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Multiple-Valued Quantum Logic Synthesis

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Multiple-Valued Quantum Logic Synthesis

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What size of (binary) Quantum Computers can be built in year 2002?

- 7 bits
Is logic synthesis for quantum computers a practical research subject?

Yes, it is a useful technique for physicists who are mapping logic operations to NMR computers.

CAD for physicists.

Isaac Chuang, IBM
5 qubit 215 Hz Q. Processor

(Vandersypen, Steffen, Breyta Yannoni, Cleve, and Chuang, 2000)

- The Molecule

- Quantum Circuit

$T_2 > 0.3 \text{ sec} ; \sim 200 \text{ gates}$

Source: IBM, Hot Chips Conference, 2000
The molecule

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<th>$T_{2,i}$</th>
<th>$J_{7i}$</th>
<th>$J_{6i}$</th>
<th>$J_{5i}$</th>
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<th>$J_{3i}$</th>
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</table>
Pulse Sequence

Init. mod. exp. QFT

~ 300 RF pulses  ||  ~ 750 ms duration
Results: Spectra

Mixture of $|0\rangle, |4\rangle$
$2^{3/4} = r = 2$
$\gcd(11^{2/2} \pm 1, 15) = 3, 5$

$15 = 3 \cdot 5$

Mixture of $|0\rangle, |2\rangle, |4\rangle, |6\rangle$
$2^{3/2} = r = 4$
$\gcd(7^{4/2} \pm 1, 15) = 3, 5$
Problem

• We would like to assume that any two quantum wires can interact, but we are limited by the realization constraints.

• Structure of atomic bonds in the molecule determines neighborhoods in the circuit.

• This is similar to restricted routing in FPGA layout - link between logic and layout synthesis known from CMOS design now appears in quantum.

• Below we are interested only in the so-called “permutation circuits” - their unitary quantum matrices are permutation matrices.
A schematics with two binary Toffoli gates

Quantum wires A and C are not neighbors

This is a result of our ESOP minimizer program, but this is not realizable in NMR for the above molecule
So I modify the schematics as follows

But this costs me two swap gates

Costs 3
Feynmans
Solution

- One solution to connection problem in VLSI has been to increase the number of values in wires.
- Have a “quantum wire” have more than two eigenstates.
- Increase from superpositions of $2^n$ to superpositions of $3^n$
- Basic gate in quantum logic is a 2*2 (2-qubit gate). We have to build from such gates.
Can we build multiple-valued Quantum Computers in year 2002?

• *In principle, yes*

Has one tried?

*No.*

Gates, yes
Qudits not qubits

• In ternary logic, the notation for the superposition is $\alpha|0> + \beta|1> + \gamma|2>$.  

• These intermediate states cannot be distinguished, rather a measurement will yield that the qudit is in one of the basis states, $|0>$, $|1>$ or $|2>$.  

• The probability that a measurement of a qudit yields state $|0>$ is $|\alpha|^2$, the probability is $|\beta|^2$ for state $|1>$ and $|\gamma|^2$ for state $\gamma$. The sum of these probabilities is one. The absolute values are required, since in general, $\alpha$, $\beta$ and $\gamma$ are
The concept of Multiple-Valued Quantum Logic

Classical Binary logic  →  Classical Multiple-Valued logic

Binary Quantum logic  →  Multiple-Valued Quantum logic
What is known?

- **Mattle** 1996 - *Trit* $|0\rangle$, $|1\rangle$, $|2\rangle$
- Chau 1997 - *qudit, error correcting quantum codes*
- Gottesman, Aharonov and Ben-Or 1999 - *MV fault tolerant gates*.
- Burlakov 1999 - *correlated photon realization of ternary qubit*.
- Muthukrishnan and Stroud 2000 - *multi-valued universal quantum logic for linear ion trapped devices*.
- Picton 2000 - *Multi-valued reversible PLA*.
- **Perkowski, Al-Rabadi, Kerntopf and Portland Quantum Logic Group** 2001 - *Galois Field quantum logic synthesis*.
What is known?

