed into the block diagonal matrix:

$$SRS^{-1} = \begin{pmatrix} R^{11}_{11} & R^{11}_{33} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ R^{33}_{11} & R^{33}_{33} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & R^{13}_{13} & R^{13}_{31} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & R^{31}_{13} & R^{31}_{31} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & R^{22}_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & R^{23}_{23} & R^{23}_{21} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & R^{21}_{23} & R^{21}_{21} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & R^{32}_{32} & R^{32}_{12} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & R^{12}_{32} & R^{12}_{12} \end{pmatrix} \tag{6.7}$$

Proof of lemma 6.1.1. Let $d > 1$ be an odd integer. Let $a = \frac{d+1}{2}$, and $a_i = 2a - i$. Let $\prec$ denote the lexicographic ordering. Define the sets of paired indices A_1 and A_2 by:

$$A_1 = \{ij | i + j = \text{odd and } ij \prec aa\} \tag{6.8}$$

$$A_2 = \{ij | i + j = \text{even and } ij \succ aa\} \tag{6.9}$$

where i and j both range from 1 to d. Consider A_1 as sorted in lexicographic order and A_2 as sorted in reverse lexicographic order. Define the permutation $S : V \otimes V \to V \otimes V$ which sends the basis vectors $e_i \otimes e_j$ for $ij \in A_1$ to the basis vector $e_k \otimes e_l$ where $kl \in A_2$ and kl is at the same position in the set A_2 as ij is in A_1.

Then under conjugation by S any $d^2 \times d^2$ X-shaped matrix is similar to a block diagonal matrix consisting of one 1×1 block with entry R^{aa}_{aa}, and $\frac{d^2-1}{2}$ 2×2 blocks given by:

$$\begin{pmatrix} R_{ij}^{ij} & R_{a_ia_j}^{ij} \\ R_{ij}^{a_ia_j} & R_{a_ia_j}^{a_ia_j} \end{pmatrix} \tag{6.10}$$

where $ij \in A = \{ij|i+j = \text{even and } ij \prec aa\} \cup \{ij|i+j = \text{odd and } ij \succ aa\}$. The determinant of R is therefore the product of R_{aa}^{aa} and the determinants of each 2×2 block. □

Theorem 6.1.3. The d dimensional bYBE has no X-shaped solutions when d is odd.

Proof. Let d be an odd integer. The X-shaped ansatz requires that:

$$R_{ij}^{ab} = \begin{cases} \text{nonzero} & i = a \text{ and } j = b \text{ (diagonal entries)} \\ \text{nonzero} & i = 1+d-a \text{ and } j = 1+d-b \text{ (antidiagonal entries)} \\ 0 & \text{otherwise} \end{cases} \tag{6.11}$$

The d-dimensional bYBE can be written in the Einstein notation as follows:

$$R_{j_2j_3}^{k_1k_2} R_{j_1k_1}^{l_1k_3} R_{k_3k_2}^{l_2l_3} = R_{j_1j_2}^{k_1k_2} R_{k_2j_3}^{k_3l_3} R_{k_1k_3}^{l_1l_2} \tag{6.12}$$

where each variable can range from 1 to d, and sums are taken over repeated variables.

Consider equation 6.12 when $j_1 = j_2 = l_1 = l_2 = \frac{d+1}{2}$ and $j_3 = l_3 = \frac{d-1}{2}$. To simplify the notation let $a = \frac{d+1}{2}$, $b = a - 1$. We will use the convention that only repeated indices labeled with a k are summed over. Equation 6.12 can then be written as follows:

$$R_{ab}^{k_1k_2}R_{ak_1}^{ak_3}R_{k_3k_2}^{ab} = R_{aa}^{k_1k_2}R_{k_2b}^{k_3b}R_{k_1k_3}^{aa} \tag{6.13}$$

Terms on the left side will vanish unless $k_1 = a$ and $k_2 = a$ or $k_1 = 1 + d - a = a$ and $k_2 = 1 + d - b = \frac{d+3}{2} = a + 1$, which we will denote by $c = a + 1$. In the first case we get $R_{ab}^{aa}R_{aa}^{ak_3}R_{k_3a}^{ab}$, which is only nonzero when $k_3 = a$. In the second case we get $R_{ab}^{ac}R_{aa}^{ak_3}R_{k_3c}^{ab}$, which is nonzero when $k_3 = a$. So on the left side of 6.13 we get only two nonzero terms: $R_{ab}^{ab}R_{aa}^{aa}R_{ab}^{ab} + R_{ab}^{ac}R_{aa}^{aa}R_{ac}^{ab}$. The terms on the right side of equation 6.13 will vanish unless $k_1 = k_2 = a$, resulting in $R_{aa}^{aa}R_{ab}^{k_3b}R_{ak_3}^{aa}$ which is only nonzero when $k_3 = a$. Therefore we get the equation:

$$R_{ab}^{ab}R_{aa}^{aa}R_{ab}^{ab} + R_{ab}^{ac}R_{aa}^{aa}R_{ac}^{ab} = R_{aa}^{aa}R_{ab}^{ab}R_{aa}^{aa} \tag{6.14}$$

$$R_{aa}^{aa}(R_{ab}^{ab}R_{ab}^{ab} + R_{ab}^{ac}R_{ac}^{ab} - R_{ab}^{ab}R_{aa}^{aa}) = 0 \tag{6.15}$$

Now consider equation 6.12 with $j_1 = j_2 = l_1 = l_2 = a$, $j_3 = b$, and $l_3 = c$:

$$R_{ab}^{k_1k_2}R_{ak_1}^{ak_3}R_{k_3k_2}^{ac} = R_{aa}^{k_1k_2}R_{k_2b}^{k_3c}R_{k_1k_3}^{aa} \tag{6.16}$$

The terms on the left will vanish unless $k_1 = a$ and $k_2 = b$ or $k_1 = 1 + d - a = a$ and $k_2 = 1 + d - b = c$. In the first case $R_{ab}^{ab}R_{aa}^{ak_3}R_{k_3b}^{ac} \neq 0$ requires $k_3 = a$, leaving only the term: $R_{ab}^{ab}R_{aa}^{aa}R_{ab}^{ac}$. In the second case $R_{ab}^{ac}R_{aa}^{ak_3}R_{k_3c}^{ac} \neq 0$ requires that $k_3 = a$. So there are two nonzero terms on the left: $R_{ab}^{ab}R_{aa}^{aa}R_{ab}^{ac}$ and $R_{ab}^{ac}R_{aa}^{aa}R_{ac}^{ac}$

The terms on the right side of equation 6.16 will vanish unless $k_1 = k_2 = k_3 = a$ which leaves only one term: $R_{aa}^{aa}R_{ab}^{ac}R_{aa}^{aa}$. Therefore for R to satisfy the d-dimensional bYBE its entries must satisfy the equation:

$$R^{ab}_{ab}R^{aa}_{aa}R^{ac}_{ab} + R^{ac}_{ab}R^{aa}_{aa}R^{ac}_{ac} = R^{aa}_{aa}R^{ac}_{ab}R^{aa}_{aa} \tag{6.17}$$

$$R^{aa}_{aa}R^{ac}_{ab}(R^{ab}_{ab} + R^{ac}_{ac} - R^{aa}_{aa}) = 0 \tag{6.18}$$

All entries are nonzero to preserve the X-shape, so the only solution is when $R^{aa}_{aa} = R^{ab}_{ab} + R^{ac}_{ac}$ substituting this into equation 6.15 results in:

$$R^{ab}_{ab}R^{ab}_{ab} + R^{ac}_{ab}R^{ab}_{ac} - R^{ab}_{ab}(R^{ab}_{ab} + R^{ac}_{ac}) = 0 \tag{6.19}$$

$$R^{ac}_{ab}R^{ab}_{ac} - R^{ab}_{ab}R^{ac}_{ac} = 0 \tag{6.20}$$

Now consider that $a + c = 2a + 1$ is odd and $a(a+1) \succ aa$, therefore ac is an index in the product for the determinant of R in lemma 6.1.1. This means that the equation is a factor of the determinant of R:

$$R^{ac}_{ac}R^{(2a-a)(2a-c)}_{(2a-a)(2a-c)} - R^{ac}_{(2a-a)(2a-c)}R^{(2a-a)(2a-c)}_{ac} \tag{6.21}$$

Recall that $c = a + 1$, $b = a - 1$, so $2a - c = 2a - (a + 1) = a - 1 = b$ resulting in:

$$R^{ac}_{ac}R^{ab}_{ab} - R^{ac}_{ab}R^{ab}_{ac} \tag{6.22}$$

This must be nonzero for R to be invertible, meaning that R cannot solve equation 6.20, and therefore an X shaped matrix cannot solve the d dimensional bYBE when d is odd.

□

A similar proof might apply to the gYBE when d is odd. We now list X-shaped solutions found for certain instances of the gYBE. In the case of the $(2,3,2)$-gYBE we classify which ones can be made unitary. The solving process is fully described only in the $(2,3,2)$ case. The other cases were solved using a very similar process, although in many cases the initial factoring process takes hundreds of steps. Table 6.1.1 provides a summary of the X-shaped solutions found in this paper. Solutions are given up to the symmetries listed in proposition 4.7.1, except with Q restricted to only permutations and diagonal matrices, therefore the numbers in table 6.1.1 should be considered as upper bounds, except in the cases of $(2,2,1)$ and $(2,3,2)$.

(m,l)	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$
$(2,1)$	4	0	91	0	-	0
$(3,1)$	19	0	-	0	-	0?
$(3,2)$	4	0	-	0?	-	0?
$(4,1)$	-	0	-	0?	-	0?
$(4,2)$	91	0	-	0?	-	0?
$(4,3)$	12	0	-	0?	-	0?

Table 6.1: Number of X-shaped solutions for different values of (d,m,l). The (2,4,1) case could not be fully classified. When $d^m > 32$ and d is even the system is too large for the author's computational resources, but could be completed in the future with access to more computational power.

The following sections list all the X-shaped solutions found for different instances of the gYBE. All variables can take any complex value unless it breaks the X-shape, results in division by 0, or is otherwise specified. To list closely related solutions we let $\pm$ denote a choice of positive or negative, all $\pm$ are assumed to take the same sign, and $\mp = -\pm$.

6.1.2 (2,2,1)-gYBE X-shaped solutions

The (2,2,1) X-shaped ansatz results in a polynomial system consisting of 32 equations in 8 unknowns. These are also the 2-dimensional aYBE X-shaped solutions after composing with the swap matrix. Let $\gamma = \sqrt{\beta^2 - 2\beta + 2}$, the four solutions are:

$$\begin{pmatrix} 1 & 0 & 0 & \alpha \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ -\frac{1}{\alpha} & 0 & 0 & 1 \end{pmatrix} \tag{6.23}$$

$$\begin{pmatrix} \beta & 0 & 0 & \alpha \\ 0 & 1 & \beta & 0 \\ 0 & \beta & 1 & 0 \\ \frac{1}{\alpha} & 0 & 0 & \beta \end{pmatrix} \tag{6.24}$$

$$\begin{pmatrix} \beta & 0 & 0 & \alpha \\ 0 & 1 & -\beta & 0 \\ 0 & -\beta & 1 & 0 \\ \frac{1}{\alpha} & 0 & 0 & \beta \end{pmatrix} \tag{6.25}$$

$$\begin{pmatrix} 2-\beta & 0 & 0 & \alpha \\ 0 & 1 & \gamma & 0 \\ 0 & \gamma & 1 & 0 \\ \frac{1}{\alpha} & 0 & 0 & \beta \end{pmatrix} \tag{6.26}$$

6.1.3 (2,3,1)-gYBE X-shaped solutions

The (2,3,1) X-shaped ansatz results in a polynomial system consisting of 64 equations in 16 unknowns.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha \\ 0 & 1 & 0 & 0 & 0 & 0 & \pm\beta & 0 \\ 0 & 0 & 1 & 0 & 0 & \beta & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{\pm 1}{\beta} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mp\beta & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{-1}{\beta} & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{\mp 1}{\beta} & 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{-1}{\alpha} & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{6.27}$$

$$\begin{pmatrix} \frac{1}{2}\left(1-\eta^2\right) & 0 & 0 & 0 & 0 & 0 & 0 & \alpha \\ 0 & 1 & 0 & 0 & 0 & 0 & -i\beta & 0 \\ 0 & 0 & -\frac{1}{2}(\eta-1)^2 & 0 & 0 & \beta & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2}(\eta-1)^2 & \frac{i(\eta-1)^2}{2\beta} & 0 & 0 & 0 \\ 0 & 0 & 0 & -i\beta & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{(\eta-1)^2}{2\beta} & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{i(\eta-1)^2}{2\beta} & 0 & 0 & 0 & 0 & -\frac{1}{2}\left(\eta-1\right)^2 & 0 \\ -\frac{(\eta-1)((\eta-1)\eta+2)}{2\alpha} & 0 & 0 & 0 & 0 & 0 & 0 & \eta \end{pmatrix} \tag{6.28}$$

$$\begin{pmatrix} \pm i & 0 & 0 & 0 & 0 & 0 & 0 & \alpha \\ 0 & 1 & 0 & 0 & 0 & 0 & -i\beta & 0 \\ 0 & 0 & 1 & 0 & 0 & \beta & 0 & 0 \\ 0 & 0 & 0 & 1 & -\frac{i}{\beta} & 0 & 0 & 0 \\ 0 & 0 & 0 & -i\beta & 1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\beta} & 0 & 0 & 1 & 0 & 0 \\ 0 & -\frac{i}{\beta} & 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{1}{\alpha} & 0 & 0 & 0 & 0 & 0 & 0 & \pm i \end{pmatrix} \tag{6.29}$$

$$\begin{pmatrix} \eta & 0 & 0 & 0 & 0 & 0 & 0 & \alpha \\ 0 & 1 & 0 & 0 & 0 & 0 & \mp\beta\eta & 0 \\ 0 & 0 & \pm\eta & 0 & 0 & \beta & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{\mp\eta}{\beta} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mp\beta\eta & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\beta} & 0 & 0 & \pm\eta & 0 & 0 \\ 0 & \frac{\mp\eta}{\beta} & 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{1}{\alpha} & 0 & 0 & 0 & 0 & 0 & 0 & \eta \end{pmatrix} \tag{6.30}$$

Let $\phi = \sqrt{(\eta-2)\eta+2}$.

$$\begin{pmatrix} 2-\eta & 0 & 0 & 0 & 0 & 0 & 0 & \alpha \\ 0 & 1 & 0 & 0 & 0 & 0 & \mp\beta\phi & 0 \\ 0 & 0 & \pm\phi & 0 & 0 & \beta & 0 & 0 \\ 0 & 0 & 0 & 1 & \mp\frac{\phi}{\beta} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mp\beta\phi & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\beta} & 0 & 0 & \pm\phi & 0 & 0 \\ 0 & \mp\frac{\phi}{\beta} & 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{1}{\alpha} & 0 & 0 & 0 & 0 & 0 & 0 & \eta \end{pmatrix} \tag{6.31}$$

$$\begin{pmatrix} \eta & 0 & 0 & 0 & 0 & 0 & 0 & \alpha \\ 0 & 1 & 0 & 0 & 0 & 0 & \pm\beta\eta & 0 \\ 0 & 0 & \pm\eta & 0 & 0 & \beta & 0 & 0 \\ 0 & 0 & 0 & 1 & \pm\frac{\eta}{\beta} & 0 & 0 & 0 \\ 0 & 0 & 0 & \pm\beta\eta & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\beta} & 0 & 0 & \pm\eta & 0 & 0 \\ 0 & \pm\frac{\eta}{\beta} & 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{1}{\alpha} & 0 & 0 & 0 & 0 & 0 & 0 & \eta \end{pmatrix} \tag{6.32}$$

$$\begin{pmatrix} 2-\eta & 0 & 0 & 0 & 0 & 0 & 0 & \alpha \\ 0 & 1 & 0 & 0 & 0 & 0 & \pm\beta\phi & 0 \\ 0 & 0 & \pm\phi & 0 & 0 & \beta & 0 & 0 \\ 0 & 0 & 0 & 1 & \pm\frac{\phi}{\beta} & 0 & 0 & 0 \\ 0 & 0 & 0 & \pm\beta\phi & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\beta} & 0 & 0 & \pm\phi & 0 & 0 \\ 0 & \pm\frac{\phi}{\beta} & 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{1}{\alpha} & 0 & 0 & 0 & 0 & 0 & 0 & \eta \end{pmatrix} \tag{6.33}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha \\ 0 & 1 & 0 & 0 & 0 & 0 & \beta & 0 \\ 0 & 0 & \pm i & 0 & 0 & \beta & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{\beta} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\beta & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\beta} & 0 & 0 & \pm i & 0 & 0 \\ 0 & -\frac{1}{\beta} & 0 & 0 & 0 & 0 & 1 & 0 \\ -\frac{1}{\alpha} & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{6.34}$$

For the next solution let $\theta = \sqrt{1+\beta\gamma(\beta\gamma-14)}$, $\Phi = \sqrt{\beta\gamma\left(\beta\gamma+\theta-6\right)+\theta-1}$,

$\Psi = -\frac{\sqrt{\beta}\sqrt{\beta\gamma+1}\sqrt{\beta\gamma-\theta+1}}{2\sqrt{2}\sqrt{\gamma}}$, $\kappa = -\frac{\sqrt{\gamma}\sqrt{\beta\gamma+1}\sqrt{\beta\gamma-\theta+1}}{2\sqrt{2}\sqrt{\beta}}$, and $\rho = \frac{1}{4}(-\beta\gamma+\theta+3)$.

$$\begin{pmatrix} -\beta\gamma \pm \frac{\Phi}{2\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & \alpha \\ 0 & 1 & 0 & 0 & 0 & 0 & \Psi & 0 \\ 0 & 0 & \rho & 0 & 0 & \beta & 0 & 0 \\ 0 & 0 & 0 & -\beta\gamma & \kappa & 0 & 0 & 0 \\ 0 & 0 & 0 & \Psi & 1 & 0 & 0 & 0 \\ 0 & 0 & \gamma & 0 & 0 & \frac{1}{4}(-3\beta\gamma-\theta+1) & 0 & 0 \\ 0 & \kappa & 0 & 0 & 0 & 0 & -\beta\gamma & 0 \\ \frac{\pm\sqrt{2}\alpha\Phi(\beta\gamma+1)-4\alpha\beta\gamma}{4\alpha^2} & 0 & 0 & 0 & 0 & 0 & 0 & 1 \mp \frac{\Phi}{2\sqrt{2}} \end{pmatrix} \quad (6.35)$$

The next solutions have a similar form as the ones above except with $\Phi = i\sqrt{\beta\gamma(\beta\gamma+\theta-6)+\theta+1}$, $\Psi = -\frac{\sqrt{\beta}\sqrt{\beta\gamma+1}\sqrt{\beta\gamma+\theta+1}}{2\sqrt{2}\sqrt{\gamma}}$, $\kappa = \frac{\sqrt{\gamma}\sqrt{\beta\gamma+1}\sqrt{\beta\gamma+\theta+1}}{2\sqrt{2}\sqrt{\beta}}$, and $\rho = \frac{1}{4}(-\beta\gamma-\theta+3)$.

$$\begin{pmatrix} -\beta\gamma \pm \frac{\Phi}{2\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & \alpha \\ 0 & 1 & 0 & 0 & 0 & 0 & \Psi & 0 \\ 0 & 0 & \rho & 0 & 0 & \beta & 0 & 0 \\ 0 & 0 & 0 & -\beta\gamma & \kappa & 0 & 0 & 0 \\ 0 & 0 & 0 & \Psi & 1 & 0 & 0 & 0 \\ 0 & 0 & \gamma & 0 & 0 & \frac{1}{4}(-3\beta\gamma+\theta+1) & 0 & 0 \\ 0 & \kappa & 0 & 0 & 0 & 0 & -\beta\gamma & 0 \\ \frac{\pm\sqrt{2}\alpha\Phi(\beta\gamma+1)-4\alpha\beta\gamma}{4\alpha^2} & 0 & 0 & 0 & 0 & 0 & 0 & 1 \mp \frac{\Phi}{2\sqrt{2}} \end{pmatrix} \quad (6.36)$$

6.1.4 (2,3,2)-gYBE X-shaped solutions

The (2,3,2) X-shaped ansatz results in a polynomial system consisting of 128 equations in 16 unknowns. What follows is a description of the procedure used to obtain all 4 distinct families of X-shaped solutions to the $(2,3,2)$-gYBE.

$$X = \begin{pmatrix} r_{11} & 0 & 0 & 0 & 0 & 0 & 0 & r_{18} \\ 0 & r_{22} & 0 & 0 & 0 & 0 & r_{27} & 0 \\ 0 & 0 & r_{33} & 0 & 0 & r_{36} & 0 & 0 \\ 0 & 0 & 0 & r_{44} & r_{45} & 0 & 0 & 0 \\ 0 & 0 & 0 & r_{54} & r_{55} & 0 & 0 & 0 \\ 0 & 0 & r_{63} & 0 & 0 & r_{66} & 0 & 0 \\ 0 & r_{72} & 0 & 0 & 0 & 0 & r_{77} & 0 \\ r_{81} & 0 & 0 & 0 & 0 & 0 & 0 & r_{88} \end{pmatrix} \tag{6.37}$$

The variable r_{22} appears in the most equations and can be scaled to 1 using the overall scaling symmetry since all variables are assumed to be nonzero. After this scaling, the following equations are in the set:

$$-r_{36}r_{63}(r_{55} - 1) = 0 \tag{6.38}$$

$$r_{36}r_{63}(r_{44} - r_{77}) = 0 \tag{6.39}$$

and therefore $r_{55} = 1$ and $r_{77} = r_{44}$. After making these substitutions 108 equations remain. This system is then small enough that a Gröbner basis can be computed using a computer algebra system. We used Mathematica to compute a Gröbner basis with a lexicographic monomial ordering, and then used the reduce function to find all solutions to the system. Initially this procedure results in 7 solutions, which are listed below, with $\alpha, \beta, \gamma, \delta$ being complex free parameters:

$$X_1 = \begin{pmatrix} \delta & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\alpha\beta^2}{\delta^2} \\ 0 & 1 & 0 & 0 & 0 & 0 & \beta & 0 \\ 0 & 0 & \delta & 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{\delta^2}{\beta} & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\alpha} & 0 & 0 & \delta & 0 & 0 \\ 0 & \frac{\delta^2}{\beta} & 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{\delta^2}{\alpha\beta^2} & 0 & 0 & 0 & 0 & 0 & 0 & \delta \end{pmatrix} \tag{6.40}$$

$$X_2 = \begin{pmatrix} -i & 0 & 0 & 0 & 0 & 0 & 0 & i\alpha\beta^2 \\ 0 & 1 & 0 & 0 & 0 & 0 & \beta & 0 \\ 0 & 0 & 1 & 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & -i & \frac{i}{\beta} & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{i}{\alpha} & 0 & 0 & -i & 0 & 0 \\ 0 & \frac{i}{\beta} & 0 & 0 & 0 & 0 & -i & 0 \\ \frac{1}{\alpha\beta^2} & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{6.41}$$

$$X_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha\beta^2 \\ 0 & 1 & 0 & 0 & 0 & 0 & -\beta & 0 \\ 0 & 0 & 1 & 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{\beta} & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta & 1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\alpha} & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{\beta} & 0 & 0 & 0 & 0 & 1 & 0 \\ -\frac{1}{\alpha\beta^2} & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{6.42}$$

$$X_5 = \begin{pmatrix} i & 0 & 0 & 0 & 0 & 0 & 0 & -i\alpha\beta^2 \\ 0 & 1 & 0 & 0 & 0 & 0 & \beta & 0 \\ 0 & 0 & 1 & 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & i & -\frac{i}{\beta} & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta & 1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{i}{\alpha} & 0 & 0 & i & 0 & 0 \\ 0 & -\frac{i}{\beta} & 0 & 0 & 0 & 0 & i & 0 \\ \frac{1}{\alpha\beta^2} & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{6.43}$$

$$X_6 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha\beta^2 \\ 0 & 1 & 0 & 0 & 0 & 0 & \beta & 0 \\ 0 & 0 & 1 & 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{\beta} & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\alpha} & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{\beta} & 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{1}{\alpha\beta^2} & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{6.44}$$

$$X_7 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\alpha\beta^2}{\gamma} \\ 0 & 1 & 0 & 0 & 0 & 0 & \beta & 0 \\ 0 & 0 & \gamma & 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & \gamma & \frac{\gamma}{\beta} & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{\gamma}{\alpha} & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{\gamma}{\beta} & 0 & 0 & 0 & 0 & \gamma & 0 \\ \frac{\gamma^2}{\alpha\beta^2} & 0 & 0 & 0 & 0 & 0 & 0 & \gamma \end{pmatrix} \tag{6.45}$$

$$X_4 = \begin{pmatrix} 2-\delta & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\alpha\beta^2}{\delta^2-2\delta+2} \\ 0 & 1 & 0 & 0 & 0 & 0 & \beta & 0 \\ 0 & 0 & 2-\delta & 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{\delta^2-2\delta+2}{\beta} & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\alpha} & 0 & 0 & \delta & 0 & 0 \\ 0 & \frac{\delta^2-2\delta+2}{\beta} & 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{\delta^2-2\delta+2}{\alpha\beta^2} & 0 & 0 & 0 & 0 & 0 & 0 & \delta \end{pmatrix} \tag{6.46}$$

The matrices X_6 and X_7 are not invertible for any choice of the parameters by lemma 6.1.1. The matrices X_2 and X_5 are from the same family after utilizing the symmetries in proposition 4.7.1. In particular, $X_5 = 2 * X_2^{-1}$ with α replaced with $i\alpha$ and β replaced with $i\beta$. We now determine if there is a choice of the parameters and overall scale factor λ, which make X_1, X_2, X_3, X_4 unitary.

