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Waterfall lasers

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Laser concepts can be applied to a broad range of physical phenomena. One of the closest parallels occurs with the fluttering oscillations that are sometimes observed in the falling sheets of water associated with fountains, dams, and natural waterfalls. In many respects these fluid feedback oscillations are similar to the electromagnetic modes of typical lasers, and recognition of this similarity led to the interpretation of the waterfall behavior. Gain profiles for the waterfall oscillations are developed, and the relationship of experimental waterfall data to the laser-like models is considered in detail. © 1997 American Institute of Physics. [S0021-8979(97)07622-6]

I. INTRODUCTION

Periodic fluttering oscillations have been reported in the sheets of water associated with some dams at least since the early 1800’s, and it is not unlikely that they had been mystifying observers of natural and artificial waterfall systems long before that. The full impact of the waterfall oscillations is best experienced from video recordings or from direct encounters. One of the notable features of these waterfall oscillations is their similarity to the oscillation modes of lasers, and some of the causes and consequences of this analogy are considered here.

There are actually several different types of instability that can affect a waterfall. A few of these lead to feedback oscillations, and a striking class of such oscillations is of particular interest here. The underlying instability mechanism for this class is related to the shear flow instability of surfaces separating fluids moving at different velocities, as first discussed by Helmholtz. In this Helmholtz mechanism small displacements of a surface are augmented by the pressure changes resulting from the relative fluid motion. Helmholtz’s theory was further developed and applied to wind-generated water waves by Kelvin; and other related problems, including the flapping of a flag in the wind, were considered by Rayleigh. The same mechanism underlies the wave patterns frequently exhibited by clouds in the sky and by internal waves in the ocean. In contrast to the ordinary water wave or flag problem, the water sheet in a waterfall is moving through the air, and it is not necessary that there be any wind or overall air motion. Also, the waterfall instability has the added complications of a height-dependent velocity and an associated nonconstant water membrane thickness due to gravitational acceleration.

The fluttering waterfall oscillations also differ from most other wind-generated waves in that the waterfall oscillations may be extremely periodic, while other types are typically irregular. Periodic behavior always suggests the existence of a feedback mechanism, and the feedback in this case is now known to be provided by the confined air chamber behind the water sheet. The large-amplitude wave motions at the bottom of the sheet tend to compress or expand the trapped air, which in turn pushes out or pulls in on the water surface at the top. The small displacements at the top are amplified by the Helmholtz mechanism as the sheet falls, and thus the laser-like oscillations are maintained. The heights of fall for which the oscillations have been observed range from about 0.2 to 8.0 m, and oscillation frequencies range between about 2.5 and 25 Hz. The number of wavelengths observed in a fall have varied between 1 and about 12. Numerical models of this effect provide good agreement with experimental observations. Approximate analytic solutions and stability criteria for this phenomenon have also been obtained, and an extension to annular fountains has been reported.

The waterfall oscillations considered here are not subtle in effect, and they have not always been appreciated. Notices of their occurrence are often accompanied by comments of astonishment at the level of sound (or infrasound) produced. The clothing of observers vibrates back and forth; and, in one early report, “The doors and windows shake in Spring- field (MA), two and a half miles from the dam.” The moving water sheet acts externally like a large-displacement loudspeaker diaphragm, and in some cases this diaphragm has had an area of hundreds of square meters.

The purpose of this study is to focus more directly on some of the laser-like features of the fluttering waterfall effect. Formulas governing the most basic mode characteristics of waterfall and conventional lasers are reviewed or developed in Sec. II, and these formulas are compared with experimental data in Sec. III. The experimental frequencies of the oscillation modes are in agreement with the theoretical models and are also similar to laser mode characteristics.

II. MODE FORMULAS

A rigorous representation for the fluttering oscillation modes of a falling water sheet requires the numerical solution of a set of nonlinear partial differential equations. The results are in the form of growing downward-propagating waves. The amplitude of the waves increases with propagation downward due to the Helmholtz mechanism, and the local wavelength increases due to the acceleration of gravity. For some purposes the full numerical solutions can be replaced by much simpler analytic approximations. Thus, the possible frequencies that satisfy the oscillation phase condition can be written:
The height of these wave forms is given in Fig. 1 for a fountain of actual oscillation wave forms of a falling water sheet. A set of oscillators would be present with the waterfall oscillators.