- Al-Rabadi, 2002 - ternary EPR and Chrestenson Gate
- De Vos 2002 - Two ternary 1*1 gates and two ternary 2*2 gates for reversible logic.
- Zilic and Radecka 2002 - Super-Fast Fourier Transform
- Bartlett et al, 2002 - Quantum Encoding in Spin Systems
- Brassard, Braunstein and Cleve, 2002 - Teleportation
- Rungta, Munro et al Qudit Entanglement.
Ternary Galois Field (GF3) operations.

<table>
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<tr>
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<th>2</th>
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</tr>
<tr>
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<td>0</td>
<td>1</td>
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(a) Addition

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<tr>
<td>2</td>
<td>2</td>
<td>0</td>
<td>1</td>
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</table>

(b) Multiplication
Reversible ternary shift operations.

<table>
<thead>
<tr>
<th>Operator Name</th>
<th>Buffer</th>
<th>Single-Shift</th>
<th>Dual-Shift</th>
<th>Self-Shift</th>
<th>Self-Single-Shift</th>
<th>Self-Dual-Shift</th>
</tr>
</thead>
<tbody>
<tr>
<td>Operator symbol &amp; equation</td>
<td>$A$</td>
<td>$A' = A+1$</td>
<td>$A'' = A+2$</td>
<td>$A''' = A + A = 2A$</td>
<td>$A^# = 2A+1$</td>
<td>$A^\wedge = 2A+2$</td>
</tr>
<tr>
<td>Gate symbol</td>
<td>🔄</td>
<td>🔄↑</td>
<td>🔄↓</td>
<td>🔄↑</td>
<td>🔄↑</td>
<td>🔄↑</td>
</tr>
<tr>
<td>$A$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>1</td>
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</table>
Conversion of one shift form to another shift form using ternary shift gates

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>( A )</td>
</tr>
<tr>
<td>( A )</td>
<td>( A )</td>
</tr>
<tr>
<td>( A' )</td>
<td>( A'' )</td>
</tr>
<tr>
<td>( A'' )</td>
<td>( A''' )</td>
</tr>
<tr>
<td>( A''' )</td>
<td>( A^# )</td>
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<tr>
<td>( A^# )</td>
<td>( A^\wedge )</td>
</tr>
<tr>
<td>( A^\wedge )</td>
<td>( A'' )</td>
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</table>
Quantum realization of ternary shift gates.

(a) Single-Shift

(b) Dual-Shift

(c) Self-Shift

(d) Self-Single-Shift

(e) Self-Dual-Shift
Optimal Solution to Ternary Miller Function

Check ternary maps
2-qubit quantum realization of Miller Gate

These techniques can be also applied for Multiple-Valued Quantum Logic
a) $A = a \oplus c$

$B = a \oplus b$

$C = ab \oplus ac \oplus bc$

b) $A = a \oplus c$

$B = \overline{a} \overline{b} \oplus \overline{a} \overline{c} \oplus bc$

$C = ab \oplus ac \oplus bc$
Design a Ternary Toffoli Gate from 2-qubit quantum primitives
Ternary controlled gates

First task is to demonstrate that a universal 3-qubit gate can be built from MV quantum primitives.

2-qubit controlled gate with controlling value $d-1 = 2$

2-qubit controlled gate with controlling values 0 and 2

Generalization of Stroud and De Vos gates
Principle of creating arbitrary reversible gates
Main Result - Galois Logic is practically quantum-realizable

\[ R = ab + c \]

Toffoli for Ternary
Main Result - Galois Logic is practically quantum-realizable

This structure realizes also a very huge family of ternary Toffoli-like gates

But is it worthy?
Complete ternary systems

- System 1. Post literal, min, max
- System 2. Power of variable, shifts of variable (two of them for ternary – these are optional), Galois ADD, Galois MUL
- System 3. Post Literals, MIN, MODSUM.
- These three are most popular, but there are many other.

Are they good for quantum?
Ternary Operator Kmaps

MODSUM, which for primary number 3 is the same as Galois Addition. Observe Latin square property, very important.