Proposition 6.1.4. λX_1 is unitary when $\mathrm{Re}(\delta) = 0$, $\delta^2 = -|\beta|^2$, $|\alpha| = 1$, $|\lambda|^2 = \frac{1}{1+|\delta|^2}$

Proof. We have that $\lambda X_1(\lambda X_1)^\dagger$ is equal to:

$$\lambda\bar{\lambda} \begin{pmatrix} \frac{a\beta^2\bar{\alpha}\bar{\beta}^2}{\delta^2\bar{\delta}^2} + \delta\bar{\delta} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\alpha\beta^2\bar{\delta}}{\delta^2} + \frac{\delta\bar{\delta}^2}{\bar{\alpha}\bar{\beta}^2} \\ 0 & \beta\bar{\beta}+1 & 0 & 0 & 0 & 0 & \frac{\bar{\delta}^2}{\bar{\beta}} + \beta & 0 \\ 0 & 0 & \alpha\bar{\alpha}+\delta\bar{\delta} & 0 & 0 & \frac{\delta}{\bar{\alpha}} + \alpha\bar{\delta} & 0 & 0 \\ 0 & 0 & 0 & \frac{\delta^2\bar{\delta}^2}{\beta\bar{\beta}} + 1 & \bar{\beta} + \frac{\delta^2}{\beta} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\bar{\delta}^2}{\bar{\beta}} + \beta & \beta\bar{\beta}+1 & 0 & 0 & 0 \\ 0 & 0 & \delta\bar{\alpha} + \frac{\bar{\delta}}{a} & 0 & 0 & \frac{1}{\alpha\bar{\alpha}} + \delta\bar{\delta} & 0 & 0 \\ 0 & \bar{\beta} + \frac{\delta^2}{\beta} & 0 & 0 & 0 & 0 & \frac{\delta^2\bar{\delta}^2}{\beta\bar{\beta}} + 1 & 0 \\ \frac{\delta^2\bar{\delta}}{\alpha\beta^2} + \frac{\delta\bar{\alpha}\bar{\beta}^2}{\bar{\delta}^2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\delta^2\bar{\delta}^2}{\alpha\beta^2\bar{\alpha}\bar{\beta}^2} + \delta\bar{\delta} \end{pmatrix} \tag{6.47}$$

Since it is required that $\bar{\beta}+\frac{\delta^2}{\beta}=0$ and $\frac{\delta}{\bar{\alpha}}+\alpha\bar{\delta}=0$, we must have $\delta^2=-\beta\bar{\beta}=-|\beta|^2$ and $\alpha\bar{\alpha}=\frac{-\delta}{\bar{\delta}}$. Therefore, $\mathrm{Re}(\delta)=0$ and $\alpha\bar{\alpha}=1$ and all off-diagonal elements will be zero when this is the case:

$$\frac{\delta^2\bar{\delta}}{\alpha\beta^2}+\frac{\delta\bar{\alpha}\bar{\beta}^2}{\bar{\delta}^2}=\delta^2\bar{\delta}+\frac{\delta\alpha\bar{\alpha}(\beta\bar{\beta})^2}{\bar{\delta}^2}=\delta^2\bar{\delta}+\frac{\delta\alpha\bar{\alpha}(-\delta^2)^2}{\bar{\delta}^2} \tag{6.48}$$

$$=\delta^2\bar{\delta}^3+\alpha\bar{\alpha}\delta^5=\bar{\delta}^3+\alpha\bar{\alpha}\delta^3=\bar{\delta}^3+\frac{-\delta}{\bar{\delta}}\delta^3=(-\delta)^4-\delta^4=0 \tag{6.49}$$

A similar computation shows that all diagonal elements are equal to $1+|\delta|^2$ and therefore λX_1 will be unitary whenever $|\lambda|^2=\frac{1}{1+|\delta|^2}$. □

Proposition 6.1.5. λX_2 is unitary when $|\alpha|=1$, $|\beta|=1$, and $|\lambda|^2=\frac{1}{2}$

Proof. We have that $\lambda X_2(\lambda X_2)^\dagger$ is equal to:

$$\lambda\bar{\lambda}\begin{pmatrix}
\alpha\beta^2\bar{\alpha}\bar{\beta}^2+1 & 0 & 0 & 0 & 0 & 0 & 0 & i\alpha\beta^2-\frac{i}{\bar{\alpha}\bar{\beta}^2} \\
0 & \beta\bar{\beta}+1 & 0 & 0 & 0 & 0 & i\beta-\frac{i}{\bar{\beta}} & 0 \\
0 & 0 & \alpha\bar{\alpha}+1 & 0 & 0 & i\alpha-\frac{i}{\bar{\alpha}} & 0 & 0 \\
0 & 0 & 0 & \frac{1}{\beta\bar{\beta}}+1 & \frac{i}{\beta}-i\bar{\beta} & 0 & 0 & 0 \\
0 & 0 & 0 & i\beta-\frac{i}{\bar{\beta}} & \beta\bar{\beta}+1 & 0 & 0 & 0 \\
0 & 0 & \frac{i}{\alpha}-i\bar{\alpha} & 0 & 0 & \frac{1}{\alpha\bar{\alpha}}+1 & 0 & 0 \\
0 & \frac{i}{\beta}-i\bar{\beta} & 0 & 0 & 0 & 0 & \frac{1}{\beta\bar{\beta}}+1 & 0 \\
\frac{i}{\alpha\beta^2}-i\bar{\alpha}\bar{\beta}^2 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\alpha\beta^2\bar{\alpha}\bar{\beta}^2}+1
\end{pmatrix} \tag{6.50}$$

To make the off-diagonal elements zero, we need $\frac{i}{\beta}-i\bar{\beta}=0$ and $\frac{i}{\alpha}-i\bar{\alpha}=0$, which only has the solutions $|\alpha|=|\beta|=1$. The diagonal elements are then all equal to 2 so λX_2 will be unitary as long as $|\lambda|^2=\frac{1}{2}$. □

Proposition 6.1.6. λX_3 is unitary when $|\alpha|=1$, $|\beta|=1$, and $|\lambda|^2=\frac{1}{2}$

Proof. We have that $\lambda X_3(\lambda X_3)^\dagger$ is equal to:

$$\lambda\bar{\lambda}\begin{pmatrix} \alpha\beta^2\bar{\alpha}\bar{\beta}^2+1 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha\beta^2-\frac{1}{\bar{\alpha}\bar{\beta}^2} \\ 0 & \beta\bar{\beta}+1 & 0 & 0 & 0 & 0 & \frac{1}{\bar{\beta}}-\beta & 0 \\ 0 & 0 & \alpha\bar{\alpha}+1 & 0 & 0 & a-\frac{1}{\bar{a}} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\beta\bar{\beta}}+1 & \bar{\beta}-\frac{1}{\beta} & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta-\frac{1}{\bar{\beta}} & \beta\bar{\beta}+1 & 0 & 0 & 0 \\ 0 & 0 & \bar{\alpha}-\frac{1}{\alpha} & 0 & 0 & \frac{1}{\alpha\bar{\alpha}}+1 & 0 & 0 \\ 0 & \frac{1}{\beta}-\bar{\beta} & 0 & 0 & 0 & 0 & \frac{1}{\beta\bar{\beta}}+1 & 0 \\ \bar{\alpha}\bar{\beta}^2-\frac{1}{\alpha\beta^2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{a\beta^2\bar{\alpha}\bar{\beta}^2}+1 \end{pmatrix} \tag{6.51}$$

To make the off-diagonal elements zero, we need $\frac{1}{\beta}-\bar{\beta}=0$ and $\frac{1}{\alpha}-\bar{\alpha}=0$ which only has the solutions $|\alpha|=|\beta|=1$. The diagonal elements are then all equal to 2 so λX_3 will be unitary as long as $|\lambda|^2=\frac{1}{2}$. □

Proposition 6.1.7. λX_4 is unitary when $|\alpha|^2=\frac{\delta-2}{\bar{\delta}}$, $|\beta|^2=\bar{\delta}^2-2\bar{\delta}+2$, $|\lambda|^2=\frac{1}{1+|\beta|^2}$, and δ is one of the following: $1+i$, $1-i$, 1, $\frac{5}{4}+\frac{\sqrt{7}}{4}$, $\frac{5}{4}-\frac{\sqrt{7}}{4}$.

Proof. Let $f = \delta^2 - 2\delta + 2$, then $\lambda X_4 (\lambda X_4)^\dagger$ is equal to:

$$\lambda\bar{\lambda}\begin{pmatrix} \frac{\alpha\beta^2\bar{\alpha}\bar{\beta}^2}{f\bar{f}} + (\delta-2)\left(\bar{\delta}-2\right) & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\alpha\beta^2\bar{\delta}}{f} - \frac{(\delta-2)\bar{f}}{\bar{\alpha}\bar{\beta}^2} \\ 0 & \beta\bar{\beta}+1 & 0 & 0 & 0 & 0 & \frac{\bar{f}}{\bar{\beta}} + \beta & 0 \\ 0 & 0 & \alpha\bar{\alpha} + (\delta-2)\left(\bar{\delta}-2\right) & 0 & 0 & \frac{2-\delta}{\bar{\alpha}} + \alpha\bar{\delta} & 0 & 0 \\ 0 & 0 & 0 & \frac{f\bar{f}}{\beta\bar{\beta}} + 1 & \bar{\beta} + \frac{f}{\beta} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\bar{f}}{\bar{\beta}} + \beta & \beta\bar{\beta} + 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{\alpha\delta\bar{a} - \bar{\delta} + 2}{a} & 0 & 0 & \frac{1}{\alpha\bar{\alpha}} + \delta\bar{\delta} & 0 & 0 \\ 0 & \bar{\beta} + \frac{f}{\beta} & 0 & 0 & 0 & 0 & \frac{f\bar{f}}{\beta\bar{\beta}} + 1 & 0 \\ \frac{\delta\bar{\alpha}\bar{\beta}^2}{\bar{f}} - \frac{f\bar{f}}{\alpha\beta^2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{f\bar{f}}{a\beta^2\bar{\alpha}\bar{\beta}^2} + \delta\bar{\delta} \end{pmatrix} \tag{6.52}$$

Because we need $\frac{2-\delta}{\bar{\alpha}} + \alpha\bar{\delta} = 0$ we can set $\alpha\bar{\alpha} = |\alpha|^2 = \frac{\delta-2}{\bar{\delta}}$. We also need $\frac{\bar{f}}{\bar{\beta}} + \beta = 0$ which will happen if $\beta\bar{\beta} = |\beta|^2 = \bar{f} = \bar{\delta}^2 - 2\bar{\delta} + 2$. Clearing the denominator and substituting into the upper-rightmost element gives us:

$$\alpha\bar{\alpha}(\beta\bar{\beta})^2\bar{\delta} - (\delta-2)f\bar{f} = \frac{(\delta-2)(\bar{\delta}^2 - 2\bar{\delta} + 2)}{\bar{\delta}} - (\delta-2)(\bar{\delta}^2 - 2\bar{\delta} + 2)(\delta^2 - 2\delta + 2) \tag{6.53}$$

Clearing the denominator again this simplifies to:

$$(\delta-2)(\bar{\delta}^2 - 2\bar{\delta} + 2) - \bar{\delta}(\delta-2)(\bar{\delta}^2 - 2\bar{\delta} + 2)(\delta^2 - 2\delta + 2) \tag{6.54}$$

$$= (\delta-2)(\bar{\delta}^2 - 2\bar{\delta} + 2)(1 - \bar{\delta}(\delta^2 - 2\delta + 2)) \tag{6.55}$$

We can eliminate the case $\delta = 2$ since that breaks the X-shape. There are then two possibilities, either $\bar{\delta}^2 - 2\bar{\delta} + 2 = 0$, or $\bar{\delta}(\delta^2 - 2\delta + 2) = 1$. In the first case, the only two solutions are $\delta = 1 + i$ or $\delta = 1 - i$. Solving the second case, we substitute $\delta = x + iy$

where x and y are real:

$$\bar{\delta}(\delta^2 - 2\delta + 2) = (x - iy)((x + iy)^2 - 2(x + iy) + 2) = 1 \tag{6.56}$$

This can then be split into real and imaginary parts:

$$x^3 - 2x^2 + xy^2 + 2x - 2y^2 - 1 = 0 \tag{6.57}$$

$$y(x^2 + y^2 - 2) = 0 \tag{6.58}$$

This has the following solutions: $y = 0$ and $x = 1$ or $x^2 + y^2 = 2$. In the case $x^2 + y^2 = 2$ we need to solve:

$$x^3 - 2(x^2 + y^2) + xy^2 + 2x - 1 = x^3 - 2(2) + xy^2 + 2x - 1 = 0 \tag{6.59}$$

$$x^2 + y^2 - 2 = 0 \tag{6.60}$$

We can set $y = \pm\sqrt{\frac{5-x^3-2x}{x}}$ as long as $x \neq 0$. In the case that $x = 0$ we must have $y = \pm\sqrt{2}$ however this does not solve the first equation. Substituting $y = \pm\sqrt{\frac{5-x^3-2x}{x}}$ into the second equation we get:

$$x^2 + \frac{5 - x^3 - 2x}{x} = 2 \tag{6.61}$$

Multiplying through by x and then subtracting $2x$ from both sides gives us:

$$x^3 + 5 - x^3 - 2x - 2x = 5 - 4x = 0 \tag{6.62}$$

Therefore, $x = \frac{5}{4}$ and $y = \pm\frac{\sqrt{7}}{4}$. □

6.1.5 (2,4,3)-gYBE X-shaped solutions

The (2,4,3) X-shaped ansatz results in a polynomial system consisting of 512 equations in 32 unknowns. To save space we omit the 0 entries from the 16×16 solutions. The first row is the diagonal, and the second row is the antidiagonal. Let $\Psi = 1+\sqrt{\beta\gamma - 1}$, and $\psi = 1 - \sqrt{\beta\gamma - 1}$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \alpha\gamma^2 & \gamma & \frac{\alpha\gamma}{\beta} & \frac{1}{\beta} & \alpha\beta\gamma & \beta & \alpha & \frac{1}{\gamma} & -\gamma & -\frac{1}{\alpha} & -\frac{1}{\beta} & -\frac{1}{\alpha\beta\gamma} & -\beta & -\frac{\beta}{\alpha\ \gamma} & -\frac{1}{\gamma} & -\frac{1}{\alpha\gamma^2} \end{pmatrix} \tag{6.63}$$

$$\begin{pmatrix} 1 & i & i & 1 & 1 & i & i & 1 & i & 1 & 1 & i & i & 1 & 1 & i \\ -i\alpha\beta\gamma & \gamma & \beta & -\frac{i}{\alpha} & -i\alpha^2\beta & \alpha & \frac{\alpha\beta}{\gamma} & -\frac{i}{\gamma} & \gamma & -\frac{i\gamma}{\alpha\beta} & -\frac{i}{\alpha} & \frac{1}{\alpha^2\beta} & \alpha & -\frac{i}{\beta} & -\frac{i}{\gamma} & \frac{1}{\alpha\beta\ \gamma} \end{pmatrix} \tag{6.64}$$

$$\begin{pmatrix} i & 1 & i & 1 & 1 & i & 1 & i & 1 & i & 1 & i & i & 1 & i & 1 \\ i\alpha\beta\gamma & \gamma & \beta & -\frac{i}{\alpha} & -i\alpha^2\beta & \alpha & -\frac{\alpha\beta}{\gamma} & -\frac{i}{\gamma} & \gamma & \frac{i\gamma}{\alpha\beta} & -\frac{i}{\alpha} & \frac{1}{\alpha^2\beta} & \alpha & -\frac{i}{\beta} & -\frac{i}{\gamma} & -\frac{1}{\alpha\beta\ \gamma} \end{pmatrix} \tag{6.65}$$

$$\begin{pmatrix} 1 & -i & -i & 1 & 1 & -i & -i & 1 & -i & 1 & 1 & -i & -i & 1 & 1 & -i \\ \gamma & -\frac{i\gamma}{\alpha\beta} & \beta & \frac{i}{\alpha} & i\alpha^2\beta & \alpha & \frac{i\alpha^2\ \beta^2}{\gamma} & -\frac{\alpha\beta}{\gamma} & -\frac{i\ \gamma}{\alpha\beta} & \frac{\gamma}{\alpha^2\beta^2} & \frac{i}{\alpha} & \frac{1}{\alpha^2\beta} & \alpha & \frac{i}{\beta} & -\frac{\alpha\beta}{\gamma} & \frac{i}{\gamma} \end{pmatrix} \tag{6.66}$$

$$\begin{pmatrix} -i & 1 & -i & 1 & 1 & -i & 1 & -i & 1 & -i & 1 & -i & -i & 1 & -i & 1 \\ \gamma & \frac{i\gamma}{\alpha\beta} & \beta & \frac{i}{\alpha} & i\alpha^2\beta & \alpha & \frac{i\alpha^2\ \beta^2}{\gamma} & \frac{\alpha\beta}{\gamma} & \frac{i\gamma}{\alpha\beta} & \frac{\gamma}{\alpha^2\beta^2} & \frac{i}{\alpha} & \frac{1}{\alpha^2\beta} & \alpha & \frac{i}{\beta} & \frac{\alpha\beta}{\gamma} & \frac{i}{\gamma} \end{pmatrix} \tag{6.67}$$

$$\begin{pmatrix} \psi & 1 & \psi & 1 & \psi & 1 & \psi & 1 & 1 & \Psi & 1 & \Psi & 1 & \Psi & 1 & \Psi \\ \frac{\beta\eta^2}{\alpha\gamma} & \frac{\beta\eta}{\alpha} & \eta & \gamma & \frac{\beta\eta}{\gamma} & \beta & \alpha & \frac{\alpha\gamma}{\eta} & \frac{\beta\eta}{\alpha} & \frac{1}{\alpha} & \gamma & \frac{\gamma}{\beta\eta} & \beta & \frac{1}{\eta} & \frac{\alpha\gamma}{\eta} & \frac{\alpha\ \gamma}{\beta\eta^2} \end{pmatrix} \tag{6.68}$$

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} & 1 & \frac{1}{2} & 1 & 1 & \frac{1}{2} & \frac{1}{2} & 1 & 1 & \frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{\beta^2\gamma^2}{\alpha} & \frac{\beta\gamma}{\alpha} & \gamma & \frac{1}{4\beta} & 4\beta^2\gamma & \beta & \alpha & \frac{\alpha}{\beta\gamma} & \frac{\beta\gamma}{\alpha} & \frac{1}{4\alpha} & \frac{1}{4\beta} & \frac{1}{4\beta^2\gamma} & \beta & \frac{1}{\gamma} & \frac{\alpha}{\beta\gamma} & \frac{\alpha}{4\beta^2\gamma^2} \end{pmatrix} \tag{6.69}$$

Let $\rho = \pm\sqrt{\beta}\sqrt{\gamma}$

$$\begin{pmatrix} \rho & 1 & \rho & 1 & \rho & 1 & -\sqrt{\beta}\sqrt{\gamma} & 1 & 1 & \rho & 1 & \rho & 1 & \rho & 1 & \rho \\ \frac{\beta\eta^2}{\alpha\gamma} & \frac{\beta\eta}{\alpha} & \eta & \gamma & \frac{\beta\eta}{\gamma} & \beta & \alpha & \frac{\alpha\gamma}{\eta} & \frac{\beta\eta}{\alpha} & \frac{1}{\alpha} & \gamma & \frac{\gamma}{\beta\eta} & \beta & \frac{1}{\eta} & \frac{\alpha\gamma}{\eta} & \frac{\alpha\ \gamma}{\beta\eta^2} \end{pmatrix} \tag{6.70}$$

$$\begin{pmatrix} 1 & \rho & \rho & 1 & \rho & 1 & 1 & \rho & \rho & 1 & 1 & \rho & 1 & \rho & \rho & 1 \\ \frac{\beta^2\eta^2}{\alpha} & \frac{\beta\eta}{\alpha} & \eta & \gamma & \frac{\beta\eta}{\gamma} & \beta & \alpha & \frac{\alpha}{\beta\eta} & \frac{\beta\eta}{\alpha} & \frac{\beta\gamma}{\alpha} & \gamma & \frac{\gamma}{\beta\eta} & \beta & \frac{1}{\eta} & \frac{\alpha}{\beta\eta} & \frac{\alpha\gamma}{\beta\eta^2} \end{pmatrix} \tag{6.71}$$

$$\begin{pmatrix} 1 & 2 & 2 & 1 & 2 & 1 & 1 & 2 & 2 & 1 & 1 & 2 & 1 & 2 & 2 & 1 \\ \frac{\beta^2\gamma^2}{\alpha} & \frac{\beta\gamma}{\alpha} & \gamma & \frac{4}{\beta} & \frac{\beta^2\gamma}{4} & \beta & \alpha & \frac{\alpha}{\beta\gamma} & \frac{\beta\gamma}{\alpha} & \frac{4}{\alpha} & \frac{4}{\beta} & \frac{4}{\beta^2\ \gamma} & \beta & \frac{1}{\gamma} & \frac{\alpha}{\beta\gamma} & \frac{4\alpha}{\beta^2\gamma^2} \end{pmatrix} \tag{6.72}$$

