Discretization of the Hellinger-Reissner Variational Form of Linear Elasticity Equations

Kevin Sweet

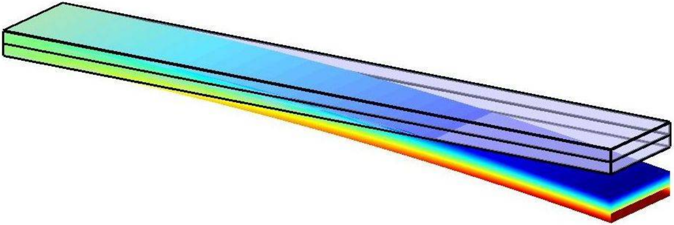
2019 PSU REU Computational Modeling Serving the City

sweetkev@pdx.edu

23 August 2019

Linear Elasticity

- Used to describe the deformation of solid objects
- Linear elasticity represents small, reversible deformations
- Linear plasticity represents small, irreversible deformations
- Application to geological phenomena (e.g. landslides)



Strong Form of Equations

Strong Form

Find a displacement vector $\vec{u} : \Omega \to \mathbb{R}^3$ and symmetric stress tensor $\sigma : \Omega \to \mathbb{R}^{3x3}_{sym}$ such that

$$\begin{cases} \mathcal{A}\sigma = \epsilon(\vec{u}) \text{ in } \Omega \\ -(\nabla \cdot \sigma) = f \text{ in } \Omega \end{cases}$$

$$\epsilon(\vec{u}) = \frac{1}{2}\left(\nabla \vec{u} + (\nabla \vec{u})^T\right)$$

$$\mathcal{A}\sigma = \frac{1}{2\mu}\left(\sigma - \frac{\lambda}{3\lambda + 2\mu}\mathrm{tr}(\sigma)\mathcal{I}\right)$$

Hellinger-Reissner Variational Form

Find $\vec{u} \in L^2(\Omega; \mathbb{R}^3)$ and $\sigma \in H(\mathrm{div}, \Omega; \mathbb{R}^{3\times 3}_{\text{sym}})$ such that

$$\begin{cases} a(\sigma, \tau) + b(\tau, \vec{u}) = 0 \\ -b(\sigma, \vec{v}) = \int_\Omega f \cdot \vec{v}\, dx \end{cases}$$

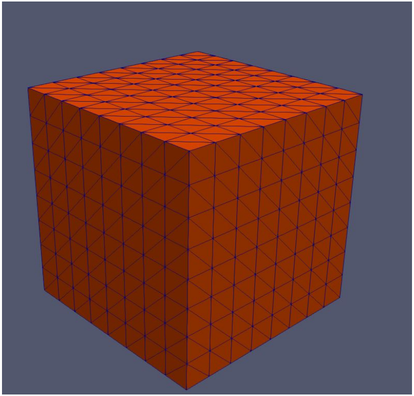
for all $\vec{v} \in L^2(\Omega; \mathbb{R}^3)$ and $\tau \in H(\mathrm{div}, \Omega; \mathbb{R}^{3\times 3}_{\text{sym}})$

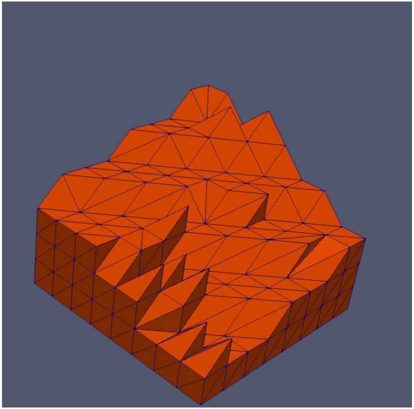
$$a(\sigma, \tau) = \int_\Omega \mathcal{A}\sigma : \tau \; dx \quad , \quad b(\tau, \vec{v}) = \int_\Omega (\nabla \cdot \tau) \cdot \vec{v}\, dx$$

$$L^2(\Omega; \mathbb{R}^3) = \left\{ \vec{v} : \Omega \to \mathbb{R}^3 : \int_\Omega \vec{v} \cdot \vec{v}\, dx < \infty \right\} \; ,$$

$$H(\mathrm{div}, \Omega; \mathbb{R}^{3\times 3}_{\text{sym}}) = \left\{ \tau : \Omega \to \mathbb{R}^{3\times 3}_{\text{sym}} : \int_\Omega \tau : \tau \; dx < \infty \text{ and } \int_\Omega (\nabla \cdot \tau) \cdot (\nabla \cdot \tau)\; dx < \infty \right\}$$