Galois Multiplication. Also has Latin square for non-zero columns and rows.
Example: Ternary Kmaps of ternary adder

<table>
<thead>
<tr>
<th></th>
<th>0</th>
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</tr>
<tr>
<td>2</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

\[ C = A \cdot B + A \cdot B + A \cdot B \]

Step 1: write from Kmap the formula for mv minterms

This is modsum3 by inspection, so \( S = A + B \). But you can also calculate is with much formula writing the same as I show for \( C \)
Step 2. Algebraic Simplifications using rules of ternary Galois Field Algebra

\[ C = 1A \ast 2B + 2A \ast 1B + 2A \ast 2B = (2A^2 + 2A) \ast (2B^2 + B) \]
\[ + (2A^2 + A) \ast (2B^2 + 2B) + (2A^2 + A) \ast (2B^2 + B) \]
\[ = 2(A^2 + A) \ast (2B^2 + B) + 2 \ast 2A^2B^2 + 2 \ast 2A^2B + 2AB^2 + 2AB + \\
2 \ast 2A^2B^2 + 2A^2B + 2AB^2 + AB = 2A^2B + 2AB^2 + 2AB \]

Example of Post literal, it has value 1 for argument value 1 and 0 otherwise.

Here Post literals are next replaced by tautological polynomials in Galois Field.
Complex Ternary Quantum Gates
Ternary Fredkin Gate build from Ternary Toffoli and Ternary Feynman gates

\[ b \oplus c \oplus ab \oplus a'c = ac \oplus ba' \]

\[ c \oplus a(b \oplus c) = c \oplus ab \oplus ac = ca' \oplus ab \]
If $f_1$ is reversible, gate is correct. What about non-reversible $f_1$, please check if the gate is still reversible.
Generalized Ternary 3*3 Toffoli Gate

Do the same exercise as in previous slide, this will help you get intuition in MV logic.
Generalized Ternary $n \times n$ Toffoli Gate

$$A_1$$

$$\cdots$$

$$A_{n-1}$$

$$f_{n-1}$$

$$A_n$$

$$A_{n-1} \oplus f_{n-1} (A_1, \ldots, A_{n-1})$$

Do the same exercise as in previous slide, this will help you get intuition in MV logic.
Is this a realizable quantum gate? -yes
Generalized Ternary 4*4 Fredkin Gate

Rewrite this part to quantum ternary notation with Galois Logic.
Generalized Ternary n*n Fredkin Gate

Do the same exercise as in previous slide, this will help you get intuition in MV logic.
Generalized Ternary 4*4 Kerntopf Gate

Do the same exercise as in previous slide, this will help you get intuition in MV logic.
Ternary Quantum Circuits
Ternary GFSOP Cascade (non-optimal)

Notation for EACH gate:
Inputs: A, B, C
Outputs: P, Q, R

How to realize ternary swap gate?
In any case, this is very costly!

All operations are Galois

A

B

C

A

B

C

AB ⊕ C

B ⊕ A (AB ⊕ C) = B ⊕ AB ⊕ AC

AC = A’B ⊕ AC

How to realize ternary swap gate?
In any case, this is very costly!
General Ternary Cascade of Kerntopf, Toffoli and Fredkin Family Gates

\[ f_2 \oplus g_2 \oplus h_2 \]

\[ \psi_1, \psi_2 \]
Example of multi-output FPRM-like GFSOP cascade of Toffoli family gates

$\psi_1 = 1 \oplus C'' \oplus A'B'C \oplus A' \cdot B$

$\psi_2 = 2 \oplus C'' \oplus A' \cdot B$
Example of ternary multi-output GFSOP cascade of Toffoli family gates

\[ \psi_1 = 1 \oplus C \oplus ABC \oplus A' B \]
\[ \psi_2 = 0 \oplus C \oplus A' B \]

This is notation for single shift
This is notation for dual shift
The **general pattern** of a cascade to implement any ternary function using ternary Toffoli gates

\[ F_1 = 1 + A'' A' C^2 + B^2 C^2 + A'' B \]

\[ F_2 = 2 + B^2 C^2 + B''' + 2C^2 \]
Simplified GFSOP array when powers are not used for some variables. Function of four variables.