6.1.6 (4,2,1)-gYBE X-shaped solutions

The (4,2,1) X-shaped ansatz (equivalent to the (2,4,2) X-shaped ansatz) results in a polynomial system consisting of 256 equations in 32 unknowns. To speed up the solving process we use the local conjugation symmetry, listed in proposition 4.7.1, to scale the element in the first row and 16th column to 1 and the element in the 11th row and 6th column to 1. Let $\rho = \frac{1}{\sqrt{5}}$.

$$\begin{pmatrix} 1 & \pm\rho & \mp\rho & \pm\rho & \mp\rho & \pm\rho & \pm3\rho & \pm\rho & \pm\rho & \mp\rho & \pm\rho & \pm\frac{3}{\sqrt{5}} & \pm\rho & \pm3\rho & \pm\rho & 1 \\ \frac{\alpha}{5} & \gamma & \beta & \alpha & \alpha^2\beta & \frac{1}{\alpha} & \frac{\gamma}{5\alpha\beta} & \frac{1}{\gamma} & \gamma & \frac{\alpha\beta}{\gamma} & \alpha & \frac{1}{5\alpha^2\beta} & \frac{1}{\alpha} & \frac{1}{5\beta} & \frac{1}{\gamma} & \frac{1}{\alpha} \end{pmatrix} \tag{6.73}$$

$$\begin{pmatrix} 1 & \pm\rho & \mp\rho & \pm\rho & \mp\rho & \pm\rho & \pm3\rho & \pm\rho & \pm\rho & \mp\rho & \pm\rho & \pm3\rho & \pm\rho & \pm3\rho & \pm\rho & 1 \\ -\frac{\alpha}{5} & \gamma & \beta & \alpha & \alpha^2\beta & -\frac{1}{\alpha} & \frac{\gamma}{5\alpha\beta} & \frac{1}{\gamma} & \gamma & \frac{\alpha\beta}{\gamma} & -\alpha & \frac{1}{5\alpha^2\beta} & \frac{1}{\alpha} & \frac{1}{5\beta} & \frac{1}{\gamma} & -\frac{1}{\alpha} \end{pmatrix} \tag{6.74}$$

$$\begin{pmatrix} 1 & \beta & \pm1 & \beta & \pm1 & \beta & \pm1 & \beta & \beta & \pm1 & \beta & \pm1 & \beta & \pm1 & \beta & 1 \\ \pm\beta^2\eta & \frac{\alpha\gamma}{\beta^2\eta} & \frac{\gamma}{\eta^2} & \eta & \gamma & \pm\frac{1}{\eta} & \alpha & \frac{\beta^2\eta}{\alpha\gamma} & \frac{\alpha\gamma}{\beta^2\eta} & \frac{\beta^2}{\alpha} & \pm\eta & \frac{\beta^2}{\gamma} & \frac{1}{\eta} & \frac{\beta^2\eta^2}{\gamma} & \frac{\beta^2\eta}{\alpha\gamma} & \pm\frac{1}{\eta} \end{pmatrix} \tag{6.75}$$

$$\begin{pmatrix} 1 & -\beta & \pm1 & -\beta & \pm1 & \beta & \pm1 & -\beta & -\beta & \pm1 & \beta & \pm1 & -\beta & \pm1 & -\beta & 1 \\ \pm\beta^2\eta & \frac{\alpha\gamma}{\beta^2\eta} & \frac{\gamma}{\eta^2} & \eta & \gamma & \pm\frac{1}{\eta} & \alpha & \frac{\beta^2\eta}{\alpha\gamma} & \frac{\alpha\gamma}{\beta^2\eta} & \frac{\beta^2}{\alpha} & \pm\eta & \frac{\beta^2}{\gamma} & \frac{1}{\eta} & \frac{\beta^2\eta^2}{\gamma} & \frac{\beta^2\eta}{\alpha\gamma} & \pm\frac{1}{\eta} \end{pmatrix} \tag{6.76}$$

$$\begin{pmatrix} 1 & -i & -i & -i & -i & \pm 1 & -i & -i & -i & -i & \pm 1 & -i & -i & -i & -i & 1 \\ -i\gamma & -\frac{\alpha\beta}{\gamma} & \frac{\beta}{\gamma^2} & \gamma & \beta & -\frac{i}{\gamma} & \alpha & -\frac{\gamma}{\alpha\beta} & -\frac{\alpha\beta}{\gamma} & \frac{1}{\alpha} & -i\gamma & \frac{1}{\beta} & \frac{1}{\gamma} & \frac{\gamma^2}{\beta} & -\frac{\gamma}{\alpha\beta} & -\frac{i}{\gamma} \end{pmatrix} \tag{6.77}$$

$$\begin{pmatrix} 1 & i & i & i & i & \pm 1 & i & i & i & i & \pm 1 & i & i & i & i & 1 \\ i\gamma & -\frac{\alpha\beta}{\gamma} & \frac{\beta}{\gamma^2} & \gamma & \beta & \frac{i}{\gamma} & \alpha & -\frac{\gamma}{\alpha\beta} & -\frac{\alpha\beta}{\gamma} & \frac{1}{\alpha} & i\gamma & \frac{1}{\beta} & \frac{1}{\gamma} & \frac{\gamma^2}{\beta} & -\frac{\gamma}{\alpha\beta} & \frac{i}{\gamma} \end{pmatrix} \tag{6.78}$$

$$\begin{pmatrix} 1 & \eta & \eta & \pm 1 & \eta & 1 & \pm 1 & \eta & \eta & \pm 1 & 1 & \eta & \pm 1 & \eta & \eta & 1 \\ \alpha & \gamma & \beta & \alpha & \frac{\alpha^2\beta}{\eta^2} & \frac{\eta^2}{\alpha} & \frac{\gamma\eta^2}{\alpha\beta} & \frac{1}{\gamma} & \gamma & \frac{\alpha\beta}{\gamma} & \alpha & \frac{\eta^2}{\alpha^2\beta} & \frac{\eta^2}{\alpha} & \frac{1}{\beta} & \frac{1}{\gamma} & \frac{\eta^2}{\alpha} \end{pmatrix} \tag{6.79}$$

$$\begin{pmatrix} 1 & \eta & \eta & \pm 1 & \eta & -1 & \pm 1 & \eta & \eta & \pm 1 & -1 & \eta & \pm 1 & \eta & \eta & 1 \\ -\alpha & \gamma & \beta & \alpha & \frac{\alpha^2\beta}{\eta^2} & -\frac{\eta^2}{\alpha} & \frac{\gamma\eta^2}{\alpha\beta} & \frac{1}{\gamma} & \gamma & \frac{\alpha\beta}{\gamma} & -\alpha & \frac{\eta^2}{\alpha^2\beta} & \frac{\eta^2}{\alpha} & \frac{1}{\beta} & \frac{1}{\gamma} & -\frac{\eta^2}{\alpha} \end{pmatrix} \tag{6.80}$$

$$\begin{pmatrix} 1 & i & i & 1 & i & i & 1 & i & i & 1 & i & i & 1 & i & i & 1 \\ \beta & \gamma & -\frac{\gamma}{\alpha\beta} & \beta & \frac{\beta\gamma}{\alpha} & -\frac{1}{\beta} & \alpha & \frac{1}{\gamma} & \gamma & -\frac{1}{\alpha} & -\beta & \frac{\alpha}{\beta\gamma} & -\frac{1}{\beta} & -\frac{\alpha\beta}{\gamma} & \frac{1}{\gamma} & -\frac{1}{\beta} \end{pmatrix} \tag{6.81}$$

$$\begin{pmatrix} 1 & \pm i & \pm i & \mp 1 & \pm i & \pm i & \mp 1 & \pm i & \pm i & \mp 1 & \pm i & \pm i & \mp 1 & \pm i & \pm i & 1 \\ \pm\beta & \gamma & -\frac{\gamma}{\alpha\beta} & \beta & \frac{\beta\gamma}{\alpha} & \mp\frac{1}{\beta} & \alpha & \frac{1}{\gamma} & \gamma & -\frac{1}{\alpha} & \mp\beta & \frac{\alpha}{\beta\gamma} & -\frac{1}{\beta} & -\frac{\alpha\beta}{\gamma} & \frac{1}{\gamma} & \mp\frac{1}{\beta} \end{pmatrix} \tag{6.82}$$

$$\begin{pmatrix} 1 & \pm i & \pm i & \pm i & \pm i & \pm i & \pm i & \pm i & \pm i & \pm i & \pm i & \pm i & \pm i & \pm i & \pm i & 1 \\ \pm i\beta & \gamma & -\frac{\gamma}{\alpha\beta} & \beta & -\frac{\beta\gamma}{\alpha} & \pm\frac{i}{\beta} & \alpha & \frac{1}{\gamma} & \gamma & \frac{1}{\alpha} & \mp i\beta & -\frac{\alpha}{\beta\gamma} & \frac{1}{\beta} & -\frac{\alpha\beta}{\gamma} & \frac{1}{\gamma} & \pm\frac{i}{\beta} \end{pmatrix} \tag{6.83}$$

$$\begin{pmatrix} 1 & -i & -i & -1 & -i & -i & -1 & -i & -i & -1 & -i & -i & -1 & -i & -i & 1 \\ -\beta & \gamma & -\frac{\gamma}{\alpha\beta} & \beta & \frac{\beta\gamma}{\alpha} & \frac{1}{\beta} & \alpha & \frac{1}{\gamma} & \gamma & -\frac{1}{\alpha} & \beta & \frac{\alpha}{\beta\gamma} & -\frac{1}{\beta} & -\frac{\alpha\beta}{\gamma} & \frac{1}{\gamma} & \frac{1}{\beta} \end{pmatrix} \tag{6.84}$$

$$\begin{pmatrix} 1 & \beta & \pm 1 & \beta & \pm 1 & \beta & \pm 1 & \beta & \beta & \pm 1 & \beta & \pm 1 & \beta & \pm 1 & \beta & 1 \\ \mp\beta^2\eta & \frac{\alpha\gamma}{\beta^2\eta} & \frac{\gamma}{\eta^2} & \eta & \gamma & \mp\frac{1}{\eta} & \alpha & \frac{\beta^2\eta}{\alpha\gamma} & \frac{\alpha\gamma}{\beta^2\eta} & \frac{\beta^2}{\alpha} & \mp\eta & \frac{\beta^2}{\gamma} & \frac{1}{\eta} & \frac{\beta^2\eta^2}{\gamma} & \frac{\beta^2\eta}{\alpha\gamma} & \mp\frac{1}{\eta} \end{pmatrix} \tag{6.85}$$

$$\begin{pmatrix} 1 & -\beta & \pm 1 & -\beta & \pm 1 & \beta & \pm 1 & -\beta & -\beta & \pm 1 & \beta & \pm 1 & -\beta & 1 & -\beta & 1 \\ \mp\beta^2\eta & \frac{\alpha\gamma}{\beta^2\eta} & \frac{\gamma}{\eta^2} & \eta & \gamma & \mp\frac{1}{\eta} & \alpha & \frac{\beta^2\eta}{\alpha\gamma} & \frac{\alpha\gamma}{\beta^2\eta} & \frac{\beta^2}{\alpha} & \mp\eta & \frac{\beta^2}{\gamma} & \frac{1}{\eta} & \frac{\beta^2\eta^2}{\gamma} & \frac{\beta^2\eta}{\alpha\gamma} & \mp\frac{1}{\eta} \end{pmatrix} \tag{6.86}$$

$$\begin{pmatrix} 1 & -i & -i & -i & -i & \pm 1 & -i & -i & -i & -i & \pm 1 & -i & -i & -i & -i & 1 \\ i\gamma & -\frac{\alpha\beta}{\gamma} & \frac{\beta}{\gamma^2} & \gamma & \beta & \frac{i}{\gamma} & \alpha & -\frac{\gamma}{\alpha\beta} & -\frac{\alpha\beta}{\gamma} & \frac{1}{\alpha} & i\gamma & \frac{1}{\beta} & \frac{1}{\gamma} & \frac{\gamma^2}{\beta} & -\frac{\gamma}{\alpha\beta} & \frac{i}{\gamma} \end{pmatrix} \tag{6.87}$$

$$\begin{pmatrix} 1 & i & i & i & i & \pm 1 & i & i & i & i & \pm 1 & i & i & i & i & 1 \\ -i\gamma & -\frac{\alpha\beta}{\gamma} & \frac{\beta}{\gamma^2} & \gamma & \beta & -\frac{i}{\gamma} & \alpha & -\frac{\gamma}{\alpha\beta} & -\frac{\alpha\beta}{\gamma} & \frac{1}{\alpha} & -i\gamma & \frac{1}{\beta} & \frac{1}{\gamma} & \frac{\gamma^2}{\beta} & -\frac{\gamma}{\alpha\beta} & -\frac{i}{\gamma} \end{pmatrix} \tag{6.88}$$

$$\begin{pmatrix} 1 & \eta & \eta & \pm 1 & \eta & 1 & \pm 1 & \eta & \eta & \pm 1 & 1 & \eta & \pm 1 & \eta & \eta & 1 \\ -\alpha & \gamma & \beta & \alpha & \frac{\alpha^2\beta}{\eta^2} & -\frac{\eta^2}{\alpha} & \frac{\gamma\eta^2}{\alpha\beta} & \frac{1}{\gamma} & \gamma & \frac{\alpha\beta}{\gamma} & -\alpha & \frac{\eta^2}{\alpha^2\beta} & \frac{\eta^2}{\alpha} & \frac{1}{\beta} & \frac{1}{\gamma} & -\frac{\eta^2}{\alpha} \end{pmatrix} \tag{6.89}$$

$$\begin{pmatrix} 1 & \eta & \eta & \pm 1 & \eta & -1 & \pm 1 & \eta & \eta & \pm 1 & -1 & \eta & \pm 1 & \eta & \eta & 1 \\ \alpha & \gamma & \beta & \alpha & \frac{\alpha^2\beta}{\eta^2} & \frac{\eta^2}{\alpha} & \frac{\gamma\eta^2}{\alpha\beta} & \frac{1}{\gamma} & \gamma & \frac{\alpha\beta}{\gamma} & \alpha & \frac{\eta^2}{\alpha^2\beta} & \frac{\eta^2}{\alpha} & \frac{1}{\beta} & \frac{1}{\gamma} & \frac{\eta^2}{\alpha} \end{pmatrix} \tag{6.90}$$

$$\begin{pmatrix} 1 & \pm i & 1 & \pm i & 1 & 1 & 1 & \pm i & \pm i & 1 & 1 & 1 & \pm i & 1 & \pm i & 1 \\ \beta & \gamma & -\frac{\gamma}{\alpha\beta} & \beta & -\frac{\beta\gamma}{\alpha} & -\frac{1}{\beta} & \alpha & \frac{1}{\gamma} & \gamma & -\frac{1}{\alpha} & \beta & \frac{\alpha}{\beta\gamma} & \frac{1}{\beta} & \frac{\alpha\beta}{\gamma} & \frac{1}{\gamma} & -\frac{1}{\beta} \end{pmatrix} \tag{6.91}$$

$$\begin{pmatrix} 1 & \pm i & -1 & \pm i & -1 & -1 & -1 & \pm i & \pm i & -1 & -1 & -1 & \pm i & -1 & \pm i & 1 \\ -\beta & \gamma & -\frac{\gamma}{\alpha\beta} & \beta & -\frac{\beta\gamma}{\alpha} & \frac{1}{\beta} & \alpha & \frac{1}{\gamma} & \gamma & -\frac{1}{\alpha} & -\beta & \frac{\alpha}{\beta\gamma} & \frac{1}{\beta} & \frac{\alpha\beta}{\gamma} & \frac{1}{\gamma} & \frac{1}{\beta} \end{pmatrix} \tag{6.92}$$

Let $\rho_1 = 1 - i\sqrt{2}$ and $\rho_2 = 1 + i\sqrt{2}$

$$\begin{pmatrix} 1 & \rho_1 & 1 & 1 & 1 & 1 & \rho_2 & \rho_2 & \rho_1 & \rho_1 & 1 & 1 & 1 & 1 & \rho_2 & 1 \\ -i\beta & \gamma & -\frac{\alpha}{\beta^2} & \beta & \alpha & -\frac{i}{\beta} & -\frac{\beta\gamma}{\alpha} & \frac{1}{\gamma} & \gamma & -\frac{\alpha}{\beta\gamma} & -i\beta & -\frac{1}{\alpha} & -\frac{1}{\beta} & \frac{\beta^2}{\alpha} & \frac{1}{\gamma} & -\frac{i}{\beta} \end{pmatrix} \tag{6.93}$$

$$\begin{pmatrix} 1 & \rho_2 & 1 & 1 & 1 & \pm i & \rho_1 & \rho_1 & \rho_2 & \rho_2 & \pm i & 1 & 1 & 1 & \rho_1 & 1 \\ i\beta & -\frac{\alpha}{\beta\gamma} & -\frac{\alpha}{\beta^2} & \beta & \alpha & \frac{i}{\beta} & \frac{1}{\gamma} & -\frac{\beta\gamma}{\alpha} & -\frac{\alpha}{\beta\gamma} & \gamma & -i\beta & -\frac{1}{\alpha} & -\frac{1}{\beta} & \frac{\beta^2}{\alpha} & -\frac{\beta\gamma}{\alpha} & \frac{i}{\beta} \end{pmatrix} \tag{6.94}$$

$$\begin{pmatrix} 1 & \rho_2 & 1 & 1 & 1 & \pm i & \rho_1 & \rho_1 & \rho_2 & \rho_2 & \pm i & 1 & 1 & 1 & \rho_1 & 1 \\ -i\beta & -\frac{\alpha}{\beta\gamma} & -\frac{\alpha}{\beta^2} & \beta & \alpha & -\frac{i}{\beta} & \frac{1}{\gamma} & -\frac{\beta\gamma}{\alpha} & -\frac{\alpha}{\beta\gamma} & \gamma & i\beta & -\frac{1}{\alpha} & -\frac{1}{\beta} & \frac{\beta^2}{\alpha} & -\frac{\beta\gamma}{\alpha} & -\frac{i}{\beta} \end{pmatrix} \tag{6.95}$$

Let $\rho = \eta^2 - 2\eta + 2$.

$$\begin{pmatrix} 1 & \eta & \eta & 1 & \eta & 1 & 1 & 2-\eta & \eta & 1 & 1 & 2-\eta & 1 & 2-\eta & 2-\eta & 1 \\ \pm\gamma & \frac{\alpha\beta}{\gamma} & \frac{\beta\rho}{\gamma^2} & \gamma & \beta & \pm\frac{\rho}{\gamma} & \alpha & \frac{\gamma}{\alpha\beta} & \frac{\alpha\beta}{\gamma} & \frac{\rho}{\alpha} & \pm\gamma & \frac{1}{\beta} & \frac{\rho}{\gamma} & \frac{\gamma^2}{\beta\rho} & \frac{\gamma}{\alpha\beta} & \pm\frac{\rho}{\gamma} \end{pmatrix} \tag{6.96}$$

Let $\Phi = \frac{1}{3}\left(1+2i\sqrt{2}\right)$, $\phi = \frac{1}{3}\left(2+i\sqrt{2}\right)$, $\kappa = \frac{2\sqrt{7-4i\sqrt{2}}\gamma}{9\alpha\beta}$, $\rho = \frac{\left(-2-4i\sqrt{2}\right)\gamma^2}{9\alpha^2\beta}$, $\eta = -\frac{1+2i\sqrt{2}}{9\alpha}$, and $\theta = \frac{1}{3}\sqrt{-2-4i\sqrt{2}}$

$$\begin{pmatrix} \Phi & \Phi & \phi & \phi & \phi & \phi & 1 & 1 & \Phi & \Phi & \phi & \phi & \phi & \phi & 1 & 1 \\ -\frac{i\alpha\beta}{2\gamma} & \gamma & \rho & -\frac{\alpha\beta}{\gamma} & \beta & \kappa & \alpha & \frac{\alpha\eta}{\gamma} & \gamma & \eta & -\frac{i\alpha\beta}{\gamma} & -\frac{2+4i\sqrt{2}}{9\beta} & \frac{2\left(\gamma+2i\sqrt{2}\gamma\right)}{9\alpha\beta} & \frac{\alpha^2\beta}{\gamma^2} & \eta & \kappa \end{pmatrix} \tag{6.97}$$

$$\begin{pmatrix} \Phi & \Phi & \phi & \phi & \phi & \pm\theta & 1 & 1 & \Phi & \Phi & \pm\theta & \phi & \phi & \phi & 1 & 1 \\ -\frac{i\beta}{2} & \gamma & -\frac{2\left(1+2i\sqrt{2}\right)\alpha}{9\beta^2} & \beta & \alpha & \alpha\kappa & -\frac{\beta\gamma}{\alpha} & \frac{\alpha\eta}{\gamma} & \gamma & \frac{\alpha+2i\sqrt{2}\alpha}{9\beta\gamma} & i\beta & \frac{2\alpha\eta}{\gamma} & -\frac{2+4i\sqrt{2}}{9\beta} & \frac{\beta^2}{\alpha} & \frac{\alpha\eta}{\gamma} & \alpha\kappa \end{pmatrix} \tag{6.98}$$

$$\begin{pmatrix} \Phi & \Phi & \phi & \phi & \phi & \pm\theta & 1 & 1 & \Phi & \Phi & \pm\theta & \phi & \phi & \phi & 1 & 1 \\ \frac{i\beta}{2} & \gamma & -\frac{2\left(1+2i\sqrt{2}\right)\alpha}{9\beta^2} & \beta & \alpha & -\alpha\kappa & -\frac{\beta\gamma}{\alpha} & \frac{\alpha\eta}{\gamma} & \gamma & \frac{\alpha+2i\sqrt{2}\alpha}{9\beta\gamma} & -i\beta & \frac{2\alpha\eta}{\gamma} & \frac{2\alpha\eta}{\beta} & \frac{\beta^2}{\alpha} & \frac{\alpha\eta}{\gamma} & -\alpha\kappa \end{pmatrix} \tag{6.99}$$

