2002

Analytically Continued Hypergeometric Expression of the Incomplete Beta Function

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Citation Details
http://pdxscholar.library.pdx.edu/phy_fac/248
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Abstract

The Incomplete Beta Function is rewritten as a Hypergeometric Function that is the analytic continuation of the conventional form, a generalization of the finite series, which simplifies the Stieltjes transform of powers of a monomial divided by powers of a binomial.

1991 Mathematics Subject Classification: 33B20, 33C05, 44A15

Key Words: Incomplete beta function, hypergeometric function, Stieltjes transforms, definite integrals

The finite hypergeometric series expression for the Incomplete Beta Function, [1]

\[ \binom{-n}{1}F1(-n,1;c;z) = (1-c)z^{1-c}(z-1)^{n+c-1}B_{1,1}(1-c,n,n+1), \]  

may be generalized to

Theorem

\[ 2F1(-\nu,1;\gamma;z) = (1-\gamma)z^{1-\gamma}(z-1)^{\nu+\gamma-1} \left[ B_{1,1}(1-\gamma-\nu,\nu+1) \ight. \
- B(1-\gamma-\nu,\nu+1) \left( 1 - \frac{(-1)^{-\nu}\sin[\pi(\gamma+\nu)]}{\sin(\pi\gamma)} \right) \]. \]  

The Incomplete Beta Function [2] is conventionally defined [3] with real parameters for statistical problems,

\[ B_x(p,q) = \int_0^x t^{p-1}(1-t)^{q-1} \, dt \quad (0 \leq x \leq 1, \quad p,q > 0), \]  

but is a smooth function of \( p,q \) or \( x \) when any or all are taken off the real axis (though it diverges as \( x \) takes on large, real values). Its hypergeometric expression [4] is likewise well-behaved for complex parameters, so we rewrite this expression in its more general form

\[ 2F1(\alpha,\beta;\beta+1;w) = \beta w^{-\beta}B_{w}(\beta,1-\alpha) = \beta w^{-\beta}B(\beta,1-\alpha)(1 - I_{1-w}(1-\alpha,\beta)) \] 
\[ = \beta w^{-\beta} [B(1-\alpha,\beta) - B_{1-w}(1-\alpha,\beta)]. \]  

One may analytically continue the left-hand side to [5]

\[ 2F1(\alpha,\beta;\beta+1;w) = (-1)^{-\alpha}(w)^{-\alpha}\frac{\Gamma(\beta+1)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\beta+1-\alpha)}2F1(\alpha,\alpha-\beta;\alpha+1-\beta;1/w) \]
\[ + (-1)^{-\beta}(w)^{-\beta}\frac{\Gamma(\beta+1)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(1)}2F1(\beta,0;\beta+1-\alpha;1/w). \]  

Then equating right-hand sides of (4) and (5) and transforming the nontrivial hypergeometric function again [6] gives
(B(1 - \alpha, \beta) - B_{1/w}(1 - \alpha, \beta)) = (-1)^{-\alpha}w^{-\alpha+\beta} \frac{1}{(\beta - \alpha)} \left(1 - \frac{1}{w}\right)^{1-\alpha} 2F_1(1 - \beta, 1; \alpha + 1 - \beta; 1/w) + (-1)^{-\beta}B(1 - \alpha, \beta) \frac{\Gamma[1 - (\alpha - \beta)]\Gamma(\alpha - \beta)}{\Gamma(1 - \alpha)\Gamma(\alpha)}, \quad (6)

Letting z = 1/w this simplifies [7] to

B_{1/z}(1 - \alpha, \beta) = z^{\alpha - \beta} \frac{1}{(\beta - \alpha)} (z - 1)^{1-\alpha} 2F_1(1 - \beta, 1; \alpha + 1 - \beta; z) + B(1 - \alpha, \beta) \left(1 + (-1)^{1-\beta} \frac{\sin[\pi\alpha]}{\sin[\pi(\alpha - \beta)]}\right), \quad (7)

Finally one substitutes \beta = \nu + 1 and \alpha = \gamma + \beta - 1 and rearranges sides to obtain Eq. (2).

In addition, if one substitutes \beta = 1 - \nu, \alpha = 2 - \mu, and \nu = \frac{\delta}{2} and analytically continues the Gauss function, [8] one may obtain a more useful form for the known [9] Stieltjes transform [10] of powers of a monomial divided by powers of a binomial,

**Corollary**

\[
\int_0^{\infty} \frac{x^{\nu-1}(\beta + x)^{1-\mu}}{\gamma + x} dx = 2 \int_0^{\infty} \frac{x^{\nu-1/2}(\beta + x^2)^{1-\mu}}{\gamma + x^2} dx = \pi \gamma^{-1}(\beta - \gamma)^{1-\mu} \csc(\nu\pi) I_{1-\frac{\nu}{2}}(\mu - 1, 1 - \nu) = \pi \gamma^{-1}(\beta - \gamma)^{1-\mu} \csc(\nu\pi) \left(1 + (-1)^{\nu} \frac{\sin[\pi(2 - \mu)]}{\sin[\pi(1 + \nu - \mu)]}\right) - \frac{\pi \csc(\nu\pi) \beta^{\nu+1-\mu}}{(\mu - 1 - \nu)(\beta - \gamma)B(\mu - 1, 1 - \nu)} \cdot 2F_1(2 - \mu, 1; 2 - \mu + \nu; \frac{\beta}{\beta - \gamma}), \quad (8)
\]

(|arg\gamma| < \pi, |arg\beta| < \pi, 0 < Re \nu < Re \mu) which is a finite series for integer \mu > 1.

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