Creating a Mesh





Discretizing the Hellinger-Reissner Form

Discretized Hellinger-Reissner Form

Find $(\sigma_h, \vec{u}_h) \in \Sigma_h \times V_h$ such that

$$\begin{cases} a(\sigma_h, \tau_h) + b(\tau_h, \vec{u}_h) = 0 \\ -b(\sigma_h, \vec{v}_h) + c(\vec{u}_h, \vec{v}_h) = \int_\Omega f \cdot \vec{v}_h \, dx \end{cases}$$

for all $\tau_h \in \Sigma_h$ and $\vec{v}_h \in V_h$.

The jump stablization term c is defined as

$$c(\vec{u}_h, \vec{v}_h) = \sum_{F \in \mathcal{F}} h_F \int_F [\![\vec{u}_h]\!] : [\![\vec{v}_h]\!]$$

Defining $[\![\vec{w}]\!]$ and Vector Spaces

For a face on the boundary of Ω

$$[\![\vec{w}]\!] := \frac{1}{2}\left(\vec{w}\vec{n}^T + \vec{n}\vec{w}^T\right)$$

For an edge that is on the interior of the mesh

$$[\![\vec{w}]\!] := \frac{1}{2}\left(\vec{w}_+(\vec{n}_+)^T + \vec{n}_+(\vec{w}_+)^T + \vec{w}_-(\vec{n}_-)^T + \vec{n}_-(\vec{w}_-)^T\right)$$

Vector Spaces

$$\Sigma_h \subset H(\mathrm{div}, \Omega; \mathbb{R}^{3\times 3}_{\mathrm{sym}}) \quad , \quad V_h \subset L^2(\Omega; \mathbb{R}^3)$$

$$\Sigma_h = \{\tau \in C(\Omega; \mathbb{R}^{3\times 3}_{\mathrm{sym}}) : \tau\big|_K \in \mathbb{P}_1(K; \mathbb{R}^{3\times 3}_{\mathrm{sym}}) \text{ for all } K \in \mathcal{T}\}$$
$$V_h = \{\vec{v} : \Omega \to \mathbb{R}^3 : \vec{v}\big|_K \in \mathbb{R}^3 \text{ for all } K \in \mathcal{T}\}$$

A Basis of V_h and the Definition of Σ_h

Basis of V_h

$$V_h = \mathrm{span}\{\phi_{K,j} : K \in \mathcal{T} \text{ and } 1 \leq j \leq 3\} \quad , \quad \phi_{K,j}(x) = \begin{cases} \mathbf{e}_j, & x \in K \\ \mathbf{0}, & x \notin K \end{cases}$$

Global numbering $\phi_{K,j} \leftrightarrow \phi_i$, where $1 \leq i = 3(K-1) + j \leq 3NT$

Spaces in Definition of Σ_h

$$\Sigma_h = \{\tau \in C(\Omega; \mathbb{R}^{3\times3}_{\text{sym}}) : \tau\big|_K \in \mathbb{P}_1(K; \mathbb{R}^{3\times3}_{\text{sym}}) \text{ for all } K \in \mathcal{T}\}$$

$$C(\Omega; \mathbb{R}^{3\times3}_{\text{sym}}) = \{\alpha : \Omega \to \mathbb{R}^{3\times3}_{\text{sym}} : \alpha_{ij} \in C(\Omega; \mathbb{R})\}$$

$$\mathbb{P}_1(K; \mathbb{R}^{3\times3}_{\text{sym}}) = \{\alpha : K \to \mathbb{R}^{3\times3}_{\text{sym}} : \alpha_{ij} \in \mathbb{P}_1(K; \mathbb{R})\}$$

A Basis of Σ_h

A basis of the space $\mathbb{R}^{3\times 3}_{\text{sym}}$ is given by

$$S_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ S_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ S_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$S_4 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ S_5 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},\ S_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

For each vertex z, we define a continuous, piecewise linear function $\ell_z : \Omega \to \mathbb{R}$. On a tetrahedron K having vertices z_1, z_2, z_3, z_4, the associated linear functions are defined locally by the relations

$$\begin{cases} \ell_{z_1}(x)\, z_1 + \ell_{z_2}(x)\, z_2 + \ell_{z_3}(x)\, z_3 + \ell_{z_4}(x)\, z_4 = x \\ \ell_{z_1}(x) + \ell_{z_2}(x) + \ell_{z_3}(x) + \ell_{z_4}(x) = 1 \end{cases} \qquad \text{for all } x \in K$$