Let $\Phi = \frac{1}{3}\left(1-2i\sqrt{2}\right)$, $\phi = \frac{1}{3}\left(2-i\sqrt{2}\right)$, $\kappa = \left(\frac{1}{3}+\frac{i}{3}\right)\sqrt{2\sqrt{2}+i}$, $\kappa = \frac{2\sqrt{7+4i\sqrt{2}}\gamma}{9\alpha\beta}$, $\rho = \frac{-1+2i\sqrt{2}}{9\gamma}$, $\eta_1 = \frac{\left(-2+4i\sqrt{2}\right)\gamma^2}{9\alpha^2\beta}$, $\eta_2 = \frac{\left(-2+4i\sqrt{2}\right)\alpha}{9\beta^2}$.

$$\begin{pmatrix} \Phi & \Phi & \phi & \phi & \phi & \phi & 1 & 1 & \Phi & \Phi & \phi & \phi & \phi & \phi & 1 & 1 \\ \pm\frac{i\alpha\beta}{2\gamma} & \gamma & \eta_1 & -\frac{\alpha\beta}{\gamma} & \beta & \pm\kappa & \alpha & \rho & \gamma & \frac{-1+2i\sqrt{2}}{9\alpha} & \pm\frac{i\alpha\beta}{\gamma} & \frac{\left(\sqrt{2}+2i\right)^2}{9\beta} & \frac{2\left(\gamma-2i\sqrt{2}\gamma\right)}{9\alpha\beta} & \frac{\alpha^2\beta}{\gamma^2} & \rho & \pm\kappa \end{pmatrix} \tag{6.100}$$

$$\begin{pmatrix} \Phi & \Phi & \phi & \phi & \phi & \pm\kappa & 1 & 1 & \Phi & \Phi & \pm\kappa & \phi & \phi & \phi & 1 & 1 \\ \frac{i\beta}{2} & \gamma & \eta_2 & \beta & \alpha & \frac{\alpha\kappa}{\gamma} & -\frac{\beta\gamma}{\alpha} & \rho & \gamma & \frac{\alpha-2i\sqrt{2}\alpha}{9\beta\gamma} & -i\beta & \frac{\left(\sqrt{2}+2i\right)^2}{9\alpha} & \frac{\left(\sqrt{2}+2i\right)^2}{9\beta} & \frac{\beta^2}{\alpha} & \frac{-1+2i\sqrt{2}}{9\gamma} & \alpha\kappa \end{pmatrix} \tag{6.101}$$

$$\begin{pmatrix} \Phi & \Phi & \phi & \phi & \phi & \phi & 1 & 1 & \Phi & \Phi & \phi & \phi & \phi & \phi & 1 & 1 \\ \pm\frac{i\alpha\beta}{2\gamma} & \gamma & \eta_1 & -\frac{\alpha\beta}{\gamma} & \beta & \pm\kappa & \alpha & \rho & \gamma & \frac{-1+2i\sqrt{2}}{9\alpha} & \pm\frac{i\alpha\beta}{\gamma} & \frac{\left(\sqrt{2}+2i\right)^2}{9\beta} & \frac{2\left(\gamma-2i\sqrt{2}\gamma\right)}{9\alpha\beta} & \frac{\alpha^2\beta}{\gamma^2} & \rho & \pm\kappa \end{pmatrix} \tag{6.102}$$

$$\begin{pmatrix} \Phi & \Phi & \phi & \phi & \phi & \pm\kappa & 1 & 1 & \Phi & \Phi & \pm\kappa & \phi & \phi & \phi & 1 & 1 \\ -\frac{i\beta}{2} & \gamma & \eta_2 & \beta & \alpha & -\frac{\alpha\kappa}{\gamma} & -\frac{\beta\gamma}{\alpha} & \rho & \gamma & \frac{\alpha-2i\sqrt{2}\alpha}{9\beta\gamma} & i\beta & \frac{\left(\sqrt{2}+2i\right)^2}{9\alpha} & \frac{\left(\sqrt{2}+2i\right)^2}{9\beta} & \frac{\beta^2}{\alpha} & \frac{-1+2i\sqrt{2}}{9\gamma} & -\alpha\kappa \end{pmatrix} \tag{6.103}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \mp\frac{\alpha\beta}{\gamma} & \gamma & \frac{\gamma^2}{\alpha^2\beta} & \frac{\alpha\beta}{\gamma} & \beta & \pm\frac{\gamma}{\alpha\beta} & \alpha & \frac{1}{\gamma} & -\gamma & -\frac{1}{\alpha} & \mp\frac{\alpha\beta}{\gamma} & -\frac{1}{\beta} & -\frac{\gamma}{\alpha\beta} & -\frac{\alpha^2\beta}{\gamma^2} & -\frac{1}{\gamma} & \pm\frac{\gamma}{\alpha\beta} \end{pmatrix} \tag{6.104}$$

$$\begin{pmatrix} 1 & 1 & 1 & \pm i & 1 & 1 & \pm i & 1 & 1 & \pm i & 1 & 1 & \pm i & 1 & 1 & 1 \\ \frac{i\alpha\beta}{\gamma} & \gamma & -\frac{\gamma^{\ 2}}{\alpha^2\beta} & -\frac{\alpha\beta}{\gamma} & \beta & \frac{i\gamma}{\alpha\beta} & \alpha & \frac{1}{\gamma} & -\gamma & \frac{1}{\alpha} & \frac{i\alpha\beta}{\gamma} & -\frac{1}{\beta} & -\frac{\gamma}{\alpha\beta} & \frac{\alpha^2\ \beta}{\gamma^2} & -\frac{1}{\gamma} & \frac{i\gamma}{\alpha\ \beta} \end{pmatrix} \tag{6.105}$$

$$\begin{pmatrix} 1 & 1 & \pm i & 1 & \pm i & \pm i & \pm i & 1 & 1 & \pm i & \pm i & \pm i & 1 & \pm i & 1 & 1 \\ \pm\frac{i\alpha\beta}{\gamma} & \gamma & \frac{\gamma^2}{\alpha^2\beta} & \frac{\alpha\beta}{\gamma} & \beta & \pm\frac{i\gamma}{\alpha\beta} & \alpha & \frac{1}{\gamma} & -\gamma & \frac{1}{\alpha} & \mp\frac{i\alpha\beta}{\gamma} & \frac{1}{\beta} & -\frac{\gamma}{\alpha\beta} & \frac{\alpha^2\beta}{\gamma^2} & -\frac{1}{\gamma} & \pm\frac{i\gamma}{\alpha\beta} \end{pmatrix} \tag{6.106}$$

$$\begin{pmatrix} 1 & 1 & 1 & \pm i & 1 & 1 & \pm i & 1 & 1 & \pm i & 1 & 1 & \pm i & 1 & 1 & 1 \\ -\frac{i\alpha\beta}{\gamma} & \gamma & -\frac{\gamma^2}{\alpha^2\beta} & -\frac{\alpha\beta}{\gamma} & \beta & -\frac{i\gamma}{\alpha\beta} & \alpha & \frac{1}{\gamma} & -\gamma & \frac{1}{\alpha} & -\frac{i\alpha\beta}{\gamma} & -\frac{1}{\beta} & -\frac{\gamma}{\alpha\beta} & \frac{\alpha^2\beta}{\gamma^2} & -\frac{1}{\gamma} & -\frac{i\gamma}{\alpha\beta} \end{pmatrix} \tag{6.107}$$

$$\begin{pmatrix} 1 & 1 & \pm i & 1 & \pm i & \pm i & \pm i & 1 & 1 & \pm i & \pm i & \pm i & 1 & \pm i & 1 & 1 \\ \mp\frac{i\alpha\beta}{\gamma} & \gamma & \frac{\gamma^2}{\alpha^2\beta} & \frac{\alpha\beta}{\gamma} & \beta & \mp\frac{i\gamma}{\alpha\beta} & \alpha & \frac{1}{\gamma} & -\gamma & \frac{1}{\alpha} & \pm\frac{i\alpha\beta}{\gamma} & \frac{1}{\beta} & -\frac{\gamma}{\alpha\beta} & \frac{\alpha^2\beta}{\gamma^2} & -\frac{1}{\gamma} & \mp\frac{i\gamma}{\alpha\beta} \end{pmatrix} \tag{6.108}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \pm i & 1 & 1 & 1 & 1 & \pm i & 1 & 1 & 1 & 1 & 1 \\ -\beta & \gamma & \frac{\alpha}{\beta^2} & \beta & \alpha & \frac{1}{\beta} & \frac{\beta\gamma}{\alpha} & \frac{1}{\gamma} & -\gamma & -\frac{\alpha}{\beta\gamma} & \beta & -\frac{1}{\alpha} & -\frac{1}{\beta} & -\frac{\beta^2}{\alpha} & -\frac{1}{\gamma} & \frac{1}{\beta} \end{pmatrix} \tag{6.109}$$

$$\begin{pmatrix} 1 & 1 & 1 & \pm i & 1 & -i & \pm i & 1 & 1 & \pm i & -i & 1 & \pm i & 1 & 1 & 1 \\ i\gamma & -\frac{\alpha\beta}{\gamma} & -\frac{\beta}{\gamma^2} & \gamma & \beta & \frac{i}{\gamma} & \alpha & -\frac{\gamma}{\alpha\beta} & \frac{\alpha\beta}{\gamma} & \frac{1}{\alpha} & -i\gamma & -\frac{1}{\beta} & \frac{1}{\gamma} & \frac{\gamma^2}{\beta} & \frac{\gamma}{\alpha\beta} & \frac{i}{\gamma} \end{pmatrix} \tag{6.110}$$

$$\begin{pmatrix} 1 & 1 & 1 & \pm i & 1 & i & \pm i & 1 & 1 & \pm i & i & 1 & \pm i & 1 & 1 & 1 \\ i\gamma & -\frac{\alpha\beta}{\gamma} & -\frac{\beta}{\gamma^2} & \gamma & \beta & \frac{i}{\gamma} & \alpha & -\frac{\gamma}{\alpha\beta} & \frac{\alpha\beta}{\gamma} & \frac{1}{\alpha} & -i\gamma & -\frac{1}{\beta} & \frac{1}{\gamma} & \frac{\gamma^2}{\beta} & \frac{\gamma}{\alpha\beta} & \frac{i}{\gamma} \end{pmatrix} \tag{6.111}$$

$$\begin{pmatrix} 1 & 1 & i & 1 & i & \pm 1 & i & 1 & 1 & i & \pm 1 & i & 1 & i & 1 & 1 \\ -i\gamma & \frac{\alpha\beta}{\gamma} & \frac{\beta}{\gamma^2} & \gamma & \beta & -\frac{i}{\gamma} & \alpha & \frac{\gamma}{\alpha\beta} & -\frac{\alpha\beta}{\gamma} & \frac{1}{\alpha} & -i\gamma & \frac{1}{\beta} & -\frac{1}{\gamma} & \frac{\gamma^2}{\beta} & -\frac{\gamma}{\alpha\beta} & -\frac{i}{\gamma} \end{pmatrix} \tag{6.112}$$

$$\begin{pmatrix} 1 & 1 & -i & 1 & -i & \pm 1 & -i & 1 & 1 & -i & \pm 1 & -i & 1 & -i & 1 & 1 \\ i\gamma & \frac{\alpha\beta}{\gamma} & \frac{\beta}{\gamma^2} & \gamma & \beta & \frac{i}{\gamma} & \alpha & \frac{\gamma}{\alpha\beta} & -\frac{\alpha\beta}{\gamma} & \frac{1}{\alpha} & i\gamma & \frac{1}{\beta} & -\frac{1}{\gamma} & \frac{\gamma^2}{\beta} & -\frac{\gamma}{\alpha\beta} & \frac{i}{\gamma} \end{pmatrix} \tag{6.113}$$

Let $\rho_1 = 1 + i\sqrt{2}$, and $\rho_2 = 1 - i\sqrt{2}$

$$\begin{pmatrix} 1 & 1 & \rho_1 & 1 & \rho_1 & 1 & \rho_2 & 1 & 1 & \rho_1 & 1 & \rho_2 & 1 & \rho_2 & 1 & 1 \\ \frac{i\alpha\beta}{\gamma} & \gamma & \frac{\gamma^2}{\alpha^2\beta} & \frac{\alpha\beta}{\gamma} & \beta & \frac{i\gamma}{\alpha\beta} & \alpha & \frac{1}{\gamma} & -\gamma & \frac{1}{\alpha} & \frac{i\alpha\beta}{\gamma} & \frac{1}{\beta} & -\frac{\gamma}{\alpha\beta} & \frac{\alpha^2\beta}{\gamma^2} & -\frac{1}{\gamma} & \frac{i\gamma}{\alpha\beta} \end{pmatrix} \tag{6.114}$$

The following have quite complicated parametrizations. Let $\Phi = \frac{1}{5}\left(1 + 2i\sqrt{6}\right)$, $\phi = \frac{1}{5}\left(3 + i\sqrt{6}\right)$, $\kappa = \frac{5\beta\sqrt{\gamma}}{\sqrt{-1+2i\sqrt{6}}\sqrt{\gamma+2i\sqrt{6}\gamma}}$, $\hat{\kappa} = \frac{5\beta\sqrt{\gamma}}{\sqrt{1+2i\sqrt{6}}\sqrt{\gamma+2i\sqrt{6}\gamma}}$, $\theta = -\frac{\beta\sqrt{\gamma}}{\sqrt{\frac{\sqrt{6}+2i}{3\sqrt{6}-14i}}\sqrt{\gamma+2i\sqrt{6}\gamma}}$, $\rho = \frac{3i\beta\sqrt{\gamma+2i\sqrt{6}\gamma}}{\sqrt{1+2i\sqrt{6}}\sqrt{\gamma}}$, and $\eta = \frac{1+2i\sqrt{6}}{25\beta}$.

$$\begin{pmatrix} \Phi & \Phi & \Phi & \phi & \Phi & \phi & \phi & 1 & \Phi & \phi & \phi & 1 & \phi & 1 & 1 & 1 \\ 3\kappa & \gamma & \frac{\gamma+2i\sqrt{6}\gamma}{25\alpha\beta} & \beta & -\frac{\beta\gamma}{\alpha} & \frac{i}{5\hat{\kappa}} & \alpha & -3\eta & -\gamma & \frac{1+2i\sqrt{6}}{25\alpha} & \kappa & \frac{\left(\sqrt{6}-3i\right)^2\alpha}{25\beta\gamma} & \eta & \frac{3\alpha\beta}{\gamma} & 3\eta & \frac{i}{5\hat{\kappa}} \end{pmatrix} \tag{6.115}$$

$$\begin{pmatrix} \Phi & \Phi & \Phi & \phi & \Phi & \phi & \phi & 1 & \Phi & \phi & \phi & 1 & \phi & 1 & 1 & 1 \\ \rho & \gamma & \frac{\gamma+2i\sqrt{6}\gamma}{25\alpha\beta} & \beta & -\frac{\beta\gamma}{\alpha} & -\frac{i}{5\hat{\kappa}} & \alpha & -3\eta & -\gamma & \frac{1+2i\sqrt{6}}{25\alpha} & \theta & \frac{\left(\sqrt{6}-3i\right)^2\alpha}{25\beta\gamma} & \eta & \frac{3\alpha\beta}{\gamma} & 3\eta & -\frac{1}{5\hat{\kappa}} \end{pmatrix} \tag{6.116}$$

Let $\Phi = \frac{1}{5}\left(1 - 2i\sqrt{6}\right)$, $\phi = \frac{1}{5}\left(3 - i\sqrt{6}\right)$, $\kappa = \frac{\sqrt{-1+2i\sqrt{6}}\sqrt{\gamma-2i\sqrt{6}\gamma}}{25\beta\sqrt{\gamma}}$, $\rho_1 = \frac{3\sqrt{-1+2i\sqrt{6}}\beta\sqrt{\gamma}}{\sqrt{\gamma-2i\sqrt{6}\gamma}}$, $\rho_2 = \frac{\gamma-2i\sqrt{6}\gamma}{25\alpha\beta}$, and $\eta = -\frac{\beta\sqrt{\gamma}}{\sqrt{\frac{\sqrt{6}-2i}{3\sqrt{6}+14i}}\sqrt{\gamma-2i\sqrt{6}\gamma}}$.

$$\begin{pmatrix} \Phi & \Phi & \Phi & \phi & \Phi & \phi & \phi & 1 & \Phi & \phi & \phi & 1 & \phi & 1 & 1 & 1 \\ \rho_1 & \gamma & \rho_2 & \beta & -\frac{\beta\gamma}{\alpha} & -\kappa & \alpha & \frac{-3\rho_2\alpha\beta}{\gamma^2} & -\gamma & \frac{1-2i\sqrt{6}}{25\alpha} & \frac{\rho_1}{3} & \frac{\left(\sqrt{6}+3i\right)^2\alpha}{25\beta\gamma} & \frac{1-2i\sqrt{6}}{25\beta} & \frac{3\alpha\beta}{\gamma} & \frac{3\rho_2\alpha\beta}{\gamma^2} & -\kappa \end{pmatrix} \tag{6.117}$$

$$\begin{pmatrix} \Phi & \Phi & \Phi & \phi & \Phi & \phi & \phi & 1 & \Phi & \phi & \phi & 1 & \phi & 1 & 1 & 1 \\ -\rho_1 & \gamma & \rho_2 & \beta & -\frac{\beta\gamma}{\alpha} & \kappa & \alpha & \frac{-3\rho_2\alpha\beta}{\gamma^2} & -\gamma & \frac{1-2i\sqrt{6}}{25\alpha} & \eta & \frac{\left(\sqrt{6}+3i\right)^2\alpha}{25\beta\gamma} & \frac{1-2i\sqrt{6}}{25\beta} & \frac{3\alpha\beta}{\gamma} & \frac{3\rho_2\alpha\beta}{\gamma^2} & \kappa \end{pmatrix} \tag{6.118}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 \end{pmatrix} \tag{6.119}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & i & -i & -1 & -i & 1 & -1 & -i & -i & 1 & -1 & -i & 1 & -i & i & 1 \end{pmatrix} \tag{6.120}$$

$$\begin{pmatrix} 1 & 1 & i & i & i & i & 1 & 1 & 1 & 1 & i & i & i & i & 1 & 1 \\ -1 & e^{\frac{\pm\pi i}{4}} & e^{\frac{\mp\pi i}{4}} & -1 & e^{\frac{\pm3\pi i}{4}} & 1 & \pm i & e^{\frac{\mp\pi i}{4}} & e^{\frac{\mp3\pi i}{4}} & \pm i & 1 & e^{\frac{\mp3\pi i}{4}} & -1 & e^{\frac{\pm\pi i}{4}} & e^{\frac{\pm3\pi i}{4}} & 1 \end{pmatrix} \tag{6.121}$$

$$\begin{pmatrix} 1 & 1 & i & i & i & i & 1 & 1 & 1 & 1 & i & i & i & i & 1 & 1 \\ 1 & e^{\frac{\pm\pi i}{4}} & e^{\frac{\mp\pi i}{4}} & 1 & e^{\frac{-3\pi i}{4}} & -1 & \mp i & e^{\frac{\mp\pi i}{4}} & e^{\frac{\mp3\pi i}{4}} & \mp i & -1 & e^{\frac{\mp3\pi i}{4}} & 1 & e^{\frac{\pm\pi i}{4}} & e^{\frac{\pm3\pi i}{4}} & -1 \end{pmatrix} \tag{6.122}$$

$$\begin{pmatrix} 1 & 1 & -i & -i & -i & -i & 1 & 1 & 1 & 1 & -i & -i & -i & -i & 1 & 1 \\ -1 & e^{\frac{\pm\pi i}{4}} & e^{\frac{\mp\pi i}{4}} & -1 & e^{\frac{\pm3\pi i}{4}} & 1 & \pm i & e^{\frac{\mp\pi i}{4}} & e^{\frac{\mp3\pi i}{4}} & \pm i & 1 & e^{\frac{\mp3\pi i}{4}} & -1 & e^{\frac{\pm\pi i}{4}} & e^{\frac{\pm3\pi i}{4}} & 1 \end{pmatrix} \tag{6.123}$$

$$\begin{pmatrix} 1 & 1 & -i & -i & -i & -i & 1 & 1 & 1 & 1 & -i & -i & -i & -i & 1 & 1 \\ 1 & e^{\frac{\pm\pi i}{4}} & e^{\frac{\mp\pi i}{4}} & 1 & e^{\frac{\pm 3\pi i}{4}} & -1 & \mp i & e^{\frac{\mp\pi i}{4}} & e^{\frac{\mp 3\pi i}{4}} & \mp i & -1 & e^{\frac{\mp 3\pi i}{4}} & 1 & e^{\frac{\pm\pi i}{4}} & e^{\frac{\pm 3\pi i}{4}} & -1 \end{pmatrix} \tag{6.124}$$

6.1.7 (3,2,1)-aYBE X-shaped / (3,2,1)-gYBE XP-shaped solutions

What follows are X-shaped solutions to the 3-dimensional aYBE which are shown below after composing with the swap matrix so that they are solutions to the (3,2,1)-gYBE. This ansatz results in 84 equations in 17 unknowns. These 2 matrices are similar, but not locally similar.

$$\begin{pmatrix} \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \beta \\ 0 & 0 & 0 & -\frac{\eta\theta}{\delta} & 0 & -\beta\eta & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \alpha & 0 & 0 \\ 0 & -\theta & 0 & 0 & 0 & 0 & 0 & -\beta\delta & 0 \\ 0 & 0 & 0 & 0 & \gamma & 0 & 0 & 0 & 0 \\ 0 & \delta & 0 & 0 & 0 & 0 & 0 & \theta & 0 \\ 0 & 0 & \alpha & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \eta & 0 & \frac{\eta\theta}{\delta} & 0 & 0 & 0 \\ \frac{1}{\beta} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha \end{pmatrix} \tag{6.125}$$

$$
\begin{pmatrix}
\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\beta \\
0 & 0 & 0 & \frac{\eta\theta}{\delta} & 0 & -\beta\eta & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & \alpha & 0 & 0 \\
0 & -\theta & 0 & 0 & 0 & 0 & 0 & -\beta\delta & 0 \\
0 & 0 & 0 & 0 & \gamma & 0 & 0 & 0 & 0 \\
0 & \delta & 0 & 0 & 0 & 0 & 0 & -\theta & 0 \\
0 & 0 & \alpha & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & \eta & 0 & \frac{\eta\theta}{\delta} & 0 & 0 & 0 \\
-\frac{1}{\beta} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha
\end{pmatrix}
\tag{6.126}
$$

Chapter 7

Permutation solutions

> "Nothing is built on stone; All is built on sand, but we must build as if the sand were stone."
>
> Jorge Luis Borges

Here we use a computational approach to find permutation solutions to the aYBE and bYBE. Stephen Jordan has computed all permutation matrix solutions to the bYBE (and aYBE) up to dimension 5 by a brute force search using Microsoft supercomputers [76]. We show how to determine all permutation solutions of fixed minimal cycle type in all dimensions.