The growth of the envelope of the downward propagating waves is dominated by an exponential term except near the top of the fall, where feedback effects are larger than the gain effects. Thus, the amplitude of the waves can be written approximately as

$$A = A_0 \exp \left[ \frac{2 \rho_w \omega}{\rho_a F} \left( y_0 - y \right) \right],$$

where $\rho_a$ is the air density, $\rho_w$ is the water density, $\omega = 2 \pi f$ is the radian frequency of the oscillations, $F$ is the flow rate, and $y$ measures distance upward from the bottom of the waterfall. This exponential amplitude growth is qualitatively the same as the electromagnetic amplitude behavior within the amplifier of a one-directional ring laser. The wavelength variations due to gravity would also have an electromagnetic parallel if, for example, the index of refraction of the ring laser medium was a function of the longitudinal position.

The longitudinal mode frequencies vary linearly with the longitudinal mode order. Slight corrections to Eq. (2) occur because of dispersion associated with the amplifying transition of a laser, and similar corrections would be present with the waterfall oscillators.

Approximate analytic solutions are also possible for the actual oscillation wave forms of a falling water sheet. A set of these wave forms is given in Fig. 1 for a fountain of height $y_0 = 0.8 \text{ m}$, flow rate $F = 7.5 \text{ m}^{-1} \text{s}^{-1}$, and various values of the longitudinal mode index $m$. The only adjustable parameter in these plots is the mode amplitude, and these results are in good agreement with experimental wave forms.

The gain is a monotonically increasing function of frequency. This is a little different from lasers where the gain is typically a Lorentzian function of frequency for homogeneously broadened media or a Gaussian function for Doppler broadened media. On the other hand, the feedback in a waterfall oscillator is generally a decreasing function of frequency, while the reflectivity in ring lasers is usually considered to be frequency independent. The decrease with frequency results in part from the smaller changes with time of the volume of the air chamber behind the water sheet for larger values of the frequency (smaller local wavelength) and also the smaller feedback displacement possible near the top of the sheet for higher frequencies. Thus, the overall effective roundtrip gain function for waterfalls, as for conventional lasers, may have a narrow maximum.

Although a rigorous formula for the frequency dependence of the feedback is not available, the general principles can be understood from an empirical formula that provides a good qualitative approximation for the frequency range of interest. Thus, we assume that the feedback decreases exponentially with frequency according to

$$R = R_0 \exp(-a \omega^n),$$

where $a$ and $n$ are constants. Combining Eqs. (5) and (6), one finds that the effective unsaturated roundtrip gain is

$$G = R_0 \exp \left[ -a \omega^n + \frac{2 \rho_w \omega}{\rho_a F} \left( y_0 - y \right) \right].$$

Setting to zero the derivative of this function with respect to $\omega$ shows that the gain maximum occurs at the frequency

$$\omega_p = \left( \frac{\rho_a}{2 \rho_w F} \right)^{1/2} \frac{y_0}{n a^{1/2}} \left( n - 1/2 \right).$$
A best estimate for $n$ is 1. The triangles are experimental data points, and the hyperbola-like curves are plots of Eq. (3) for various values of $m$. The upward sloping lines are rough fits of Eq. (8) for three values of $n$.

The height dependence of the gain maximum implied by Eq. (8) depends on the value of the parameter $n$. For some specific half-integer-spaced $n$ values, Eq. (8) takes on the forms

$$\omega_p = \left( \frac{\rho_a}{2\rho_v F} \right)^{1/2} \frac{y_0}{a} n = 1,$$

$$\omega_p = \left( \frac{\rho_a}{2\rho_v F} \right)^{1/2} \frac{2y_0}{3a} n = 3/2,$$

$$\omega_p = \left( \frac{\rho_a}{2\rho_v F} \right)^{1/3} \frac{y_0^{2/3}}{2a} n = 2.$$

A best estimate for $n$ will be obtained in the next section by comparing these formulas with experimental data.

### III. RESULTS

A set of experimental results showing the variation of the oscillation frequency with waterfall height is given in Fig. 2. The triangles are experimental data points, and the associated hyperbola-like curves are plots of Eq. (3). These data were obtained using a laboratory waterfall fountain built in the Department of Physics at Portland State University.

This fountain is 1 m in width and has a fall height that is adjustable up to 1.65 m. The flow rate in this example is $1.91 \text{ m}^{-1} \text{ s}^{-1}$, and the frequency values were obtained from spectral analysis of the fluttering vibrations.

The results shown in Fig. 2 are in good agreement with the theory discussed above. One implication of the figure is that two oscillation frequencies are sometimes possible for the same set of experimental conditions. This behavior is mainly due to hysteresis near the mode transitions, and as with lasers it is only at these transitions that a single-mode theory does not fully represent the experimental data.