A Basis of Σ_h, Representing Functions in V_h and Σ_h

Basis of Σ_h

$$\Sigma_h = \mathrm{span}\{\psi_i = \ell_z S_m : \ i = 6(z-1) + m\,,\ 1 \le z \le NV\,,\ 1 \le m \le 6\}$$

Representation by Coefficient Vectors Knowing the basis functions of V_h and Σ_h allows $\vec{u}_h$ and σ_h to be rewritten as

$$\sigma_h = \sum_{j=1}^{6NV} x_j \psi_j \in \Sigma_h \leftrightarrow \mathbf{x} \in \mathbb{R}^{6NV}$$

$$\vec{u}_h = \sum_{j=1}^{3NT} y_j \phi_j \in V_h \leftrightarrow \mathbf{y} \in \mathbb{R}^{3NT}$$

Devloping the Linear System

Linear System

Find $\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \in \mathbb{R}^{6NV+3NT}$ such that

$$\begin{pmatrix} A & B^T \\ -B & C \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{f} \end{pmatrix}$$

where $A \in \mathbb{R}^{6NVx6NV}$, $B \in \mathbb{R}^{3NTx6NV}$, $C \in \mathbb{R}^{3NTx3NT}$ are given by

$$\begin{aligned} a_{ij} &= a(\psi_j, \psi_i),\ 1 \le i, j \le 6NV \\ b_{ij} &= b(\psi_j, \phi_i),\ 1 \le i \le 3NT\ \ 1 \le j \le 6NV \\ c_{ij} &= c(\phi_j, \phi_i),\ 1 \le i, j \le 3NT \end{aligned}$$

and $\mathbf{f} \in \mathbb{R}^{3NT}$ is given by $\mathbf{f}_k = \int_\Omega f \cdot \phi_k \, dx,\ 1 \le k \le 3NT$

Example of Compressed Sparse Row (CSR) Format

$$\begin{pmatrix} 0 & 8 & 2 & 0 \\ 1 & 0 & 5 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 8 & 7 \end{pmatrix} \qquad \begin{aligned} \text{values} &= \begin{pmatrix} 8 & 2 & 1 & 5 & 2 & 8 & 7 \end{pmatrix} \\ \text{columns} &= \begin{pmatrix} 2 & 3 & 1 & 3 & 4 & 3 & 4 \end{pmatrix} \\ \text{row index} &= \begin{pmatrix} 1 & 3 & 5 & 6 & 8 \end{pmatrix} \end{aligned}$$

- Non-zero entries listed (in order) row by row in `values` array
- Sparsity structure, i.e. number and column position of nonzeros in each row, encoded in `row index` and `columns` arrays

Total Number of Non-Zero Entries *nnz* in Our Matrix

$$\begin{pmatrix} A & B^T \\ -B & C \end{pmatrix} \quad , \qquad nnz = nnz_A + 2nnz_B + nnz_C$$

Computing the Linear System: Matrix A

$$a_{6(i-1)+m,6(j-1)+n} = a(\ell_i S_m, \ell_j S_n) = \int_K \mathcal{A}(\ell_i S_m) : (\ell_j S_n)\, dx \quad 1 \le i,j \le 4 \quad 1 \le m,n \le 6$$

$$\int_K \mathcal{A}(\ell_i S_m) : (\ell_j S_n)\, dx = \frac{1}{2\mu}\left(S_m : S_n - \frac{\lambda}{3\lambda + 2\mu}\mathrm{tr}(S_m)\mathcal{I} : S_n\right)\int_K \ell_i \ell_j\, dx$$

$$\frac{1}{2\mu}\left(S_m : S_n - \frac{\lambda}{3\lambda + 2\mu}\mathrm{tr}(S_m)\mathcal{I} : S_n\right) = \begin{cases} \frac{\lambda+\mu}{\mu(3\lambda+2\mu)}, & 1 \le m,n \le 3 \text{ and } m = n \\ -\frac{\lambda}{2\mu(3\lambda+2\mu)}, & 1 \le m,n \le 3 \text{ and } m \ne n \\ \frac{1}{\mu}, & 4 \le m,n \le 6 \text{ and } m = n \\ 0, & \text{else} \end{cases}$$

$$\int_K \ell_i \ell_j = \begin{cases} \frac{|K|}{10}, & i = j \\ \frac{|K|}{20}, & i \ne j \end{cases}$$