Let d be the dimension of the vector space V in the definition of the bYBE/aYBE, and let $\{e_1, \ldots, e_d\}$ be a basis for V. Any permutation on $V \otimes V$ can be decomposed into d^2-1 transpositions. A single transposition T acting on $V \otimes V$ can be represented by a set of four integers $\{a_1, a_2, b_1, b_2\}$ that index the basis vectors being exchanged. Each index variable ranges from 1 to d. Using this representation the transposition T is determined by:

$$T(e_{a_1} \otimes e_{a_2}) = e_{b_1} \otimes e_{b_2} \tag{7.1}$$

$$T(e_{b_1} \otimes e_{b_2}) = e_{a_1} \otimes e_{a_2} \tag{7.2}$$

$$T(e_i \otimes e_j) = e_i \otimes e_j \qquad \text{otherwise} \tag{7.3}$$

An arbitrary permutation can $R : V \otimes V \to V \otimes V$ can be decomposed as $R = T_1 T_2 \dots T_s$, where $s \leq d^2 - 1$, and T_j is a transposition for each $j = 1, \dots, s$. We will assume that this decomposition is chosen such that s is as small as possible given the permutation R, we will refer to this as "the minimal cycle type" of the permutation. We can therefore represent the permutation R as an ordered set of $4s$ integers, each ranging from 1 to d:

$$R = \{a_{11}, a_{12}, b_{11}, b_{12}, a_{21}, a_{22}, b_{21}, b_{22}, \dots, a_{s1}, a_{s2}, b_{s1}, b_{s2}\} \tag{7.4}$$

where $T_j = \{a_{j1}, a_{j2}, b_{j1}, b_{j2}\}$. Under this representation, composition of two permutations corresponds to concatenating their representations. For example consider the d dimensional swap matrix P_d which is defined by $P_d(e_i \otimes e_j) = e_j \otimes e_i$ for all $i, j \in \{1, \dots, d\}$. The swap matrix can be represented by:

$$P_d = \{1, 2, 2, 1, 1, 3, 3, 1, \dots 1, d, d, 1, 2, 3, 3, 2, \dots, d-1, d, d, d-1\} \tag{7.5}$$

$$P_d = \bigcup_{i \neq j \in \{1, \dots, d\}} \{i, j, j, i\} \tag{7.6}$$

The the 4-tuples representing a single transposition must stay in order when taking the union above. Tor the swap matrix each basis vector only appears once so this order happens to not matter in this case. In general the order of the integers appearing in this kind of representation of a permutation will depend on the permutation. We can also write P_d as a product of individual transpositions:

$$P_d = \prod_{i \neq j \in \{1, \dots, d\}} \{i, j, j, i\} \tag{7.7}$$

Using this representation we can find all permutation solutions in every dimension that decomposes into a fixed number of transpositions. For each dimension d finding all permutation solutions which decomposes into a product of s transpositions amounts to checking d^{4s} possible cases. It turns out that, using the representation above, permutations that decompose into a fixed number of transpositions s can be checked in a finite number of cases depending only on s and completely independent of the dimension d. Consider the equivalence relation between two permutation representations from equation 7.4 given by:

$$\{a_1, a_2, a_3, \ldots, a_{4s}\} \sim \{b_1, b_2, b_3, \ldots, b_{4s}\} \text{ when } a_j = a_i \leftrightarrow b_j = b_i \tag{7.8}$$

which means that the location of matching indices is the same in the two representations. Some examples of permutations that are equivalent under $\sim$:

$$\{1, 2, 3, 3\} \sim \{5, 4, 1, 1\} \tag{7.9}$$

$$\{6, 2, 2, 3, 3, 6, 8, 7\} \sim \{2, 5, 5, 6, 6, 2, 8, 4\} \tag{7.10}$$

For a fixed value of s the number of possible equivalence classes corresponds to the number of set partitions of the $4s$ integers. The number of set partitions of $4s$ integers corresponds to the bell number Bell($4s$).

Example 7.0.1. Consider the two permutations below:

$$A = \{6, 2, 2, 3, 3, 6, 8, 7\} \sim B = \{2, 5, 5, 6, 6, 2, 8, 4\} \tag{7.11}$$

Let i be an integer in the representation of A and denote the set of matching variables in which i falls by $[i]_A$. Similarly for B denote the set of matching variables in which i falls by $[i]_B$. We then have $[6]_A = 1$, $[2]_A = 2$, $[3]_A = 3$, $[8]_A = 4$, and $[7]_A = 5$. And for the second permutation $[2]_B = 1$, $[5]_B = 2$, $[6]_B = 3$, $[8]_B = 4$, $[4]_B = 5$. Define the matrix Q by it's action on the basis vectors of V:

$$Qe_i = e_{[i]_A} \qquad \text{for } i \in A \tag{7.12}$$

Then conjugating A by $Q \otimes Q$ results in:

$$(Q \otimes Q)\{6, 2, 2, 3, 3, 6, 8, 7\}(Q^{-1} \otimes Q^{-1}) \tag{7.13}$$

$$= \{1, 2, 2, 3, 3, 1, 4, 5\} \tag{7.14}$$

This can be seen by considering the action on the relevant basis vectors:

$$(Q \otimes Q)\{6, 2, 2, 3, 3, 6, 8, 7\}(Q^{-1} \otimes Q^{-1})(e_4 \otimes e_5) \tag{7.15}$$

$$= (Q \otimes Q)\{6, 2, 2, 3, 3, 6, 8, 7\}(e_8 \otimes e_7) \tag{7.16}$$

$$= (Q \otimes Q)(e_3 \otimes e_6) \tag{7.17}$$

$$= (e_3 \otimes e_1) \tag{7.18}$$

$$(Q \otimes Q)\{6, 2, 2, 3, 3, 6, 8, 7\}(Q^{-1} \otimes Q^{-1})(e_3 \otimes e_1) \tag{7.19}$$

$$= (Q \otimes Q)\{6, 2, 2, 3, 3, 6, 8, 7\}(e_3 \otimes e_6) \tag{7.20}$$

$$= (Q \otimes Q)(e_8 \otimes e_7) \tag{7.21}$$

$$= (e_4 \otimes e_5) \tag{7.22}$$

$$(Q \otimes Q)\{6, 2, 2, 3, 3, 6, 8, 7\}(Q^{-1} \otimes Q^{-1})(e_2 \otimes e_3) \tag{7.23}$$

$$= (Q \otimes Q)\{6, 2, 2, 3, 3, 6, 8, 7\}(e_2 \otimes e_3) \tag{7.24}$$

$$= (Q \otimes Q)(e_6 \otimes e_2) \tag{7.25}$$

$$= (e_1 \otimes e_1) \tag{7.26}$$

$$(Q \otimes Q)\{6, 2, 2, 3, 3, 6, 8, 7\}(Q^{-1} \otimes Q^{-1})(e_1 \otimes e_2) \tag{7.27}$$

$$= (Q \otimes Q)\{6, 2, 2, 3, 3, 6, 8, 7\}(e_6 \otimes e_2) \tag{7.28}$$

$$= (Q \otimes Q)(e_2 \otimes e_3) \tag{7.29}$$

$$= (e_2 \otimes e_3) \tag{7.30}$$

We can also conjugate B to obtain $\{1, 2, 2, 3, 3, 1, 4, 5\}$, therefore A and B are locally conjugate to each other.

Theorem 7.0.2. Given two permutations A and B, such that $A \sim B$, if A is a solution to the d-dimensional bYBE or aYBE then B is a solution to the d-dimensional bYBE or aYBE, respectively. Therefore if one representative from an equivalence class is a solution, then all the permutations in that class are also solutions.

Proof. Let $A \sim B$ be two permutations. Let c be the number of distinct integers in the representation of each permutation. Then A and B must be solutions in dimension c, (and potentially in dimensions larger than c, as shown in theorem 7.0.6). There are then c sets of matching variables in the representations of both A and B. Enumerate the corresponding sets of matching variables by $1, \ldots c$. Let i be an integer in the representation of A and denote the set of matching variables in which i falls by $[i]_A$. Similarly for B denote the set of matching variables in which i falls by $[i]_B$. Define the matrix Q by it's action on the basis vectors of V:

$$Qe_{[i]_A} = e_i \qquad \text{for } i = 1, \ldots, c \tag{7.31}$$

This is an invertible permutation on V. Conjugating A by $Q \otimes Q$ results in:

$$\{[a_1]_A, [a_2]_A, \ldots, [a_{4s}]_A\}$$

Similarly for B we define the matrix S by:

$$Se_{[i]_B} = e_i \qquad \text{for } i = 1, \ldots, c \tag{7.32}$$

Conjugating B by $S \otimes S$ results in:

$$\{[b_1]_B, [b_2]_B, \ldots, [b_{4s}]_B\}$$

Since $A \sim B$ we have that $a_i = a_j$ if an only if $b_i = b_j$. Therefore $[a_i]_A = [b_i]_B$ and A and B are conjugate as follows:

$$(Q \otimes Q)A(Q^{-1} \otimes Q^{-1}) = (S \otimes S)B(S^{-1} \otimes S^{-1}) \tag{7.33}$$

$$A = (Q^{-1} \otimes Q^{-1})(S \otimes S)B(S^{-1} \otimes S^{-1})(Q \otimes Q) \tag{7.34}$$

$$A = (Q^{-1}S \otimes Q^{-1}S)B(S^{-1}Q \otimes S^{-1}Q) \tag{7.35}$$

Under the fifth symmetry in proposition 4.7.1 the local conjugation of a solution of the bYBE or aYBE is also a solution. □

Corollary 7.0.3. Every permutation that has a representation that is a product of s transpositions and contains c distinct integers in its representation can be conjugated to obtain a permutation whose representation only contains the integers $1, \ldots, c$.

Proof. Let c be the number of distinct variables in the representation of a permutation that is the product of s transpositions given by:

$$\{a_1, a_2, \ldots, a_{4s}\}$$

Once a dimension d is chosen, c can range from 1 to d. The product can be conjugated by $Q \otimes Q$ as in the proof of theorem 7.0.2 to obtain a representation where only the integers $1, \ldots, c$ remain. We can further assume that a_1 is equal to 1. □

Corollary 7.0.4. There are a finite number of permutation solutions to the bYBE and aYBE up to the symmetries in proposition 4.7.1, across all dimensions d, which can be decomposed into s transpositions.

Proof. This follows directly from corollary 7.0.3. There are a maximum of Bell($4s$) equivalence classes under the equivalence relation $\sim$. By corollary 7.0.3 every permutation within an equivalence class is considered equivalent under the symmetries in proposition 4.7.1. □

The symmetries from proposition 4.7.1 can be reinterpreted in terms of the representation of a permutation in equation 7.4 as follows.

Proposition 7.0.5. Recall from proposition 4.7.1 that if $R = T_1 ... T_s$ is a permutation matrix solution to the aYBE or bYBE in dimension d then so are:

1. $R^{-1} = R^T$

2. $Q \otimes QR(Q \otimes Q)^{-1}$ where Q is any $d \times d$ permutation matrix.

3. PRP where $P : |ij\rangle \to |ji\rangle$

These can be translated into symmetries of the representation of R in equation 7.4 as:

1. $T_s ... T_1 = \{t_{s1}, t_{s2}, m_{s1}, m_{s2}, ..., t_{11}, t_{12}, m_{11}, m_{12}\}$

2. Following corollary 7.0.3, let A be the set of distinct integers appearing in the representation of R, then any bijective mapping $\phi : A \to \{1, \dots, d\}$ defines another solution represented by:

$$\{\phi(t_{s1}), \phi(t_{s2}), \phi(m_{s1}), \phi(m_{s2}), ..., \phi(t_{11}), \phi(t_{12}), \phi(m_{11}), \phi(m_{12})\}$$

3. $PRP = PT_1 ... T_s P = \{t_{12}, t_{11}, m_{12}, m_{11}, ..., t_{s2}, t_{s1}, m_{s2}, m_{s1}\}$ (t_{i1} switches places with t_{i2} and m_{i1} switches places with m_{i2} for $i = 1, \dots, s$).

4. An additional symmetries of the transposition representation: Since any single transposition is symmetric t_{i1} can be exchanged with m_{i1} while also exchanging t_{i2} with m_{i2} for $i = 1, \dots, s$.

Theorem 7.0.6. If R is a permutation solution to the aYBE in dimension d with a representation as a product of s transpositions given by $R = \{a_1, a_2, \dots, a_{4s}\}$ then $\{a_1, a_2, \dots, a_{4s}\}$ also represents a solution when interpreted as a permutation in any dimension $D \geq d$.

Proof. Let R be a permutation solution to the aYBE in dimension d with a representation as a product of s transpositions given by $R = \{a_1, a_2, \dots, a_{4s}\}$. Let V be a D dimensional vector space with basis $\{e_i \mid i = 1, \dots, D\}$. Let A be the permutation with the same representation, reinterpreted in dimension $D > d$, meaning that A acts as the identity on the basis vectors $e_i \otimes e_j$ when one of i, j is larger than d. When both $i, j \leq d$ then A acts according to the representation of R. We need to show that the aYBE in equation 4.27 is satisfied by A:

$$A_{12} A_{13} A_{23} = A_{23} A_{13} A_{12} \tag{7.36}$$

The operators on the left and right side of equation 7.36 agree on all basis vectors $e_i \otimes e_j \otimes e_k$ when all of i, j, k are less than or equal to d. There are two cases to consider: one of i, j, k is greater than d, two or more of i, j, k are greater than d. If two or more of i, j, k are greater than d, then

$$A_{12}A_{13}A_{23}(e_i \otimes e_j \otimes e_k) = A_{23}A_{13}A_{12}(e_i \otimes e_j \otimes e_k)$$

since both sides act as the identity on these basis vectors. In the case that one of i, j, k is greater than d we have three sub-cases to consider: $i > d$, $j > d$, and $k > d$. If $i > d$ then A_{12} and A_{13} will act as the identity on any basis vector with i as the first tensor factor, so equation 7.36 becomes $A_{23} = A_{23}$. If $j > d$ we then A_{12} and A_{23} will act as the identity and equation 7.36 becomes $A_{13} = A_{13}$. If $k > d$ we then A_{23} and A_{13} will act as the identity and equation 7.36 becomes $A_{12} = A_{12}$. Therefore A is a solution to the aYBE. □

Lemma 7.0.7. Suppose A is an invertible non-identity solution to the bYBE in dimension d that decomposes into the product of s transpositions with the representation $A = \{a_1, a_2, \ldots, a_{4s}\}$, then the representation $\{a_1, a_2, \ldots, a_{4s}\}$ must contain all the integers $1, \ldots, d$.

Proof. Suppose A is an invertible solution to the bYBE in dimension d that decomposes into the product of s transpositions. Let c be the number of unique integers in the representation $A = \{a_1, a_2, \ldots, a_{4s}\}$. If $c < d$ then let $c < i \leq d$. Then $A(e_i \otimes e_j) = e_i \otimes e_j$ and $A(e_j \otimes e_i) = e_j \otimes e_i$ for all $j = 1, \ldots, d$. Let $j, k \in \{a_1, a_2, \ldots, a_{4s}\}$ we then have on one side of the bYBE:

$$A_{12}A_{23}A_{12}(e_i \otimes e_j \otimes e_k) = A_{12}A_{23}(e_i \otimes e_j \otimes e_k) \tag{7.37}$$

$$= A_{12}(e_i \otimes A(e_j \otimes e_k)) \tag{7.38}$$

$$= e_i \otimes A(e_j \otimes e_k) \tag{7.39}$$

The other side of the bYBE is:

$$A_{23}A_{12}A_{23}(e_i \otimes e_j \otimes e_k) = A_{23}A_{12}(e_i \otimes A(e_j \otimes e_k)) \tag{7.40}$$

$$= A_{23}(e_i \otimes A(e_j \otimes e_k)) \tag{7.41}$$

$$= (e_i \otimes A^2(e_j \otimes e_k)) \tag{7.42}$$

The bYBE will only be satisfied if $A(e_j \otimes e_k) = A^2(e_j \otimes e_k)$ for all $j, k = 1, \ldots, c$. Since A is invertible it must be the identity. □

Corollary 7.0.8. A representation $\{a_1, a_2, \ldots, a_{4s}\}$ for an invertible non-identity solution to the bYBE in dimension d does not represent a solution in any other dimension.

Proof. Repeat the argument in the proof of theorem 7.0.7 with the condition that $i > d$ instead of $c < i < d$. □

Theorem 7.0.9. Suppose A is an invertible non-identity solution to the d-dimensional bYBE that decomposes into the product of s transpositions with the representation $A = \{a_1, a_2, \ldots, a_{4s}\}$, then every integer $\{1, \ldots, d\}$ appears at least two times. Moreover, each integer must appear once in the first tensor factor and once in the second tensor factor of a transposition. That is, for all $i \in \{1, \ldots d\}$, $A(e_i \otimes e_j) \neq (e_i \otimes e_j)$ for some $j \in \{1, \ldots d\}$ and $A(e_j \otimes e_i) \neq (e_j \otimes e_i)$ for some $j \in \{1, \ldots d\}$.

Proof. Let A be an invertible solution to the d-dimensional bYBE that decomposes into the product of s transpositions with the representation $A = \{a_1, a_2, \ldots, a_{4s}\}$. Suppose $i \in \{a_1, a_2, \ldots, a_{4s}\}$ appears only once in the representation. Without loss of generality we can assume that i appears in the first tensor factor of a transposition.

That is, $A(e_i \otimes e_j) \neq (e_i \otimes e_j)$ for some unique $j \in A$, and $A(e_j \otimes e_i) = (e_j \otimes e_i)$ for all $j \in A$. Let $j, k \in \{a_1, a_2, \ldots, a_{4s}\}$ such that $i \neq j$ and $i \neq k$. We then have on one side of the bYBE:

$$A_{12}A_{23}A_{12}(e_j \otimes e_k \otimes e_i) = A_{12}A_{23}(A(e_j \otimes e_k) \otimes e_i) \tag{7.43}$$

$$= A_{12}(A(e_j \otimes e_k) \otimes e_i) \tag{7.44}$$

$$= (A^2(e_j \otimes e_k) \otimes e_i) \tag{7.45}$$

The other side of the bYBE is:

$$A_{23}A_{12}A_{23}(e_j \otimes e_k \otimes e_i) = A_{23}A_{12}(e_j \otimes e_k \otimes e_i) \tag{7.46}$$

$$= A_{23}(A(e_j \otimes e_k) \otimes e_i) \tag{7.47}$$

$$= (A(e_j \otimes e_k) \otimes e_i) \tag{7.48}$$

The bYBE will only be satisfied if $A(e_j \otimes e_k) = A^2(e_j \otimes e_k)$ for all $j, k = 1, \ldots, c$. Since A is invertible it must be the identity. Now suppose that i appears twice and it appears only in the first tensor factor within the transposition(s) it appears. The same argument above applies and this can only happen if A is the identity. □

Theorem 7.0.10. If A is an invertible permutation solution to the d-dimensional bYBE that is not the identity. Then if $A(e_a \otimes e_b) = e_c \otimes e_d$ it must be the case that for all $i \in \{1, \ldots, d\}$ either $A(e_b \otimes e_i) \neq (e_b \otimes e_i)$ or $A(e_d \otimes e_i) \neq (e_d \otimes e_i)$.

Proof. Let A be an invertible permutation solution to the d-dimensional bYBE. Suppose $A(e_a \otimes e_b) = e_c \otimes e_d \neq e_a \otimes e_b$. Let $i \in \{1, \ldots, d\}$ such that $A(e_b \otimes e_i) = e_b \otimes e_i$ and $A(e_d \otimes e_i) = e_d \otimes e_i$. We then have:

$$A_{12}A_{23}A_{12}(e_a \otimes e_b \otimes e_i) = A_{12}A_{23}(e_c \otimes e_d \otimes e_i) \tag{7.49}$$

$$= A_{12}(e_c \otimes e_d \otimes e_i) \tag{7.50}$$

$$= (e_a \otimes e_b \otimes e_i) \tag{7.51}$$

$$A_{23}A_{12}A_{23}(e_a \otimes e_b \otimes e_i) = A_{23}A_{12}(e_a \otimes e_b \otimes e_i) \tag{7.52}$$

$$= A_{23}(e_c \otimes e_d \otimes e_i) \tag{7.53}$$

$$= (e_c \otimes e_d \otimes e_i) \tag{7.54}$$

Since A is a solution to the bYBE it must be the case that $a = c$ and $b = d$, a contradiction. □

Theorem 7.0.11. Every solution R to the d-dimensional bYBE induces a solution in dimension $d > D$ as follows. Let P_d be the d dimensional swap matrix whose representation is given in equation 7.4. Let P_D be the D dimensional swap matrix whose representation is given in equation 7.4. Then the representation given by concatenating the representations of R, P_d, and P_D represents a solution in dimension D.

Proof. Let R be a solution to the d-dimensional bYBE. Then RP_d is a solution to the d-dimensional aYBE by proposition 4.7.1. By theorem 7.0.6 the representation of RP_d given by equation 7.4 can be reinterpreted as a solution in dimension D. When RP_d is interpreted as a matrix in dimension D it can be composed with P_D to obtain a solution to the D-dimensional bYBE by proposition 4.7.1. This solution is represented by the concatenating the representations of R, P_d, and P_D. □

We can also express RP_dP_D as follows:

$$RP_dP_D = R \prod_{\substack{i=1,\ldots,D \\ j=d+1,\ldots,D \\ i\neq j}} \{i, j, j, i\} \tag{7.55}$$

The following theorem outlines how two permutation solutions to the aYBE or bYBE can be used to generate additional solutions.

Theorem 7.0.12.

1. If A is a permutation solution to the aYBE with a representation that contains c_1 unique integers and B is a permutation solution to the aYBE with a representation that contains c_2 unique integers then AB is a solution to the d dimensional aYBE, where $d \geq \max(c_1, c_2)$. (In the case of the bYBE, no permutations are disjoint).

   Proof. Both the domain and codomain of A and B are disjoint and are spanned by basis vectors of the form $e_i \otimes e_j$, which are indexed by disjoint sets of indices. Therefore the aYBE will be satisfied on by AB on all basis vectors of

   $$\text{Span}\{e_i \otimes e_j \mid i, j = 1, \ldots, d\}$$

   □

2. A is a permutation solutions to the d-dimensional aYBE such $A = BC$ where B and C are permutations that have disjoint representations, then B and C are both solutions.

   Proof. After restricting A and B to their respective domains, they will still satisfy the aYBE. □

3. If A is a permutation solution to the aYBE with a representation that contains c_1 unique integers and B is a permutation solution to the aYBE with a representation that contains c_2 unique integers then there exists a dimension $dim(V) = d \geq \max(c_1, c_2)$, and $Q \in \mathrm{Aut}(V)$ such that $A(Q \otimes Q)B(Q^{-1} \otimes Q^{-1})$ is a solution in dimension d.