\[
F = 1 + 2a \ a' d^2 + c \ d^2 + 2 \ a''c
\]

\[
G = 2 + c \ d^2 + 2b + 2d^2
\]
\begin{align*}
F_1 &= A'B \oplus A'B'C \\
F_2 &= A'B \oplus A'B'C' \\
F_3 &= A'B \oplus B'C' \oplus AC'
\end{align*}

(a) Realization of multi-output ESOP

Macrogeneration introduces many Feynman gates that originate from swaps
(a) Realization of multi-output GFSOP

\[ F_1 = A'B'' \oplus AB'C \]
\[ F_2 = A'B'' \oplus AB'C \oplus BC' \]
\[ F_3 = A'B'' \oplus BC' \oplus AC' \]

(b) Realization of single-output GFSOP

\[ F = AC \oplus AD'' \oplus B'C \oplus B'D'' \oplus CD \oplus A'B'' \]
MV Quantum Design Structures and Approaches

• 1. GFSOP
• 2. Multiple-Valued Reed-Muller
• 3. Canonical Forms over Galois Logic (equivalents of PPRM, FPRM, GRM, etc)
• 4. Multiple-Valued Maitra Cascades and Wave Cascades.
• 5. Other cascades of specific type of elements
• 6. Cascades of general gates
Design Issues

• **1.** Local mirroring
• **2.** Variable ordering *versus* gate ordering
• **3.** Return to zero and folding
• **4.** Realization of complex multiple-valued reversible gates (permutation gates) using directly 1-qubit and 2-qubit quantum primitives
“Return to Zero” and Folding

Technique of local mirror can improve your ternary circuits, reduce the number of zeros in inputs. Here is the explanation for ternary logic.

This is a signal AB

This is a signal no longer used, it is converted to zero

AB + 2* AB = 0

Zero created for input to next block
Molecule - Driven **Layout** and **Logic Synthesis**

Allowed gate neighborhood for 2 qubit gates
Using Local Mirrors and Return-to-zero factorization

\[ F = a \cdot e \cdot (b \oplus cd \oplus f) \oplus bc(ae \oplus d) \oplus cd \oplus f \]
Toffoli gates = 4  
Swap gates = 3  
Shift gates = 6 (c)

Toffoli gates = 6  
Swap gates = 6  
Shift gates = 5 (a)

Toffoli gates = 8  
Swap gates = 15  
Shift gates = 5 (b)

Toffoli gates = 6  
Swap gates = 12  
Shift gates = 12 (d)
System for mixed quantum logic NMR

- **Ternary expression**
  - GFSOP
  - Factorized
  - Khan gates
  - Ternary swap

- **Binary expression**
  - ESOP
  - Factorized
  - Complex gates
  - Binary swap

- **DD**
  - Lattice
  - Complex gate cascade

- **Planar**
  - Evolutionary Gate Synthesizer

- **Complex quantum gate library**

- **Molecule description**

- **Macro to Tof**
- **Optimization Tof**

- **Macro to 2-qubit**
  - Optimization 2-qubit

- **Macro NMR**
  - NMR operators
Open Problems

1. How to select the best gates for permutation circuit synthesis.

2. Simplest practical realization of a ternary Toffoli-like gate

3. Best realization, in quantum circuit sense (simplicity and ease of realization), of other Galois gates and non-Galois standard MV operators such as minimum, maximum, truncated sum and others.

4. Synthesis algorithms for MV reversible circuit families:
   - GFSOP,
   - nets,
   - lattices,
   - PLAs
   - MV counterparts of Maitra cascades and wave cascades
   - other reversible cascades
Conclusion

• Practical algorithms for MV quantum circuits. Quantum permutation circuits design (for NMR) is not the same as standard reversible logic.

• **CAD Tools** for quantum physicists: *link levels of design*.

• Evolutionary Approaches versus GFSOP-like approaches

• MV Quantum Simulation

• MV Quantum Circuits Verification

• Designing MV counterparts of Deutcht, Shorr, Grover and other original MV quantum algorithms

• Generalization to MV Of efficient Garbage-less quantum gates by Barenco, DiVincenzo, etc.

• NMR realization of ternary logic.

• MV Quantum Computational Intelligence