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The Multiplicities of a Dual-thin $Q$-polynomial Association Scheme

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Abstract
Let $Y = (X, \{R_i\}_{0 \leq i \leq D})$ denote a symmetric association scheme, and assume that $Y$ is $Q$-polynomial with respect to an ordering $E_0, ..., E_D$ of the primitive idempotents. Bannai and Ito conjectured that the associated sequence of multiplicities $m_i$ ($0 \leq i \leq D$) of $Y$ is unimodal. Talking to Terwilliger, Stanton made the related conjecture that $m_i \leq m_{i+1}$ and $m_i \leq m_{D-i}$ for $i < D/2$. We prove that if $Y$ is dual-thin in the sense of Terwilliger, then the Stanton conjecture is true.

1 Introduction
For a general introduction to association schemes, we refer to [1], [2], [5], or [9]. Our notation follows that found in [3].

Throughout this article, $Y = (X, \{R_i\}_{0 \leq i \leq D})$ will denote a symmetric, $D$-class association scheme. Our point of departure is the following well-known result of Taylor and Levingston.

1.1 Theorem. [7] If $Y$ is $P$-polynomial with respect to an ordering $R_0, ..., R_D$ of the associate classes, then the corresponding sequence of valencies $k_0, k_1, \ldots, k_D$
is unimodal. Furthermore,

\[ k_i \leq k_{i+1} \quad \text{and} \quad k_i \leq k_{D-i} \quad \text{for} \quad i < D/2. \]

Indeed, the sequence is log-concave, as is easily derived from the inequalities \( b_{i-1} \geq b_i \) and \( c_i \leq c_{i+1} \) \((0 < i < D)\), which are satisfied by the intersection numbers of any \(P\)-polynomial scheme (cf. [5, p. 199]).

In their book on association schemes, Bannai and Ito made the dual conjecture.

1.2 Conjecture. [1, p. 205] If \(Y\) is \(Q\)-polynomial with respect to an ordering \(E_0, ..., E_D\) of the primitive idempotents, then the corresponding sequence of multiplicities

\[ m_0, m_1, \ldots, m_D \]

is unimodal.

Bannai and Ito further remark that although unimodality of the multiplicities follows easily whenever the dual intersection numbers satisfy the inequalities \(b_{i-1}^* \geq b_i^*\) and \(c_i^* \leq c_{i+1}^*\) \((0 < i < D)\), unfortunately these inequalities do not always hold. For example, in the Johnson scheme \(J(k^2, k)\) we find that \(c_{k-1}^* > c_k^*\) whenever \(k > 3\).

Talking to Terwilliger, Stanton made the following related conjecture.

1.3 Conjecture. [8] If \(Y\) is \(Q\)-polynomial with respect to an ordering \(E_0, ..., E_D\) of the primitive idempotents, then the corresponding multiplicities satisfy

\[ m_i \leq m_{i+1} \quad \text{and} \quad m_i \leq m_{D-i} \quad \text{for} \quad i < D/2. \]

Our main result shows that under a suitable restriction on \(Y\), these last inequalities are satisfied.

To state our result more precisely, we first review a few definitions. Let \( \text{Mat}_X(\mathbb{C}) \) denote the \( \mathbb{C} \)-algebra of matrices with entries in \( \mathbb{C} \), where the rows and columns are indexed by \(X\), and let \( A_0, ..., A_D \) denote the associate matrices for \(Y\). Now fix any \(x \in X\), and for each integer \(i \) \((0 \leq i \leq D)\), let \(E_i^* = E_i(x)\) denote the diagonal matrix in \( \text{Mat}_X(\mathbb{C})\) with \(yy\) entry

\[
(E_i^*)_{yy} = \begin{cases} 
1 & \text{if } xy \in R_i, \\
0 & \text{if } xy \not\in R_i.
\end{cases} \quad (y \in X).
\]

The Terwilliger algebra for \(Y\) with respect to \(x\) is the subalgebra \(T = T(x)\) of \( \text{Mat}_X(\mathbb{C})\) generated by \(A_0, ..., A_D\) and \(E_0^*, ..., E_D^*\). The Terwilliger algebra was first introduced in [9] as an aid to the study of association schemes. For any \(x \in X\), \(T = T(x)\) is a finite dimensional, semisimple \( \mathbb{C} \)-algebra, and is noncommutative in general. We refer to [3] or [9] for more details. \(T\) acts faithfully on the vector space \(V := \mathbb{C}^X\) by matrix multiplication. \(V\) is endowed with the inner product \(\langle , \rangle\) defined by \(\langle u, v \rangle := u^*v\) for all \(u, v \in V\). Since \(T\) is semisimple, \(V\) decomposes into a direct sum of irreducible \(T\)-modules.

Let \(W\) denote an irreducible \(T\)-module. Observe that \(W = \sum E_i^*W\) (orthogonal direct sum), where the sum is taken over all the indices \(i \) \((0 \leq i \leq D)\) such that \(E_i^*W \neq 0\). We set

\[ d := |\{i : E_i^*W \neq 0\}| - 1, \]
and note that the dimension of $W$ is at least $d + 1$. We refer to $d$ as the diameter of $W$. The module $W$ is said to be thin whenever $\dim(E_i^* W) \leq 1$ ($0 \leq i \leq D$). Note that $W$ is thin if and only if the diameter of $W$ equals $\dim(W) - 1$. We say $Y$ is thin if every irreducible $T(x)$-module is thin for every vertex $x \in X$.