Proof. This follows directly from the above, theorem 7.0.6, and proposition 7.0.5. □

Theorem 7.0.13. Up to the symmetries in proposition 7.0.5 there are only two solutions to the bYBE across all dimensions that can be written as a single transposition.

Proof. By theorems 7.0.9 and 7.0.2 any single transposition solution must be locally conjugate to one of the following representations:

$$A = \{1, 1, 2, 2\} \tag{7.56}$$

$$B = \{1, 2, 2, 1\} = P_2 \tag{7.57}$$

Both of these representations are solutions to the 2-dimensional bYBE. The second representation is the dimension 2 swap matrix P_2 which is a solution. For the first representation we show that the bYBE is satisfied. We can reinterpret A as a matrix which exchanges $e_1 \otimes e_1 \leftrightarrow e_2 \otimes e_2$.

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

We then compute A_{12} and A_{23}:

$$A_{12} = A \otimes I = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A_{23} = I \otimes A = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

It is straightforward to check that the bYBE is satisfied:

$$A_{12}A_{23}A_{12} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = A_{23}A_{12}A_{23}$$

It's interesting to note that:

$$B_{12}B_{23}B_{12} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = B_{23}B_{12}B_{23}$$

□

Theorem 7.0.14. Up to the symmetries in proposition 7.0.5 there are exactly 2 solutions to the bYBE that can be written as a product of two transpositions. These solutions have the representations:

$$\{1, 2, 2, 1, 1, 3, 3, 1\} \tag{7.58}$$

$$\{1, 2, 2, 3, 2, 1, 3, 2\} \tag{7.59}$$

Up to the symmetries in proposition 7.0.5 there are exactly 11 solutions to the bYBE that can be written as a product of three transpositions. These solutions have the representations:

$$\{1, 1, 1, 2, 1, 1, 2, 1, 1, 1, 1, 2\}$$

$$\{1, 1, 1, 2, 1, 1, 2, 1, 2, 1, 2, 2\}$$

$$\{1, 1, 1, 2, 1, 1, 2, 2, 1, 2, 2, 2\}$$

$$\{1, 1, 2, 2, 1, 3, 3, 1, 2, 3, 3, 2\}$$

$$\{1, 1, 2, 2, 1, 3, 3, 2, 2, 3, 3, 1\}$$

$$\{1, 1, 2, 3, 1, 2, 3, 3, 2, 2, 3, 1\}$$

$$\{1, 2, 2, 1, 1, 3, 3, 1, 2, 3, 3, 2\}$$

$$\{1, 2, 2, 1, 1, 3, 3, 2, 2, 3, 3, 1\}$$

$$\{1, 2, 2, 1, 1, 3, 3, 1, 1, 4, 4, 1\}$$

$$\{1, 2, 2, 1, 1, 3, 4, 1, 1, 4, 3, 1\}$$

$$\{1, 2, 2, 3, 2, 1, 4, 2, 2, 4, 3, 2\}$$

Proof. One can generate all 4140 possible representations in the case of two transpositions, and 4213597 possible representations in the case of three. These representations can then easily be checked if they satisfy the bYBE on all relevant basis vectors. □

Theorem 7.0.15. Up to the symmetries in proposition 7.0.5 there are only 2 solutions to the aYBE that can be written as a single transposition. These have the following representations:

$$\{1, 2, 2, 1\} \tag{7.60}$$

$$\{1, 2, 3, 2\} \tag{7.61}$$

Up to the symmetries in proposition 7.0.5 there are only 9 solutions to the aYBE that can be written as a product of two transpositions. These have the following representations:

$$\{1, 1, 1, 2, 2, 1, 2, 2\} \tag{7.62}$$

$$\{1, 1, 2, 2, 1, 2, 2, 1\} \tag{7.63}$$

$$\{1, 2, 1, 3, 1, 2, 1, 4\} \tag{7.64}$$

$$\{1, 2, 1, 3, 1, 4, 1, 5\} \tag{7.65}$$

$$\{1,2,1,3,2,1,3,1\} \tag{7.66}$$

$$\{1,2,1,3,2,4,3,4\} \tag{7.67}$$

$$\{1,2,1,3,4,1,5,1\} \tag{7.68}$$

$$\{1,2,1,3,4,2,4,3\} \tag{7.69}$$

$$\{1,2,3,4,1,4,3,2\} \tag{7.70}$$

Up to the symmetries in proposition 7.0.5 there are only 27 solutions to the aYBE that can be written as a product of three transpositions. These have the following representations:

$$\{1,1,1,2,1,3,2,3,2,1,2,2\} \tag{7.71}$$

$$\{1,1,1,2,2,1,2,2,3,1,3,2\} \tag{7.72}$$

$$\{1,1,2,2,1,2,2,1,1,3,2,3\} \tag{7.73}$$

$$\{1,1,2,3,1,2,2,1,1,3,2,2\} \tag{7.74}$$

$$\{1,2,1,3,2,1,2,3,3,1,3,2\} \tag{7.75}$$

$$\{1,2,1,3,2,1,3,1,2,3,3,2\} \tag{7.76}$$

$$\{1,2,2,1,1,3,3,1,2,3,3,2\} \tag{7.77}$$

$$\{1,2,1,3,1,4,4,1,2,4,3,4\} \tag{7.78}$$

$$\{1,2,1,3,1,4,4,1,4,2,4,3\} \tag{7.79}$$

$$\{1,2,1,3,2,1,3,1,1,4,4,1\} \tag{7.80}$$

$$\{1,2,1,3,2,1,3,1,2,4,3,4\} \tag{7.81}$$

$$\{1,2,1,3,1,2,1,4,1,2,1,5\} \tag{7.82}$$

$$\{1,2,1,3,1,4,1,5,2,1,3,1\} \tag{7.83}$$

$$\{1,2,1,3,1,4,5,4,5,2,5,3\} \tag{7.84}$$

$$\{1,2,1,3,2,4,2,5,3,4,3,5\} \tag{7.85}$$

$$\{1,2,1,3,2,4,3,4,2,5,3,5\} \tag{7.86}$$

$$\{1,2,1,3,2,4,3,5,2,5,3,4\} \tag{7.87}$$

$$\{1,2,1,3,4,2,4,3,5,2,5,3\} \tag{7.88}$$

$$\{1,2,1,3,4,2,5,3,4,3,5,2\} \tag{7.89}$$

$$\{1,2,1,3,1,2,1,4,1,5,1,6\} \tag{7.90}$$

$$\{1,2,1,3,1,2,1,4,5,1,6,1\} \tag{7.91}$$

$$\{1,2,1,3,1,4,1,5,2,6,3,6\} \tag{7.92}$$

$$\{1,2,1,3,1,4,1,5,6,2,6,3\} \tag{7.93}$$

$$\{1,2,1,3,2,4,3,4,4,5,4,6\} \tag{7.94}$$

$$\{1,2,1,3,4,1,5,1,4,6,5,6\} \tag{7.95}$$

$$\{1,2,1,3,1,4,1,5,1,6,1,7\} \tag{7.96}$$

$$\{1,2,1,3,1,4,1,5,6,1,7,1\} \tag{7.97}$$

Proof. There are 15 possible single transposition representations:

$\{1,1,1,1\}$	$\{1,1,1,2\}$	$\{1,1,2,1\}$
$\{1,1,2,2\}$	$\{1,1,2,3\}$	$\{1,2,1,1\}$
$\{1,2,1,2\}$	$\{1,2,1,3\}$	$\{1,2,2,1\}$
$\{1,2,2,2\}$	$\{1,2,2,3\}$	$\{1,2,3,1\}$
$\{1,2,3,2\}$	$\{1,2,3,3\}$	$\{1,2,3,4\}$

One can also generate all 4140 possible representations in the case of two transpositions, and 4213597 possible representations in the case of three. These representations can then easily be checked if they satisfy the aYBE on all relevant basis vectors, and the redundant ones removed according to proposition 7.0.5. □

This method generalizes to longer products of transpositions. Let s be the number of transpositions in the product. Then $4s$ variables are required to represent the product's action on each basis vector. The number of possible representation assumptions is the $4s$ Bell number, $B[4s]$. The number of inputs to consider for each assumption is $(4s+1)^3$. Therefore the total number of cases to consider for a product of s transpositions is: $B[4s](4s+1)^3$. It's well known that any permutation can be factored into a product of transpositions, and that every permutation of d elements can be written using $d-1$ transpositions or less. Therefore this method can produce all permutation solutions in dimension d by considering all $B[4(d^2-1)](4(d^2-1)+1)^3$ possible combinations of assumptions. This is greater than the brute force method of checking all $d^2!$ permutation matrices. Finding which representations are solutions of the bYBE amounts to checking whether if $R_{12}R_{23}R_{12}$ agrees with $R_{23}R_{12}R_{23}$ on each basis vector of $V \otimes V \otimes V$. For a given representation we only need to check the basis vectors with indices ranging from 1 to c, where c is the number of matching variables in the original representation.

Theorem 7.0.16. This theorem allows us to compose certain types of transposition solutions that have some overlapping indices. Let B be the transposition representation:

$$\{a, b, a, c\} \tag{7.98}$$

This represents a solution to the d-dimensional aYBE, where $d > 2$, by theorem 7.0.15, where the form in theorem 7.0.15 is equal to B after conjugating by swap and

$Q \otimes Q$. Consider the following related representations:

$$B = \{a, b, a, c\} \tag{7.99}$$

$$PBP = \{b, a, c, a\} \tag{7.100}$$

$$(Q \otimes Q)B(Q^{-1} \otimes Q^{-1}) = \{l, m, l, n\} \tag{7.101}$$

$$(Q \otimes Q)PBP(Q^{-1} \otimes Q^{-1}) = \{m, l, n, l\} \tag{7.102}$$

Notice that two indices appear once and one index always appears twice. These representations can be composed to form new solutions as follows. Let $B_1, B_2, \ldots, B_s$ be transpositions that can be represented by one of the above forms such that if $i \in B_j$ appears twice in B_j then it appears twice in any of the other transpositions that it appears, and if $i, j \in B_k$ such that $i \neq j$ then $i \in B_p$ if and only if $j \in B_p$, and i, j only appear once in each transposition that they appear, we will call these paired indices. Denote the paired indices by single letters i and it's pairing $\hat{\imath}$. Further assume that if $R(e_i \otimes e_j) \neq e_i \otimes e_j$ then $e_i \otimes e_j$ is only transformed by a single transposition, B_k. For example we can write the forms above as:

$$\{a, b, a, \hat{b}\} \tag{7.103}$$

$$\{b, a, \hat{b}, a\} \tag{7.104}$$

$$\{l, m, l, \hat{m}\} \tag{7.105}$$

$$\{m, l, \hat{m}, l\} \tag{7.106}$$

The following composition

$$R = B_1 B_2 \ldots B_s \tag{7.107}$$

is a solution to the D-dimensional aYBE where $D \geq c$, and c is the number of unique indices appearing in the representation of R.

Example 7.0.17. The following representation represents a solution to the d-dimensional aYBE where $d \geq 9$.

$$\begin{aligned}\{1, 2, 1, 3,\\ 1, 4, 1, 5,\\ 1, 6, 1, 7,\\ 2, 1, 3, 1,\\ 9, 2, 9, 3,\\ 9, 6, 9, 7\}\end{aligned}$$

Proof of Theorem 7.0.16. Let $R = B_1 B_2 \dots B_s$ be a permutation with a representation that follows the assumptions of theorem 7.0.16. Let c be the number of unique indices appearing in the representation of R. We need to show that

$$R_{12}R_{13}R_{23}(e_i \otimes e_j \otimes e_k) = R_{23}R_{13}R_{12}(e_i \otimes e_j \otimes e_k)$$

holds for all $i, j, k \in \{1, \dots, c\}$. There are four possible cases to consider:

1. $R(e_i \otimes e_j) \neq e_i \otimes e_j$ and $R(e_j \otimes e_k) \neq e_j \otimes e_k$
2. $R(e_i \otimes e_j) \neq e_i \otimes e_j$ and $R(e_j \otimes e_k) = e_j \otimes e_k$
3. $R(e_i \otimes e_j) = e_i \otimes e_j$ and $R(e_j \otimes e_k) \neq e_j \otimes e_k$
4. $R(e_i \otimes e_j) = e_i \otimes e_j$ and $R(e_j \otimes e_k) = e_j \otimes e_k$

In the first case we have two sub-cases:

(a) $R(e_i \otimes e_j) = e_{\hat{\imath}} \otimes e_j$ and $R(e_j \otimes e_k) = e_j \otimes e_{\hat{k}}$ in which case we have:

$$R_{12}R_{13}R_{23}(e_i \otimes e_j \otimes e_k) = R_{23}R_{13}R_{12}(e_i \otimes e_j \otimes e_k)$$
$$R_{12}R_{13}(e_i \otimes e_j \otimes e_{\hat{k}}) = R_{23}R_{13}(e_{\hat{\imath}} \otimes e_j \otimes e_k)$$

It must be the case that $R(e_i \otimes e_{\hat{k}}) = (e_i \otimes e_{\hat{k}})$ and $R(e_{\hat{\imath}} \otimes e_k) = (e_{\hat{\imath}} \otimes e_k)$ since both indices i and k are paired. We then have:

$$(e_{\hat{\imath}} \otimes e_j \otimes e_{\hat{k}}) = (e_{\hat{\imath}} \otimes e_j \otimes e_{\hat{k}})$$

(b) $R(e_i \otimes e_j) = e_i \otimes e_{\hat{\jmath}}$ and $R(e_j \otimes e_k) = e_{\hat{\jmath}} \otimes e_k$ in which case both i and k are repeated and we have:

$$R_{12}R_{13}R_{23}(e_i \otimes e_j \otimes e_k) = R_{23}R_{13}R_{12}(e_i \otimes e_j \otimes e_k)$$
$$R_{12}R_{13}(e_i \otimes e_{\hat{\jmath}} \otimes e_k) = R_{23}R_{13}(e_i \otimes e_{\hat{\jmath}} \otimes e_k)$$
$$R_{12}(e_i \otimes e_{\hat{\jmath}} \otimes e_k) = R_{23}(e_i \otimes e_{\hat{\jmath}} \otimes e_k)$$
$$(e_i \otimes e_j \otimes e_k) = (e_i \otimes e_j \otimes e_k)$$

In the second case we have two sub-cases: $R(e_i \otimes e_j) = e_i \otimes e_{\hat{\jmath}}$ or $R(e_i \otimes e_j) = e_{\hat{\imath}} \otimes e_j$.

(a) In the case that $R(e_i \otimes e_j) = e_i \otimes e_{\hat{\jmath}}$ we have:

$$R_{12}R_{13}R_{23}(e_i \otimes e_j \otimes e_k) = R_{23}R_{13}R_{12}(e_i \otimes e_j \otimes e_k)$$
$$R_{12}R_{13}(e_i \otimes e_j \otimes e_k) = R_{23}R_{13}(e_i \otimes e_{\hat{\jmath}} \otimes e_k)$$

Then either $R(e_i \otimes k) = e_i \otimes e_{\hat{k}}$ or $R(e_i \otimes e_k) = e_i \otimes e_k$. If $R(e_i \otimes k) = e_i \otimes e_{\hat{k}}$

then

$$R_{12}(e_i \otimes e_j \otimes e_{\hat{k}}) = R_{23}(e_i \otimes e_{\hat{j}} \otimes e_{\hat{k}})$$
$$(e_i \otimes e_{\hat{j}} \otimes e_{\hat{k}}) = (e_i \otimes e_{\hat{j}} \otimes e_{\hat{k}})$$

If $R(e_i \otimes e_k) = e_i \otimes e_k$ then

$$R_{12}(e_i \otimes e_j \otimes e_k) = R_{23}(e_i \otimes e_{\hat{j}} \otimes e_k)$$
$$(e_i \otimes e_{\hat{j}} \otimes e_k) = (e_i \otimes e_{\hat{j}} \otimes e_k)$$

(b) In the case that $R(e_i \otimes e_j) = e_{\hat{i}} \otimes e_j$ we have:

$$R_{12}R_{13}R_{23}(e_i \otimes e_j \otimes e_k) = R_{23}R_{13}R_{12}(e_i \otimes e_j \otimes e_k)$$
$$R_{12}R_{13}(e_i \otimes e_j \otimes e_k) = R_{23}R_{13}(e_{\hat{i}} \otimes e_j \otimes e_k)$$

If $R(e_i \otimes e_k) = e_i \otimes e_k$ then we get:

$$R_{12}(e_i \otimes e_j \otimes e_k) = R_{23}(e_{\hat{i}} \otimes e_j \otimes e_k)$$
$$(e_{\hat{i}} \otimes e_j \otimes e_k) = (e_{\hat{i}} \otimes e_j \otimes e_k)$$

If $R(e_i \otimes e_k) \neq e_i \otimes e_k$ then it must be the case that $R(e_i \otimes e_k) = e_{\hat{i}} \otimes e_k$ and $R(e_{\hat{i}} \otimes e_k) = (e_i \otimes e_k)$ giving us:

$$R_{12}(e_{\hat{i}} \otimes e_j \otimes e_k) = R_{23}(e_i \otimes e_j \otimes e_k)$$
$$(e_i \otimes e_j \otimes e_k) = (e_i \otimes e_j \otimes e_k)$$

The third case can be handled almost identically to the second case. In the last case we have three sub-cases:

(a) $R(e_i \otimes e_k) = e_i \otimes e_k$ in which case we have:

$$
\begin{aligned}
R_{12}R_{13}R_{23}(e_i \otimes e_j \otimes e_k) &= R_{23}R_{13}R_{12}(e_i \otimes e_j \otimes e_k) \\
R_{12}R_{13}(e_i \otimes e_j \otimes e_k) &= R_{23}R_{13}(e_i \otimes e_j \otimes e_k) \\
R_{12}(e_i \otimes e_j \otimes e_k) &= R_{23}(e_i \otimes e_j \otimes e_k) \\
(e_i \otimes e_j \otimes e_k) &= (e_i \otimes e_j \otimes e_k)
\end{aligned}
$$

(b) $R(e_i \otimes e_k) = e_{\hat{i}} \otimes e_k$ in which case i is paired and k is repeated.

$$
\begin{aligned}
R_{12}R_{13}R_{23}(e_i \otimes e_j \otimes e_k) &= R_{23}R_{13}R_{12}(e_i \otimes e_j \otimes e_k) \\
R_{12}R_{13}(e_i \otimes e_j \otimes e_k) &= R_{23}R_{13}(e_i \otimes e_j \otimes e_k) \\
R_{12}(e_{\hat{i}} \otimes e_j \otimes e_k) &= R_{23}(e_{\hat{i}} \otimes e_j \otimes e_k)
\end{aligned}
$$

By the assumption $R(e_j \otimes e_k) = e_j \otimes e_k$, R_{23} will act as the identity on the right. By the assumption $R(e_i \otimes e_j) = e_i \otimes e_j$, $R(e_{\hat{i}} \otimes e_j) = (e_{\hat{i}} \otimes e_j)$, otherwise it would imply that $R(e_i \otimes e_j) \neq e_i \otimes e_j$, so R_{12} acts as the identity on the left. Therefore the equality is satisfied in this case.

(c) $R(e_i \otimes e_k) = e_i \otimes e_{\hat{k}}$ in which case i is repeated and k is paired.

$$R_{12}R_{13}R_{23}(e_i \otimes e_j \otimes e_k) = R_{23}R_{13}R_{12}(e_i \otimes e_j \otimes e_k)$$

$$R_{12}R_{13}(e_i \otimes e_j \otimes e_k) = R_{23}R_{13}(e_i \otimes e_j \otimes e_k)$$

$$R_{12}(e_i \otimes e_j \otimes e_{\hat{k}}) = R_{23}(e_i \otimes e_j \otimes e_{\hat{k}})$$

By a similar logic to the previous case, both sides are equal.

□

Chapter 8

Upper triangular solutions

> "It is my experience that proofs involving matrices can be shortened by 50% if one throws the matrices out."
>
> Emil Artin

8.1 Hietarinta's method for dimension 3

The 9×9 invertible upper-triangular solutions to the 3-dimensional aYBE have been classified fully by Hietarinta in [39]. The only unitary upper triangular matrices are diagonal, however, the upper triangular solutions might be transformable into a non-trivial unitary solution as described by H. A. Dye in [25]. What follows is a description of the solving process used by [39]. In order to keep R upper triangular the matrix Q in proposition 4.7.1 is restricted to be upper triangular. We can examine the effect of the Q-transformation by organizing R into 3×3 blocks:

$$R = \begin{pmatrix} A & B & C \\ 0 & E & F \\ 0 & 0 & G \end{pmatrix}$$

$$(Q \otimes Q)R(Q \otimes Q)^{-1} = \begin{pmatrix} QAQ^{-1} & B' & C' \\ 0 & QEQ^{-1} & F' \\ 0 & 0 & QGQ^{-1} \end{pmatrix}$$

Therefore the 3×3 diagonal blocks undergo a similarity transformation by Q. Since Q is upper triangular we cannot always diagonalize the blocks, however, they can be brought to a Belitskiĭ canonical form.[52] In the 3×3 invertible case there are 5 Belitskiĭ forms:

$$C_1 = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \qquad C_2 = \begin{pmatrix} a & b & 0 \\ 0 & a & 0 \\ 0 & 0 & c \end{pmatrix}$$

$$C_3 = \begin{pmatrix} a & 0 & b \\ 0 & c & 0 \\ 0 & 0 & a \end{pmatrix} \qquad C_4 = \begin{pmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & 0 & b \end{pmatrix}$$

$$C_5 = \begin{pmatrix} a & b & 0 \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix}$$

These were obtained by following Belitskiĭs algorithm. The diagonal elements are left unchanged by the similarity transformation so initially there are 5 possible cases (3 eigenvalues, 2 eigenvalues, or 1 eigenvalue and their possible arrangements). A given element above the diagonal can be transformed to zero provided that the eigenvalues in its row and column are different. If they are the same then the element can be scaled to 1.

Example 8.1.1. For example when there are 3 eigenvalues:

$$QAQ^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{a_{23}}{a_{22}-a_{33}} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{a_{23}}{a_{22}-a_{33}} \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} & a'_{11} & a'_{13} \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix} = A'$$

$$QA'Q^{-1} = \begin{pmatrix} 1 & \frac{a_{12}}{a_{11}-a_{22}} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a'_{11} & a'_{13} \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix} \begin{pmatrix} 1 & -\frac{a_{12}}{a_{11}-a_{22}} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} a_{11} & 0 & a'_{13} \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix} = A''$$

$$QA''Q^{-1} = \begin{pmatrix} 1 & 0 & \frac{a_{13}}{a_{11}-a_{33}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & 0 & a'_{13} \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix} \begin{pmatrix} 1 & 0 & -\frac{a_{13}}{a_{11}-a_{33}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}$$

For the case where there is only one eigenvalue and the upper-diagonal entries are the same but nonzero (if there were then this is a special case of a different canonical form) we can use

$$Q = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & \frac{a_{12}+a_{13}}{a_{12}} \\ 0 & 0 & 1 \end{pmatrix}$$

to transform the upper right element to 0.