Computing the Linear System: Matrix A Cont.

$$\frac{|K|}{20}\begin{pmatrix}2 & 1 & 1 & 1\\ 1 & 2 & 1 & 1\\ 1 & 1 & 2 & 1\\ 1 & 1 & 1 & 2\end{pmatrix} \otimes \begin{pmatrix}u & v & v & 0 & 0 & 0\\ v & u & v & 0 & 0 & 0\\ v & v & u & 0 & 0 & 0\\ 0 & 0 & 0 & w & 0 & 0\\ 0 & 0 & 0 & 0 & w & 0\\ 0 & 0 & 0 & 0 & 0 & w\end{pmatrix}$$

$$u = \frac{\lambda + \mu}{\mu(3\lambda + 2\mu)} \quad v = -\frac{\lambda}{2\mu(3\lambda + 2\mu)} \quad w = \frac{1}{\mu}$$

$$nnz_A = \sum_{z=1}^{NV} 12 v_z$$

where v_z is the number of vertices that interact with vertex z.

Computing the Linear System: Matrix B

$$b_{i,6(m-1)+j} = b(\ell_j S_m, \phi_i) = \int_\Omega (\nabla \cdot \ell_j S_m) \cdot \phi_i \, dx \quad 1 \le i \le 3, \quad 1 \le j \le 4, \quad 1 \le m \le 6$$

$$\int_\Omega (\nabla \cdot \ell_j S_m) \cdot \phi_i \, dx = |K| (S_m \nabla \ell_j) \cdot \phi_i$$

$$|K| \begin{pmatrix} \frac{\partial \ell_j}{\partial x} & 0 & 0 & \frac{\partial \ell_j}{\partial y} & \frac{\partial \ell_j}{\partial z} & 0 \\ 0 & \frac{\partial \ell_j}{\partial y} & 0 & \frac{\partial \ell_j}{\partial x} & 0 & \frac{\partial \ell_j}{\partial z} \\ 0 & 0 & \frac{\partial \ell_j}{\partial z} & 0 & \frac{\partial \ell_j}{\partial x} & \frac{\partial \ell_j}{\partial y} \end{pmatrix}$$

$$nnz_B = \sum_{K=1}^{NT} 3(3)(4) = \sum_{K=1}^{NT} 36 = 36NT$$

Computing the Linear System: Matrix C

$$[\![\phi_{K,i}]\!] : [\![\phi_{K,j}]\!] = \begin{cases} \frac{1}{2}(\delta_{ij} + (e_i^T n_+)(e_j^T n_+)), & i = j \\ \frac{1}{2}(e_i^T n_+)(e_j^T n_+), & i \neq j \end{cases}$$

$$[\![\phi_{K,i}]\!] : [\![\phi_{\widehat{K},j}]\!] = \begin{cases} -\frac{1}{2}(\delta_{ij} + (e_i^T n_+)(e_j^T n_+)), & i = j \\ -\frac{1}{2}(e_i^T n_+)(e_j^T n_+), & i \neq j \end{cases}$$

$$1 \leq K, \widehat{K} \leq NT \quad 1 \leq i, j \leq 3$$

$$nnz_C = \sum_{K=1}^{NT} 9t_K$$

where t_K is the number of tetrahedra that interact with tetrahedron K.

Code Example

```
subroutine CSR_arrays(this,vaptr,vertadj,tetadj,G_irw,G_col,G_val,rhs)
  !! Outputs:
  ! allocate G_irw,G_col,G_val,rhs
  !! Initialize and fill G_irw,G_col

  type(tetmesh), intent(in)                                   :: this
  integer,intent(in),  dimension(4,this%numTetrahedra)        :: tetadj
  integer,intent(in), allocatable                             :: vaptr(:)
  integer,intent(in), allocatable                             :: vertadj(:)
  integer,intent(out) , allocatable                           :: G_irw(:)
  integer,intent(out) , allocatable                           :: G_col(:)
  real(RealPrec),intent(out) , allocatable                    :: G_val(:)
  real(RealPrec),intent(out) , allocatable                    :: rhs(:)
  integer, dimension(6*this%numVertices)                      :: vert_tet_adj

  integer                                                     :: A_nnz,B_nnz,C_nnz,i,j,l,k,i0,ii,m,n,a
  integer, allocatable                                        :: vertsort(:)
  integer, allocatable                                        :: tetsort(:)
  integer, dimension(2,3)                                     :: index1 = [1,2,1,3,2,3]
  integer, dimension(3,3)                                     :: index2 = [1,4,5,2,4,6,3,5,6]


  !
  ! -------Number of non-zeros in the LHS-RHS matrix
  !
  A_nnz=(vaptr(this%numVertices+1)-1)*12
  B_nnz=this%numTetrahedra*36
  C_nnz=this%numTetrahedra*9
  do i = 1, this%numTetrahedra
     do j=1,4
        if (tetadj(j,i) .gt. 0 ) then
           C_nnz=C_nnz+9
        end if
     enddo
  enddo
  !
  ! -------allocate LHS-RHS CSR arrays
  !
  allocate ( G_val(A_nnz+B_nnz*2+C_nnz), G_col(A_nnz+B_nnz*2+C_nnz), G_irw((6*this%numVertices)+(3*this%numTetrahedra)+1))
  allocate (  rhs((6*this%numVertices)+(3*this%numTetrahedra)))
```