Similarly, note that $W = \sum E_i W$ (orthogonal direct sum), where the sum is over all $i$ ($0 \leq i \leq D$) such that $E_i W \neq 0$. We define the dual diameter of $W$ to be

$$d^* := |\{i : E_i^* W \neq 0\}| - 1,$$

and note that $\dim W \geq d^* + 1$. A dual thin module $W$ satisfies $\dim(E_i^* W) \leq 1$ ($0 \leq i \leq D$). So $W$ is dual thin if and only if $\dim(W) = d^* + 1$. Finally, $Y$ is dual thin if every irreducible $T(x)$-module is dual thin for every vertex $x \in X$.

Many of the known examples of $Q$-polynomial schemes are dual thin. (See [10] for a list.) Our main theorem is as follows.

1.4 Theorem. Let $Y$ denote a symmetric association scheme which is $Q$-polynomial with respect to an ordering $E_0, \ldots, E_D$ of the primitive idempotents. If $Y$ is dual-thin, then the multiplicities satisfy

$$m_i \leq m_{i+1} \quad \text{and} \quad m_i \leq m_{D-i} \quad \text{for} \ i < D/2.$$

The proof of Theorem 1.4 is contained in the next section.

We remark that if $Y$ is bipartite $P$- and $Q$-polynomial, then it must be dual-thin and $m_i = m_{D-i}$ for $i < D/2$. So Theorem 1.4 implies the following corollary. (cf. [4, Theorem 9.6]).

1.5 Corollary. Let $Y$ denote a symmetric association scheme which is bipartite $P$- and $Q$-polynomial with respect to an ordering $E_0, \ldots, E_D$ of the primitive idempotents. Then the corresponding sequence of multiplicities

$$m_0, m_1, \ldots, m_D$$

is unimodal.

1.6 Remark. By recent work of Ito, Tanabe, and Terwilliger [6], the Stanton inequalities (Conjecture 1.3) have been shown to hold for any $Q$-polynomial scheme which is also $P$-polynomial. In other words, our Theorem 1.4 remains true if the words “dual-thin” are replaced by “$P$-polynomial”.

2 Proof of the Theorem

Let $Y = (X, \{R_i\}_{0 \leq i \leq D})$ denote a symmetric association scheme which is $Q$-polynomial with respect to the ordering $E_0, \ldots, E_D$ of the primitive idempotents. Fix any $x \in X$ and let $T = T(x)$ denote the Terwilliger algebra for $Y$ with respect to $x$. Let $W$ denote any irreducible $T$-module. We define the dual endpoint of $W$ to be the integer $t$ given by

$$t := \min\{i : 0 \leq i \leq D, \ E_i W \neq 0\}. \quad (2)$$
We observe that $0 \leq t \leq D - d^*$, where $d^*$ denotes the dual diameter of $W$.

**2.1 Lemma.** [9, p.385] Let $Y$ be a symmetric association scheme which is $Q$-polynomial with respect to the ordering $E_0, \ldots, E_D$ of the primitive idempotents. Fix any $x \in X$, and write $E_i^* = E_i^*(x)$ $(0 \leq i \leq D)$, $T = T(x)$. Let $W$ denote an irreducible $T$-module with dual endpoint $t$. Then

(i) $E_i W \neq 0$ \iff $t \leq i \leq t + d^*$ \quad $(0 \leq i \leq D)$.

(ii) Suppose $W$ is dual-thin. Then $W$ is thin, and $d = d^*$. ■

**2.2 Lemma.** [3, Lemma 4.1] Under the assumptions of the previous lemma, the dual endpoint $t$ and diameter $d$ of any irreducible $T$-module satisfy

$$2t + d \geq D.$$

**Proof of Theorem 1.4.** Fix any $x \in X$, and let $T = T(x)$ denote the Terwilliger algebra for $Y$ with respect to $x$. Since $T$ is semisimple, there exists a positive integer $s$ and irreducible $T$-modules $W_1, W_2, \ldots, W_s$ such that

$$V = W_1 + W_2 + \cdots + W_s \quad \text{(orthogonal direct sum)}. \quad (3)$$

For each integer $j$, $1 \leq j \leq s$, let $t_j$ (respectively, $d^*_j$) denote the dual endpoint (respectively, dual diameter) of $W_j$. Now fix any nonnegative integer $i < D/2$. Then for any $j$, $1 \leq j \leq s$,

$$E_i W_j \neq 0 \ \Rightarrow \ t_j \leq i \quad \text{(by Lemma 2.1(i))}$$
$$\Rightarrow \ t_j < i + 1 \leq D - i \leq D - t_j \quad \text{(since } i < D/2)$$
$$\Rightarrow \ t_j < i + 1 \leq D - i \leq t_j + d^*_j \quad \text{(by Lemmas 2.1(ii), 2.2)}$$
$$\Rightarrow \ E_{i+1} W_j \neq 0 \ \text{and} \ E_{D-i} W_j \neq 0 \quad \text{(by Lemma 2.1(i))}.$$ 

So we can now argue that, since $Y$ is dual thin,

$$\dim(E_i V) = |\{j : 0 \leq j \leq s, E_i W_j \neq 0\}|$$
$$\leq |\{j : 0 \leq j \leq s, E_{i+1} W_j \neq 0\}|$$
$$= \dim(E_{i+1} V).$$

In other words, $m_i \leq m_{i+1}$. Similarly,

$$\dim(E_i V) = |\{j : 0 \leq j \leq s, E_i W_j \neq 0\}|$$
$$\leq |\{j : 0 \leq j \leq s, E_{D-i} W_j \neq 0\}|$$
$$= \dim(E_{D-i} V)$$

This yields $m_i \leq m_{D-i}$. ■
References