It may be seen from the figure that the frequency at which the gain is highest increases with the height of the waterfall, as discussed above. Superimposed on the data are plots of the three frequency functions $f=b y_0^2$ (with $b=25 \text{ m}^{-2} \text{ s}^{-1}$), $f=b y_0$ (with $b=19 \text{ m}^{-1} \text{ s}^{-1}$), and $f=b y_0^{2/3}$ (with $b=17 \text{ m}^{-2/3} \text{ s}^{-1}$), which are simplified forms of the three formulas given in Eqs. (9)–(11). The experimental height dependence of the gain maximum evidently lies between the first and third of these frequency functions, and to be specific we will adopt Eq. (10) as a representation of the frequency of the gain maximum.

The tendency implied by Eq. (10) for the frequency of the gain maximum to increase as the flow rate decreases is also in agreement with experimental data and was evident in the data of Ref. 8. The experimental value for the flow rate in Fig. 2 ($1.91 \text{ m}^{-1} \text{ s}^{-1} = 1.9\times10^3 \text{ m}^2 \text{ s}^{-1}$) together with the straight line in the figure, suggest that Eq. (10) can be written in the form

$$f_p = 0.828 F^{-1/2} y_0,$$

where the waterfall height $y_0$ is measured in units of m, the flow rate $F$ is in $\text{ m}^2 \text{ s}^{-1}$, and the resulting frequency is in Hz.

Pending more detailed experiments, Eq. (12) may be considered a preliminary general formula for the frequency of the gain maximum when a water sheet falls through air.

Using Eq. (12), it is also possible to obtain an estimate of the effective unsaturated roundtrip gain profile. First, Eqs. (7) and (10) may be combined to obtain the gain function

$$G = R_0 \exp \{ 3 (\omega/\omega_p)^{1/2} - (\omega/\omega_p)^{3/2} \},$$

where the coefficient $g$ is given by

$$g = \left( 2 \frac{\rho_a \omega_p}{\rho_v F} \right)^{1/2} \frac{y_0}{3}.$$

In conventional frequency units Eq. (13) can be written

$$G = G_p \exp \{ 3 (f/f_p)^{1/2} - (f/f_p)^{3/2} - 2 \},$$

where $G_p = R_0 \exp(2g)$ is the peak value of the gain function at the frequency $f = f_p$. With Eqs. (12) the coefficient $g$ can be written explicitly in terms of the waterfall height:

$$g = \left( 4 \frac{\pi \rho_a 0.828 y_0}{\rho_v F^{3/2}} \right)^{1/2} \frac{y_0}{3} = 3.764 \times 10^{-2} F^{-3/4} y_0^{3/2},$$

where we have used the densities $\rho_a = 1.225 \text{ kg/m}^3$ and $\rho_v = 1.0 \times 10^3 \text{ kg/m}^3$.

As an example of a gain profile estimation, we consider the same flow rate as in Fig. 2 ($1.91 \text{ m}^{-1} \text{ s}^{-1} = 1.9\times10^3 \text{ m}^2 \text{ s}^{-1}$) together with a height of 0.8 m. Then Eq. (16) implies the coefficient value $g = 3.0$, and Eq. (15) is plotted in Fig. 3 for three reasonable values of this coefficient ($g = 1, 3, 5$). The gain profile is seen to be similar to the Lorentzian (homogeneous line broadening) and Gaussian (Doppler line broadening) gain profiles of laser studies. While the profile in the waterfall case is broader in a relative sense than most laser profiles, it is somewhat comparable to the broadest bandwidth lasers such as those based on titanium:sapphire. The gain profile of these lasers can support 3 fs pulses at a wavelength of 0.8 microns.
A direct comparison of the waterfall and conventional laser frequency models is shown in Fig. 4. Figure 4(a) shows the theoretical oscillation frequency for a fluttering waterfall based on Eqs. (3) and (12) with \( F = 3.01 \text{ m}^{-1} \text{s}^{-1} \). For simplicity the possibility of hysteresis is not included, and the frequency is single valued. In agreement with the experimental data of Fig. 2, the frequency of a given mode tends to decrease with increasing waterfall height. Figure 4(b) shows the corresponding results of Eq. (4) for an ordinary laser. The axis label \( f' \) in Fig. 4(b) represents the actual mode frequency \( f \) normalized to the constant center frequency of the laser transition, and the label \( L' \) represents the actual length of the oscillator cavity \( L \) normalized to the wavelength associated with the transition frequency. To be specific, it has been assumed in Eq. (4) that the phase \( \phi \) is equal to zero.