Discretization Error

The main theroem in [Chen, 2017] is the following:

Theorem

Let $(\sigma, \vec{u}) \in H(\mathrm{div}, \Omega; \mathbb{R}^{3\times3}_{sym}) \times L^2(\Omega; \mathbb{R}^3)$ *be the exact solution of the problem and* $(\sigma_h, \vec{u}_h) \in \Sigma_h \times V_h$ *the discrete solution of the stabilized mixed finite element method. If* $\sigma \in H^2(\Omega; \mathbb{R}^{3\times3}_{sym})$ *and* $\vec{u} \in H^1(\Omega; \mathbb{R}^3)$*, then*

$$\|\sigma - \sigma_h\|_{H(\mathrm{div},\mathcal{A})} + \|\vec{u} - \vec{u}_h\|_{0,c} \leq Ch(\|\sigma\|_2 + \|\vec{u}\|_1) \ ,$$

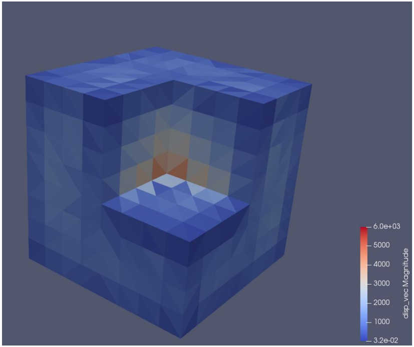
where C *is a constant that is independent of* h*,* σ *and* $\vec{u}$*.*

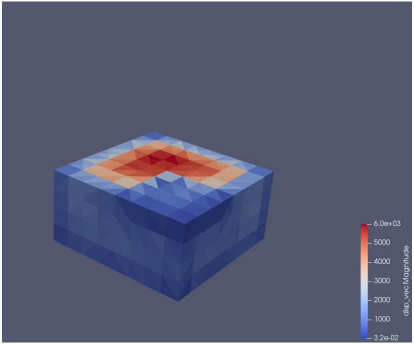
Error Comparison

Table: Convergence of approximations of the stress tensor and displacement vector from [Chen, 2017].

h	$\|\sigma - \sigma_h\|_{H(\mathrm{div}),\mathcal{A}}$	order	$\|\vec{u}_h\|_C$	order	$\|\vec{u} - \vec{u}_h\|_0$	order
2^{-1}	4.1723E+00	—	4.0747E-01	—	2.4720E-01	—
2^{-2}	2.3595E+00	0.82	3.5554E-01	0.20	1.7403E-01	0.51
2^{-3}	1.2849E+00	0.88	2.5527E-01	0.48	1.1168E-01	0.64
2^{-4}	6.8023E-01	0.92	1.5243E-01	0.74	6.3889E-02	0.81
2^{-5}	3.5167E-01	0.95	8.3310E-02	0.87	3.4309E-02	0.90

Calculated Displacement Vector





Acknowledgements

- My faculty mentor Dr. Jeff Ovall
- The REU Site is supported by the National Science Foundation under grant no. 1758006

Questions?

References

Chen, Long and Hu, Jun and Huang, Xuehai (2017)
Stabilized Mixed Finite Element Methods for Linear Elasticity on Simplicial Grids in $\mathbb{R}^n$

Fouts, Bram (2018)
Derivation of the Hellinger-Reissner Variational Form of the Linear Elasticity Equations, and a Finite Element Discretization