Belitskiĭ algorithm is more general than this and it works by going through the off-diagonal elements one by one to make them either 0 or 1 while taking care to not change a previously transformed entry. The algorithm and the canonical forms for up to the 5×5 case are described in [52].

Hietarinta's solution method can then be described as follows:

1. Assume that the upper left block of R takes one of the canonical forms

2. Since R is invertible and the diagonal elements must be nonzero the most common eigenvalue in that block can be scaled to 1.

3. A repeated off-diagonal element in the canonical forms can also be transformed to 1 using the Q transformation

Some cases can be eliminated using the fact that two solutions are considered identical if they are equivalent under the symmetries listed above. For each case Hietarinta manually factored the simpler equations into sub-cases and solved using a computer algebra system.

8.2 The 4 dimensional upper triangular Belitskiĭ canonical forms under upper triangular similarity

Two matrices A and B are considered t-similar if there exists a 4×4 upper triangular invertible matrix Q such that $A = QBQ^{-1}$. We find a generating set for the set of invertible 4×4 upper triangular matrices over $\mathbb{C}$ under t-similarity. These matrices can be used to split the problem of finding the 16×16 upper triangular invertible matrices satisfying aYBE into sub-cases. Belitskiĭs algorithm results in the following generating set of 48 matrices after considering all possible cases:

1 distinct eigenvalue

$$C_1 = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{pmatrix}$$

$$C_2 = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 1 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{pmatrix} \quad C_3 = \begin{pmatrix} a & 0 & 0 & 1 \\ 0 & a & 1 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{pmatrix} \quad C_4 = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 1 & 0 \\ 0 & 0 & a & 1 \\ 0 & 0 & 0 & a \end{pmatrix}$$

$$C_5 = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 1 \\ 0 & 0 & 0 & a \end{pmatrix} \quad C_6 = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 1 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{pmatrix} \quad C_7 = \begin{pmatrix} a & 1 & 1 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 1 \\ 0 & 0 & 0 & a \end{pmatrix}$$

2 distinct eigenvalues

$$C_8 = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & b \end{pmatrix}$$

$$C_9 = \begin{pmatrix} a & 1 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & b & 1 \\ 0 & 0 & 0 & b \end{pmatrix} \quad C_{10} = \begin{pmatrix} a & 1 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & b \end{pmatrix} \quad C_{11} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & b & 1 \\ 0 & 0 & 0 & b \end{pmatrix}$$

$$C_{12} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \end{pmatrix}$$

$$C_{13} = \begin{pmatrix} a & 0 & 1 & 0 \\ 0 & b & 0 & 1 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \end{pmatrix} \quad C_{14} = \begin{pmatrix} a & 0 & 1 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \end{pmatrix} \quad C_{15} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 1 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \end{pmatrix}$$

$$C_{16} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix}$$

$$C_{17} = \begin{pmatrix} a & 0 & 0 & 1 \\ 0 & b & 1 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix} \quad C_{18} = \begin{pmatrix} a & 0 & 0 & 1 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix} \quad C_{19} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 1 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix}$$

$$C_{20} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & b \end{pmatrix}$$

$$C_{21} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 1 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & b \end{pmatrix} \quad C_{22} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 1 & 0 \\ 0 & 0 & b & 1 \\ 0 & 0 & 0 & b \end{pmatrix} \quad C_{23} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & 1 \\ 0 & 0 & 0 & b \end{pmatrix}$$

$$C_{24} = \begin{pmatrix} b & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & b \end{pmatrix}$$

$$C_{25} = \begin{pmatrix} b & 0 & 1 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & b \end{pmatrix} \quad C_{26} = \begin{pmatrix} b & 0 & 1 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & b & 1 \\ 0 & 0 & 0 & b \end{pmatrix} \quad C_{27} = \begin{pmatrix} b & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & b & 1 \\ 0 & 0 & 0 & b \end{pmatrix}$$

$$C_{28} = \begin{pmatrix} b & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \end{pmatrix}$$

$$C_{29} = \begin{pmatrix} b & 1 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \end{pmatrix} \quad C_{30} = \begin{pmatrix} b & 1 & 0 & 0 \\ 0 & b & 0 & 1 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \end{pmatrix} \quad C_{31} = \begin{pmatrix} b & 0 & 0 & 0 \\ 0 & b & 0 & 1 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \end{pmatrix}$$

$$C_{32} = \begin{pmatrix} b & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix}$$

$$C_{33} = \begin{pmatrix} b & 1 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix} \quad C_{34} = \begin{pmatrix} b & 1 & 0 & 0 \\ 0 & b & 1 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix} \quad C_{35} = \begin{pmatrix} b & 0 & 0 & 0 \\ 0 & b & 1 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix}$$

3 distinct eigenvalues

$$C_{36} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \end{pmatrix} \quad C_{37} = \begin{pmatrix} a & 1 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \end{pmatrix}$$

$$C_{38} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & a \end{pmatrix} \quad C_{39} = \begin{pmatrix} a & 0 & 0 & 1 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & a \end{pmatrix}$$

$$C_{40} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & a \end{pmatrix} \quad C_{41} = \begin{pmatrix} a & 0 & 1 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & c \end{pmatrix}$$

$$C_{42} = \begin{pmatrix} b & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & a \end{pmatrix} \quad C_{43} = \begin{pmatrix} b & 0 & 0 & 0 \\ 0 & a & 0 & 1 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & a \end{pmatrix}$$

$$C_{44} = \begin{pmatrix} b & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{pmatrix} \quad C_{45} = \begin{pmatrix} b & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & a & 1 \\ 0 & 0 & 0 & a \end{pmatrix}$$

$$C_{46} = \begin{pmatrix} b & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & c \end{pmatrix} \quad C_{47} = \begin{pmatrix} b & 0 & 0 & 0 \\ 0 & a & 1 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & c \end{pmatrix}$$

4 distinct eigenvalues

$$C_{48} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix}$$

These canonical forms were obtained through the following procedure. Recall we are assuming the matrix is invertible therefore all eigenvalues are nonzero. After picking the number of distinct eigenvalues and their arrangement on the diagonal, any element above the diagonal can be brought to 0 (by a similarity transformation) if the eigenvalues in it's row and column are distinct.

Let $E_{ij}(a)$ be the elementary matrix with ones on the diagonal and with the element in row i and column j equal to a and zero everywhere else. Then multiplication on the left represents the row operation of adding a times row j to row i. Multiplication on the right represents the column operation of adding a times column i to column j. Therefore $E_{ij}(a)BE_{ij}^{-1}(a)$ results in element (i,j) being transformed to: $B_{ij} + aB_{jj} - aB_{ii}$. Choosing $a = \frac{B_{ij}}{B_{ii}-B_{jj}}$ allows us to transform element (i,j) to 0.

So after making elements in the row and column of distinct eigenvalues 0 we are left with sub matrix blocks that all have the a single eigenvalue. It's left to consider what canonical forms these can be brought to under an upper triangular similarity transformation. Each of these blocks is of the form $\lambda I + N$ where N is nilpotent. These cases are covered in [52].

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Classifying all invertible upper triangular solutions to the 4-dimensional aYBE amounts to classifying all solutions with the upper left block given by one of the 48 Canonical forms above. The upper triangular ansatz results in a polynomial system of $16^3 = 4,096$ equations in 256 variables. We consider the case when the upper left block has the same form as C_{34} above. This reduces the number of equations to 1974 and the number of variables to 136. The system contains many equations that factor and can be simultaneously solved quickly using a computer algebra system. Using algorithm 3.3 we were able to compute all 21 solutions in this case. In many cases two of the solutions differ by only a single entry in the lower right. We first define the following auxiliary variables which allow the matrices to stay within the margins of the page:

$$
\begin{aligned}
b_1 &= -\frac{1}{4}a_3\left(8a_1 + (a_3 - 1)\, a_3\right)\\
b_2 &= 1 - \frac{1}{2}a_3\left(a_3 + 3\right)\\
b_3 &= \frac{1}{2}a_3\left(a_3 + 1\right)\\
b_4 &= \frac{1}{4}a_3\left(3a_3\left(a_3 + 1\right) - 2\right) - a_1\\
b_5 &= \frac{1}{2}a_2\left(a_3 + 2\right)\\
b_6 &= a_1 - \frac{1}{4}\left(a_3 - 1\right) a_3\left(3a_3 - 2\right)\\
b_7 &= \left(a_3 - 1\right)\left(2a_3 - 1\right)\\
b_8 &= \frac{1}{4}a_3\left(3a_3\left(a_3 + 1\right) - 2\right) - a_2\\
b_9 &= \frac{1}{4}a_3\left(8a_2 + a_3\left(5 - a_3\left(6a_3 + 7\right)\right)\right)\\
b_{10} &= a_2 - \frac{3}{4}a_3\left(a_3^2 + a_3 - 2\right)\\
b_{11} &= -\frac{1}{4}a_3\left(8a_2 + (a_3 - 1)\, a_3\right)\\
b_{12} &= -\frac{1}{4}a_6\left(8a_3 + (a_6 - 1)\, a_6\right)\\
b_{13} &= a_2 - \frac{1}{4}\left(a_3 1\right) a_3\left(3a_3 - 2\right)\\
b_{14} &= \frac{1}{4}a_6\left(3a_6\left(a_6 + 1\right) - 2\right) - a_3\\
b_{15} &= a_{10}\left(a_3^2 a_{10}^2 - \left(a_4 + a_3\left(a_5 + 1\right)\right) a_{10} + a_5 + a_8 - a_9\right)\\
b_{16} &= a_3 - \frac{1}{4}\left(a_6 - 1\right) a_6\left(3a_6 - 2\right)
\end{aligned}
$$

$$b_{17} = a_8 \left(a_3^2 a_8^2 - a_4 a_8 + a_1 + a_5 - a_7 - 1\right)$$

$$b_{18} = -\frac{1}{8}\left(a_3 + 1\right)\left(a_3 + 3\right)$$

$$b_{19} = a_{10}\left(a_5 - a_3 a_{10} + 1\right)$$

$$\begin{pmatrix}
1 & 1 & 0 & 0 & -1 & -a_3 & a_1 & 0 & 1 & a_3 - a_1 & b_1 & 0 & 0 & 0 & 0 & -\frac{a_1 a_2}{a_3 - 1} \\
0 & 1 & 1 & 0 & 0 & -a_3 & b_2 & 0 & 0 & b_3 & b_4 & 0 & 0 & 0 & 0 & b_5 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 - 2a_3 & 0 & 0 & 0 & a_3\left(2a_3 - 1\right) & 0 & 0 & 0 & 0 & a_2 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & a_3 & 0 & 0 & 0 & -b_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & a_3 & \frac{1}{2}\left(a_3 - 1\right) a_3 & 0 & -1 & -b_3 & b_6 & 0 & 0 & 0 & 0 & -\frac{1}{2} a_2 a_3 \\
0 & 0 & 0 & 0 & 0 & 1 & a_3 & 0 & 0 & -a_3 & a_3 - 2a_3^2 & 0 & 0 & 0 & 0 & -a_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 - 2a_3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & a_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2a_3 - 1 & b_7 & 0 & 0 & 0 & 0 & a_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2a_3 - 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -a_3 & -\frac{1}{2}\left(a_3 - 1\right) a_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -a_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} \tag{8.1}$$

$$
\begin{pmatrix}
1 & 1 & 0 & 0 & -1 & -a_3 & a_1 & 0 & 1 & a_3-a_1 & b_1 & 0 & 0 & 0 & 0 & -\frac{a_1 a_2}{a_3-1} \\
0 & 1 & 1 & 0 & 0 & -a_3 & b_2 & 0 & 0 & b_3 & b_4 & 0 & 0 & 0 & 0 & b_5 \\
0 & 0 & 1 & 0 & 0 & 0 & 1-2a_3 & 0 & 0 & 0 & a_3\left(2a_3-1\right) & 0 & 0 & 0 & 0 & a_2 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & a_3 & 0 & 0 & 0 & -b_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & a_3 & \frac{1}{2}\left(a_3-1\right)a_3 & 0 & -1 & -b_3 & b_6 & 0 & 0 & 0 & 0 & -\frac{1}{2}a_2 a_3 \\
0 & 0 & 0 & 0 & 0 & 1 & a_3 & 0 & 0 & -a_3 & a_3-2a_3^2 & 0 & 0 & 0 & 0 & -a_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1-2a_3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & a_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2a_3-1 & b_7 & 0 & 0 & 0 & 0 & a_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2a_3-1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -a_3 & -\frac{1}{2}\left(a_3-1\right)a_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -a_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix}
\tag{8.2}
$$

$$
\begin{pmatrix}
1 & 1 & 0 & 0 & -1 & -a_3 & b_8 & 0 & 1 & b_{10} & b_9 & 0 & 0 & 0 & 0 & a_1 \\
0 & 1 & 1 & 0 & 0 & -a_3 & b_2 & 0 & 0 & b_3 & a_2 & 0 & 0 & 0 & 0 & a_1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1-2a_3 & 0 & 0 & 0 & a_3\left(2a_3-1\right) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & \frac{1}{2}\left(a_3+1\right) & 0 & 0 & 0 & b_{18} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & a_3 & \frac{1}{2}\left(a_3-1\right)a_3 & 0 & -1 & -b_3 & a_3\left(2a_3-1\right)-a_2 & 0 & 0 & 0 & 0 & -a_1 \\
0 & 0 & 0 & 0 & 0 & 1 & a_3 & 0 & 0 & -a_3 & a_3-2a_3^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1-2a_3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & \frac{1}{2}\left(a_3+1\right) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2a_3-1 & b_7 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2a_3-1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & \frac{1}{2}\left(-a_3-1\right) & \frac{1}{8}\left(1-a_3^2\right) & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & \frac{1}{2}\left(-a_3-1\right) & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\tag{8.3}
$$

$$
\begin{pmatrix}
1 & 1 & 0 & 0 & -1 & -a_3 & b_8 & 0 & 1 & b_{10} & b_9 & 0 & 0 & 0 & 0 & a_1 \\
0 & 1 & 1 & 0 & 0 & -a_3 & b_2 & 0 & 0 & b_3 & a_2 & 0 & 0 & 0 & 0 & a_1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1-2a_3 & 0 & 0 & 0 & a_3\left(2a_3-1\right) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & \frac{1}{2}\left(a_3+1\right) & 0 & 0 & 0 & b_{18} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & a_3 & \frac{1}{2}\left(a_3-1\right)a_3 & 0 & -1 & -b_3 & a_3\left(2a_3-1\right)-a_2 & 0 & 0 & 0 & 0 & -a_1 \\
0 & 0 & 0 & 0 & 0 & 1 & a_3 & 0 & 0 & -a_3 & a_3-2a_3^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1-2a_3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & \frac{1}{2}\left(a_3+1\right) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2a_3-1 & b_7 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2a_3-1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & \frac{1}{2}\left(-a_3-1\right) & \frac{1}{8}\left(1-a_3^2\right) & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & \frac{1}{2}\left(-a_3-1\right) & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix}
\tag{8.4}
$$

$$
\begin{pmatrix}
1 & 1 & 0 & 0 & -1 & -a_3 & a_2 & 0 & 1 & a_3-a_2 & b_{11} & 0 & 0 & 0 & 0 & a_1 \\
0 & 1 & 1 & 0 & 0 & -a_3 & b_2 & 0 & 0 & b_3 & \frac{1}{4}a_3\left(3a_3\left(a_3+1\right)-2\right)-a_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1-2a_3 & 0 & 0 & 0 & a_3\left(2a_3-1\right) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & a_3 & \frac{1}{2}\left(a_3-1\right)a_3 & 0 & -1 & -b_3 & b_{13} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & a_3 & 0 & 0 & -a_3 & a_3-2a_3^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1-2a_3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2a_3-1 & b_7 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2a_3-1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\tag{8.5}
$$

$$
\begin{pmatrix}
1 & 1 & 0 & 0 & -1 & -a_3 & a_2 & 0 & 1 & a_3-a_2 & b_{11} & 0 & 0 & 0 & 0 & a_1 \\
0 & 1 & 1 & 0 & 0 & -a_3 & b_2 & 0 & 0 & b_3 & \frac{1}{4}a_3\left(3a_3\left(a_3+1\right)-2\right)-a_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1-2a_3 & 0 & 0 & 0 & a_3\left(2a_3-1\right) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & a_3 & \frac{1}{2}\left(a_3-1\right)a_3 & 0 & -1 & -b_3 & a_2-\frac{1}{4}\left(a_3-1\right)a_3\left(3a_3-2\right) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & a_3 & 0 & 0 & -a_3 & a_3-2a_3^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1-2a_3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2a_3-1 & b_7 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2a_3-1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix}
\tag{8.6}
$$

$$
\begin{pmatrix}
1 & 1 & 0 & 0 & -1 & -a_6 & a_3 & 0 & 1 & a_6-a_3 & b_{12} & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & -a_6 & 1-\frac{1}{2}a_6\left(a_6+3\right) & 0 & 0 & \frac{1}{2}a_6\left(a_6+1\right) & b_{14} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1-2a_6 & 0 & 0 & 0 & a_6\left(2a_6-1\right) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & a_4 & 0 & 0 & 0 & a_5 & 0 & 0 & 0 & \frac{a_5\left(a_5-a_4\right)}{2a_4} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & a_6 & \frac{1}{2}\left(a_6-1\right)a_6 & 0 & -1 & -\frac{1}{2}a_6\left(a_6+1\right) & b_{16} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & a_6 & 0 & 0 & -a_6 & a_6-2a_6^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1-2a_6 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & a_4 & 0 & 0 & 0 & a_5 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2a_6-1 & \left(a_6-1\right)\left(2a_6-1\right) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2a_6-1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_1 & -\frac{a_1a_5}{a_4} & \frac{a_1a_5\left(a_4+a_5\right)}{2a_4^2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_1 & -\frac{a_1a_5}{a_4} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_2
\end{pmatrix}
\tag{8.7}
$$

$$
\begin{pmatrix}
1 & 1 & 0 & 0 & a_5 & a_6 & a_7 & 0 & a_8 & a_1 & a_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & a_5 & a_6 & 0 & 0 & a_8 & a_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & a_5 & 0 & 0 & 0 & a_8 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & a_{10} & 0 & 0 & 0 & b_{19} & 0 & 0 & 0 & b_{15} & 0 & 0 & a_9 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & a_5 & a_6 & a_7 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & a_5 & a_6 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & a_5 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{10} & 0 & 0 & 0 & b_{19} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{10} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{a_{10}} & a_3 & a_4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{a_{10}} & a_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{a_{10}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\tag{8.8}
$$

$$
\begin{pmatrix}
1 & 1 & 0 & 0 & a_5 & a_6 & a_7 & 0 & a_8 & a_1 & a_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & a_5 & a_6 & 0 & 0 & a_8 & a_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & a_5 & 0 & 0 & 0 & a_8 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & a_{10} & 0 & 0 & 0 & b_{15} & 0 & 0 & a_9 & 0 & & & & \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & a_5 & a_6 & a_7 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & a_5 & a_6 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & a_5 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{10} & 0 & 0 & 0 & b_{19} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{10} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{a_{10}} & a_3 & a_4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{a_{10}} & a_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{a_{10}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix}
\tag{8.9}
$$

$$\begin{pmatrix}
1 & 1 & 0 & 0 & -1 & -1 & a_5 & 0 & a_1+a_5 & a_1 & a_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & a_1+a_5 & a_6 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & a_1+a_5 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & a_8 & 0 & 0 & 0 & -a_3a_8^2 & 0 & 0 & 0 & b_{17} & 0 & 0 & a_7 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & -1 & a_1+a_5-a_6 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & a_8 & 0 & 0 & 0 & -a_3a_8^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_8 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{a_8} & a_3 & a_4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{a_8} & a_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{a_8} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} \tag{8.10}$$

$$\begin{pmatrix}
1 & 1 & 0 & 0 & -1 & -1 & a_5 & 0 & a_1+a_5 & a_1 & a_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & a_1+a_5 & a_6 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & a_1+a_5 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & a_8 & 0 & 0 & 0 & -a_3a_8^2 & 0 & 0 & 0 & b_{17} & 0 & 0 & a_7 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & -1 & a_1+a_5-a_6 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & a_8 & 0 & 0 & 0 & -a_3a_8^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_8 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{a_8} & a_3 & a_4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{a_8} & a_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{a_8} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix} \tag{8.11}$$

$$
\begin{pmatrix}
1 & 1 & 0 & 0 & a_6 & a_7 & a_9 & 0 & a_{10} & a_1 & a_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & a_6 & a_7 & 0 & 0 & a_{10} & a_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & a_6 & 0 & 0 & 0 & a_{10} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & a_{12} & 0 & 0 & 0 & a_{13} & 0 & 0 & 0 & a_{11} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & a_6 & a_7 & a_9 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & a_6 & a_7 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & a_6 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{12} & 0 & 0 & 0 & a_{13} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{12} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_5 & a_3 & a_4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_5 & a_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_5 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_8
\end{pmatrix} \tag{8.12}
$$

$$
\begin{pmatrix}
1 & 1 & 0 & 0 & -1 & -1 & a_6 & 0 & a_1+a_6 & a_1 & a_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & a_1+a_6 & a_7 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & a_1+a_6 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & a_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_8 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & -1 & a_1+a_6-a_7 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & a_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_9 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_4 & 0 & a_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_5
\end{pmatrix}
\tag{8.13}
$$

$$
\begin{pmatrix}
1 & 1 & 0 & 0 & -1 & -1 & a_7 & 0 & a_1+a_7 & a_1 & a_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & a_1+a_7 & a_8 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & a_1+a_7 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{a_5 a_{10}}{a_3} & 0 & 0 & 0 & a_{10} & 0 & 0 & 0 & a_9 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & -1 & a_1+a_7-a_8 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{a_5 a_{10}}{a_3} & 0 & 0 & 0 & a_{10} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{a_5 a_{10}}{a_3} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_5 & a_3 & a_4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_5 & a_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_5 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_6
\end{pmatrix}
\tag{8.14}
$$

$$\begin{pmatrix} 1 & 1 & 0 & 0 & a_4 & a_5 & a_6 & 0 & a_7 & a_1 & a_2 & 0 & 0 & 0 & 0 & a_3 \\ 0 & 1 & 1 & 0 & 0 & a_4 & a_5 & 0 & 0 & a_7 & a_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & a_4 & 0 & 0 & 0 & a_7 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & -a_4 & 0 & 0 & 0 & -a_7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & a_4 & a_5 & a_6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & a_4 & a_5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & a_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -a_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{8.15}$$

$$\begin{pmatrix} 1 & 1 & 0 & 0 & a_4 & a_5 & a_6 & 0 & a_7 & a_1 & a_2 & 0 & 0 & 0 & 0 & a_3 \\ 0 & 1 & 1 & 0 & 0 & a_4 & a_5 & 0 & 0 & a_7 & a_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & a_4 & 0 & 0 & 0 & a_7 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & -a_4 & 0 & 0 & 0 & -a_7 & 0 & 0 & a_8 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & a_4 & a_5 & a_6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & a_4 & a_5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & a_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -a_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & a_8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \tag{8.16}$$

$$
\begin{pmatrix}
1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & a_1 \\
0 & 1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & a_2 \\
0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & a_3 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & a_3 - a_2 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & -a_3 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & a_3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\tag{8.17}
$$

$$
\begin{pmatrix}
1 & 1 & 0 & 0 & -1 & -1 & a_3 & 0 & a_4 & a_4 - a_3 & a_1 & 0 & 0 & 0 & 0 & a_2 \\
0 & 1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & a_4 & a_5 & 0 & 0 & 0 & 0 & a_6 \\
0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & a_4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -a_4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & -1 & a_4 - a_5 & 0 & 0 & 0 & 0 & -a_6 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\tag{8.18}
$$

$$\begin{pmatrix}
1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & a_1 \\
0 & 1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & a_2 \\
0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & a_3 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & a_3 - a_2 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & -a_3 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & a_3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix} \tag{8.19}$$

$$\begin{pmatrix}
1 & 1 & 0 & 0 & -1 & -1 & a_3 & 0 & a_4 & a_4 - a_3 & a_1 & 0 & 0 & 0 & 0 & a_2 \\
0 & 1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & a_4 & a_5 & 0 & 0 & 0 & 0 & a_6 \\
0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & a_4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -a_4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & -1 & a_4 - a_5 & 0 & 0 & 0 & 0 & -a_6 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix} \tag{8.20}$$

$$\begin{pmatrix}
1 & 1 & 0 & 0 & -1 & -1 & a_4 & 0 & a_5 & a_5-a_4 & a_1 & 0 & 0 & 0 & 0 & a_2 \\
0 & 1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & a_5 & a_6 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & a_5 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -a_5 & 0 & 0 & a_3 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & -1 & a_5-a_6 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & a_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix} \tag{8.21}$$

Conjecture 8.2.1. The only invertible upper-triangular solutions to the d-dimensional bYBE are scalar multiples of the identity matrix.