It is clear from Fig. 4 that, for both the waterfall and laser oscillations, the frequency of a given longitudinal mode decreases with increasing length (or height) until the frequency dependent roundtrip gain becomes lower than that for the next higher frequency mode. At that point a transition occurs to the higher frequency mode. It should be noted that the curves in Fig. 4 show only the dominant frequency behavior of lasers and waterfalls. The more complicated results observed near mode transitions in the two systems are also closely related. The hysteresis in waterfalls that was mentioned above has a close analogy in conventional lasers, and for example hysteresis is sometimes observed in semiconductor lasers near mode transitions. Noise-like fluctuations near these transitions have also been observed in both waterfalls and lasers. In conventional lasers, below-threshold side modes are continuously excited at a very low level by spontaneous emission or thermal noise, whereas in waterfall fountains these modes can be driven by wave noise from the upper pool and by splash noise entering the feedback chamber behind the water sheet.

The emphasis here has been on the longitudinal modes of waterfalls and more conventional lasers. Many lasers also exhibit higher-order transverse modes in which time dependent phase and amplitude variations occur across the profile of the wave field. In waterfalls too there may be transverse mode structure in the form of amplitude and phase variations across the face of the falls. Higher-order transverse mode behavior tends to start occurring in stripe-geometry semiconductor lasers, for example, when the stripe width is greater than a few microns, while in waterfalls the corresponding effects occur in low-flow systems for widths greater than a few meters. With higher flow rates single transverse mode operation in waterfalls may be found to widths of greater than ten meters, but as with diode lasers a more complicated transverse structure seems to be inevitable for very wide falls. In broad area diode lasers defects in the amplifying region or reflecting facets can cause fixed transverse nonuniformity or dead regions in the laser output, and analogous behavior commonly results from weir defects in waterfalls.

IV. DISCUSSION

As the reader will have noticed, we are applying familiar laser terminology outside of what would seem to be its usual domain, and a few words of explanation may be helpful. The acronym MASER was introduced by C. H. Townes and his colleagues in 1954 to summarize the operating principles of the devices that they had developed (Microwave Amplification by Stimulated Emission of Radiation). The related acronym LASER was introduced by G. Gould in 1957. In their most technical meanings these acronyms would seem to refer to a particular physical process, stimulated emission of
radiation, for obtaining the amplification of electromagnetic radiation at optical or microwave frequencies. However, both historically and in modern usage broader definitions of the maser and laser concepts are often employed.

In spite of the apparently explicit meaning of both of the names maser and laser, the distinction between stimulated emission and other amplification mechanisms is not always completely clear. Thus, conventional stimulated emission of electromagnetic radiation during transitions between the discrete energy states of an atom or molecule is sometimes replaced by some other process such as stimulated Compton, Raman, or Brillouin scattering in systems that are still called lasers. The words maser or laser have also been previously used when the amplified waves are not electromagnetic, and this possibility is of interest for our considerations of waterfall oscillations. Thus, exciton lasing in semiconductors has recently been observed, and atomic field lasing in Bose–Einstein condensations has also been reported. In addition it has been suggested that neutrino lasing may have played an important role in the early evolution of the universe.

Of particular interest here are those lasers in which the wave phenomenon is mechanical. For such systems a compound name such as acoustic maser, phonon maser, or phaser is sometimes employed, and again these terms are not restrictive to any particular amplification mechanism. As an illustration, we may quote Chiao and Townes: “The amplification of hypersonic waves due to stimulated Brillouin scattering may be viewed as phonon maser action.” If the word hypersonic here is replaced by infrasonic and the name Brillouin is replaced by Helmholtz, we find that the fluttering waterfall oscillations are not so different from other forms of mechanical maser oscillations. All are based on distributed amplification with feedback, and all involve similar gain profiles and resonant mode behavior.

It may be noted that the oscillations under consideration here are not associated in a previously established way with quanta of field energy (photons, phonons, etc.) as other electromagnetic or acoustic lasers might seem to be. However, the lack of a quantum theoretical basis does not exclude the use of laser language and interpretations. In connection with free-electron lasers, for example, one finds statements like the following: “In fact, the quantum theory of a free-electron laser is extremely tedious, and neither desirable nor necessary.” Laser concepts were the basis for the first interpretation of the long-recognized waterfall oscillations, and a familiarity with laser principles may lead to an understanding of other as-yet-unexplained oscillating systems.

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