Lemma 8.2.1. The diagonal elements of an invertible upper-triangular solution to the d-dimensional bYBE are all equal.

Proof. Let R be an invertible upper triangular matrix. The upper triangular assumption corresponds to the condition that $R_{ab}^{cd} = 0$ whenever $ab \prec cd$, where $\prec$ denotes lexicographic ordering. Since R is invertible the diagonal elements must be nonzero, the diagonal elements are indexed by $R_{ab}^{ab} \neq 0$. The bYBE can be written in Einstein notation as (see appendix A.2):

$$R_{j_2 j_1}^{k_1 k_2} R_{j_3 k_1}^{l_1 k_3} R_{k_3 k_2}^{l_2 l_3} = R_{j_3 j_2}^{k_2 k_3} R_{k_3 j_1}^{k_1 l_3} R_{k_2 k_1}^{l_1 l_2} \tag{8.22}$$

Consider two of the diagonal elements of R given by R^{ab}_{ab} and R^{cd}_{cd}. We will show that $R^{ab}_{ab} = R^{cd}_{cd}$ by showing that each of the following are satisfied:

$$R^{ab}_{ab} = R^{ca}_{ca} = R^{dc}_{dc} = R^{cd}_{cd} \tag{8.23}$$

Consider the polynomial appearing in the bYBE indexed by $j_1 = b$, $j_2 = a$, $j_3 = c$, $l_1 = c$, $l_2 = a$, $l_3 = b$:

$$R^{k_1k_2}_{ab} R^{ck_3}_{ck_1} R^{ab}_{k_3k_2} = R^{k_2k_3}_{ca} R^{k_1b}_{k_3b} R^{ca}_{k_2k_1} \tag{8.24}$$

For the left hand side of equation 8.24 not to vanish we need each term to be nonzero. In order for $R^{k_1k_2}_{ab}$ not to vanish we need $k_1k_2 \prec ab$, meaning $k_1 \leq a$. In order for $R^{ck_3}_{ck_1}$ not to vanish we need $ck_3 \prec ck_1$, meaning $k_3 \leq k_1$. In order for $R^{ab}_{k_3k_2}$ not to vanish we need $ab \prec k_3k_2$, meaning $a \leq k_3$. Therefore we need $a \leq k_3 \leq k_1 \leq a$, so $k_1 = k_3 = a$. The left hand side then becomes:

$$R^{ak_2}_{ab} R^{ca}_{ca} R^{ab}_{ak_2} \tag{8.25}$$

This will only be nonzero if $b \leq k_2 \leq b$ so $k_2 = b$, making the left hand side equal to:

$$R^{ab}_{ab} R^{ca}_{ca} R^{ab}_{ab} \tag{8.26}$$

For the right hand side of equation 8.24 not to vanish we need each term of $R^{k_2k_3}_{ca} R^{k_1b}_{k_3b} R^{ca}_{k_2k_1}$ to be nonzero. In order for $R^{k_2k_3}_{ca}$ not to vanish we need $k_2k_3 \prec ca$, meaning $k_2 \leq c$. In order for $R^{k_1b}_{k_3b}$ not to vanish we need $k_1b \prec k_3b$, meaning $k_3 \leq k_1$. In order for $R^{ca}_{k_2k_1}$ not to vanish we need $ca \prec k_2k_1$, meaning $c \leq k_2$. Therefore $k_2 = c$

which forces $a \leq k_1 \leq k_3 \leq a$. So the right hand side becomes:

$$R^{ca}_{ca} R^{ab}_{ab} R^{ca}_{ca} \tag{8.27}$$

The full equation is therefore:

$$R^{ab}_{ab} R^{ca}_{ca} R^{ab}_{ab} - R^{ca}_{ca} R^{ab}_{ab} R^{ca}_{ca} = 0 \tag{8.28}$$

Which only has the nonzero solution $R^{ab}_{ab} = R^{ca}_{ca}$. The above argument can be repeated to show

$$R^{ab}_{ab} = R^{ca}_{ca} = R^{dc}_{dc} = R^{cd}_{cd} \tag{8.29}$$

□

Chapter 9

Future directions

> "The future ain't what it used to be."
>
> Yogi Berra

What follows are some questions for further research that build off of the results in this dissertation.

1. Any solution to the d-dimensional bYBE gives rise to a knot and link invariant [81], what invariants arise from the solutions we found?

2. In section 6 only some of the solutions were classified into unitary solutions, which of the others can be made unitary?

3. In section 6 many of the solutions have quite complicated parametrizations, is there a simpler way to represent these? Is there a generating function for the X-shaped solutions in a given dimension?

4. Can the proof that there are no X-shaped solutions to the d-dimensional bYBE when d is odd be extended to prove conjecture 6.1.1, that there are also no X-shaped solutions to the odd dimensional gYBEs?

5. In section 7 the underlying vector space is assumed to be finite, can the results be extended to the infinite setting?

6. What kind of algebraic structures arise from the permutation solutions found in section 7? For example, which skew braces (defined in section 4) give rise to the solutions in section 7?

7. Can the techniques used in section 7 be extended to the gYBEs?

8. Can the techniques used in section 7 be adapted to find square-free, degenerate, or involutive solutions?

9. Is there a basis free approach to our conjecture 8.2.1 that there are only diagonal invertible upper triangular solutions to the d-dimensional bYBE.

10. The X-shaped and upper triangular ansatzes produce rich sets of solutions, are there other ansatzes that can be classified using algorithm 1 and produce unitary solutions?

11. With the impending implementation of quantum computing technologies can a quantum algorithm be developed to compute X-shaped or upper triangular solutions in higher dimensions? As a short term goal is there a NISQ (Noisy intermediate-scale quantum) algorithm that can be implemented with current quantum technology?

12. There is a generalization of the aYBEs, referred to as Zamolodchikov's Tetrahedron Equations [36], do any of the methods in this dissertation apply to these equations?

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Appendix A

Selected additional proofs

> "Your appendix needs to be removed"
>
> Dr. Rippey to the author in 2011

A.1 Proof of symmetries

What follows is a proof of the symmetries listed in proposition 4.7.1.

Proof. Let R be a solution to the (d, m, l)-gYBE and let λ be a nonzero scalar, and Q a non-singular $d \times d$ matrix.

1.

$$(\lambda R \otimes I_V^{\otimes l})(I_V^{\otimes l} \otimes \lambda R)(\lambda R \otimes I_V^{\otimes l}) = (I_V^{\otimes l} \otimes \lambda R)(\lambda R \otimes I_V^{\otimes l})(I_V^{\otimes l} \otimes \lambda R)$$

$$\lambda^3(R \otimes I_V^{\otimes l})(I_V^{\otimes l} \otimes R)(R \otimes I_V^{\otimes l}) = \lambda^3(I_V^{\otimes l} \otimes R)(R \otimes I_V^{\otimes l})(I_V^{\otimes l} \otimes R)$$

$$(R \otimes I_V^{\otimes l})(I_V^{\otimes l} \otimes R)(R \otimes I_V^{\otimes l}) = (I_V^{\otimes l} \otimes R)(R \otimes I_V^{\otimes l})(I_V^{\otimes l} \otimes R)$$

2.

$$(R^{-1} \otimes I_V^{\otimes l})(I_V^{\otimes l} \otimes R^{-1})(R^{-1} \otimes I_V^{\otimes l}) = (I_V^{\otimes l} \otimes R^{-1})(R^{-1} \otimes I_V^{\otimes l})(I_V^{\otimes l} \otimes R^{-1})$$

$$(R \otimes I_V^{\otimes l})^{-1}(I_V^{\otimes l} \otimes R)^{-1}(R \otimes I_V^{\otimes l})^{-1} = (I_V^{\otimes l} \otimes R)^{-1}(R \otimes I_V^{\otimes l})^{-1}(I_V^{\otimes l} \otimes R)^{-1}$$

$$((R \otimes I_V^{\otimes l})(I_V^{\otimes l} \otimes R)(R \otimes I_V^{\otimes l}))^{-1} = ((I_V^{\otimes l} \otimes R)(R \otimes I_V^{\otimes l})(I_V^{\otimes l} \otimes R))^{-1}$$

$$(R \otimes I_V^{\otimes l})(I_V^{\otimes l} \otimes R)(R \otimes I_V^{\otimes l}) = (I_V^{\otimes l} \otimes R)(R \otimes I_V^{\otimes l})(I_V^{\otimes l} \otimes R)$$

3. The same as 2 with R^{-1} replaced by R^*

4. The same as 2 with R^{-1} replaced by R^T

5. To limit the need for parenthesis, denote $\boldsymbol{Q} = Q^{-1}$ and consider the left hand side of the gYBE:

$$(Q^{\otimes m} R \boldsymbol{Q}^{\otimes m} \otimes I_V^{\otimes l})(I_V^{\otimes l} \otimes Q^{\otimes m} R \boldsymbol{Q}^{\otimes m})(Q^{\otimes m} R \boldsymbol{Q}^{\otimes m} \otimes I_V^{\otimes l}) \tag{A.1}$$

$$=(Q^{\otimes m} R \otimes I_V^{\otimes l})(\boldsymbol{Q}^{\otimes m} \otimes I_V^{\otimes l})(I_V^{\otimes l} \otimes Q^{\otimes m})(I_V^{\otimes l} \otimes R \boldsymbol{Q}^{\otimes m})(Q^{\otimes m} R \boldsymbol{Q}^{\otimes m} \otimes I_V^{\otimes l}) \tag{A.2}$$

(A.3)

To simplify further, we look at the term $(\boldsymbol{Q}^{\otimes m} \otimes I_V^{\otimes l})(I_V^{\otimes l} \otimes Q^{\otimes m})$ there are then two cases to consider. In the **first case** $m > l$ and in the **second case** $m \leq l$. In the **first case** we can write:

$$(\boldsymbol{Q}^{\otimes m} \otimes I_V^{\otimes l})(I_V^{\otimes l} \otimes Q^{\otimes m}) = (\boldsymbol{Q}^{\otimes l} \otimes I_V^{\otimes m-l} \otimes Q^{\otimes l})$$

And in the **second case**, we can write:

$$(\boldsymbol{Q}^{\otimes m} \otimes I_V^{\otimes l})(I_V^{\otimes l} \otimes Q^{\otimes m}) = (\boldsymbol{Q}^{\otimes m} \otimes I_V^{\otimes l-m} \otimes Q^{\otimes m})$$

Substituting the **first case** into equation A.1 above we get:

$$
\begin{aligned}
&(Q^{\otimes m}R\otimes I_V^{\otimes l})(\boldsymbol{Q}^{\otimes l}\otimes I_V^{\otimes m-l}\otimes Q^{\otimes l})(I_V^{\otimes l}\otimes R\boldsymbol{Q}^{\otimes m})(Q^{\otimes m}R\boldsymbol{Q}^{\otimes m}\otimes I_V^{\otimes l})\\
=&(Q^{\otimes m}R\otimes I_V^{\otimes l})(\boldsymbol{Q}^{\otimes l}\otimes (I_V^{\otimes m-l}\otimes Q^{\otimes l})R\boldsymbol{Q}^{\otimes m})(Q^{\otimes m}\otimes I_V^{\otimes l})(R\boldsymbol{Q}^{\otimes m}\otimes I_V^{\otimes l})\\
=&(Q^{\otimes m}R\otimes I_V^{\otimes l})(I_V^{\otimes l}\otimes (I_V^{\otimes m-l}\otimes Q^{\otimes l})R\boldsymbol{Q}^{\otimes m}(Q^{\otimes m-l}\otimes I_V^{\otimes l}))(R\boldsymbol{Q}^{\otimes m}\otimes I_V^{\otimes l})\\
=&(Q^{\otimes m}R\otimes I_V^{\otimes l})(I_V^{\otimes l}\otimes (I_V^{\otimes m-l}\otimes Q^{\otimes l})R(I_V^{\otimes m-l}\otimes \boldsymbol{Q}^{\otimes l}))(R\boldsymbol{Q}^{\otimes m}\otimes I_V^{\otimes l})\\
=&(Q^{\otimes m}R\otimes I_V^{\otimes l})(I_V^{\otimes l}\otimes I_V^{\otimes m-l}\otimes Q^{\otimes l})(I_V^{\otimes l}\otimes R)(I_V^{\otimes l}\otimes I_V^{\otimes m-l}\otimes \boldsymbol{Q}^{\otimes l})(R\boldsymbol{Q}^{\otimes m}\otimes I_V^{\otimes l})\\
=&(Q^{\otimes m}R\otimes I_V^{\otimes l})(I_V^{\otimes m}\otimes Q^{\otimes l})(I_V^{\otimes l}\otimes R)(I_V^{\otimes m}\otimes \boldsymbol{Q}^{\otimes l})(R\boldsymbol{Q}^{\otimes m}\otimes I_V^{\otimes l})\\
=&(Q^{\otimes m}R\otimes Q^{\otimes l})(I_V^{\otimes l}\otimes R)(R\boldsymbol{Q}^{\otimes m}\otimes \boldsymbol{Q}^{\otimes l})\\
=&(Q^{\otimes m}\otimes Q^{\otimes l})(R\otimes I_V^{\otimes l})(I_V^{\otimes l}\otimes R)(R\otimes I_V^{\otimes l})(\boldsymbol{Q}^{\otimes m}\otimes \boldsymbol{Q}^{\otimes l})\\
=&Q^{\otimes m+l}(R\otimes I_V^{\otimes l})(I_V^{\otimes l}\otimes R)(R\otimes I_V^{\otimes l})\boldsymbol{Q}^{\otimes m+l}
\end{aligned}
$$

By a similar argument, the right hand side of the gYBE can be simplified to:

$$
Q^{\otimes m+l}(I_V^{\otimes l}\otimes R)(R\otimes I_V^{\otimes l})(I_V^{\otimes l}\otimes R)\boldsymbol{Q}^{\otimes m+l}
$$

Conjugating by $Q^{\otimes m}$, we get the gYBE. We can similarly handle the **second case**, when $m \leq l$, by substituting into equation A.1:

$$
\begin{aligned}
&(Q^{\otimes m}R\otimes I_V^{\otimes l})(\boldsymbol{Q}^{\otimes m}\otimes I_V^{\otimes l-m}\otimes Q^{\otimes m})(I_V^{\otimes l}\otimes R\boldsymbol{Q}^{\otimes m})(Q^{\otimes m}R\boldsymbol{Q}^{\otimes m}\otimes I_V^{\otimes l})\\
=&(Q^{\otimes m}R\otimes I_V^{\otimes l})(\boldsymbol{Q}^{\otimes m}\otimes I_V^{\otimes l-m}\otimes Q^{\otimes m}R\boldsymbol{Q}^{\otimes m})(Q^{\otimes m}\otimes I_V^{\otimes l})(R\boldsymbol{Q}^{\otimes m}\otimes I_V^{\otimes l})\\
=&(Q^{\otimes m}R\otimes I_V^{\otimes l})(I_V^{\otimes m}\otimes I_V^{\otimes l-m}\otimes Q^{\otimes m}R\boldsymbol{Q}^{\otimes m})(R\boldsymbol{Q}^{\otimes m}\otimes I_V^{\otimes l})
\end{aligned}
$$

$$=(Q^{\otimes m}R\otimes I_V^{\otimes l})(I_V^{\otimes m}\otimes I_V^{\otimes l-m}\otimes Q^{\otimes m})(I_V^{\otimes m}\otimes I_V^{\otimes l-m}R)$$
$$(I_V^{\otimes m}\otimes I_V^{\otimes l-m}\otimes \boldsymbol{Q}^{\otimes m})(R\boldsymbol{Q}^{\otimes m}\otimes I_V^{\otimes l})$$
$$=(Q^{\otimes m}R\otimes I_V^{\otimes l-m}\otimes Q^{\otimes m})(I_V^{\otimes l}\otimes R)(R\boldsymbol{Q}^{\otimes m}\otimes I_V^{\otimes l-m}\otimes \boldsymbol{Q}^{\otimes m})$$
$$=(Q^{\otimes m}\otimes I_V^{\otimes l-m}\otimes Q^{\otimes m})(R\otimes I_V^{\otimes l})(I_V^{\otimes l}\otimes R)(R\otimes I_V^{\otimes l})(\boldsymbol{Q}^{\otimes m}\otimes I_V^{\otimes l-m}\otimes \boldsymbol{Q}^{\otimes m})$$

And the right hand side of the gYBE can be manipulated to:

$$(Q^{\otimes m}\otimes I_V^{\otimes l-m}\otimes Q^{\otimes m})(I_V^{\otimes l}\otimes R)(R\otimes I_V^{\otimes l})(I_V^{\otimes l}\otimes R)(\boldsymbol{Q}^{\otimes m}\otimes I_V^{\otimes l-m}\otimes \boldsymbol{Q}^{\otimes m})$$

Therefore, if R is a solution to the gYBE so is $Q^{\otimes m}R\boldsymbol{Q}^{\otimes m}$.

6. Suppose R is a solution to the bYBE. Since R^{-1} is also a solution to the bYBE we have that $R^{-1}P$ is a solution to the aYBE. It follows that $(R^{-1}P)^{-1} = PR$ is a solution to the aYBE and therefore PRP is a solution to the bYBE.

□

A.2 Forms of the aYBE and bYBE

Here we show how to obtain the form of the aYBE in equation 4.28 from equation 4.27. In equation 4.27, R_{ab} acts on the factors a and b, and does not affect the third factor. For example R_{13} does not affect the middle factor:

$$R_{13}(e_i\otimes e_j\otimes e_k) = \sum_{ab} R_{ik}^{ab}(e_a\otimes e_j\otimes e_b)$$

The left hand side of equation 4.27 acts on $(e_{j_1}\otimes e_{j_2}\otimes e_{j_3})$ as follows:

$$
\begin{aligned}
R_{12}R_{13}R_{23}(e_{j_1}\otimes e_{j_2}\otimes e_{j_3}) &= R_{12}R_{13}\sum_{k_2,k_3} R^{k_2k_3}_{j_2j_3}(e_{j_1}\otimes e_{k_2}\otimes e_{k_3})\\
&= R_{12}\sum_{k_2,k_3,k_1,l_3} R^{k_2k_3}_{j_2j_3}R^{k_1l_3}_{j_1k_3}(e_{k_1}\otimes e_{k_2}\otimes e_{l_3})\\
&= \sum_{k_2,k_3,k_1,l_3,l_1,l_2} R^{k_2k_3}_{j_2j_3}R^{k_1l_3}_{j_1k_3}R^{l_1l_2}_{k_1k_2}(e_{l_1}\otimes e_{l_2}\otimes e_{l_3})
\end{aligned}
$$

The action of the right hand side of equation 4.27 can similarly be written:

$$
\begin{aligned}
R_{23}R_{13}R_{12}(e_{j_1}\otimes e_{j_2}\otimes e_{j_3}) &= R_{23}R_{13}\sum_{k_1,k_2} R^{k_1k_2}_{j_1j_2}(e_{k_1}\otimes e_{k_2}\otimes e_{j_3})\\
&= R_{23}\sum_{k_1,k_2,l_1,k_3} R^{k_1k_2}_{j_2j_2}R^{l_1k_3}_{k_1j_3}(e_{l_1}\otimes e_{k_2}\otimes e_{k_3})\\
&= \sum_{k_1,k_2,l_1,k_3,l_2,l_3} R^{k_1k_2}_{j_1j_2}R^{l_1k_3}_{k_1j_3}R^{l_2l_3}_{k_2k_3}(e_{l_1}\otimes e_{l_2}\otimes e_{l_3})
\end{aligned}
$$

Using the Einstein notation convention of summing over repeated indices we can write equation 4.27 as:

$$
R^{k_1k_2}_{j_1j_2}R^{l_1k_3}_{k_1j_3}R^{l_2l_3}_{k_2k_3} = R^{k_2k_3}_{j_2j_3}R^{k_1l_3}_{j_1k_3}R^{l_1l_2}_{k_1k_2} \tag{A.4}
$$

Similarly the bYBE in equation 4.3 can be converted into Einstein summation notation by using the fact that RP must solve the bYBE if R solves the aYBE. We then have

$$RP(e_i \otimes e_j) = R(e_j \otimes e_i) = \sum_{k,l} R^{kl}_{ji}(e_k \otimes e_l)$$

Therefore the bYBE can be written be exchanging the bottom two indicies in the equation for the aYBE:

$$R^{k_1k_2}_{j_2j_1} R^{l_1k_3}_{j_3k_1} R^{l_2l_3}_{k_3k_2} = R^{k_2k_3}_{j_3j_2} R^{k_1l_3}_{k_3j_1} R^{l_1l_2}_{k_2k_1} \tag{A.5}$$